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INTEGRAL FORMULATIONS OF STATIONARY TRANSPORT EQUATIONS IN PLANE-PARALLEL MEDIA

C. V. M. van der Mee[1-5]

Istituto di Matematica Applicata "G. Sansone"
Via di S. Marta 3, I-50139 Firenze, Italy

ABSTRACT

A review is given of integral formulations of a variety of boundary value problems in abstract kinetic theory. Apart from the introduction of the boundary value problems and their equivalent integral formulations, we pay special attention to their most important applications. In the first place we shall discuss representations of solutions in generalized H-, X- and Y-functions. In the second place we shall develop the projection and semigroup formalism using only the integral formulation and not using selfadjoin-

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3) This paper was written while the author was visiting the University of Florence in Summer 1985.
4) Permanent Address: Leaving from: Dept. of Mathematics, Texas Tech University, Lubbock, Texas 79409. Going to: Dept. of Mathematics and Computer Science, Clarkson University, Potsdam, New York, 13676.
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tness of the scattering operators. In the third place we shall discuss the impact of the equivalence theory in combination with the Fredholm alternative on the existence and uniqueness theory. We conclude this work with a discussion of methods of proving existence and uniqueness of solutions of stationary transport equations.

1. Introduction

Integral formulations of boundary value problems in kinetic theory have been known since 1921, when Milne[1] derived an integral equation of Wiener-Hopf type to study radiative transfer in a stellar atmosphere. For isotropic scattering these equations were thoroughly analyzed by Hopf[2]. They have triggered the development of the theory of Wiener-Hopf equations[3]. A systematic study of these equations and of the closely related H-, X- and Y-equations was made by Busbridge[4]. Her virtually complete analysis of the isotropic scattering case was extended to anisotropic scattering by Maslennikov[5] and Feldman[6,7], who were primarily interested in the asymptotic behavior of the solution deep inside a semiinfinite atmosphere (or reactor, if one considers the parallel example from neutron transport). Using cone preservation techniques criticality properties of neutron transport processes, also for inhomogeneous media and non-plane-parallel reactors, were studied by Nelson[8].

The present interest in integral formulations of transport processes stems from the recently developed theory of abstract kinetic equations, which was built up to a large extent by Beals[9-11], Greenberg[12,13], van der Mee[12-15], Protopopescu[11] and Zweifel[12], as a far fetched generalization of the work of Hangelbroek and Lekkerkerker[16,17] on the half-space problem of isotropic neutron transport. On restricting ourselves to models where integral formulations are apparent, we consider the abstract vector differential equation

$$(T\psi)'(x) = -(I - B)\psi(x) \ , \quad 0 < x < \tau \ , \tag{1}$$

where T is an injective selfadjoint operator and B a compact operator, both defined on a Hilbert space H. We make the regularity assumption[18]

$$\exists\ \alpha > 0 : \text{Ran } B \subset \text{Ran } |T|^{\alpha} \cap D(|T|^{1+\alpha}) \ . \tag{2}$$

In this way we obtain a repertoire of boundary value problems, both for finite slabs (τ finite) and half-spaces ($\tau = \infty$), both without reflecting walls and with reflection taken into account on one or two surfaces. It is then possible to prove rigorously that each boundary value problem is equivalent to a vector-valued integral equation, which is of convolution type if boundary reflection is neglected (see Refs. 14, 15 and 19), and to cut down the dimensionality of the space on which the integral equation is defined to the rank of the compact operator B (see Refs. 15 and 19). In most applications B has finite rank.

Having in stock a rigorous proof of the equivalence of boundary value problem and integral equation, there are various tasks that can be performed with relative ease. In the first place, on generalizing a method used for two-group neutron transport by Burniston et al.[20] and extended to a huge class of multigroup and continuously energy-dependent equations by Kelley[21], one can solve the half-space problem without reflecting boundary explicitly in terms of generalizations of Chandrasekhar's H-functions (see Refs. 15 and 19) and obtain formulas also obtainable with much more effort using resolvent integration[22]. The results can be extended to finite slab problems without reflecting boundaries (see Ref. 19). In the second place, one may exploit the equivalence to obtain existence and uniqueness results for the boundary value problem from those for the integral equation. (It should be noted that the explicit formulas referred to above require a priori knowledge of the existence and uniqueness result for the boundary value problem, in order for the generalized H-equations to be solvable). In this way one may obtain existence and uniqueness for half-space problems with $spr(B) < 1$, where $spr(B)$ stands for the spectral radius of B [23]. In the third place, one may exploit such equivalence to develop the projection and semigroup formalism accompanying the study of the boundary value problems[24] in cases where the nonselfadjointness and inexplicitness of B prevent one from using (a generalization of) the Spectral Theorem for selfadjoint operators. The projections thus constructed may then be applied in the proof of the existence of suitable Wiener-Hopf factorizations[15]. Finally, the equivalence can be exploited to derive existence results for the boundary value problem quickly from uniqueness results while using (a generalization of) the Fredholm alternative[25].

In Section 2 we introduce boundary value problems and their equivalent integral formulations. In Sections 3 and 4 we obtain the explicit solution formulas and the projection and semigroup formalism, respectively. In Section 5 we discuss the impact of the Fredholm alternative. We conclude the paper with a discussion of methods for obtaining well-posedness of stationary kinetic equations.

To a considerable extent this paper is a review article. On writing this article, certain parts of the literature have been disregarded, in part because they do not seem to fit into the adopted framework. In the first place, we mention the recent work of Maslova[26,27] on linearized Boltzmann equations with and without reflecting boundaries, where an almost complete treatment was given of the existence and uniqueness theory for such problems, written in integral form , though in a concrete setting. In the second place, we mention the work of Bardos, Caflish and Nikolaenko[28] on stationary linearized Boltzmann equations, where extensive use is made of energy estimates and a close relationship to the corresponding time--dependent problem appears. For this work we refer to the contribution of B. Nikolaenko to this conference. Finally, we have avoided the terminology of half-range completeness and orthogonality and other Caseology type notions. In our opinion the possibility of making eigenfunction expansions is a nonessential feature of the existence and uniqueness theory.

2. Boundary Value Problems and Integral Formulations

Throughout this article, except in the last section, T will be an injective selfadjoint operator and B a compact operator defined on the (real or complex) Hilbert space H, satisfying condition (2). Throughout we put $A = I - B$. On denoting the resolution of the identity of T by $\sigma(.)$ and writing $Q_+ = \sigma([0,\infty))$ and $Q_- = \sigma((-\infty,0])$, we may define the propagator function $H(z)$ by

$$H(z) = \begin{cases} +T^{-1}e^{-zT^{-1}}Q_+ = +\int_0^\infty t^{-1}\, e^{-z/t}\,\sigma(dt) & \text{for } z > 0 \\ -T^{-1}e^{-zT^{-1}}Q_- = -\int_{-\infty}^0 t^{-1}\, e^{-z/t}\,\sigma(dt) & \text{for } z < 0 \end{cases},$$

and similarly the semigroups $e^{\mp xT^{-1}}Q_\pm$ for $x > 0$.

We now discuss five boundary value problems connected with Eq. (1). For semiinfinite media, $\tau = \infty$, we have the half-space problem without reflection

$$(T\psi)'(x) = -(I - B)\psi(x)\ , \qquad 0 < x < \infty \tag{3a}$$

$$Q_+\psi(0) = \varphi_+ \tag{3b}$$

$$\|\psi(x)\|_H = O(1) \quad (\text{as} \quad x \to \infty) , \tag{3c}$$

which is relevant to neutron transport and to radiative transfer in half-spaces, as well as to some idealized or special problems in rarefied gas dynamics. On incorporating reflection by the wall, one gets the usual gas dynamics situation for a semiinfinite tube, i.e.

$$(T\psi)'(x) = -(I-B)\psi(x) , \quad 0 < x < \infty \tag{4a}$$

$$Q_+\psi(0) = RJQ_-\psi(0) + \varphi_+ \tag{4b}$$

$$\|\psi(x)\|_H = O(1) \quad (\text{as} \quad x \to \infty) . \tag{4c}$$

Here J is a signature operator $(J = J^* = J^{-1})$ satisfying $TJ = -JT$ (i.e. $J[D(T)] = D(T)$) and $JB = BJ$, and R is a bounded operator on H satisfying $R[D(T)] \subset D(T)$. By a solution to Eqs. (4) we then mean a continuous function $\psi : [0,\infty) \to D(T) \subset H$ such that $T\psi$ is strongly differentiable on $(0,\infty)$ and Eqs. (4a)-(4c) are fulfilled. In a similar way one defines a solution to Eqs. (3). It is also possible, and sometimes physically reasonable, to study boundary value problems, where the condition (3c) or (4c) is replaced by either the normal solution Ansatz

$$\exists\, n \in \mathbb{N} : \|\psi(x)\|_H = O(x^n) \quad (\text{as} \quad x \to \infty) , \tag{3/4d}$$

or by the condition

$$\lim_{x\to\infty} \|\psi(x)\|_H = 0 \quad (\text{as} \quad x \to \infty) . \tag{3/4e}$$

For finite layers, $\tau \in (0,\infty)$, we can formulate three boundary value problems of physical significance. On not accounting for reflection one has

$$(T\psi)'(x) = -(I-B)\psi(x) \ , \qquad 0 < x < \tau \tag{5a}$$

$$Q_+\psi(0) = \varphi_+ \tag{5b}$$

$$Q_-\psi(\tau) = \varphi_- \ , \tag{5c}$$

relevant to radiative transfer with a totally absorbing planetary surface $(\varphi_- = 0)$, neutron transport, and Poiseuille and Couette flow problems. In planetary atmosphere problems where the surface is reflecting, one has the boundary value problem

$$(T\psi)'(x) = -(I-B)\psi(x) \ , \qquad 0 < x < \tau \tag{6a}$$

$$Q_+\psi(0) = \varphi_+ \tag{6b}$$

$$Q_-\psi(\tau) = JRQ_+\psi(\tau) + \varphi_- \ , \tag{6c}$$

where usually $\varphi_- = 0$. In rarefied gas dynamics one usually studies the problem

$$(T\psi)'(x) = -(I-B)\psi(x) \ , \qquad 0 < x < \tau \tag{7a}$$

$$Q_+\psi(0) = RJQ_-\psi(0) + \varphi_+ \tag{7b}$$

$$Q_-\psi(\tau) = RJQ_+\psi(\tau) + \varphi_- \ . \tag{7c}$$

In all cases, J and R are as above. It is possible to add an inhomogeneous term $f(x)$ to the right-hand sides of Eqs. (3a), (4a), (5a), (6a) and (7a). For the last three problems we mean by a solution a continuous function $\psi : [0,\tau] \to D(T) \subset H$ such that $T\psi$ is differentiable on $(0,\tau)$ and satisfies the boundary value problem.

At this specific point, and prior to discussing the equivalent integral equations, we would like to justify why $T\psi$ was taken differentiable and not ψ. This notion of differentiability has been stressed in particular in previous conference contributions of R. J. Hangelbroek and H. G. Kaper. In fact, such a notion is needed to get a full equivalence proof with a vector valued integral equation, but then one must require φ_+, $\varphi_- \in D(T)$. On the other hand, if φ_+, $\varphi_- \in H$ (rather than $D(T)$), one must require ψ to be differentiable in the strong sense on $(0,\tau)$ with derivative satisfying $\psi'(x) \in D(T)$ $(x \in (0,\tau))$. The solution will still satisfy the integral equation, but the converse proof that every solution of the integral equation satisfies the boundary value problem breaks down. For bounded T, most important to neutron transport and radiative transfer, the two notions will lead to the same set of solutions. If the compactness of B is dropped and a weaker solution concept is warranted (see Refs. 9-12), there is no equivalent integral equation.

Let us formulate the integral equations equivalent to the above boundary value problems. Basically, in order to derive the integral equation from the boundary value problems one first writes the vector-valued differential equation in the form

$$(T\psi)'(x) + \psi(x) = B\psi(x) , \quad 0 < x < \tau ,$$

and solves the latter using the boundary conditions. Implementing this procedure for Eqs. (3) we obtain

$$\psi(x) - \int_0^\infty H(x-y)B\psi(y)dy = e^{-xT^{-1}}\varphi_+ , \quad 0 < x < \infty . \tag{8}$$

Conversely, every bounded[29] solution can be proven bounded and continuous on $[0,\infty)$. Moreover, for such a solution ψ one can prove $T\psi$ strongly differentiable on $(0,\infty)$ and satisfying Eqs. (3a)-(3c). This procedure was first followed by Van der Mee[14] for bounded T, and later for unbounded T (see Refs. 25, 29). When dealing with Eqs. (4), one writes down Eq. (8) with φ_+ replaced by $RJQ_-\psi(0)+\varphi_+$ and the net result turns out to be

$$\psi(x) - \int_0^\infty H(x-y)B\psi(y)dy = e^{-xT^{-1}}[\varphi_+ + RJQ_-\psi(0)] , \quad 0<x<\infty .$$

On premultiplying by Q_- and inserting $x=0$, we get

$$Q_-\psi(0) = \int_0^\infty H(-y)B\psi(y)dy ,$$

which on substitution yields the integral equation

$$\psi(x) - \int_0^\infty [H(x-y) + e^{-xT^{-1}} Q_+RJH(-y)]B\psi(y)dy = e^{-xT^{-1}}\varphi_+ , \qquad (9)$$

$$0<x<\infty .$$

As a result we have obtained an integral equation with a kernel whose first term is of convolution type and whose second term is separated. The equivalence proof holds true if $\varphi_+ \in Q_+[D(T)]$. The result can be extended in a straightforward way, if one adds a term $f(x)$ to the right-hand sides of Eqs. (3a) and (4a) which is uniformly Hölder continuous on $[0,\infty)$ and satisfies $\int_1^\infty (\|f(t)\|_H/t)dt < \infty$.

Next, let us turn to the finite layer problems. For Eqs. (5) the method is entirely the same as for Eqs. (4) (see Refs. 14, 15, 19). One obtains the convolution integral equation

$$\psi(x) - \int_0^\tau H(x-y)B\psi(y)dy = e^{-xT^{-1}}\varphi_+ + e^{(\tau-x)T^{-1}}\varphi_- , \qquad (10)$$

$$0<x<\tau .$$

On replacing φ_- by $JRQ_+\psi(\tau)+\varphi_-$, premultiplying by Q_+ and substituting $x=\tau$, one obtains easily

$$Q_+\psi(\tau) = \int_0^\tau H(\tau - y)B\psi(y)dy + e^{-\tau T^{-1}}\varphi_+ ,$$

whence

$$\psi(x) - \int_0^\tau [H(x-y) + e^{(\tau-x)T^{-1}} JR\ (\tau-y)]\,B\psi(y)dy = \tag{11}$$

$$= e^{-xT^{-1}}\varphi_+ + e^{(\tau-x)T^{-1}}[\varphi_- + JRe^{-\tau T^{-1}}\varphi_+] .$$

Finally, in order to convert Eqs. (7) to an integral equation, we replace φ_+ and φ_- by $RJQ_-\psi(0) + \varphi_+$ and $RJQ_+\psi(\tau) + \varphi_-$, and premultiply the expression (10) for $x = 0$ by Q_- and for $x = \tau$ by Q_+, respectively. We obtain

$$Q_+\psi(\tau) - e^{-\tau T^{-1}} RJQ_-\psi(0) = e^{-\tau T^{-1}}\varphi_+ + \int_0^\tau H(\tau-y)B\psi(y)dy$$

and

$$Q_-\psi(0) - e^{\tau T^{-1}} RJQ_+\psi(\tau) = e^{\tau T^{-1}}\varphi_- + \int_0^\tau H(-y)B\psi(y)dy .$$

These equations can be written concisely as

$$(I - e^{-\tau|T|^{-1}} RJ)\{Q_+\psi(\tau) + Q_-\psi(0)\} = e^{-\tau|T|^{-1}}\varphi + \int_0^\tau K(y)B\psi(y)dy ,$$

where $\varphi = \varphi_+ + \varphi_-$ and $K(y) = H(\tau-y) + H(-y)$. If $(I-\exp(-\tau|T|^{-1})RJ)$ is invertible, which occurs, for instance, if $\|R\|_H < 1$, and if $\|R\|_H \le 1$ and T is bounded, we obtain the integral equation

$$\psi(x) - \int_0^\tau [H(x-y) +$$

$$+ \{e^{-xT^{-1}}Q_+ + e^{(\tau-x)T^{-1}}Q_-\}RJ(I-e^{-\tau|T|^{-1}}RJ)^{-1}\{H(\tau-y)+H(-y)\}]B\psi(y)dy =$$

$$= \{e^{-xT^{-1}}Q_+ + e^{(\tau-x)T^{-1}}Q_-\}[I + RJ(I - e^{-\tau|T|^{-1}}RJ)^{-1}e^{-\tau|T|^{-1}}]\varphi, \tag{12}$$

where $x \in (0,\tau)$. It should be assumed that $\varphi_\pm \in D(T)$ in order to have equivalence. If an inhomogeneous term $f(x)$ is added to Eqs. (5a), (6a) or (7a), it should be uniformly Hölder continuous on $[0,\tau]$. In that case Eqs. (10), (11) and (12) can be extended in a straightforward way.

In most applications where B is compact, B has finite rank. In these cases one may reduce the dimensionality of the space on which Eqs. (8)-(12) are defined to a finite number, namely the rank of B. Let us choose a closed subspace M of H containing Ran B^*. Let $\pi : H \to M$ and $j : M \to H$ be operators such that πj is the identity on M and $j\pi$ the orthogonal projection of H onto M. On introducing $\chi(x) = \pi\psi(x)$ and using $Bj\pi = B$, the solutions of Eq. (8) may be expressed in terms of $\chi(x)$ as

$$\psi(x) = e^{-xT^{-1}}\varphi_+ + \int_0^\infty H(x-y)Bj\chi(y)dy , \tag{13}$$

where

$$\chi(x) - \int_0^\infty \pi H(x-y)Bj\chi(y)dy = \pi e^{-xT^{-1}}\varphi_+ , \quad 0 < x < \infty , \tag{14}$$

is a Wiener-Hopf equation in the (usually finite-dimensional) space M. Similar procedures may be followed for Eqs. (9)-(12).

Finally, in order to conclude this section, we discuss the solution spaces on which to study Eqs. (8)-(12). Let us denote by $L_p(H)_0^\tau$ the (real or complex) Banach space of all strongly measurable[30] functions $\psi : (0,\tau) \to H$ that are bounded with respect to the norm

$$\|\psi\|_p = \begin{cases} \left[\int_0^\tau \|\psi(x)\|_H^p dx\right]^{1/p} , & 1 \le p < \infty \\ \text{ess sup}\,\{\|\psi(x)\|_H \;/\; x \in (0,\tau)\} , & p = \infty . \end{cases}$$

By $C(H)_0^\tau$ we denote the (real or complex) Banach space of all continuous functions $\psi : [0,\tau] \to H$ if τ is finite, or all bounded and continuous functions $\psi : [0,\infty) \to H$ if $\tau = \infty$, endowed with the norm

$$\|\psi\|_c = \sup\{\|\psi(x)\|_H \;/\; x \in (0,\tau)\} \;.$$

Then the convolution equations (8) and (10), occurring because reflection processes are neglected, can be written as

$$(I - L_\tau)\psi = \omega \;, \tag{15}$$

where $(L_\tau\psi)(x) = \int_0^\tau H(x-y)B\psi(y)dy$ is a bounded operator on all spaces $L_p(H)_0^\tau$ and $C(H)_0^\tau$ $(p \in [1,\infty],\; \tau \in (0,\infty])$. An upper bound for the norm of L_τ on all of these spaces is given by

$$\|L_\tau\| \leq \int_{-\tau}^{\tau} \|H(z)B\|\, dz \;; \tag{16}$$

the finiteness of the upper bound is guaranteed by condition (2) (see Refs. 15 and 19). The right-hand side $\omega \in C(H)_0^\tau \subset L_\infty(H)_0^\tau$. In extending the result to Eqs. (9), (11) and (12), where boundary reflection processes come to the fore, we have to restrict ourselves to the spaces $L_\infty(H)_0^\tau$ and $C(H)_0^\tau$ and write these equations as

$$(I - N_\tau)\psi = \omega \;, \tag{17}$$

where N_τ is bounded and $\omega \in C(H)_0^\tau \subset L_\infty(H)_0^\tau$. We easily obtain

$$\|N_\tau\| \leq \int_{-\tau}^{\tau} \|H(z)B\|\, dz\; \{1 + \|R\|\, K\} \;,$$

where $K = 1$ for Eqs. (9) and (11) and $K = \|(1 - e^{-\tau|T|^{-1}}RJ)^{-1}\|$ for Eq. (12). One will bring about some technical trouble in extending (17) to other function spaces.

3. Representations of Solutions

In this section we review some of the methods for obtaining representations for the solutions of the boundary value problems, in particular for the half-space problem. On departing from (14), we first write down the (modified) Laplace transforms

$$\hat{\chi}_{\pm}(\lambda) = \pm \int_0^{\pm\infty} e^{x/\lambda} \chi(x)dx \ , \quad \Lambda(\lambda) = I - \int_{-\infty}^{\infty} e^{x/\lambda} \pi H(x) Bj dx \ ,$$

where $\chi(x) = \int_0^{\infty} \pi H(x-y) Bj\chi(y)dy$ for $x \in (-\infty, 0)$. In this way we convert Eq. (14) (in fact, its extension to the real line $x \in (-\infty,\infty)$) to the Riemann-Hilbert problem

$$\Lambda(\lambda)\ \hat{\chi}_{+}(\lambda) + \hat{\chi}_{-}(\lambda) = \lambda\pi T(\lambda - T)^{-1}\varphi_{+} \ , \quad \mathrm{Re}\lambda = 0 \ , \qquad (18)$$

which can be solved in principle by Wiener-Hopf factorization of the <u>dispersion function</u> $\Lambda(\lambda)$. Let us assume, for the moment, the existence of two functions $H_{\ell}(\lambda)$ and $H_{r}(\lambda)$, which both of them, together with their inverses, depend continuously on λ in the closed right half-plane (including infinity) and analytically in the open right half-plane, satisfy $H_{\ell}(0^{+}) = H_{r}(0^{+}) = I$ and obey the factorization law

$$\Lambda(\lambda)^{-1} = H_{\ell}(-\lambda)H_{r}(\lambda) \ , \quad \mathrm{Re}\lambda = 0 \ . \qquad (19)$$

We may then rewrite (18) as

$$H_{\ell}(-\lambda)^{-1}\hat{\chi}_{+}(\lambda) + H_{r}(\lambda)\hat{\chi}_{-}(\lambda) = \lambda H_{r}(\lambda)\pi T(\lambda - T)^{-1}\varphi_{+} \ , \quad \mathrm{Re}\,\lambda = 0 \ .$$

It is then possible[15] to compute $\hat{\chi}_{\pm}(\lambda)$ explicitly, substitute the result in (13) and obtain

$$\psi(0) = \varphi_+ + \int_{-\infty}^{0}\int_{0}^{\infty} \frac{\gamma}{\gamma-\mu}\,\sigma(d\mu)BjH_\ell(-\mu)H_r(\gamma)\pi\sigma(d\gamma)\varphi_+ \, . \tag{20}$$

For $\psi(x)$ with $x \in (0,\infty)$ a more complicated expression arises[19]. It can then be shown that H_ℓ and H_r satisfy the following non-linear integral equations[15]:

$$H_\ell(z)^{-1} = I - z\int_0^\infty (z+t)^{-1}H_r(t)\pi\sigma(dt)Bj \tag{21a}$$

$$H_r(z)^{-1} = I - z\int_0^\infty (z+t)^{-1}\pi\sigma(-dt)BjH_\ell(t) \, , \tag{21b}$$

where $\sigma(.)$ is the resolution of the identity of T. These equations generalize the H-equations of Chandrasekhar[31].

In order to obtain necessary and sufficient conditions for the solution formula (20) to hold true, we first discuss the existence of the factorization (19). A necessary condition for its existence clearly is that $\Lambda(\lambda)$ is invertible for all extended imaginary λ. Since

$$\Lambda(\lambda) = I - \lambda\pi(\lambda - T)^{-1}Bj = \pi(T-\lambda)^{-1}(T-\lambda A)j \quad ,$$

it is necessary that $T^{-1}A$ does not have zero or imaginary eigenvalues. However, it is not a sufficient condition. However, on assuming unique solvability of Eqs. (3), there is a unique operator that maps $\varphi_+ \in Q_+[D(T)]$ into $\psi(0) \in D(T)$ and extends to a bounded operator, say E_+, on H. We may then write down the factors H_ℓ and H_r explicitly[15] as

$$H_\ell(-\lambda) = I - \lambda\pi(T-\lambda A)^{-1}\hat{E}_+Bj \tag{22a}$$

$$H_r(\lambda) = I - \lambda\pi(I-E_+)(T-\lambda A)^{-1}Bj \tag{22b}$$

$$H_\ell(-\lambda)^{-1} = I - \lambda\pi E_+(\lambda - T)^{-1}Bj \tag{22c}$$

$$H_r(\lambda)^{-1} = I - \lambda\pi(\lambda - T)^{-1}(I - \hat{E}_+)Bj \quad , \tag{22d}$$

where $\hat{E}_+ = TE_+T^{-1}$ can be proven bounded on H, and formula (20) follows. In the case when the condition that $T^{-1}A$ has no zero or imaginary eigenvalues is violated, while there still is a projection operator E_+ singling out a solution ψ for each incident "flux" φ_+, one may write down Eqs. (22), check (19) and (21) directly and obtain the solution formula (20). However, $H_\ell(-\lambda)$ and $H_r(\lambda)$ cannot be analytically continued to every extended imaginary λ.

At this point we would like to make some comments on history, related problems and generalizations. The factorization formulas (22) are more sophisticated versions of analogs obtained by Van der Mee[24,14,32] using a previously developed cascade decomposition of linear systems[32,33]. Formula (20) and its generalization to cases where $T^{-1}A$ has zero or imaginary eigenvalues — the corresponding factors (22) do not follow using the method of Refs. 32 and 33 — were obtained in the above way in Ref. 15. Clearly, Eq. (20) generalizes many similar expressions obtained previously by resolvent integration[22]. The method of obtaining a formula like Eq. (20) has been used previously by Burniston et al.[20] for two-group neutron transport and by Kelley[21] for multigroup and continuously energy-dependent models. Recently, Eq. (20) has been generalized to multigroup models with nondiagonal cross-section matrices by Willis et al.[34]; in this case T is not selfadjoint nor allows a Spectral Theorem.

In the case when the finite-slab problem (5) without reflection is uniquely solvable, one may proceed by invariant imbedding and obtain a generalization of Eq. (20), in fact, formulas for

$\psi(0)$ and $\psi(\tau)$, which contain four X- and four Y-functions related by nonlinear integral equations (see Ref. 19). These functions generalize the X- and Y-functions of Chandrasekhar[31]. It should be remarked that the generalized H-equations (21), as well as the generalized X- and Y-equations, may be nonuniquely solvable. For simplified radiative transfer problems where these functions are scalar, Mullikin[35] has indicated constraints that should specify the physically relevant solution uniquely. It is not clear sofar how this work should be generalized to the present setting.

4. The Projection and Semigroup Formalism

In Refs. 24, 14 and 32 the factorization formulas (22), or rather certain less sophisticated versions, were obtained by interpreting E_+ as a projection onto a suitable invariant subspace of $T^{-1}A$ along the invariant subspace $\operatorname{Ran} Q_-$ of T^{-1}. This interpretation enabled us to derive these formulas using the cascade decomposition method of Bart et al.[32,33]. Such an interpretation is apparent if $A = I - B$ is a strictly positive selfadjoint operator, since $T^{-1}A$ then is selfadjoint with respect to the inner product $(h,k)_A = (Ah,k)$ (see Refs. 9-17; the inner product goes back to Ref. 16). In this section we shall find a way of extending this interpretation to the general case.

Let us first discuss the case when A is strictly positive selfadjoint. Then $A^{-1}T$ is selfadjoint with respect to the (equivalent) inner product

$$(h,k)_A = (Ah,k) \ , \tag{23}$$

and we may then define P_+ and P_- as the projections, orthogonal with respect to (23), onto the maximal positive and negative (with respect to (23)) $A^{-1}T$-invariant subspaces. In this way we have used the Spectral Theorem. It can then be shown that the operator $V = Q_+P_+ + Q_-P_-$ is invertible[36], whence $E_+ = V^{-1}Q_+$ is the bounded projection of H onto the range of P_+ along the range of Q_-.

In order to write down Eqs. (22), let us assume that $T^{-1}A$ does not have zero or imaginary eigenvalues. Since (23) no longer is an inner product if A is nonselfadjoint, we must find another way of defining P_+ and P_-. For this reason we consider the convolution equation

$$\psi_\varphi(x) - \int_{-\infty}^{\infty} H(x-y)B\psi_\varphi(y)dy = \omega_\varphi(x) \quad , \qquad 0 \neq x \in \mathbb{R} , \qquad (24)$$

where

$$\omega_\varphi(x) = \begin{cases} + e^{-xT^{-1}} Q_+\varphi , & x > 0 \\ - e^{-xT^{-1}} Q_-\varphi , & x < 0 \end{cases} . \qquad (25)$$

On solving Eq. (24) on the space $L_\infty(H)_{-\infty}^{\infty}$, we take (modified) Laplace transforms on both sides and obtain the algebraic equation

$$W(\lambda)\hat{\psi}_\varphi(\lambda) = \hat{\omega}_\varphi(\lambda) \quad , \qquad \mathrm{Re}\lambda = 0 ,$$

where

$$\hat{\psi}_\varphi(\lambda) = \int_{-\infty}^{\infty} e^{x/\lambda}\psi_\varphi(x)dx \quad , \qquad \hat{\omega}_\varphi(\lambda) = \int_{-\infty}^{\infty} e^{x/\lambda}\omega_\varphi(x)dx$$

and

$$W(\lambda) = I - \int_{-\infty}^{\infty} e^{x/\lambda}H(x)Bdx = (T-\lambda)^{-1}(T-\lambda A) \ .$$

In the case where A is strictly positive selfadjoint, it is straightforward to check that

$$\psi_\varphi(x) = \begin{cases} + e^{-xT^{-1}A} P_+\varphi , & x > 0 \\ - e^{-xT^{-1}A} P_-\varphi , & x < 0 . \end{cases} \tag{26}$$

However, since $\int_{-\infty}^{\infty} \|H(x)B\| \, dx < \infty$, it is possible to prove that[37,38]

$$W(\lambda)^{-1} = I + \int_{-\infty}^{\infty} e^{x/\lambda} \ell(x) dx , \quad \text{Re}\lambda = 0 ,$$

for some strongly measurable function ℓ satisfying $\int_{-\infty}^{\infty} \|\ell(x)\| \, dx < \infty$, whence $\psi(x)$ is bounded and continuous on $(-\infty,\infty)$ except for a jump discontinuity with

$$(\psi_\varphi(0^+) - \psi_\varphi(0^+)) = (\omega_\varphi(0^+) - \omega_\varphi(0^-)) = \varphi .$$

One may now use (26) to introduce P_+ and P_- by

$$P_\pm\varphi = \pm\psi_\varphi(0^\pm).$$

It can then be shown[15,19] that P_+ and P_- are complementary bounded projections of H commuting with $T^{-1}A$ and that

$$\sigma(T^{-1}A|_{\text{Ran } P_\pm}) \subset \{0\} \cup \{\lambda \in \mathbb{C} / \pm\text{Re}\lambda > 0\} .$$

Also, the restriction of $\pm T^{-1}A$ to Ran $P_\pm$ generates a bounded strongly continuous semigroup $\{V_\pm(x)\}_{x>0}$, which can also be defined by

$$V_\pm(x)P_\pm\varphi = \pm\psi_\varphi(\pm x) , \quad x \in (0,\infty) .$$

We may then go on proving that the boundary value problem (3) is uniquely solvable if

$$\text{Ran } P_+ \oplus \text{Ran } Q_- = H .$$

If the latter condition is satisfied, we define E_+ as the projection of H onto $\operatorname{Ran} P_+$ along $\operatorname{Ran} Q_-$, and obtain for the solution of Eqs. (3)

$$\psi(x) = e^{-xT^{-1}A} E_+\varphi_+ , \quad 0 \le x < \infty . \tag{27}$$

Next, let us consider the more general case when $T^{-1}A$ has either zero or imaginary eigenvalues. On defining, for some operator S and some $\lambda \in \mathbb{C}$, the λ-root manifold $Z_\lambda(S)$ by

$$Z_\lambda(S) = \bigcup_n \operatorname{Ker} (S-\lambda)^n ,$$

we assume that $T^{-1}A$ has at most finitely many zero or imaginary eigenvalues, all of finite algebraic multiplicity. (If T is bounded, the assumption holds true automatically). We then introduce the finite-dimensional manifolds Z_0 and $\hat{Z}_0$ by

$$Z_0 = \bigoplus_{\operatorname{Re}\lambda=0} Z_\lambda(T^{-1}A) , \qquad \hat{Z}_0 = \bigoplus_{\operatorname{Re}\lambda=0} Z_\lambda(AT^{-1}) .$$

Then $A[Z_0] \subset \hat{Z}_0$ and $T[Z_0] = \hat{Z}_0$. On defining the maximal $T^{-1}A$ and AT^{-1} invariant subspaces of H on which the restrictions of these operators do not have zero or imaginary eigenvalues by Z_1 and $\hat{Z}_1$, respectively, we obtain

$$Z_0 \oplus Z_1 = H , \qquad \hat{Z}_0 \oplus \hat{Z}_1 = H ,$$

as well as

$$T[Z_0] = \hat{Z}_0 , \qquad \overline{T[Z_1]} = A[Z_1] = \hat{Z}_1 .$$

On choosing an invertible β on Z_0 without imaginary eigenvalues, we may put

$$A_\beta = T\beta^{-1}P_0 + AP_1 , \qquad B_\beta = I - A_\beta ,$$

where P_0 and P_1 are the complementary projections with ranges Z_0 and Z_1, respectively. Then A_β is invertible, B_β is compact and

$$A_\beta^{-1}T = \beta \oplus (A^{-1}T)|_{Z_1}$$

does not have imaginary eigenvalues. If condition (2) is fulfilled, we have

$$\exists\ \alpha > 0 :\ \ \text{Ran}\ B_\beta \subset \text{Ran}|T|^\alpha \cap D(|T|^{1+\alpha})$$

under minor additional hypotheses on B [39]. We then define $P_{+,\beta}$ and $P_{-,\beta}$ as the operators P_+ and P_- connected with $A_\beta^{-1}T$. In terms of these we then define the β-independent projections

$$P_{1,\pm} = P_1 P_{\pm,\beta} = P_{\pm,\beta}\, P_1 .$$

If there is a subspace N_+ of Z_0 satisfying the two conditions

$$\text{Ran}\ P_{1,+} \oplus N_+ \oplus \text{Ran}\ Q_- = H$$

and

$$N_+ \subset \bigoplus_{\text{Re}\lambda=0} \text{Ker}\ (T^{-1}A - \lambda) \subset Z_0 \quad ,$$

we may define E_+ as the projection of H onto $\text{Ran}\ P_{1,+} \oplus N_+$ along $\text{Ran}\ Q_-$, whence (27) provides a solution to Eqs. (3). For details we refer to Ref. 19.

At this point we would like to make some remarks of a historical nature. For one-speed neutron transport in non-multiplying media and in an L_p-space setting, the above projection and semigroup formalism was developed by Van der Mee [24]. It generalizes the formalism for positive selfadjoint A that can be developed using the Spectral Theorem.[9,17] The above more general formalism has been worked out in detail in Refs. 15 and 19. Recently, Bart et al.[40] have developed a Hille-Yosida type theory of generators of exponentially decaying bisemigroups, i.e. of expressions of the type (25) or (26) that decrease exponentially in the norm as $x \to \pm\infty$. If $T^{-1}A$ does not have zero or imaginary eigenvalues and if T is bounded, the expressions (25) and (26) are bisemigroups of the type they studied.

5. The Fredholm Alternative

Recently, Willis et al.[25] have proposed a method for obtaining existence results for transport equations from uniqueness results, by using the Fredholm alternative. The Fredholm alternative can be phrased as follows: If $I - K$ is a Fredholm operator of index zero (for instance, if K or some power of K is compact), then the equation

$$(I - K)\varphi = \omega \tag{28}$$

has precisely one solution φ for every vector ω in the underlying Banach space, if $\mathrm{Ker}(I-K) = \{\varphi \mid (I-K)\varphi = 0\} = \{0\}$. Thus, uniqueness of solutions to Eq. (28) implies existence.

Because B is a compact operator on H and $\int_{-\tau}^{\tau} \|H(x)B\|\, dx < \infty$ [41], it can be shown that the convolution operator

$$(L_\tau\psi)(x) = \int_0^\tau H(x-y)B\psi(y)dy\ , \quad x \in (0,\tau)\ ,$$

is compact on the Banach spaces $L_p(H)_0^\tau$ $(1 \le p \le \infty)$ and $C(H)_0^\tau$. In view of the equivalence of the finite-slab boundary value problem (5) and the convolution equation (10) on $L_\infty(H)_0^\tau$ (which is the vector equation (15) on $L_\infty(H)_0^\tau$), it suffices to prove that Eqs. (5) with zero incident "fluxes" $\varphi_+ = \varphi_- = 0$ have only the zero solution in order to have existence and uniqueness of Eqs. (5). However, on extending these results to Eqs. (6) and Eqs. (7), one should remark that the operators

$$(N_\tau^{(uni)}\psi)(x) = \int_0^\tau [H(x-y) + e^{(\tau-x)T^{-1}} JRH(\tau-y)]B\psi(y)dy$$

and

$$(N_\tau^{(bi)}\psi)(x) = \int_0^\tau [H(x-y) + \{e^{-xT^{-1}} Q_+ + e^{(\tau-x)T^{-1}} Q_-\}RJ \cdot$$
$$\cdot (I - e^{-\tau|T|^{-1}} RJ)^{-1}\{H(\tau-y) + H(-y)\}]B\psi(y)dy$$

can be shown to be compact on $L_\infty(H)_0^\tau$ and $C(H)_0^\tau$ also, provided R is bounded on H and, in the case of $N_\tau^{(bi)}$, $(I-\exp(-\tau|T|^{-1})RJ)$ is invertible. Indeed, both $N_\tau^{(uni)}$ and $N_\tau^{(bi)}$ can be written as the sum of L_τ and of M_τ, where

$$(M_\tau\psi)(x) = k(x) \int_0^\tau z(y)B\psi(y)dy \ , \quad 0 \le x \le \tau \ ,$$

where $k(x)$ is strongly continuous on $[0,\tau]$, $z(y)$ is almost everywhere bounded and continuous, and $\int_0^\tau \|z(y)B\| \, dy < \infty$. One may then go on to show that M_τ, and therefore $N_\tau^{(uni)}$ and $N_\tau^{(bi)}$, are compact on $L_\infty(H)_0^\tau$ and $C(H)_0^\tau$. Hence, Eqs. (6) and Eqs. (7) are unique solvable, if the corresponding problems with zero incident "fluxes" $\varphi_+ = \varphi_- = 0$ have the zero solution only, R is bounded on H and, in the case of Eqs. (7), $(I-\exp(-\tau|T|^{-1})RJ)$ is invertible on H.

There is one important special case where uniqueness can be proved. Let us first consider the unilaterally reflective slab problem (6), where

$$\operatorname{Re} A = \frac{1}{2}(A+A^*) \geq 0 , \quad \operatorname{Ker}(\operatorname{Re} A) = \operatorname{Ker} A . \tag{29}$$

Under conditions of zero incident "fluxes" $\varphi_+ = \varphi_- = 0$, we have

$$0 \geq -2\int_0^\tau ((\operatorname{Re} A)\psi(x),\psi(x))_H dx = \int_0^\tau \{((T\psi)'(x),\psi(x))_H + (\psi(x),(T\psi)'(x))_H\}dx =$$

$$= (T\psi(\tau),\psi(\tau)) - (T\psi(0),\psi(0)) = (TQ_+\psi(\tau),Q_+\psi(\tau)) - (TQ_-\psi(0),Q_-\psi(0)) +$$

$$+ (TJRQ_+\psi(\tau),JRQ_+\psi(\tau)) \geq (Q_+\psi(\tau),Q_+\psi(\tau))_T - (RQ_+\psi(\tau),RQ_+\psi(\tau))_T$$

where

$$(h,k)_T = (|T|\, h,k) = (T(Q_+ - Q_-)h,k) . \tag{30}$$

Hence, if R is bounded on H and satisfies the intertwining formulas

$$R[D(T)] \subset D(T) , \quad TR = \hat{R}T \tag{31}$$

for some bounded $\hat{R}$ on H, while

$$\|Rh\|_T \leq \|h\|_T , \quad h \in D(T) , \tag{32}$$

then Eqs. (6) are uniquely solvable as a result of the Fredholm alternative. Indeed, on assuming (31) and (32), one obtains

$$0 \geq -2\int_0^\tau ((\operatorname{Re} A)\psi(x),\psi(x))_H dx \geq 0 ,$$

and consequently [cf. (29)] $(\operatorname{Re} A)\psi(x) \equiv 0$ and $A\psi(x) \equiv 0$. As a result of (6a) we then get $\psi(x) \equiv h \in \operatorname{Ker} A$, whence, on recalling $\varphi_+ = \varphi_- = 0$,

$$h = Q_+\psi(0) + Q_-\psi(\tau) = JRQ_+\psi(\tau) = JRQ_+h \in (\operatorname{Ker} A) \cap \operatorname{Ran} Q_- .$$

Next, we easily compute, since $\operatorname{Ker} A \subset \operatorname{Ran} B \subset D(T)$ [cf. (2)],

$$0 \le \|h\|_T^2 = \|Q_-h\|_T^2 = \|Q_-\psi(\tau)\|_T^2 = \|JRQ_+\psi(\tau)\|_T^2$$
$$\le \|Q_+\psi(\tau)\|_T^2 = \|Q_+h\|_T^2 = 0 \ ,$$

whence $h = 0$ and $\psi(x) \equiv 0$. For the bilaterally reflective slab problem (7) one proceeds in the same way, provided conditions (31) and (32) are satisfied and $(I - \exp(-\tau|T|^{-1})RJ)$ is invertible[42]. The net result will then be $\psi(x) \equiv h \in \operatorname{Ker} A$, whence, on using the Ansatz $\varphi_+ = \varphi_- = 0$,

$$h = Q_+\psi(0) + Q_-\psi(\tau) = RJQ_-\psi(0) + RJQ_+\psi(\tau) = RJh \ .$$

For scattering laws where

$$\|Rh\|_T < \|h\|_T \ , \quad h \in D(T)\setminus\{0\}, \tag{33}$$

we then obtain $h = 0$ and $\psi(x) \equiv 0$, and therefore[42] existence and uniqueness. In this way one may retrieve the existence and uniqueness theory for the equation of transfer of polarized light[43,44] in finite optical layers, since in this case conditions (29), (31) and (32) are satisfied.

In order to deal with applications of the Fredholm alternative to the half-space problem (3), where A satisfies condition (29), we first show that under condition (29) the operator $T^{-1}A$ cannot have nonzero imaginary eigenvalues. Indeed, if $Ah = \lambda Th$ for $\operatorname{Re}\lambda = 0$, then

$$0 \le 2((\operatorname{Re} A)h,h) = (Ah,h) + (h,Ah) = (\lambda Th,h) + (h,\lambda Th) = 0 \ ,$$

because $\bar{\lambda} = -\lambda$; since $\operatorname{Re} A \ge 0$, we have $(\operatorname{Re} A)h = 0$, whence $Ah = 0$ and therefore $\lambda = 0$ or $h = 0$. Hence, if condition (29) is satisfied and A is invertible, then

$$W(\lambda) = I - \int_{-\infty}^{\infty} e^{x/\lambda} H(x) B dx = (T-\lambda)^{-1}(T-\lambda A), \quad \mathrm{Re}\lambda = 0 \ ,$$

is invertible for all extended imaginary λ, and therefore the Wiener-Hopf operator $(L_\infty\psi)(x) = \int_0^\infty H(x-y)B\psi(y)dy$ has the property that $(I-L_\infty)$, i.e. the operator relevant to the left-hand side of Eq. (8), is a Fredholm operator. We may now repeat the previous uniqueness argument [for $\tau=\infty$, $R=0$, for B and for B^* separately] and show that Eqs. (3) and the corresponding adjoint problem [i.e. with B replaced by B^*] have at most one solution. Therefore[45], $(I-L_\infty)$ is a Fredholm operator of index zero. The Fredholm alternative then yields the unique solvability of Eqs. (3). In order to obtain the same result for Eqs.(4), one assumes that A satisfies (29) and is invertible, and that R satisfies (31) and (33). The uniqueness argument then yields that Eqs. (4) have at most one solution. On introducing the operator

$$(\hat{N}_\infty\psi)(x) = \int_0^\infty [H(x-y) + e^{-xT^{-1}} Q_+ RJH(-y)]B\psi(y)dy \ ,$$

it can be shown that $(\hat{N}_\infty - L_\infty)$ is a compact operator on $L_\infty(H)_0^\infty$ and $C(H)_0^\infty$, since $e^{-xT^{-1}}Q_+$ is strongly continuous and bounded on $[0,\infty)$, RJ is bounded on H, B is compact and $\int_0^\infty \|H(-y)B\|\, dy < \infty$. In combination with the fact that $(I-L_\infty)$ is a Fredholm operator of index zero, we have available the Fredholm alternative. As a consequence, under the above hypotheses Eqs. (4) are uniquely solvable. The above method breaks down in the following cases:

(i) A is still invertible, but R satisfies (32) instead of (33), since in this case the uniqueness argument breaks down, and

(ii) $\mathrm{Ker}\, A \neq \{0\}$, because in this case $(I-L_\infty)$, and therefore $(I-\hat{N}_\infty)$, is not a Fredholm operator.

Research on the Fredholm alternative method by A.H. Ganchev and W. Greenberg, for half-spaces with invertible and noninvertible A, is in progress[46].

6. Discussion

We conclude this article with a discussion of four strategies for developing an existence and uniqueness theory of stationary kinetic equations. For these four situations we shall point out in particular the limitations of the method, and we shall discuss some open problems.

The first method originates from Hangelbroek and Lekkerkerker[16,17]. On stating the half-space problem (3) in the form of the boundary value problem (3a)-(3c), the major effort consists of constructing the total boundary "flux" $\psi(0)$ from the incident "flux" φ_+. Since the boundedness condition (3c) holds true, one must have $\psi(0) \in \operatorname{Ran} P_+$ (if $T^{-1}A$ does not have zero or imaginary eigenvalues) or $\psi(0) \in \operatorname{Ran} P_{1,+} \oplus \oplus \{\operatorname{Ker}(T^{-1}A - \lambda)/\operatorname{Re}\lambda = 0\}$ (more generally). The solution is then written as

$$\psi(x) = e^{-xT^{-1}A}\psi(0) , \qquad 0 \le x < \infty ,$$

where $P_+\psi(0) = \varphi_+$. Existence and uniqueness of the solution then amounts to proving that

$$[\operatorname{Ran} P_{1,+} \oplus \oplus \{\operatorname{Ker}(T^{-1}A - \lambda)/\operatorname{Re}\lambda = 0\} \oplus \operatorname{Ran} Q_- = H . \qquad (34)$$

In the most typical cases when $T^{-1}A$ does not have zero or imaginary eigenvalues, one has to prove the existence of the bounded projection of H onto $\operatorname{Ran} P_+$ along $\operatorname{Ran} Q_-$. Most typically, this is done by proving that $V = Q_+P_+ + Q_-P_-$ is invertible and by putting $E_+ = V^{-1}Q_+$ for this projection. Two strategies have emerged to prove the invertibility of V. The first one consists of proving that $\operatorname{Ker} V = \{0\}$ and $(I - V)$ is compact. It requires the compactness of B, condition (2) and the strict positive selfadjointness of A [36]. The second strategy consists of proving that V is invertible on an extension of $D(T)$, namely the completion of $D(T)$ with respect to (30)[9,10], where the strict positive selfadjointness of A is required.

The second method consists of writing the boundary value problem in integral form and studying the spectral radius of the integral operator. This method has proven itself quite effective if B leaves invariant the positive cone of H (which then is a Banach lattice) and $\operatorname{spr}(B) \leq 1$. For multigroup neutron transport the Ansatz $\operatorname{spr}(B) < 1$ is sufficient to prove the unique solvability of Eqs. (6) for the multigroup case[19], using monotonicity. This method has been used in a variety of situations, also for reflective boundaries, inhomogeneous media and non-plane-parallel spatial domains[47].

The third method consists of applying the Fredholm alternative to the integral formulation of a problem with at most one solution, in order to obtain unique solvability. It is most effective for finite spatial domains and dissipative reflection by the boundaries. The method has been discussed extensively in Section 5.

The fourth method consists of viewing the transport equation as a vector equation

$$A\psi = f \tag{35}$$

on $L_p(D \times V)$, where D is the spatial domain, V the velocity domain, f the inhomogeneous term (accounting for internal sources) and $p \in [1,\infty)$. Equation (35) is augmented by a reflective boundary condition of the form

$$\psi_+ = K\,\psi_- + \varphi_+ \,, \tag{36}$$

where ψ_+ and ψ_- are the incoming and outgoing traces of ψ on the boundary and φ_+ is the incident "flux". The existence of "traces" of functions in the domain of the free streaming operator (i.e. A if external forces and scattering are neglected) under various hypotheses on K has been studied thoroughly by Voigt[48] and these results have been generalized by Beals and Protopopescu[49] to more general transport operators. The method of showing unique solvability consists of three steps: (i) solving Eq. (36) for some L_p-traces $\tilde{\psi}_\pm$ for given φ_+, (ii) extending $\tilde{\psi}_+$ and $\tilde{\psi}_-$ to a function $\tilde{\psi} \in L_p(D\times V)$ satisfying $A\tilde{\psi} \in L_p(D\times V)$, and (iii) solving the equation $A\hat{\psi} = A\tilde{\psi} + f$ under the boundary condition $\hat{\psi}_+ = K\hat{\psi}_-$; the solution will then be $\psi = \hat{\psi} + \tilde{\psi}$. In order to prove unique solvability for Eqs. (35) and (36) it is sufficient to prove that $0 \notin \sigma(A_K)$, where A_K is defined by

$$A_K\psi = A\psi,\ D(A_K) = \{\psi \mid A\psi \in L_p(D\times V),\ \psi_+ = K\psi_-\} \,.$$

Under reasonable assumptions on A and K, the latter operator will generate a strongly continuous semigroup on $L_p(D\times V)$ [cf. Ref. 49 and the references quoted there]. However, one must assume $\|K\| < 1$ in order to have L_p-traces of solutions $\psi \in L_p(D\times V)$. For $\|K\| = 1$ there exist a number of examples[48], where solutions $\psi \in L_p(D\times V)$ of the corresponding time-dependent transport problem do not have L_p-traces. A method as sketched above has been worked out by C. Bardos and various co-authors for the linearized Boltzmann equation with $K = 0$, in finite plane-parallel media and in combination with or in anticipation of the investigation of nonlinear problems. For work of this nature we refer to their contributions to these proceedings.

Summarizing, it seems obvious that the first method is entirely restricted to plane-parallel homogeneous media. For the bulk of the proofs to go through one must also assume that the scattering term is selfadjoint and the medium is not multiplying. The other three methods are not limited to plane-parallel homogeneous media. All three of them require some sort of nonmultiplying medium assumption, for the second method to apply the contraction principle to the integral equation, for the third one to prove uniqueness of the solution, and for the fourth method to derive $0 \notin \sigma(A_K)$ from the (strict) contractivity of the time evolution semigroup. For all four methods there is considerable difficulty in proving existence and uniqueness for conservative media and $\|K\| = 1$. While the first three methods, especially the first one, allow various abstract generalizations if the spatial domain is plane-parallel, the last method allows an easier incorporation of external forces and a more transparant route to a joint treatment of stationary and time-dependent problems.

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42. The condition that $(I - \exp(-\tau|T|^{-1})RJ)$ is invertible on H, is fulfilled if $\|R\|_T < 1$. Indeed, in this case the above operator is invertible on the completion H_T of D(T) with respect to (30), while it also satisfies $T\{I - \exp(-\tau|T|^{-1})RJ\} = \{I + \exp(-\tau|T|^{-1})\hat{R}J\}T$ on D(T), whence $\text{spr}[\exp(-\tau|T|^{-1})]$ is the same on H and H_T.

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48. J. Voigt, "Functional-Analytic Treatment of the Initial Boundary Value Problem for Collisionless Gases", Munich, Habilitationsschrift, 1980.

49. R. Beals and V. Protopopescu, "Abstract Time-Dependent Equations", J. Math. Anal. Appl., to appear.

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