Multigroup transport equations with nondiagonal cross-section matrices

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Multigroup transport equations with nondiagonal cross-section matrices are studied using the Wiener–Hopf method. Formulas for the solution and the exit distribution are given in terms of the factorization of the symbol of the Wiener–Hopf equation. Unlike the formulas for a diagonal cross-section matrix, these formulas involve derivatives of the H-functions. For the case of two groups, the H-functions are computed explicitly.

I. INTRODUCTION

Multigroup transport equations with nondiagonal and possibly nondiagonalizable cross-section matrices have been proposed as a model of, for example, neutron transport in reactors.[1,2] In this paper, transport equations with nondiagonalizable cross-section matrices are studied by making use of the Wiener–Hopf method. In Sec. II an integral equation equivalent to the transport equation is derived along with expressions connecting the solutions of the integral equation to the solutions of the transport equation. In Secs. III and IV we outline the Wiener–Hopf method. In Sec. V the Wiener–Hopf factorization is constructed explicitly for the two-group case. For the general N-group problem, we are not able to construct the factorization; the best that we are able to do is derive the generalized Chandrasekhar H-equations and to set up a numerical scheme for computing the H-functions. This work will be published in another paper, where we consider a more general scattering matrix. Finally, in Secs. VI and VII we determine the exit distribution and the solution in terms of the H-functions. In these two sections we do not limit ourselves to the two-group problem; instead we consider the N-group problem in anticipation of the above-mentioned generalization.

Briefly, transport equations with nondiagonal cross-section matrices occur when the energy dependence of the cross section is expanded in terms of orthogonal functions, and then the method of weighted residuals is applied to determine equations for the coefficients of the expansion. The method of weighted residuals is discussed by Stacey[1] and by Ames,[3] where different choices of the orthogonal functions and the weights are considered and the physical reasons behind the choices are given. If this procedure is followed for the problem of radiative transfer with the assumption of a uniform or picket fence model,[4] then the resulting vector equation has the form

$$\mu\,\partial_x F(x,\mu) + \Sigma F(x,\mu) = \frac{1}{2} C \int_{-1}^{+1} F(x,\mu')\,d\mu', \qquad (1)$$

where the matrix C is noninvertible. A derivation of these results can be found in Siewert and Zweifel,[4] the only difference being that the cross-section matrix Σ is no longer necessarily diagonal. If Σ is diagonalizable, then a similarity transformation will reduce Eq. (1) to the problem considered in Ref. 4. More generally, Eq. (1) is solvable for the case that the matrices Σ and C are simultaneously upper triangularizable. In such a case, the problem reduces to a system of uncoupled inhomogeneous scalar Wiener–Hopf equations.

In the following, the simplest equation of the form (1) that does not satisfy either one of the two above conditions will be studied. In particular, the two-group equation defined by

$$\Sigma = \begin{vmatrix} 1 & \alpha \\ 0 & 1 \end{vmatrix}, \quad C = \begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix}, \qquad (2)$$

will be studied with $\alpha \neq 0$ and $c_{21} \neq 0$. A similarity transformation can always be applied to set $\alpha = 1$, but for bookkeeping purposes it is convenient to keep α as a parameter so that the limit $\alpha \to 0$ is apparent. A direct calculation shows that, for $\alpha \neq 0$ and $c_{21} \neq 0$, Σ and C are not simultaneously upper triangularizable, whence the conditions $\alpha \neq 0$, $c_{21} \neq 0$. In this paper we will study Eq. (1) with Σ and C defined by Eq. (2), along with half-space boundary conditions given by

$$F(0,\mu) = \Phi(\mu), \quad \mu > 0, \qquad (3a)$$

$$F(x,\mu) \to 0, \quad x \to \infty. \qquad (3b)$$

Equation (3b) holds true for each component separately.

II. AN EQUIVALENT INTEGRAL EQUATION

Equation (1) is studied using the Wiener–Hopf method. To carry out the procedure, an equivalent integral equation is sought. If G is defined by

$$G(x) = \int_{-1}^{+1} F(x,\mu)\,d\mu, \qquad (4)$$

then an integral equation for G can be derived analogously to the one-speed case.[5] The result is

$$G(x) = U(x) + \frac{1}{2} \int_0^\infty \mathrm{Ei}_\Sigma(|x-s|)\,C G(s)\,ds, \qquad (5a)$$

where

$$U(x) = \int_0^1 e^{-x\Sigma/\mu} \Phi(\mu)\,d\mu. \qquad (5b)$$

The function Ei_Σ is defined in terms of the exponential inte-

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gral[6] and its derivative by

$$\mathrm{Ei}_\Sigma(z) = \int_0^1 \mu^{-1} e^{-x\Sigma/\mu}\, d\mu = \begin{vmatrix} E_1(z) & \alpha z E_1'(z) \\ 0 & E_1(z) \end{vmatrix}. \tag{5c}$$

Once G is known, F can be computed using the formulas

$$F(x,\mu) = -\frac{1}{2\mu}\int_x^\infty e^{-(x-s)\Sigma/\mu} CG(s)ds, \quad \mu<0, \tag{6a}$$

and, for $\mu>0$,

$$F(x,\mu) = e^{-\Sigma x/\mu}\Phi(\mu) + \frac{1}{2\mu}\int_0^x e^{-(x-s)\Sigma/\mu} CG(s)ds. \tag{6b}$$

The matrix-valued function $e^{-\Sigma/\mu}$ is easy to compute if the Jordan decomposition of Σ given by

$$\Sigma = I + M, \quad M^2 = 0, \tag{7}$$

is used. It is easy to check that

$$e^{-x\Sigma/\mu} = \begin{vmatrix} 1 & \alpha x/\mu \\ 0 & 1 \end{vmatrix} e^{-x/\mu}. \tag{8}$$

III. THE WIENER–HOPF METHOD OF SOLUTION

Following the standard notation, define the functions $G^\pm$ and $U^\pm$ by

$$G^\pm(x) = \begin{cases} G(x), & \pm x>0, \\ 0, & \pm x<0, \end{cases} \tag{9}$$

and similarly for $U^\pm$. With these definitions, Eq. (5a) can be written as a convolution equation on $(-\infty,\infty)$, namely

$$G^+(x) + G^-(x) = U^+(x) + \frac{1}{2}\int_{-\infty}^{+\infty} \mathrm{Ei}_\Sigma(|x-s|)CG^+(s)ds. \tag{10a}$$

The Fourier transform of Eq. (10a) yields

$$W(\lambda)\hat{G}^+(\lambda) + \hat{G}^-(\lambda) = \hat{U}^+(\lambda). \tag{10b}$$

Here the Fourier transform of a function F is denoted as $\hat{F}$, where

$$\hat{F}(\lambda) = \int_{-\infty}^{+\infty} e^{i\lambda x} F(x)dx.$$

The matrix-valued function W is the symbol of the Wiener–Hopf equation (10a) and is given by

$$W(\lambda) = I - ((1/\lambda)\tan^{-1}\lambda)C + [1/(1+\lambda^2)]MC. \tag{11}$$

The nilpotent matrix M has already been introduced in Eq. (7).

IV. FACTORIZATION OF THE SYMBOL

The crucial step in the Wiener–Hopf method is the construction of the Wiener–Hopf factorization of the symbol. This paper will only consider the canonical Wiener–Hopf (WH) factorization. A canonical WH factorization is a pair of functions $W^\pm$ such that

$$W(\lambda) = W^-(\lambda)W^+(\lambda), \quad \lambda\in\mathbb{R}_\infty = \mathbb{R}\cup\{\pm\infty\}, \tag{12}$$

where the matrix function W^+ (W^-) is analytic in the open upper (lower) half-plane, and continuous and invertible in the closed upper (lower) half-plane. As in the one-speed case, a factorization of the form (12) does not exist for all possible choices of Σ and C. In fact, in the one-speed case, a canonical factorization exists only for $c<1$ (see Ref. 5). A necessary condition for the existence of $W^\pm$ is that $W(\lambda)$ is invertible for $\lambda\in\mathbb{R}_\infty$, i.e., det $W(\lambda) \pm 0$ for $\lambda\in\mathbb{R}_\infty$. For this reason one should study the zeros of det W. Explicitly, det W is given by

$$\det W(\lambda) = 1 - \mathrm{tr}\, C[\lambda^{-1}\tan^{-1}\lambda] + \alpha c_{21}[1+\lambda^2]^{-1}. \tag{13}$$

Here, tr C denotes the trace of C, and the assumption that det $C=0$ has been used. Observe that the dispersion function has branch points at $\pm i$. We will always choose the branch cuts to be the lines $z=it$, $|t|\geqslant 1$. Therefore, the dispersion function is analytic in the region $\mathbb{C}\setminus\{z\in\mathbb{C}: z=it, |t|\geqslant 1, t\in\mathbb{R}\}$. Note that

$$\lim \det W(\lambda) = 1, \quad |\lambda|\to\infty, \tag{14}$$

holds inside the region of analyticity. Furthermore, det W satisfies the symmetries

$$[\det W(\lambda)]^* = \det W(\lambda^*), \tag{15a}$$

$$\det W(-\lambda) = \det W(\lambda). \tag{15b}$$

The superscript * denotes complex conjugation. These symmetries imply that λ_0 is a zero of the dispersion function if and only if both λ_0^* and $-\lambda_0$ are zeros of the dispersion function. Therefore the dispersion function must have an even number of zeros. The symmetries [Eqs. (15a) and (15b)] along with the behavior of det W at infinity [Eq. (14)] allow one to compute the number of zeros of the dispersion function by computing the change of the argument of det W along the branch cuts, the so-called Nyquist method,[5] just as is done in the one-group case.[5] We apply the

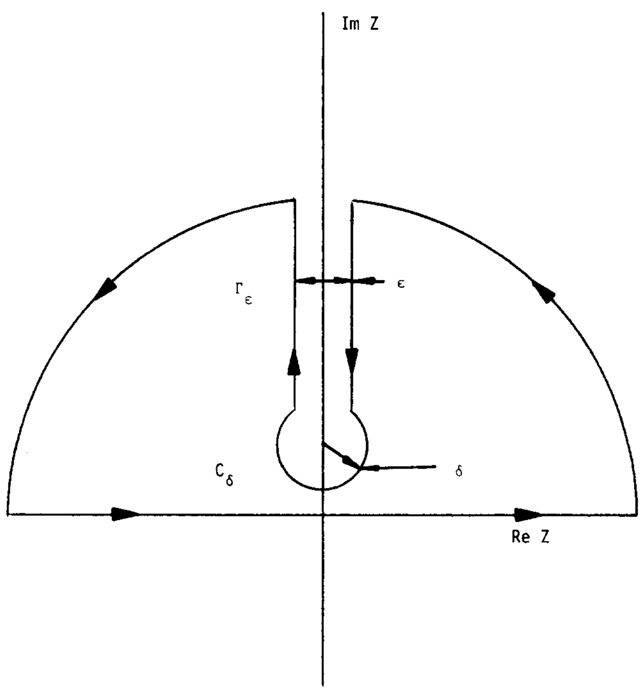

FIG. 1. Contour for computing Δ arg det W.

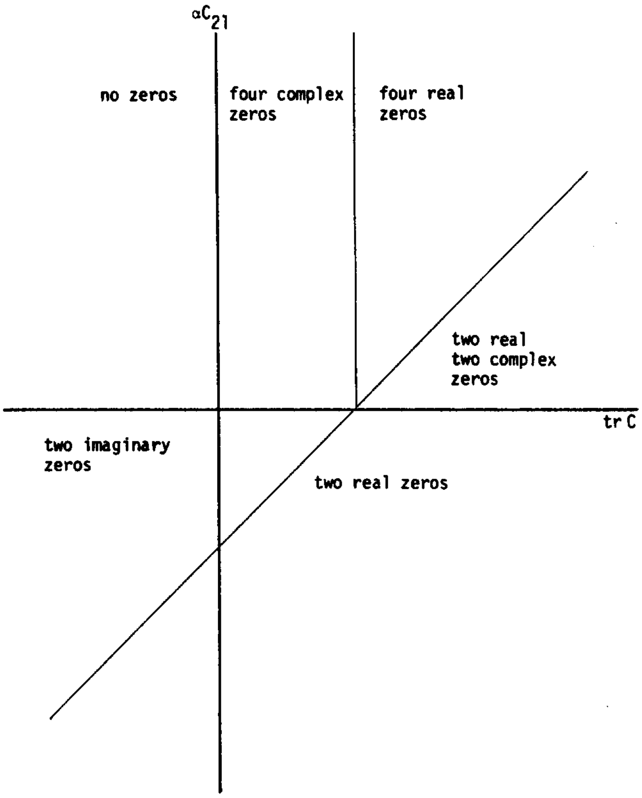

FIG. 2. The zeros of det W in the tr C, αC_{21} plane.

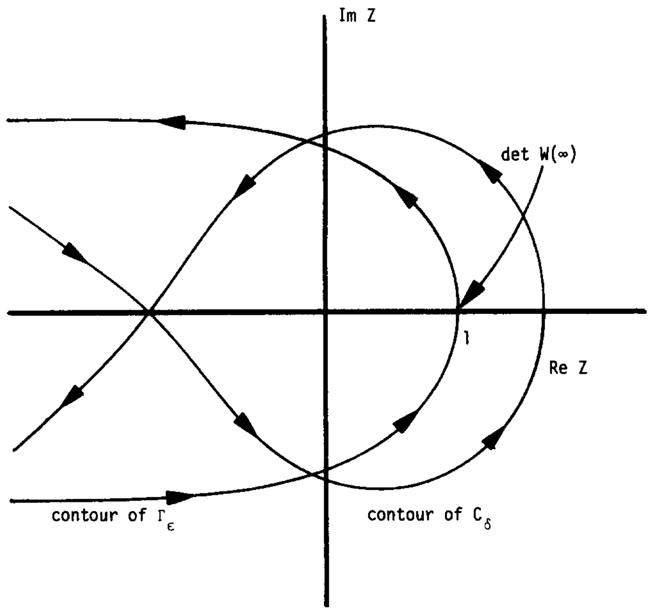

FIG. 3. Nyquist diagram for $\alpha C_{21} > 0$ and tr $C > 0$.

argument principle to the contour in Fig. 1. This problem divides into three special cases: (i) tr $C = 0$, (ii) $\alpha c_{21} = 0$, and (iii) both tr $C \neq 0$ and $\alpha c_{21} \neq 0$. The case tr $C = 0$ is solved easily by algebra, and the dispersion function for $\alpha c_{21} = 0$ is identical to the one-group dispersion function so that the number of zeros is known.[7] These results are summarized in Fig. 2. Case (iii) requires special attention. Unlike the one-group dispersion function, i.e., the case $\alpha c_{21} = 0$, the dispersion function now has poles at the branch points due to the term $\alpha c_{21}[1 + \lambda^2]^{-1}$ [see Eq. (13)]. For this case, the change in the argument when rounding the branch points is now important. For this reason, the change in the argument of the dispersion function (denoted by Δ arg det W) along the contour in Fig. 1 will be considered in the limit as ϵ and $\delta \downarrow 0$. First we study Δ arg det W along the straight lines Γ_ϵ by taking the limit $\epsilon \downarrow 0$ while keeping δ a constant, then we study Δ arg det W along the circle C_δ by taking the limit as $\delta \downarrow 0$. Along the lines Γ_ϵ the real and imaginary parts of the boundary values of det W are given by

$$\text{Re}\det(\pm 0 + iy) = 1 - \frac{\text{tr}\, C}{2y} \ln \left| \frac{1+y}{1-y} \right| + \frac{\alpha c_{21}}{1-y^2}, \tag{16a}$$

$$\text{Im}\det W(\pm 0 + iy) = \pm (\pi\, \text{tr}\, C)/2y. \tag{16b}$$

[Note that Eq. (16b) proves that det W is nonvanishing on the contour Γ as required by the argument principle.] With these formulas, the Nyquist diagram for the contour Γ_ϵ can be sketched; for the case $\alpha c_{21} > 0$ and tr $C > 0$ the result is shown in Fig. 3. The diagrams for the other possible choices of signs of αc_{21} and tr C are similar. To complete the Nyquist diagrams, the contour C_δ must now be considered. Along the C_δ, the pole term $(1 + \lambda^2)^{-1}$ dominates, and the contour approaches a circle at infinity as $\delta \downarrow 0$. With this information, the Nyquist diagrams can be sketched (see Fig. 3), and the number of zeros of the dispersion function can be deduced. Now that the number of zeros of the dispersion function is known, the remaining task is to determine whether the zeros are purely real, purely imaginary, or neither. The graphs of the real and imaginary parts of the dispersion function are easy to sketch, so it is easy to determine if the dispersion function has a real zero. These results are also summarized in Fig. 2. Thus we can conclude that $W(\lambda)$, $\lambda \in \mathbf{R}_\infty$, is invertible for $1 + \alpha c_{21} > \text{tr}\, C$, and tr $C > 1$. As we previously mentioned, these conditions give a necessary condition for the existence of a WH factorization. In the next section, these conditions will be shown to be sufficient by explicit calculation of the factorization of W.

V. CONSTRUCTION OF THE WIENER–HOPF FACTORIZATION

The matrix valued function to be factorized is

$$W(\lambda) = I - (\lambda^{-1} \tan^{-1} \lambda) C + (1 + \lambda^2)^{-1} MC; \tag{17}$$

the matrix M has been defined in Eq. (7). In general it is not known how to construct the Wiener–Hopf factorization of matrices, but Cebotarev[8] has shown how to factorize any upper triangular matrix. The matrix (17) can be made upper triangular by a similarity transformation with constant elements. One possible transformation is given by

$$S = \begin{vmatrix} -c_{22} & c_{21} \\ c_{21} & 0 \end{vmatrix} \begin{vmatrix} 1 & A \\ 0 & 1 \end{vmatrix}, \tag{18a}$$

where

$A = c_{21} (\text{tr}\, C)^{-1}$, if tr $C \neq 0$, and

$$A=0, \quad \text{if } \operatorname{tr} C=0. \tag{18b}$$

The matrix S is always invertible, because $\det S = -c_{21}^2$, which is nonvanishing by assumption. The particular choice for S has been made with forethought, so that the transformed matrices MC and C are especially simple. Explicitly the transformed matrices are

$$S^{-1}(I+M)CS = \begin{vmatrix} 0 & 0 \\ 0 & \alpha c_{21} \end{vmatrix}, \quad \operatorname{tr} C=0, \tag{19a}$$

and, for $\operatorname{tr} C \neq 0$,

$$S^{-1}(I+M)CS = \begin{vmatrix} 0 & -\alpha c_{21}^2 (\operatorname{tr} C)^{-1} \\ 0 & \operatorname{tr} C + \alpha c_{21} \end{vmatrix}. \tag{19b}$$

The transformed matrix $S^{-1}CS$ is given by the same expression, but with $\alpha = 0$. It is tempting to think that the similarity transformation [Eq. (18a)] applied to the original equation will result in a similar simplification, but this is not the case. The reason is that although C and MC are simultaneously upper triangularizable, C and Σ are not.

The Wiener–Hopf factorization now can be computed. If $S^{-1}WS$ is denoted by $\widetilde{W}$, then

$$\widetilde{W} = \begin{vmatrix} 1 & K(\lambda) \\ 0 & \det W(\lambda) \end{vmatrix}, \tag{20a}$$

where

$$K(\lambda) = -c_{21}\lambda^{-1}\tan^{-1}\lambda, \quad \operatorname{tr} C = 0, \tag{20b}$$

$$K(\lambda) = -\alpha c_{21}^2 (\operatorname{tr} C)^{-1}(1+\lambda^2)^{-1}, \quad \operatorname{tr} C \neq 0. \tag{20c}$$

The function $\widetilde{W}$ is an upper triangular matrix function of second order and the procedure for getting its Wiener–Hopf factorization when it exists has been developed by Cebotarev.[8] Here we follow the method of Ref. 9. First we note that the factors of an upper triangular matrix can be taken to be upper triangular, so we set

$$\widetilde{W}(\lambda) = X(\lambda)Y(\lambda), \tag{21}$$

with X (Y) analytic and invertible in the lower (upper) half-plane. If the elements of the matrices X and Y are denoted by X_{ij} and Y_{ij}, respectively, then the following system of equations results when Eq. (21) is substituted into Eq. (17) and the corresponding matrix elements are equated:

$$1 = X_{11}Y_{11}, \tag{22a}$$

$$1 - (\operatorname{tr} C)\lambda^{-1}\tan^{-1}\lambda + \alpha c_{21}(1+\lambda^2)2^{-1} = X_{22}(\lambda)Y_{22}(\lambda), \tag{22b}$$

and

$$-(c_{21} - A \operatorname{tr} C)\lambda^{-1}\tan^{-1}\lambda - \sigma c_{21}A(1+\lambda^2)^{-1} = X_{11}(\lambda)Y_{12}(\lambda) + X_{12}Y_{22}(\lambda). \tag{22c}$$

These equations do not uniquely determine X and Y, since XU and $U^{-1}Y$ satisfy Eqs. (22a)–(22c) whenever X and Y do, where U is any invertible matrix. However, it is consistent to impose the conditions

$$X_{ij}(\infty) = Y_{ij}(\infty) = \delta_{ij}. \tag{23}$$

With these conditions, Eq. (22a) uniquely determines X_{11} and Y_{11} to be

$$X_{11}(\lambda) = Y_{11}(\lambda) = 1, \tag{24}$$

while the solution to Eq. (22b) is given by

$$X_{22}(\lambda) = \exp\left\{\frac{1}{2\pi i}\int_{-\infty+i/2}^{+\infty+i/2} \frac{B(z)}{z-\lambda}\,dz\right\}, \tag{25a}$$

where

$$B(z) = \ln[1 - [(\operatorname{tr} C)/z]\tan^{-1}z + \alpha c_{21}(1+z^2)^{-1}]. \tag{25b}$$

The expression for Y_{22} is the same, except that the limits of integration are replaced by $\infty - \frac{1}{2}$ and $-\infty - i/2$. Finally, we determine Y_{12} and X_{12}. To do this, divide Eq. (22c) by Y_{22}, and define the left-hand side of Eq. (22c) to be $L(\lambda)$. Then

$$L(\lambda)/Y_{22}(\lambda) = Y_{12}(\lambda)/Y_{22}(\lambda) + X_{12}(\lambda). \tag{26}$$

The left-hand side of this equation is known, while the right-hand side is the sum of two functions, one analytic in the upper half-plane, the other one analytic in the lower half-plane. To solve for Y_{12} it is only necessary to write LY_{22}^{-1} as the sum of two functions:

$$L(\lambda)/Y_{22}(\lambda) = L^{+}(\lambda) + L^{-}(\lambda), \tag{27}$$

with L^{+} (L^{-}) analytic in the upper (lower) half-plane. Therefore

$$L^{+}(\lambda) = \frac{1}{2\pi i}\int_{-\infty-i/2}^{+\infty-i/2} \frac{L(z)/Y_{22}(z)}{z-\lambda}\,dz, \tag{28a}$$

$$L^{-}(\lambda) = \frac{1}{2\pi i}\int_{-\infty+i/2}^{+\infty+i/2} \frac{L(z)/Y_{22}(z)}{z-\lambda}\,dz. \tag{28b}$$

Now with the definitions

$$Y_{12}(\lambda) = Y_{22}(\lambda)L^{+}(\lambda), \tag{29a}$$

$$X_{12}(\lambda) = L^{-}(\lambda), \tag{29b}$$

the matrices X and Y have all the properties required of a WH factorization.

VI. THE EXIT DISTRIBUTION

Once the canonical Wiener–Hopf factorization has been computed, an expression for the exit distribution, i.e., $F(0,\mu)$ for $\mu < 0$, can be written in terms of the factors of $W(\lambda)$. Unlike for the one-speed case, the exit distribution will involve derivatives of the factors of $W(1/i\lambda)$. The method followed in this section parallels the one given by van der Mee.[10] First, the exit distribution for the two-speed problem defined by Eq. (29) will be derived; then the formulas will be generalized to the N-group problem.

Following Gohberg and Krein,[7] there exists a resolvent kernel $\gamma(\cdot,\cdot)$ so that the general solution to the Wiener–Hopf equation

$$G(x) = \int_0^\infty K(x-y)G(y)dy + U(x) \tag{30a}$$

can be written as

$$G(x) = U(x) + \int_0^\infty \gamma(x,y)U(y)dy, \tag{30b}$$

and the general solution to the transposed equation

$$G(x) = \int_0^\infty G(y)K(y-x)dx + U(x) \qquad (31a)$$

can be written as

$$G(x) = U(x) + \int_0^\infty U(y)\gamma(y,x)dy. \qquad (31b)$$

Note that the resolvent kernels for Eq. (30a) and Eq. (31a) are identical. Returning to Eq. (30a), the exit distribution can be written in terms of G by the formula

$$F(0,\mu) = -\frac{1}{2\mu}\int_0^\infty e^{y\Sigma/\mu}CG(y)dy, \quad \mu<0. \qquad (32)$$

Introducing the resolvent kernel $\gamma(\cdot,\cdot)$ this can be rewritten as

$$F(0,\mu) = -\frac{1}{2\mu}\int_0^\infty\int_0^\infty e^{y\Sigma/\mu} \times C[\delta(y-z)+\gamma(y,z)]U(z)dz\,dy. \qquad (33)$$

If the expression for $U(z)$ in terms of the incident flux is used in Eq. (33), then

$$F(0,\mu) = -\frac{1}{2\mu}\int_0^\infty\int_0^\infty\int_0^1 e^{y\Sigma/\mu}C[\delta(y-z) + \gamma(y,z)]e^{-z\Sigma/s}\Phi(s)ds\,dz\,dy. \qquad (34)$$

This equation relates the exit distribution to the incident distribution by making use of the resolvent kernel. To write Eq. (34) in terms of the factors of W, it is necessary to write

$$\int_0^\infty\int_0^\infty e^{y\Sigma/\mu}C[\delta(y-z)+\gamma(y,z)]e^{-z\Sigma/s}dz\,dy \qquad (35)$$

in terms of the factors of W. This will be accomplished in two parts. First we have the following lemma.

Lemma 1:

$$\int_0^\infty\int_0^\infty e^{y/\mu}C[\delta(y-z)+\gamma(y,z)]e^{-z\Sigma/s}dz\,dy = H_1(-\mu)[[s\mu/(\mu-s)]H_r(s) - s(\mu/(\mu-s))^2(H_r(s)+(\mu-s)H_r'(s))M], \qquad (36)$$

where

$$W^{-1}(1/i\mu) = H_l(-\mu)H_r(\mu)$$

is a canonical factorization with H_l and H_r analytic in the open right half-plane and continuous and invertible in the closed right half-plane.

Proof: Let $G(x;s)$ be a solution to the matrix Wiener–Hopf equation

$$G(x;s) = \int_0^\infty K(|x-y|)G(y;s)dy + e^{-x\Sigma/s}. \qquad (37)$$

In this equation the variable s is considered to be a parameter. Note that the left-hand side of Eq. (36) is

$$\int_0^\infty e^{y/\mu}G(y;s)dy = \hat{G}^+(\mu;s). \qquad (38)$$

If Eq. (37) is extended to the entire real line in the usual way and the Laplace transform is defined by

$$\tilde{G}(\lambda) = \int_{-\infty}^\infty dx\, e^{x/\lambda}G(x), \quad \mathrm{Re}(\lambda)=0, \qquad (39)$$

while $Z(\lambda) = W(1/i\lambda)$, then the Laplace transform of the integral equation is

$$Z(\lambda)\tilde{G}^+(\lambda) + \tilde{G}^-(\lambda) = [s\lambda/(\lambda-s)]I - s(\lambda/(\lambda-s))^2M. \qquad (40)$$

The functions G^+ and G^- have already been defined by Eqs. (40a)–(40d), and the matrix M was introduced in Eq. (45a). Now assume that the factorization of $Z(\lambda)$ is given by

$$Z^{-1}(\mu) = H_l(-\mu)H_r(\mu),$$

where the functions H_l and H_r are analytic and invertible on the open right half-plane, and continuous and invertible on the closed right half-plane. Using the above factorization, Eq. (40) may be rewritten as

$$H_l^{-1}(-\mu)\tilde{G}^+(\mu) + H_r(\mu)\tilde{G}^-(\mu) = H_r(\mu)\,[[s\mu/(\mu-s)]I - s(\mu/(\mu-s))^2M]. \qquad (41)$$

If the right-hand side of Eq. (41) can be written as the sum of two terms, one analytic and invertible in the right half-plane, the other one analytic and invertible in the left half-plane, then Liouville's theorem can be invoked to solve for $\tilde{G}^+$ and $\tilde{G}^-$. Due to the second-order pole in Eq. (41), it is necessary to introduce the first derivatives of the H-functions into this splitting. By inspection, the splitting is given by the sum of

$$[s\mu/(\mu-s)][H_r(\mu) - H_l(s)] - s(\mu/(\mu-s))^2[H_r(\mu) - H_r(s) - (\mu-s)H_r'(s)]M, \qquad (42a)$$

which is analytic in the right half-plane, and the expression

$$\frac{s\mu}{\mu-s}H_r(s) - s\left(\frac{\mu}{\mu-s}\right)^2[H_r(s)+(\mu-s)H_r'(s)]M, \qquad (42b)$$

which is analytic in the left half-plane. An application of Liouville's theorem then proves Lemma 1. Note that, for $M=0$, Eq. (42b) reduces to the result given in Ref. 9. Using Lemma 1 it is now possible to write Eq. (35) in terms of the H-functions. To do this it is expedient to define

$$\Gamma(\mu,s) = \text{right-hand side of Eq. (36)}. \qquad (43)$$

Now substitute the explicit formula for $\exp(-y\Sigma/\mu)$ into Eq. (35). The result is

$$\int_0^\infty\int_0^\infty e^{x/\mu}\left[I+\frac{x}{\mu}M\right]C[\delta(x-z)+\gamma(x,z)]e^{-z\Sigma/\mu}dz\,dx. \qquad (44)$$

The contribution due to the term $e^{x/\mu}$ gives $C\Gamma(\mu,s)$, while the term $(x/\mu)e^{x/\mu}$ gives rise to first derivatives of the function Γ. It is easily checked that

$$\int_0^\infty \int_0^\infty \frac{x}{\mu} e^{x/\mu} C[\delta(x-z)+\gamma(x,z)]e^{-z\Sigma/\mu}\,dz\,dx$$
$$=\mu C\,\partial_\mu \Gamma(\mu,s). \tag{45}$$

Therefore,

$$F(0,\mu) = -\frac{1}{2\mu}[I-\mu\,\partial_\mu M]C\int_0^1 \Gamma(\mu,s)\Phi(s)ds. \tag{46}$$

It is routine to generalize the exit distribution formula [Eq. (46)] to the N-group problem. If $\Sigma = D + M$ is the Jordan decomposition of Σ with D the diagonal matrix given by $\mathrm{diag}\{\sigma_i\}_{i=1}^N$, then the right-hand side of Eq. (40) is replaced by

$$\sum_{m=0}^{N-1}(-1)^m \mathrm{diag}\left\{s\left(\frac{\mu}{\mu\sigma_i - s}\right)^{m+1}\right\}_{i=1}^N M^m. \tag{47}$$

Now it is necessary to write

$$H_r(\mu)\sum_{m=0}^{N-1}(-1)^m\,\mathrm{diag}\left\{s\left(\frac{\mu}{\mu\sigma_i - s}\right)^2\right\}_{i=1}^N M^m \tag{48}$$

as the sum of two terms, just as was done for the two-group case. Note that Eq. (48) has poles at $\mu = s/\sigma_i$, which are in the right half-plane. Denoting the ith column of a matrix A by $[A]_{(i)}$ and noting that

$$\left[H_r(\mu)\mathrm{diag}\left\{s\left(\frac{\mu}{\mu\sigma_i - s}\right)^{m+1}\right\}\right]_{(i)}$$
$$=[H_r(\mu)]_{(i)}s(\mu/(\mu\sigma_i - s))^{m+1}, \tag{49}$$

Eq. (48) can easily be written as the sum of two terms, one analytic in the right half-plane, the other one analytic in the left half-plane. This is accomplished by writing Eq. (49) as the sum of

$$[Q_n^+(\mu,s)]$$
$$=\left[H_r(\mu)-\sum_{m=0}^{N-1}\frac{1}{m!}\left(\mu\sigma_i - \frac{\sigma_i}{s}\right)^m H_r^{(m)}\left(\frac{s}{\sigma_i}\right)\right]_{(i)}$$
$$\times s(\mu/(\mu\sigma_i - s))^{m+1}, \tag{50}$$

which is analytic in the right half-plane, and

$$[Q_n^-(\mu,s)]_{(i)} = \left[\sum_{m=0}^{N-1}\frac{1}{m!}\left(\mu\sigma_i - \frac{\sigma_i}{s}\right)^m H_r^{(m)}\left(\frac{s}{\sigma_i}\right)\right]_{(i)}$$
$$\times s(\mu/(\mu\sigma_i - s))^{m+1}, \tag{51}$$

which is analytic in the left half-plane, where $H_r^{(m)}$ and $H_l^{(m)}$ are the mth derivatives of H_r and H_l, respectively. Therefore the generalization of Lemma 1 to the N-group problem is

$$\Gamma(\mu,s) = H_l(-\mu)\sum_{m=0}^{N-1} Q_m^-(\mu,s)M^m, \tag{52}$$

and the exit distribution $[F(0,\mu)]_{(i)}$ is given by

$$-\frac{1}{2\mu}\sum_{m=0}^{N-1}\frac{1}{m!}\left(\frac{\mu}{\sigma_i}\right)^m(\partial_\mu)^m$$
$$\times\int_0^1\left[M^m\Gamma\left(\frac{\mu}{\sigma_i},s\right)\right]_{(i)}\Phi(s)\,ds. \tag{53}$$

Not only can the exit distribution be written in terms of the factors of the symbol, but the solution for any value of x can also be written in a similar fashion. This can be done by making use of Eq. (30b), which relates $F(x,\mu)$ to $G(x)$, and the results of this section. First we note that

$$\hat{G}^+(\mu) = \int_0^1 \Gamma(\mu,s)\Phi(s)\,ds. \tag{54}$$

From this expression it is possible to recover the function G. Now that G is known, the solution $F(x,\mu)$ for $x<0$ can be computed by making use of Eq. (30b).

VII. CONCLUSION

Formulas for the exit distribution and the solution to a multigroup transport equation with a nondiagonal cross-section matrix have been derived in terms of generalized Chandrasekhar H-functions. For the special case of two groups with a noninvertible scattering matrix, the H-functions were computed explicitly. Unfortunately, for $N>2$ we are not able to construct the factorization explicitly, so we are forced to derive a nonlinear integral equation which the H-functions satisfy and to set up a numerical scheme for solving them. This work will be published elsewhere.

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