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A CLASS OF LINEAR KINETIC EQUATIONS IN A KREIN SPACE SETTING

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Krein space methods are used to derive the unique solvability of a class of abstract kinetic equations on a half-space with accretive collision operators. At the same time a new proof is provided for the case of a positive self-adjoint collision operator. A Fokker-Planck type example is worked out as a new application.

1. Introduction

A multitude of linear kinetic equations in a half-space describing such diverse physical processes as neutron transport, radiative transfer, rarified gas dynamics, electron scattering, etc., can be incorporated in a single abstract transport equation,

$$\frac{d}{dx} T \psi(x) = -A\psi(x), \quad 0 < x < \infty. \tag{1.1}$$

Here x is the spatial variable. The phase space density of the particles is described by $\psi(x)$; more precisely, for each x, $\psi(x)$ is an element in a Hilbert space H, where H is typically a space of functions of velocity variables. The operators T and A on H are linear. The left hand side of equation (1.1) describes the free streaming and the operator A describes the collisions. In most physical situations the operator T is self-adjoint and injective, so the maximal positive/negative spec-

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tral projectors $Q_{\pm}$ for T are well defined. The equation (1.1) is usually supplemented by boundary conditions in the following form. At x=0 one assumes that the incoming flux φ_+ is given, i.e.,

$$Q_+\psi(0) = \varphi_+, \tag{1.2a}$$

and a certain behavior at infinity, appropriate to the problem at hand, is prescribed, e.g.,

$$\|\psi(x)\| = o(x^n),\ x \to \infty. \tag{1.2b}$$

The operator T is bounded for many models in neutron transport, radiative transfer and electron scattering and is in general unbounded for models in gas dynamics. The operator A is bounded, in fact the "identity plus a compact operator", for many models in neutron transport and radiative transfer, and for BGK models in gas dynamics. For models of electron scattering the operator A is unbounded, more precisely it is a Sturm-Liouville operator describing diffusion in velocity space. For various linearizations of the Boltzmann equation A may or may not be bounded. For a model of strongly anisotropic neutron transport A has the form of a "Sturm-Liouville plus a compact operator". The compact operator contributing to A is in general nonsymmetric.

Under quite general circumstances, one may show that the operator $T^{-1}A$ generates a bisemigroup with separating projectors $P_{\pm}$ (see next section, for details also [9], [8], [11] and [2]). If this is the case, then every solution of the boundary value problem has the form $exp(-xT^{-1}A)h$ for some vector $h \in P_+H$ such that $Q_+h=\varphi_+$. Hence the boundary value problem is uniquely solvable if and only if Q_+ maps P_+H bijectively onto Q_+H. Moreover, the invertibility of the operator $V=Q_+P_++Q_-P_-$ is equivalent to the unique solvability of the above boundary value problem and its counterpart for $x \in (-\infty,0)$.

In their pioneering work Hangelbroek [14] and Lekkerkerker [19] viewed the isotropic one-speed neutron transport problem as a boundary value problem of the type described above and introduced the operators $Q_{\pm}$, $P_{\pm}$ and V. A complete investigation of the case when A is a positive self-adjoint operator of the form "identity plus a compact operator" was carried out by van der Mee [20] for bounded T and by Greenberg et al. for unbounded T [12]. A different approach was proposed by Beals [3], who sought weak solutions of the abstract kinetic boundary value problem. This work was extended to unbounded positive self-adjoint collision operators in [13] and [4]. In this approach one has to work with several different Hilbert spaces H_A, H_T, H_S, which are obtained by completion of a suitable dense subspace of H in the topologies given by the scalar products $(A\cdot,\cdot)$, $(|T|\cdot,\cdot)$, $(A|S|\cdot,\cdot)$, respectively. When A is a Sturm-Liouville operator and T is a multiplication operator, one has an indefinite Sturm-Liouville problem; the corresponding boundary value problems have been investigated in [3], [16] and [15].

The first to use spaces with indefinite metrics in transport theory were Ball and Greenberg [1]. This work was extended in [10]. In these papers A is assumed to have a finite dimensional negative part, hence the scalar product $(A\cdot,\cdot)$ induces a Pontryagin space structure. If A has a nontrivial null space, one first has to separate off the zero root linear manifold $Z_0(T^{-1}A)$. Then the question of unique solvability reduces to the analysis of the structure of this finite dimensional subspace with respect to the indefinite scalar product $(T\cdot,\cdot)$. This was carried out in [20], [3], [4] and [13] for positive self-adjoint A and in [9] and [8] for accretive nonsymmetric A. In this paper we extend the analysis of [9] to include weak solutions and unbounded collision operators. We extensively use the Krein space structure of the whole space H_T with respect to $(T\cdot,\cdot)$. A study of the geometry of Krein spaces, Proposition 2.2, is crucial in proving the unique solvability and replaces the usual Fredholm argument concerning V. In this way we not only can treat nonsymmetric unbounded operators A but also provide a new and more transparent way of looking at the case of self-adjoint collision operators A. In order not to overburden the analysis with technicalities we consider only the case of bounded T.

In the next section we have collected a few definitions and facts about Krein spaces [5], positive operators in Krein spaces ([5], [17] and [18]), and perturbation of bisemigroups ([9], [8]). In Section 3 we extend the operator $T^{-1}A$ from H to H_T and present a $T^{-1}A$-invariant decomposition of H_T with $Z_0(T^{-1}A)$ being one of the summands. In Section 4 we treat the case of strictly positive A (see [3], [4] and [13]) in a Krein space setting, using the Spectral Theorem for definitizable operators and Proposition 2.2. Using a perturbation theorem for bisemigroups, we extend in Section 5 the analysis to accretive operators A of the form "Sturm-Liouville plus a compact operator". The case of a collision operator with nontrivial null space is treated in Section 6. Finally, in Section 7 we give an example of a model for neutron transport with strong anisotropy (cf. [21], [6]).

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2. Krein Spaces, Positive Operators and Bisemigroups

In this section, as a prerequisite to the sequel, we summarize some properties of Krein spaces and analytic bisemigroups.

Let H_T be an arbitrary Krein space with indefinite scalar product $(\cdot,\cdot)_T$, fundamental projectors $Q_\pm$, and fundamental decomposition $H_T=Q_+H_T\oplus Q_-H_T$. This means that $(\cdot,\cdot)_T$ is a nondegenerate sesquilinear form on H_T, Q_+H_T is a Hilbert space with scalar product $(\cdot,\cdot)_T$ and Q_-H_T is a Hilbert space with scalar product $-(\cdot,\cdot)_T$. The operator $Q= Q_+-Q_-$ is an involution, i.e. $Q^2=\mathbb{I}$, called a fundamental symmetry. The topology of H_T is defined by the norm induced by the positive definite scalar product $(Q\cdot,\cdot)_T$. With respect to this scalar product Q is unitary. If K is an operator on H_T, its adjoint with respect to $(\cdot,\cdot)_T$, the so-called the H_T-adjoint, will be denoted as $K^\#$. If K^* is the adjoint of K with respect to $(Q\cdot,\cdot)_T$, then $K^\#=QK^*Q$. A vector f in H_T is called positive, negative or neutral if $(f,f)_T>0$, <0 or $=0$, respectively. A subspace M in H_T is called positive if it does not contain negative vectors, M is called positive definite if except for the zero vector it contains only positive vectors and M is maximal positive if it is positive and is not the proper subspace of a positive subspace. One has the analogous definitions for negative, negative definite and maximal negative subspaces. The orthogonal companion of M is $M^{\angle}=\{f\in H_T:\ (f,g)_T=0 \text{ for all } g\in M\}$. The isotropic part of a subspace M is $M^0=M\cap M^{\angle}$. If M is a positive subspace then M^0 consists precisely of the neutral vectors in M. A decomposition $H_T=M_+\oplus M_-$ with $M_\pm$ closed positive/negative definite subspaces is called a fundamental decomposition of H_T. The corresponding projectors $P_\pm$: $H\to M_\pm$ are called fundamental projectors.

The following simple geometric fact about Krein spaces will be crucial in the analysis of the unique solvability in the next sections.

PROPOSITION 2.1. (*see* [5], *Theorem* 4.1) *A positive subspace* M_+ *of* H_T *is maximal positive if and only if* $Q_+M_+=Q_+H_T$, *or equivalently,* Q_+ *maps* M_+ *bijectively onto* Q_+H_T.

In Section 6 we will need the following simple fact.

PROPOSITION 2.2. *Suppose* $M_\pm$ *are closed positive/negative subspaces of* H_T, $M_+ \oplus M_- = H_T$ *and* M_- *is negative definite. Then* M_+ *is a maximal positive subspace.*

Next we will state the spectral theorem for positive operators in a Krein space (see [18] and [5]). An H_T-self-adjoint operator K is called positive if its resolvent set is nonempty and $(Kf,f)_T \geq 0$ for all $f \in D(K)$. Let $\mathcal{R}$ be the semiring which consists of all bounded intervals and their complements in $\mathbb{R}$ with endpoints different from zero.

SPECTRAL THEOREM (*see* [18], *Theorem* 3.1). *If* K *is an* H_T-*positive operator, then there exists a map* F *from* $\mathcal{R}$ *into the set of bounded,* H_T-*self-adjoint operators in* H_T *such that* $F(E)F(\hat{E}) = F(E \cap \hat{E})$; $F(E \cup \hat{E}) = F(E) + F(\hat{E})$ *for disjoint* E *and* $\hat{E}$ *in* $\mathcal{R}$; $F(E)H_T$ *is a positive/negative subspace if* $E \subseteq \mathbb{R}_\pm$; F(E) *is in the double commutant of the resolvent of* K; *if* E *is a bounded interval, then* $F(E)H_T \subseteq D(K)$ *and* $K|_{F(E)H_T}$ *is a bounded operator*; $\sigma(K|_{F(E)H_T}) \subseteq \bar{E}$.

A point t is a *critical point* for the operator K if $F(E)H_T$ is an indefinite subspace for every $E \in \mathcal{R}$ with $t \in E$. The only possible critical points for a positive operator K are zero and infinity. If K has a bounded inverse it may have a critical point only at infinity. If the limits *lim* $F((t_0,t))$ as $t \to +\infty$ and *lim* $F((t,t_0))$ as $t \to -\infty$ exist, we call infinity a regular critical point. In this case $F((0,\infty))H_T$ and $F((-\infty,0))H_T$ are maximal positive/negative definite subspaces forming a fundamental decomposition of the Krein space. In the same way one defines regularity of the critical point at zero.

A strongly continuous *bisemigroup* E(t) on a Hilbert space H is a function E from $\mathbb{R}\backslash\{0\}$ into L(H), the bounded operators on H, with the following properties:

(i) $E(t)E(s) = \pm E(t+s)$ if $sgn(t) = sgn(s) = \pm 1$ and $E(t)E(s) = 0$ if $sgn(t) = -sgn(s)$

(ii) $E(\cdot)$ is strongly continuous and has strong limits as $(\pm t) \downarrow 0$.

It is easy to check that

$$\Pi_\pm = \operatorname*{s\text{-}lim}_{(\pm t)\downarrow 0} (\pm E(t))$$

are bounded projectors, called separating projectors, and that $\Pi_+\Pi_- = 0 = \Pi_-\Pi_+$. In the definition of a bisemigroup we require also

(iii) $\Pi_+ + \Pi_- = \mathbb{I}$,

where $\mathbb{I}$ denotes the identity operator. This is equivalent to saying that $\pm E(t)\Pi_\pm$, $\pm t \geq 0$, are strongly continuous right/ left semigroups on Ran $\Pi_\pm$. An operator S is the generator of E(t) if $\Pi_\pm$ leaves D(S) invariant and $S\Pi_\pm h = \Pi_\pm Sh$, $\forall h \in D(S)$, and if $E(t) = \pm exp(-tS)\Pi_\pm$, $\pm t > 0$. We will write E(t;S) for the bisemigroup generated by S. The bisemigroup will be called bounded holomorphic, strongly decaying holomorphic, or exponentially decaying holomorphic if both of the semigroups $\pm E(t)\Pi_\pm$, $\pm t > 0$, have the respective properties.

For an angle $0 < \theta \leq \frac{1}{2}\pi$ we denote sectors about the real axis by $\Sigma_{\theta\pm}$ with $\Sigma_{\theta\pm} = \{z \in \mathbb{C}: |\arg(\pm z)| < \theta\}$ and $\Sigma_\theta = \Sigma_{\theta+} \cup \Sigma_{\theta-}$. Assume that S is an injective normal operator on a Hilbert space H with spectral measure $dF(\lambda)$, i.e.

$$S = \int_{\sigma(S)} \lambda \, dF(\lambda).$$

Assume also that $\sigma(S) \subseteq \overline{\Sigma}_{\frac{\pi}{2}-\theta_1}$ for some $0 < \theta_1 < \frac{1}{2}\pi$ and that zero is either in the resolvent set or in the continuous spectrum of S. It is immediate to check that S is the generator of a strongly decaying, holomorphic bisemigroup of angle at least θ_1, with as separating projectors $\Pi_\pm = F(\sigma(S) \cap \{\pm \mathrm{Re}\, z \geq 0\})$. If S^{-1} is a bounded operator, the bisemigroup is exponentially decaying. Besides the assumption on S made above, suppose also that the following conditions hold:

(i) $B = \mathbb{I} - A$ is compact.

(ii) B is Hölder continuous with respect to S at zero and infinity: there exist numbers $\alpha, \gamma > 0$ and bounded operators D_1, D_2 such that $B = |S|^{-\alpha} D_1$ and $B = |S|^{\gamma} D_2$, where $|S| = S(\Pi_+ - \Pi_-)$, or

(ii′) B is trace class.

(iii) The spectrum of $S^\infty = SA$ is contained in a sector around the real axis: $\sigma(S^\infty) \subseteq \overline{\Sigma}_{\frac{\pi}{2}-\theta_2}$ for some $0 < \theta_2 < \frac{1}{2}\pi$.

(iv) Ker A = {0}.

Here we assume either (ii) or (ii′).

THEOREM 2.3 (see [9]). *With the above assumptions on* S *and* B, S^∞ *generates a holomorphic bisemigroup* $E^\infty(t)$ *with separating projectors* $\Pi^\infty_\pm$. *For any* $t \in \mathbb{R}\setminus\{0\}$ *the difference operators* $E(t) - E^\infty(t)$ *and* $\Pi_\pm - \Pi^\infty_\pm$ *are compact and the bisemigroup*

$E^{\varpi}(t)$ *is strongly decaying. If* $\sigma(S)$ *has a gap at zero (i.e.,* S^{-1} *is bounded), then* $E^{\varpi}(t)$ *is exponentially decaying.*

3. Extensions and Decompositions

Let H be a Hilbert space and assume that T is a *bounded*, self-adjoint and injective operator on H, hence T^{-1} is densely defined with domain TH. Let $Q_{\pm}$ be the maximal positive/negative spectral projectors for T, i.e. $Q_{\pm}^2=Q_{\pm}$, $TQ_{\pm}=Q_{\pm}T$ and $\sigma(TQ_{\pm})\subseteq\{\lambda\in\mathbb{R}:\pm\lambda\geq0\}$. Set $Q=Q_+-Q_-$, obviously a unitary involution, i.e. $Q^*=Q=Q^{-1}$. By definition, the absolute value of T is $|T|=TQ$, a positive operator. If $(\cdot,\cdot)$ is the scalar product in H and $k\geq0$, let the Hilbert space H_k be the space $|T|^{k/2}H$ with the scalar product $(\cdot,\cdot)_k=(|T|^{-k}\cdot,\cdot)$. For $-k\geq0$ let $(\cdot,\cdot)_{-k}=(|T|^k\cdot,\cdot)$ and denote by H_{-k} the completion of H with respect to the norm $\|\cdot\|_{-k}$. In particular, we have the chain of Hilbert spaces

$$H_{-2} \xrightarrow{\equiv} H_{-1} \xrightarrow{\equiv} H \xrightarrow{\equiv} H_1 \xrightarrow{\equiv} H_2 \tag{3.1}$$

where the arrows represent the unitary isomorphisms given by $|T|^{1/2}$. Because $\|\cdot\|_k$ majorizes $\|\cdot\|_h$ if $k\geq h$, we also have the chain of continuous imbeddings

$$H_{-2} \xrightarrow{\supseteq} H_{-1} \xrightarrow{\supseteq} H \xrightarrow{\supseteq} H_1 \xrightarrow{\supseteq} H_2. \tag{3.2}$$

By construction and by the injectivity of T the imbeddings are dense. We remark that H_2 is the domain of T^{-1} in H and H_{-1} is denoted by H_T elsewhere in the literature (cf. [3], [13]).

Let K, $\hat{K}$ be operators in H. Assume that

$$T\,D(K) \subseteq D(\hat{K}) \subseteq H_2 \subseteq H_1 \tag{3.3}$$

and

$$TKf = \hat{K}Tf \quad \text{for every } f \in D(K). \tag{3.4}$$

Assume that $\hat{K}$ is a closed operator in H. Then $\hat{K}$ is a closed operator in H_1 if it has the same domain as in H. Indeed, let $f_n\in D(\hat{K})$, $\|f_n-f\|_1\to0$, and $\|\hat{K}f_n-g\|_1\to0$ for some $g \in H_1$. Because $\|\cdot\|_1$ majorizes $\|\cdot\|$ we have the same convergences in H. But we know that $\hat{K}$ is closed in H, hence $f \in D(\hat{K})$ and $\hat{K}f = g$. We now easily see that $g \in H_1$, hence $\hat{K}$ is closed in H_1. Define an extension in H_{-1} of the operator K in H by

$$T^{-1} \circ \hat{K} \circ T \tag{3.5}$$

where $\hat{K}$ is viewed as an operator in H_1 and T and T^{-1} are viewed as isometries between H_1 and H_{-1}. Denoting this extension again by K will cause no confusion. Thus K in H_{-1} is a closed operator. We also assume that $\operatorname{Ker} \hat{K} = T(\operatorname{Ker} K)$ as kernels of operators on a dense domain in H so that the kernels of K in H and H_{-1} will be the same. If K is densely defined in H, then its extension to H_{-1} is densely defined. Indeed, if M is a set in H and $clos_{-k}M$ is its closure in H_{-k}, we have $M \subseteq clos_0 M \subseteq clos_{-1} M$. Taking $clos_{-1}$ once more we get $clos_{-1}M = clos_{-1}\, clos_0 M$, so if $clos_0 M = H$ we also get $clos_{-1}M = H_{-1}$.

If K and $\hat{K}$ are bounded operators on H and $TK = \hat{K}T$ then K extends to a bounded operator on H_{-1}. Indeed, it is immediate that K has a bounded extension in H_{-2}. Using the usual interpolation between H_{-2} and H based on the norm estimate $\|h\|_{-1} \le \|h\|_0^{1/2} \|h\|_{-2}^{1/2}$, $\forall\ h \in H$, we get that K is bounded on H_{-1}.

Now consider an operator A in H which is Fredholm, accretive (i.e. $2\operatorname{Re} A = A + A^* \ge 0$) and satisfies $\operatorname{Ker} A = \operatorname{Ker}(\operatorname{Re} A)$. The proof of the following lemmas is easy and is contained in [9] and [8]. For convenience we sketch some of the proofs.

LEMMA 3.1. *We have*

(a) $f \in H$ *and* $(Af,f) = 0$ *imply* $f \in \operatorname{Ker} A$,

(b) $\operatorname{Ker} A = \operatorname{Ker} A^*$, $\operatorname{Ran} A = \operatorname{Ran} A^*$,

(c) $H = \operatorname{Ker} A \oplus \operatorname{Ran} A$.

Set $K = T^{-1}A$ and $\hat{K} = AT^{-1}$ (note the difference in notation from [9]).

LEMMA 3.2. *The operators* K *and* $\hat{K}$ *are densely defined and closed in* H.

Proof: We will only show that $\hat{K}$ is closed. The density of D(K) was proved in a straightforward way in [8], while the rest is rather obvious.

Let $f_n \in D(AT^{-1}) = T(D(A))$, so $f_n = Th_n$ for some $h_n \in D(A)$. Assume that $f_n \to f$ and $\hat{K}f_n = Ah_n \to f'$. Because Ran A is closed we can write $f' = Ah$ for some $h \in D(A)$. Thus $A(h_n - h) \to 0$. By the Fredholmness of A (more precisely by Lemma 3.1(c)) one can view A as an operator from $\mathrm{Ran}\, A$ onto $\mathrm{Ran}\, A$ with a bounded inverse, thus uniquely specifying h_n and h, whence $h_n \to h$. But T is bounded, so $Th_n \to Th$. Therefore $f = Th \in D(\hat{K})$ and $f' = Ah = \hat{K}Th = \hat{K}f$. □

The boundedness of K and $\hat{K}$ on H enables us to prove that the zero root manifold $Z_0(K)$, which is the union of the kernels of K^n, is the same on H and on H_{-1}. Indeed, $\mathrm{Ker}\, \hat{K} = T(\mathrm{Ker}\, K)$ on H_1 implies that $\mathrm{Ker}\, K$ is the same on H_1 and on H_{-1}. If we assume $K^n f = 0$ for some $f \in H_{-1}$ and $n \in \mathbb{N}$, we find $K^{n-1} f \in \mathrm{Ker}\, K \subseteq H_1 \subseteq H$. Since K has a bounded inverse on H, we find $f \in D(K^n) \subseteq H$, which settles our statement.

LEMMA 3.3. *The only possible eigenvalues of* $T^{-1}A$ *on the imaginary axis are at the origin.*

<u>Proof:</u> If λ is imaginary, $h \in D(T^{-1}A)$ and $T^{-1}Ah = \lambda h$, we have

$$2((\mathrm{Re}\, A)h,h) = (\lambda Th,h) + (h,\lambda Th) = \lambda(Th,h) + \bar{\lambda}(h,Th) = 0$$

implying $\lambda Th = Ah = (\mathrm{Re}\, A)h = 0$ and hence $\lambda h = 0$, which proves the lemma. □

LEMMA 3.4. *The Jordan chains of* $T^{-1}A$, $T^{-1}A^*$, AT^{-1} *and* A^*T^{-1} *at* $\lambda = 0$ *have length at most two.*

<u>Proof:</u> If $T^{-1}Ah = k$, $T^{-1}Ak = g$ and $T^{-1}Ag = 0$, we find

$$2((\mathrm{Re}\, A)k,k) = (Tg,k) + (k,Tg) = (g,Ah) + (Ah,g) = (A^*g,h) + (h,A^*g) = 0,$$

by virtue of Lemma 3.1(b). Hence $Tg = Ak = (\mathrm{Re}\, A)k = 0$ and therefore $g = 0$. □

LEMMA 3.5. *We have*
(a) $K\, Z_0(K) = (T^{-1}A^*)\, Z_0(T^{-1}A^*)$,
(b) $\mathit{dim}\, Z_0(K) = \mathit{dim}\, Z_0(T^{-1}A^*)$.

Using the remarks at the beginning of this section we extend the operator $K = T^{-1}A$ to H_{-1}. So we may consider K as a closed, densely defined operator in H_{-1}. Its zero root manifold $Z_0(K)$ will be identified with the one in H.

The space H_{-1} becomes a Krein space, denoted H_T, if we introduce the indefinite scalar product $(\cdot,\cdot)_T=(Q\cdot,\cdot)_{-1}=(T\cdot,\cdot)$. Let $K^\#$ be the H_T-adjoint of K. The operator $K^\#$ is an extension of $T^{-1}A^*$ and $Z_0(K^\#)=Z_0(T^{-1}A^*)$. Let $Z_1(K)$ be the H_T-orthogonal companion of $Z_0(K^\#)$, i.e. $Z_1(K)=(Z_0(K^\#))^{\angle}$. It is immediate that $Z_1(K)$ is K-invariant.

LEMMA 3.6. $Z_0(K) \cap Z_1(K) = \{0\}$.

Proof: Let M be a subspace such that $M\subseteq H\subseteq H_T$; then $TM^{\angle}\subseteq M^{\perp}$. Suppose $h\in (Z_0(K^\#))^{\angle}\subseteq(\mathrm{Ker}\,A^*)^{\angle}= (\mathrm{Ker}\,A)^{\angle}$. Then $Th\in(\mathrm{Ker}\,A)^{\perp}=\mathrm{Ran}\,A^*$. Hence $h=K^\#g$ for some g. Because both $Z_0(K)$ and $Z_1(K)$ are K-invariant and $KZ_0(K) \subseteq \mathrm{Ker}\,A = \mathrm{Ker}\,A^*$, we need only show $\mathrm{Ker}\,A^*\cap Z_1(K)=\{0\}$. So assume also $h\in\mathrm{Ker}\,(A^*)\subseteq Z_0(K^\#)$. This implies that $g\in Z_0(K^\#)$. Thus we have $0=(Th,g)=(A^*g,g)$. By Lemma 3.1 this implies $A^*g=0$, so $h= K^\#g=0$. □

THEOREM 3.7. *There is a K-invariant decomposition of* H_T: $H_T = Z_0(K) \oplus Z_1(K)$.

Proof: Using $\mathit{codim}\, M^{\angle}\leq \mathit{dim}\, M$ we get $\mathit{codim}\, Z_1(K)\leq \mathit{dim}\, Z_0(K^\#)=\mathit{dim}\, Z_0(K)$, the equality coming from Lemma 3.5. But above we obtained $Z_0(K)\cap Z_1(K)=\{0\}$, hence the decomposition holds. □

4. Strictly Positive Collision Operators

In this section we assume that A is positive self-adjoint and Fredholm and $\mathrm{Ker}\,A=\{0\}$. Even though A will in general be unbounded, the assumptions we made force A^{-1} to be a bounded operator on H. By the considerations of the previous section $A^{-1}T$ has a bounded extension to H_T and $K=T^{-1}A$ has a closed, densely defined extension to H_T and a bounded inverse in H_T. The operator $A^{-1}T$ is H_T-self-adjoint and, in fact, H_T-positive. Indeed, $(A^{-1}Tf,f)_T=(A^{-1}(Tf),(Tf))\geq 0$ by the assumption that A is positive self-adjoint on H. Thus we may view K as an H_T-self-adjoint, H_T-positive operator with an H_T-bounded inverse. Hence K has at most one critical point, namely at infinity. Since in most physical models T has both positive and negative spectrum around zero, we must treat the case that infinity is a critical point of K.

In order to proceed further and use the functional calculus for definitizable operators in a Krein space we assume that infinity is a regular critical point of K. The regularity of the critical point infinity for a positive, boundedly invertible operator in a Krein space has been investigated in detail in [7]. In particular, infinity is a regular critical point for the operator K if and only if the norms $\|\cdot\|_T$ and $\|\cdot\|_S$ are equivalent (see [3], [4], [7], [13] or [11]). Here $(\cdot,\cdot)_S=(T(P_+-P_-)\cdot,\cdot)$ where $P_\pm$ are the positive\negative spectral projections of S. For example, when T is multiplication by a piecewise continuous function satisfying a Hölder type condition at the sign changes and A is a Sturm-Liouville operator, it is shown in [4] that the two norms are equivalent and thus infinity is a regular critical point. The two norms are also equivalent if A is a bounded operator (cf. [3], [11]).

Assuming K has no singular critical points the Spectral Theorem for definitizable operators [18] provides us with separating projectors $P_\pm$, i.e. $P_\pm D(K)\subseteq D(K)$, $P_\pm Kh=KP_\pm h$ for $h\in D(K)$, $P_++P_-=\mathbb{I}$ and $\sigma(KP_\pm)\subseteq\{\lambda\in\mathbb{R}:\pm\lambda\geq0\}$. From the functional calculus of the operator K we obtain exponentially decaying holomorphic semigroups $exp(-xK)P_\pm$, $\pm x\geq0$. Moreover, $P=P_+-P_-$ is a fundamental symmetry of the Krein space H_T and $P_\pm H_T$ are maximal positive/negative definite subspaces of H_T.

Now consider the boundary value problem

$$\frac{d}{dx}T\psi(x) = -A\psi(x),\ 0<x<\infty, \tag{4.1a}$$

$$Q_+\psi(0) = \varphi_+, \tag{4.1b}$$

$$\|\psi(x)\|_{-1} \to 0,\ x\to\infty, \tag{4.1c}$$

where $\varphi_+\in Q_+H_T$ is the given incoming flux. All solutions of the boundary value problem (4.1) are of the form

$$\psi(x) = exp(-xK)h \tag{4.2}$$

for some $h\in P_+H_T$ such that $Q_+h=\varphi_+$ (see [11] and [12]). Thus the unique solvability is reduced to the question whether Q_+ maps P_+H_T bijectively onto Q_+H_T. As noted, P_+H_T is a maximal positive subspace in H_T, so the bijectivity of Q_+ as a map from P_+H_T onto Q_+H_T follows from Proposition 2.1.

5. Invertible, Accretive Collision Operators

Assume that the collision operator A can be written as a difference $A=A_1-A_2$ where

(i) A_1 is a positive self-adjoint operator in H with a bounded inverse,

(ii) the extension K_1 of the operator $T^{-1}A_1$ to the Krein space H_T has a regular critical point at infinity,

(iii) A is an accretive operator in H, i.e. $2\mathrm{Re}\, A=A+A^*\geq 0$,

(iv) $\mathrm{Ker}\, A = \mathrm{Ker}\,(\mathrm{Re}\, A) = \{0\}$,

(v) $A_1^{-1}A_2$ is a trace class operator in H_{-1}.

Following the discussion of the previous section we obtain that K_1 is similar to a self-adjoint operator. Let $P_{1\pm}$ be the separating projectors for K_1 in H_T, i.e. the maximal positive/negative spectral projectors for K_1. We also have exponentially decaying holomorphic semigroups $\exp(-xK_1)P_{1\pm}$, $\pm x\geq 0$. Let K be the extension of $T^{-1}A$ to H_T. We want to define maximal positive/negative spectral projectors $P_\pm$ for K in H_T and exponentially decaying semigroups $\exp(-xK)P_\pm$. To accomplish this one applies the perturbation theorem for bisemigroups directly. Write $T^{-1}A=T^{-1}A_1(\mathbb{I}-A_1^{-1}A_2)$. Condition (iv) assures that $T^{-1}A$ has no eigenvalues on the imaginary axis. This with assumption (v) gives us that K is the generator of an exponentially decaying holomorphic bisemigroup if K_1 is.

Consider again the boundary value problem (4.1). Once more all solutions will be of the form $\psi(x)=\exp(-xK)h$ for some $h\in P_+H_T$ such that $Q_+h=\varphi_+$. So the question of unique solvability is equivalent to the bijectivity of Q_+ as a map from P_+H_T onto Q_+H_T. To answer this question in the affirmative using Proposition 2.1, we need to show that P_+H_T is a maximal positive subspace. Here the accretivity assumption (iii) becomes crucial.

PROPOSITION 5.1. *The subspaces* $P_\pm H_T$ *are positive/negative definite.*

Proof: By virtue of assumptions (i) and (iv), we have that A^{-1} is a bounded operator on H. By the same reasoning as before we get that $K+K^\#$ is a strictly H_T-positive operator. Now take any $g\in P_+H_T$ and set $f(x)=\exp(-xK)g$, $x>0$. Because KP_+ generates a holomorphic semigroup we have that $f(x)\in D(K)$ for $x>0$. So we have

$$0 > -((K+K^\#)f(x),f(x))_T = \frac{d}{dx}(f(x),f(x))_T, \quad x > 0.$$

Integrating both sides (a trick adapted from [23]) we obtain

$$0 > -\lim_{\tau\to\infty}\int_0^\tau ((K+K^{\#})f(x),f(x))_T\,dx = \lim_{\tau\to\infty}(f(\tau),f(\tau))_T - (g,g)_T = -(g,g)_T.$$

Thus $(g,g)_T>0$ and P_+H_T is positive definite. □

From the fact that $P_++P_-=\mathbb{I}$, P_+H_T is a positive subspace, P_-H_T is a negative definite subspace and Proposition 2.2 holds true, we conclude that P_+H_T is a maximal positive subspace. So applying Proposition 2.1 once more, we obtain unique solvability.

6. Accretive Collision Operators with a Nontrivial Kernel

For the operator A assume that (i)-(iii) and (v) hold and change (iv) to read:

(iv) $\operatorname{Ker} A = \operatorname{Ker}(\operatorname{Re} A)$.

In Section 3 we obtained the decomposition $H_T=Z_0(K)\oplus Z_1(K)$. Let P_1 be the projector of H_T onto $Z_1(K)$ along $Z_0(K)$ and let $P_0=\mathbb{I}-P_1$ be the complementary projector. We now define the spectral projectors $P_\pm$ for the restriction of K to $Z_1(K)$ and to obtain the corresponding semigroups. Let us write $A'=TP_0+AP_1=A+(T-A)P_0=A_1+(A_2+(T-A)P_0)$. Here, P_0 and P_1 are the restrictions of the corresponding projectors to H. Note that $K' = T^{-1}A' = P_0 + KP_1$. Obviously we have

$$K'P_0 = P_0K' = P_0,$$

$$K'P_1 = P_1K' = P_1K = KP_1.$$

Under the assumptions on A and T we observe that A' has a bounded inverse and K' has no eigenvalues on the imaginary axis. Moreover, since $A_2'=A_2+(T-A)P_0$ is a finite rank perturbation of A_2, the trace class condition (v) will be satisfied. By the same arguments as in Section 5 we conclude that K' is the generator of an exponentially decaying holomorphic bisemigroup with separating projectors $P'_\pm$. Now the operators $P_\pm=P'_\pm P_1$ are projectors, commuting with K and such that $P_++P_-=$

$Z_0(K^{\#})^0$. Then $(Tf,g) = 0\ \forall g \in \text{Ker } A^* = \text{Ker } A$, so that $Tf \in \text{Ran } A^*$. Now write $Tf=A^*h$; then $h\in Z_0(K^{\#})$ and $(Tf,h) = 0$. We now easily see that $(h,Ah) = (A^*h,h) = (Tf,h) = 0$, so that $((\text{Re } A)h,h) = 0$ and thus $(\text{Re } A)h = 0$. By Condition (iv) we get successively $Ah=0$, $A^*h=0$, $Tf=0$ and $f=0$, which proves $(Z_0(K^{\#}))^0 = \{0\}$. On the other hand, from the definitions of the isotropic part of a subspace and of the subspace $Z_1(K)$ we have $\{0\}=(Z_1(K))^0=Z_1(K)\cap (Z_1(K))^{\angle}=Z_1(K)\cap Z_0(K^{\#})$. Hence *codim* $Z_1(K)=$*codim* $(Z_0(K^{\#})^{\angle}\leq$*dim* $Z_0(K^{\#})$, which completes the proof. □

Because of the H_T-orthogonality of $Z_0(K^{\#})$ and $Z_1(K)$, we may decompose $Z_0(K^{\#})=M_+\oplus M_-$ into a positive and a negative subspace to obtain that $H_T=(M_+\oplus P_+H_T)\oplus(M_-\oplus P_-H_T)$ is a decomposition of H_T into a positive and a negative subspace.

PROPOSITION 6.3. *There exists a decomposition* $Z_0(K^{\#})=M_+\oplus M_-$ *with* M_+ *a positive subspace contained in* Ker A *and* M_- *a negative definite subspace.*

Proof: First we show $(\text{Ker } A)^0=K^{\#}Z_0(K^{\#})$. Suppose $f\in(\text{Ker } A)^0=(\text{Ker } A)\cap(\text{Ker } A)^{\angle}$. Then $Tf\in T(\text{Ker } A)^{\angle}\subseteq(\text{Ker } A)^{\perp}=\text{Ran } A^*$, so $f=K^{\#}g$ for some g, and hence $f\in K^{\#}Z_0(K^{\#})$. Conversely, suppose $f\in K^{\#}Z_0(K^{\#})$. Then $Tf=A^*g$ for some g, and $(f,u)_T=(Tf,u)=(A^*g,u)=(g,Au)=0$ for every $u\in\text{Ker } A$.

Because $(\text{Ker } A)/(\text{Ker } A)^0$ is a Krein space, we can choose an H_T-orthogonal, linearly independent set of nonneutral vectors $\{z_1,\cdots,z_n\}\subseteq\text{Ker } A$ such that $(\text{Ker } A)^0\oplus\ \{z_1,\cdots,z_n\}=\text{Ker } A$. The same argument as the one in the proof of Proposition 6.2 shows that $Z_0(K^{\#})$ is a Krein space, so we can choose an H_T-orthogonal, linearly independent set of nonneutral vectors $\{y_1,\cdots,y_k\}\subseteq Z_0(K^{\#})$ which is H_T-orthogonal to $\{z_1,\cdots,z_n\}$ and such that $\text{Ker } A\oplus$*span*$\{y_1,\cdots,y_k\}=Z_0(K^{\#})$. One may assume that all the y_i's are negative (or positive), because if they are not, one can adjust them to be negative (or positive) without spoiling any of the other properties by the following trick [13]. If ς is a real number and $x_i=K^{\#}y_i$, then

$$((y-\varsigma x),(y-\varsigma x))_T = (y,y)_T + \varsigma^2(x,x)_T -$$

$$- \varsigma\{(x,y)_T - (y,x)_T\} = (y,y)_T - 2\varsigma((\text{Re } A)y,y)$$

can be made negative (or positive) by choosing an appropriate ς, since $((\text{Re } A)y,y)>0$.

Without loss of generality we may assume that $z_1,\cdots,z_m$ span a negative definite subspace and $z_{m+1},\cdots,z_n$ span a positive definite subspace of Ker A. Note that m is the dimension of any maximal negative subspace in the Krein space $(\text{Ker } A)/(\text{Ker } A)^0$ and thus is independent of the particular choice of the z_i's. Finally, set $M_+ = (\text{Ker } A)^0 \oplus \mathit{span}\{z_{m+1},\cdots,z_n\}$ and $M_- = \mathit{span}\{y_1,\cdots,y_\varkappa\} \oplus \mathit{span}\{z_1,\cdots,z_m\}$. Then, by construction, M_+ is a positive subspace contained in Ker A, M_- is negative definite, and $Z_0(K^\#) = M_+ \oplus M_-$. □

Note that the subspace M_+ is a maximal positive subspace in both $Z_0(K^\#)$ and Ker A. Thus its dimension is independent of the choice of the space M_+. Denote this invariant by m_+. Using Proposition 2.1, we conclude that $M_+ \oplus P_+ H_T$ is a maximal positive subspace in H_T, so from Proposition 2.2 we obtain that Q_+ maps $M_+ \oplus P_+ H_T$ bijectively onto $Q_+ H_T$.

Define the measure of nonexistence δ to be the codimension in $Q_+ H_T$ of the space of boundary values φ_+ for which the boundary value problem is solvable, and the measure of nonuniqueness γ to be the dimension of the solution space of the corresponding homogeneous ($\varphi_+=0$) boundary value problem. From the above considerations, we obtain the main theorem of this section.

THEOREM 6.4. *For the boundary value problem* (6.1)-(6.3a), *we have* $\delta = m_+$ *and* $\gamma = 0$. *For the boundary value problem* (6.1)-(6.3b), *we have* $\delta = 0$ *and* $\gamma = \mathit{dim}\,(\text{Ker } A) - m_+$. *For the boundary value problem* (6.1)-(6.3c), *we have* $\delta = 0$ *and* $\gamma = \mathit{dim}\,(Z_0(K)) - m_+$.

7. The Boltzmann-Fokker-Planck Equation

The Boltzmann-Fokker-Planck equation was formulated by Ligou to describe the transport of charged particles in hot plasmas. It was subsequently used in [21], [6] and [22] to describe the transport of very fast neutrons where the predominance of forward scattering renders the usual Legendre series expansion useless. Adopting a multigroup scheme, we obtain for $i=1,\cdots,N$

$$\mu \frac{\partial \psi_i}{\partial x} + (\sigma_i + S_i)\psi_i(x,\mu) = S_{i-1}\psi_{i-1}(x,\mu) + T_i \frac{\partial}{\partial \mu}(1-\mu^2)\frac{\partial \psi_i}{\partial \mu} + \\ + \sum_{\ell=0}^{\infty} \sum_{j=1}^{N} (\ell+\tfrac{1}{2}) \Sigma_{s,\ell}^{j,i} P_\ell(\mu) \int_{-1}^{1} P_\ell(\mu')\psi_j(x,\mu')\,d\mu', \tag{7.1}$$

where internal sources are neglected, $\sum_{s,\ell}^{j,\ell} \to 0$ as $\ell \to \infty$ and all constants involved are nonnegative. The cross-sections $\sigma_1, \cdots, \sigma_N$ and the constants $T_1, \cdots, T_N$ are positive. We study the corresponding half-space problem on the Hilbert space H consisting of the direct sum of N copies of $L_2[-1,1]$ with inner product

$$(\underline{h},\underline{k}) = \sum_{i=1}^{N} (\sigma_i + S_i) \int_{-1}^{1} h_i(\mu') \overline{k_i(\mu')} \, d\mu',$$

where $\underline{h}$ and $\underline{k}$ are the column vectors with entries h_i and k_i, respectively. Introducing the diagonal matrix $\mathbb{D}$ with diagonal entries $\sigma_i + S_i$ and appropriate other matrices and vectors, we easily write Eq. (7.1) in the vector form

$$\mathbb{D}^{-1} \mu \frac{\partial \Psi}{\partial x} + \Psi(x,\mu) = \mathbb{E} \Psi(x,\mu) + \mathbb{T} \frac{\partial}{\partial \mu}\left[(1-\mu^2) \frac{\partial \Psi}{\partial \mu}\right] +$$

$$+ \sum_{\ell=0}^{\infty} (\ell + \tfrac{1}{2}) \mathbb{D}^{-1} \textstyle\sum_{s,\ell} P_\ell(\mu) \displaystyle\int_{-1}^{1} P_\ell(\mu') \Psi(x,\mu') \, d\mu' \tag{7.2}$$

with the usual half-space boundary conditions. Apart from that, we require the solution to be bounded at $\mu = \pm 1$ in order to single out a self-adjoint boundary condition on the Sturm-Liouville operator. On H we now define the operator T as the premultiplication by $\mathbb{D}^{-1}\mu$ so that $Q_\pm$ is the restriction to $\pm[0,1]$. We then define

$$(A_1 \underline{h})(\mu) = \underline{h}(\mu) - \mathbb{T} \frac{\partial}{\partial \mu}\left[(1-\mu^2) \frac{\partial}{\partial \mu} \underline{h}(\mu)\right]$$

on the appropriate domain of functions $\underline{h} = \{h_i\}_{i=1}^N$ that are bounded at $\mu = \pm 1$, with the derivatives interpreted in the distributional sense. Then A_1 is a strictly positive self-adjoint operator with fully discrete spectrum

$$\sigma(A_1) = \{1 - n(n+1)(\sigma_i + S_i)^{-1} T_i : n = 0,1,2,\cdots \text{ and } i = 1,2,\cdots,N\},$$

whence A_1^{-1} is a trace class operator. We also define the bounded operator A_2 by

$$(A_2 \underline{h})(\mu) = \mathbb{E} \underline{h}(\mu) + \mathbb{T} \sum_{\ell=0}^{\infty} (\ell + \tfrac{1}{2}) \mathbb{D}^{-1} \textstyle\sum_{s,\ell} P_\ell(\mu) \displaystyle\int_{-1}^{1} P_\ell(\mu') \underline{h}(\mu') \, d\mu',$$

and A as the difference of A_1 and A_2. It is now straightforward to impose such conditions on the coefficients as to make the operator A accretive with Ker A= Ker (Re A). On doing so one obtains the unique solvability of the half-space problem on the Hilbert space H_T, which is the weighted direct sum of N copies of $L_2([-1,1];|\mu|d\mu)$.

Note added in proof: Recently two of the authors have developed a theory of abstract kinetic equations where T is injective, A is a compact perturbation of the identity and $\mathrm{Ran}(\mathbb{I}-A)\subseteq D(T)$. Under these assumptions the author proved $T^{-1}A$ to generate an analytic bisemigroup. As a result, assumptions (ii) and (ii') can be dropped in the derivation of Theorem 2.3, while condition (v) of Section 4 that the operator $A_1^{-1}A_2$ be trace class may be weakened to $A_1^{-1}A_2$ compact.

References

1. J.A. Ball, W. Greenberg, "A Pontryagin Space Analysis of the Supercritical Transport Equation," Transp. Theory Stat. Phys. 4, 143-154 (1975).

2. H. Bart, I. Gohberg and M.A. Kaashoek, "Wiener-Hopf Equations with Symbols Analytic in a Strip". In: I. Gohberg and M.A. Kaashoek (Eds.), "Constructive Methods of Wiener-Hopf Factorization", Basel, Birkhäuser OT 21, 1986, pp. 39-74.

3. R. Beals, "An Abstract Treatment of Some Forward-Backward Problems of Transport and Scattering," J. Funct. Anal. 34, 1-20 (1979).

4. R. Beals, "Indefinite Sturm-Liouville Problems and Half-Range Completeness," J. Diff. Eqs. 56, 391-407 (1985).

5. J. Bognár, "Indefinite Inner Product Spaces," Springer, New York (1974).

6. M. Caro, J. Ligou, "Treatment of Scattering Anisotropy of Neutrons Through the Boltzmann-Fokker-Planck Equation," Nucl. Sci. Eng. 83, 242-252 (1983).

7. B. Curgus, "On the Regularity of the Critical Point Infinity of Definitizable Operators," Int. Eqs. Oper. Theory 8, 462-487 (1985).

8. A.H. Ganchev, Ph. D. Thesis, Virginia Tech, Blacksburg, 1986.

9. A.H. Ganchev, W. Greenberg, C.V.M. van der Mee, "Abstract Kinetic Equations with Accretive Collision Operators," Int. Eqs. Oper. Theory, in press.

10. W. Greenberg, C.V.M. van der Mee, "Abstract Kinetic Equations Relevant to Supercritical Media," J. Funct. Anal. 57, 111-142 (1984).

11. W. Greenberg, C.V.M. van der Mee, V. Protopopescu, "Boundary Value Problems in Abstract Kinetic Theory," Birkhäuser OT 23, Basel, 1987.

12. W. Greenberg, C.V.M. van der Mee, W. Walus, "Strong Solutions of Stationary

Equations in Abstract Kinetic Theory," Int. Eqs. Oper. Theory, in press.

13. W. Greenberg, C.V.M. van der Mee, P.F. Zweifel, "Generalized Kinetic Equations," Int. Eqs. Oper. Theory 7, 60-95 (1984).

14. R.J. Hangelbroek, Linear Analysis and Solution of Neutron Transport Problems," Transp. Theor. Stat. Phys. 5, 1-85 (1976).

15. H.G. Kaper, M.K. Kwong, C.G. Lekkerkerker, A. Zettl, "Full and Partial Range Eigenfunction Expansions for Sturm-Liouville Problems with Indefinite Weights," Proc. Roy. Soc. Edinburgh 98A, 69-88 (1984).

16. M. Klaus, C.V.M. van der Mee, V. Protopopescu, "Half-range Solutions of Indefinite Sturm-Liouville Problems," J. Funct. Anal. 70, 254-288 (1987).

17. M.G. Krein, H. Langer, "The Spectral Function of a Self-Adjoint Operator in a Space with Indefinite Metric," Dokl. Akad. Nauk 4, 39-42 (1963).

18. H. Langer, "Spectral Functions of Definitizable Operators in Krein Spaces". In: "Functional Analysis," Lect. Notes in Math. 948, Springer, Berlin, 1982.

19. C.G. Lekkerkerker, "The Linear Transport Equation. The Degenerate Case c=1," Proc. Royal Soc. Edinburgh 75A, 259-282 and 283-295 (1975/76).

20. C.V.M. van der Mee, "Semigroup and Factorization Methods in Transport Theory," Ph.D. Thesis, Free University, Amsterdam, 1981.

21. K. Przybylski, J. Ligou, "Numerical Analysis of the Boltzmann Equation Including Fokker-Planck Terms," Nucl. Sci. Eng. 81, 92-109 (1982).

22. R. Sanchez, N.J. McCormick, "Solutions to Inverse Problems for the Boltzmann-Fokker-Planck Equation," Transp. Theory Stat. Phys. 12, 129-155 (1983).

23. B.L. Willis, P.F. Zweifel, C.V.M. van der Mee, "Whither Existence Theory?," Transp. Theory Stat. Phys. 14, 669-678 (1985).

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