TRANSPORT THEORY AND STATISTICAL PHYSICS, 15(6&7), 897-916 (1986)

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POLARIZED LIGHT TRANSFER: EXISTENCE AND UNIQUENESS OF SOLUTIONS AND SPECTRAL PROPERTIES OF TRANSFER OPERATORS

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ABSTRACT

We shall discuss the existence and uniqueness of solutions to the equation of transfer of polarized light for a finite or semiinfinite plane-parallel planetary atmospher following two different approaches. In one approach the problem is converted into a vector integral equation, of convolution type if reflection by the planetary surface is ignored, and this equation is analyzed using positive cone preservation techniques. In the second approach the boundary value problem is treated directly using positive selfadjointness of the real part of the stationary transfer operator, where an important role is played by inequalities satisfied by certain expansion coefficients.

We shall also discuss the positivity and analyticity properties of the reflection and transmission operators. A justification will be given for the adding method used to obtain numerical results.

Finally, we shall extend the results to vertically inhomogeneous atmospheres.

1. Introduction

In this article we shall discuss the equation of transfer of polarized light [1,2] for a homogeneous plane-

 0041-1450/86/1506-0897$3.50/0

parallel atmosphere of optical thickness b,

$$u \frac{\partial}{\partial \tau} \underset{\sim}{I}(\tau,u,\phi)+ \underset{\sim}{I}(\tau,u,\phi) = \frac{a}{4\pi} \int_{-1}^{1}\int_{0}^{2\pi} \underset{\sim}{Z}(u,u',\phi-\phi')\underset{\sim}{I}(\tau,u',\phi)d\phi'du'. \quad (1)$$

Here $\tau\epsilon(0,b)$ is the optical depth, $a\epsilon(0,1]$ the albedo of single scattering, $u\epsilon[-1,1]$ the direction cosine of propagation and $\phi\epsilon[0,2\pi]$ the azimuthal angle of propagation. Moreover, $\underset{\sim}{I}=(I,Q,U,V)$ is the four-vector of Stokes parameter I,Q,U and V, that specify the intensity and state of polarization of the light as a function of optical depth and direction. By $\underset{\sim}{Z}(u,u',\phi-\phi)$ we denote the phase matrix, which allows the product representation

$$\underset{\sim}{Z}(u,u',\phi-\phi') = \underset{\sim}{L}(\pi-\sigma_2)\underset{\sim}{F}(\theta)\underset{\sim}{L}(-\sigma_1) \quad (2a)$$

with $\underset{\sim}{L}(\alpha)$ being the rotation matrix

$$\underset{\sim}{L}(\alpha) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\alpha & \sin 2\alpha & 0 \\ 0 & -\sin 2\alpha & \cos 2\alpha & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (2b)$$

and $F(\theta)$ being the scattering matrix

$$\underset{\sim}{F}(\theta) = \begin{bmatrix} a_1(\theta) & b_1(\theta) & 0 & 0 \\ b_1(\theta) & a_2(\theta) & 0 & 0 \\ 0 & 0 & a_3(\theta) & b_2(\theta) \\ 0 & 0 & -b_2(\theta) & a_4(\theta) \end{bmatrix} . \quad (2c)$$

Here $u=-\cos\vartheta$ and $u'=-\cos\vartheta'$, while for $0<\phi'-\phi<\pi$ (resp. $0<\phi-\phi'<\pi$) ϑ,ϑ' and θ are the sides and σ_1,σ_2 and $\phi'-\phi$ (resp. $-\sigma_1,-\sigma_2,\phi-\phi'$) are the respective opposite angles of a spherical triangle. Throughout we follow the conventions for polarization parameters of Chandrasekhar [3] and Van de Hulst [4], and the notational system of Ref. 2.

The physical requirement that the degree of polarization $p=(Q^2+U^2+V^2)^{\frac{1}{2}}/I$ belongs to $[0,1]$ is reflected by the mathematical requirement [3,1]

$$I \geq (Q^2+U^2+V^2)^{\frac{1}{2}} \geq 0 \tag{3}$$

on the "physical" solutions to Eq.(1). At the same time the scattering matrix $\underset{\sim}{F}(\theta)$ and the phase matrix $\underset{\sim}{Z}(u,u',\phi-\phi')$ must transform vectors $\underset{\sim}{I} = (I,Q,U,V)$ satisfying condition (3) into vectors of the same type. We also have the normalization

$$\int_{-1}^{1} a_1(\theta)d\cos\theta = 2. \tag{4}$$

In this article we shall discuss the existence and uniqueness of the solution to several boundary value problems to Eq.(1), as well as some other problems. For finite optical layers, $b\in(0,\infty)$, we impose the boundary conditions

$$\underset{\sim}{I}(0,u,\phi) = \underset{\sim}{J}_+(u,\phi) \quad , \quad u>0 \tag{5a}$$

$$\underset{\sim}{I}(b,-u,\phi) = \frac{1}{\pi}\int_0^1\int_0^{2\pi} u'\underset{\sim}{R}_g(u,u',\phi-\phi')\underset{\sim}{I}(b,u',\phi')d\phi' du', \; u<0, \tag{5b}$$

where the first condition specifies sunlight incident to the top and the second condition (partial) reflection by the bottom of the atmosphere. Here $\underset{\sim}{R}_g(u,u',\phi-\phi')$, the reflection matrix of the ground, leaves invariant the set of vectors $\underset{\sim}{I} = (I,Q,U,V)$ satisfying condition (3) and obeys the symmetry laws [5]

$$\underset{\sim}{R}_g(u,u',\phi-\phi') = \underset{\sim}{P}\,\tilde{\underset{\sim}{R}}_g(u',u,\phi'-\phi)\underset{\sim}{P} \text{ (reciprocity symmetry)} \tag{6}$$

with tilde above a matrix denoting transposition, and

$$\underset{\sim}{R}_g(u,u',\phi-\phi') = \underset{\sim}{D}\,\underset{\sim}{R}_g(u,u',\phi'-\phi)\underset{\sim}{D} \text{ (mirror symmetry)}, \tag{7}$$

where $\underset{\sim}{P} = \mathrm{diag}(1,1,-1,1)$ and $\underset{\sim}{D} = \mathrm{diag}(1,1,-1,-1)$. Moreover, due to energy conservation,

$$0 \leq \frac{1}{\pi} \int_0^1 \int_0^{2\pi} u'[\underset{\sim}{R}_g(u,u',\phi-\phi')]_{11} d(\phi-\phi')du' \leq 1 ; \qquad (8)$$

for dark planetary surfaces $\underset{\sim}{R}_g(u,u',\phi-\phi') \equiv 0$. Next, for semi-infinite optical layers, $b=\infty$, we impose the boundary conditions

$$\underset{\sim}{I}(0,u,\phi) = \underset{\sim}{J}_+(u,\phi) \quad , \quad u>0 \qquad (9a)$$

$$\underset{\sim}{I}(\tau,u,\phi) = O(1)\,(\tau \to \infty), \qquad (9b)$$

where the first condition again specifies sunlight incident at the top.

In the next section we discuss the functional formulation and the existence and uniqueness results, as well as the way in which to derive them. Next, we explain the well-known adding method [6,7] and sketch the convergence proof for the series appearing during its implementation. Finally, we extend our results to vertically inhomogeneous atmospheres.

Except for the inhomogeneous atmospheres, the results are based on Refs. 8-12.

2. FUNCTIONAL FORMULATION AND WELL-POSEDNESS

Let us introduce the functional spaces H_p, H_c and $H_{1,2}$ as follows. First let Ω be the unit sphere in $\mathbb{R}^3$ endowed with the Lebesgue surface measure, and let $(u,\phi)\in[-1,1]\times[0,2\pi]$ parametrize the points $\omega\in\Omega$ [13]. Then H_p and H_c are the direct sums of four copies of the (real or complex) spaces $L_p(\Omega)$ and $C(\bar{\Omega})$, endowed with the norms

$$\|\underset{\sim}{I}\|_p = \left[\int_{-1}^{1}\int_0^{2\pi}\{|I(u,\phi)|^p+|Q(u,\phi)|^p+|U(u,\phi)|^p+|V(u,\phi)|^p\}d\phi du\right]^{1/p}$$

and

$$\|\underset{\sim}{I}\|_c = \text{Max}\{\max_{\omega\in\bar{\Omega}}|I(\omega)|,\ \max_{\omega\in\bar{\Omega}}|Q(\omega)|,\ \max_{\omega\in\bar{\Omega}}|U(\omega)|,\ \max_{\omega\in\bar{\Omega}}|V(\omega)|\},$$

where $1\leq p<\infty$. By $H_{1.2}$ we denote the direct sum of four copies of $L_2(\Omega)$ completed with respect to the norm

$$\|\underset{\sim}{I}\|_{1.2} = \int_{-1}^{1}\int_0^{2\pi}\{|I(u,\phi)|+[|Q(u,\phi)|^2+|U(u,\phi)|^2+|V(u,\phi)|^2]^{\frac{1}{2}}\, d\phi du.$$

As a consequence of the fact that (i) physical solutions $\underset{\sim}{I}=(I,Q,U,V)$ satisfy condition (3), and (ii) I is the intensity and therefore has a finite L_1-norm, the space $H_{1.2}$ seems to be naturally appropriate to the polarized light problem.

Next, let us introduce the following operators on $L_p(\Omega)$ $(1\leq p<\infty)$:

$$(T\,\underset{\sim}{I})(u,\phi)=u\underset{\sim}{I}(u,\phi),\ (B\,\underset{\sim}{I})(u,\phi)=\frac{1}{4\pi}\int_{-1}^{1}\int_0^{2\pi}\underset{\sim}{Z}(u,u',\phi-\phi')\underset{\sim}{I}(u',\phi')d\phi' du'$$

$$(Q_{\pm}\underset{\sim}{I})(u,\phi) = \begin{cases}\underset{\sim}{I}(u,\phi), & (\pm u)>0\\ 0, & (\pm u)<0\end{cases} \quad ; \quad A = 1 - aB$$

$$(R\,\underset{\sim}{I})(u,\phi) = \frac{1}{\pi}\int_0^1\int_0^{2\pi}u'\underset{\sim}{R}_g(u,u',\phi-\phi')\underset{\sim}{D}\,\underset{\sim}{I}(u',\pi-\phi')d\phi' du'$$

$$(J\,\underset{\sim}{I})(u,\phi) = \underset{\sim}{D}\,\underset{\sim}{I}(-u,\pi-\phi),$$

where $\underset{\sim}{D} = \text{diag}(1,1,-1,1)$. Then Eq.(1) with boundary conditions (5) can be written as

$$(T\,\underset{\sim}{I})'(\tau) = -A\,\underset{\sim}{I}(\tau) \quad , \quad \tau\in(0,b) \qquad (10a)$$

$$Q_+\underset{\sim}{I}(0) = \underset{\sim}{J}_+ \qquad (10b)$$

$$Q_-\underset{\sim}{I}(b) = R\,J\,Q_+\underset{\sim}{I}(b), \qquad (10c)$$

while Eq.(1) with boundary conditions (9) can be written as

$$(T\,\underset{\sim}{I})'(\tau) = -A\,\underset{\sim}{I}(\tau) \quad , \ \tau\in(0,\infty) \tag{11a}$$

$$Q_+\underset{\sim}{I}(0) = \underset{\sim}{J}_+ \tag{11b}$$

$$\|\underset{\sim}{I}(\tau)\|_p = O(1) \quad (\tau\to\infty). \tag{11c}$$

In (10a) and (11a) the derivative is a strong derivative with respect to τ. Both problems can be formulated in the same way on $H_{1.2}$. The formulation of Eqs. (11) is a far fetched generalization of the statement of the half-space problem for isotropic neutron transport by Hangelbroek [14].

By defining the "propagation function" $H(\tau)$ by

$$(H(\tau)\underset{\sim}{I})(u,\phi) = \begin{cases} |u|^{-1}e^{-\tau/u}\,\underset{\sim}{I}(u,\phi) , & \tau u>0 \\ 0 , & \tau u<0, \end{cases}$$

the boundary value problem (10) can be written as [9]

$$\underset{\sim}{I}(\tau)-a\int_0^b [H(\tau-\tau')+e^{(b-\tau)T^{-1}}Q_-J\,R\,H(b-\tau')]\,B\underset{\sim}{I}(\tau')d\tau' = \tag{12}$$

$$= [e^{-\tau T^{-1}}+e^{(b-\tau)T^{-1}}Q_-J\,R\,e^{-bT^{-1}}]\,\underset{\sim}{J}_+ , \quad \tau\in(0,b).$$

The boundary value problem (11) can be written as [8]

$$\underset{\sim}{I}(\tau) - a\int_0^\infty H(\tau-\tau')B\,\underset{\sim}{I}(\tau')d\tau' = e^{-\tau T^{-1}}\underset{\sim}{J}_+ \ , \ \tau\in(0,\infty). \tag{13}$$

More precisely, every essentially bounded [15] vector function $\underset{\sim}{I}:(0,b)\to H_p$ satisfying Eq.(12) (resp.13)) is bounded and continuous on $[0,b]$ (resp. $[0,\infty)$) and satisfies Eqs. (10) (resp. (11)). Conversely, every bounded solution of Eqs. (10) (resp.

(11)) satisfies Eq. (12) (resp.(13)). By introducing the Banach space $L_q(H_p)_0^b$ of strongly measurable L_q-functions from (0,b) into H_p, also for $q = \infty$, and by defining the bounded operators

$$(L_b \underset{\sim}{I})(\tau) = \int_0^b H(\tau-\tau')B\,\underset{\sim}{I}(\tau')d\tau', \quad \tau\in(0,b)$$

$$(M_b \underset{\sim}{I})(\tau) = \int_0^b e^{(b-\tau)T^{-1}} Q_- J\,R\,H(b-\tau')B\,\underset{\sim}{I}(\tau')d\tau', \tau\in(0,b),$$

and

$$N_b = L_b + M_b,$$

we can convert Eq.(12) to the vector equation

$$(1-aN_b)\,\underset{\sim}{I} = \underset{\sim}{\omega} \qquad (14)$$

on $L_q(H_p)_0^b$ and Eq.(13) to the vector equation

$$(1-aL_\infty)\underset{\sim}{I} = \underset{\sim}{\omega} \qquad (15)$$

on $L_q(H_p)_0^\infty$, where in both cases $\underset{\sim}{\omega}$ depends on b, $\underset{\sim}{J}_+$ and R. It should be assumed that R is bounded on H_p. One should also assume the regularity condition

$$\exists r>1: \int_{-1}^{1} a_1(\theta)^r d(\cos\theta)<\infty. \qquad (16)$$

All statements in this paragraph apply equally well to $H_{1.2}$.

Now that the original boundary value problem has been reduced to a simple vector equation on $L_q(H_p)_0^b$, where $b\in(0,\infty]$, it is appropriate to emphasize the lattice positivity structure of the problem. First of all, as observed by Germogenova and Konovalov [16], the set of four-vector functions

$$K_p = \{\underset{\sim}{I} = (I,Q,U,V)/I\geq(Q^2+U^2+V^2)^{\frac{1}{2}} \text{ almost everywhere}\}$$

is a positive cone on H_p that is normal and reproducing[17]. With respect to this cone, the operators $B, H(\tau), Q_\pm, J$ and

R (the latter when bounded on H_p) are positive, i.e. they leave invariant K_p. As a result, L_b,M_b and N_b are positive operators on $L_q(H_p)_0^b$, i.e. they leave invariant the cone $L_q(K_p)_0^b$ of strongly measurable L_q-functions of (0,b) into K_p. Moreover, if $\underset{\sim}{J}_{+} \epsilon\ K_p$, we have $\underset{\sim}{\omega} \epsilon L_q(K_p)_0^b$ for the right-hand sides $\underset{\sim}{\omega}$ of Eqs.(14) and (15). We may therefore apply the theory of positive operators on Banach spaces having a cone [18].

In order to deal with the existence and uniqueness of solutions, it is important to determine the spectral radii $r(N_b)$ and $r(L_\infty)$ of N_b and L_∞, respectively. Ignoring reflection by the planetary surface first (R=0, thus N_b=L_b), the compactness of L_b for finite b is exploited to prove that $r(L_b)$, and also its limit $r(L_\infty)$ for $b \to \infty$, do not depend on the functional setting offered by H_p or $H_{1.2}$. Next, it is shown, using that B has unit norm [16], that $r(L_\infty)$=1 on H_2. By the independence of $r(L_\infty)$ on the functional setting, we have $r(L_\infty)$=1 on H_p as well as $H_{1.2}$. Using the analyticity of $r(L_b)$ as a function of b and the strict monotonicity of the norm $H_{1.2}$ in combination with the independence of $r(L_b)$ on the functional setting, we may prove that $r(L_b)<1$ for finite b. Hence, for $a\epsilon(0,1]$ the boundary value problem (1)-(5) without reflection ($\underset{\sim}{R}_g(u,u',\phi-\phi')\equiv 0$) is uniquely solvable in H_p and in $H_{1.2}$. For $a\epsilon(0,1)$ we have obtained the unique solvability of the semiinfinite layer problem (1)-(9). It requires a detailed structural analysis of the eigenspace of B at the eigenvalue λ=1 to extend the latter result to a=1. The details can be found in Ref. 8.

Next, let us consider the finite layer problem (1)-(5) with reflection. In this case it is proved that $r(N_b)$ increases by strict monotonicity from zero to a finite value r_∞, as b increasing from zero to infinity. This is in fact done by also viewing $r(N_b)$ as the spectral radius of a vector integral operator originating from Eq.(1), where the reflection and the incidence of sunlight both take place at $\tau=0$, which is a physically irrelevant situation. However, this theoretical situation clearly has a limit case for $b \to \infty$, whence r_∞ exists and is finite. Using results of van der Mee and Protopopescu [19] on half-space problems with reflecting boundary conditions, it then follows easily that $r_\infty=1$, whence $r(N_b)<1$. We may therefore conclude that Eqs.(1)-(5) are well-posed on both H_p and $H_{1,2}$, provided R is bounded on this space as well as on H_2 [20]. The details can be found in Ref. 9.

Recently a different method has been developed to prove the unique solvability of Eqs. (1)-(5). It hinges upon the observation that because of the compactness of L_b for finite b one only has to show that Eqs. (10) (with R=0) have at most one solution. Since the result does not depend on the functional setting, one may work on H_2 where $A=1-aB$ has a positive real part, $\mathrm{Re}A = \frac{1}{2}(A+A^*)\geq 0$, and satisfies $\mathrm{Ker}\, A = \mathrm{Ker}\, (\mathrm{Re}A)$ [21]. A straightforward argument modeled on the uniqueness proof for the solution of the neutron transport equation for a submultiplying medium, given by Case and Zweifel [23], then yields uniqueness of the solution of the finite layer problem (10) without reflection by a planetary surface. The Fredholm alternative then implies existence also . The details can be found in Ref. 10 as well as in P.F. Zweifel's contribution to this conference.

Some of the work has been included in Section VI.2 of Ref. 22.

2. REFLECTION AND TRANSMISSION, AND ADDING EQUATIONS

The use of reflection and transmission matrices (or functions, if polarization effects are neglected) is virtually as old as radiative transfer itself and can be found in the monographs of Chandrasekhar [3], Sobolev [24] and Van de Hulst[7]. When they are viewed as kernels of linear operators yielding the reflected and transmitted intensities from the incident fluxes, one naturally arrives at the conception of reflection and transmission operators, such as appears in Refs. 25 and 26 and in the contribution of R.J. Hangelbroek to this conference. These operators govern the input-output characteristics of the planetary atmosphere system and may be combined into one "transfer" operator, as has been done by Ribarič[27] for the nuclear reactor system.

Let us consider the finite layer problem (10) without reflection by the ground, and let us include in the model a fictituous incident flux at the bottom. We then obtain the boundary value problem

$$(T\,\underset{\sim}{I})'(\tau) = -A\,\underset{\sim}{I}(\tau) \quad , \tau\in(0,b) \tag{17a}$$

$$Q_+\,\underset{\sim}{I}(0) = \underset{\sim}{J}_+ \tag{17b}$$

$$Q_-\,\underset{\sim}{I}(b) = \underset{\sim}{J}_- , \tag{17c}$$

where $\underset{\sim}{J}_\pm = Q_\pm \underset{\sim}{J}$. Under the hypothesis (16) this problem is uniquely solvable in H_p and in $H_{1,2}$, and the solution is continuous on $[0,b]$ in the strong topology of the underlying Banach function space. Moreover, if $\underset{\sim}{J}\in K_p$, then $\underset{\sim}{I}(\tau)\in K_p$ for $\tau\in[0,b]$, with a similar statement applying to $H_{1,2}$. We may therefore define unique reflection operators $R_{\pm b}$ and transmission operators $T_{\pm b}$ by

$$R_{\pm b}\,Q_\mp = T_{\pm b}\,Q_\mp = 0 \tag{18a}$$

$$\underset{\sim}{I}(0) = (R_{+b} + T_{-b})\underset{\sim}{J} \tag{18b}$$

$$\underset{\sim}{I}(b) = (R_{-b}+T_{+b})\underset{\sim}{J} . \tag{18c}$$

As a result R_{+b} and R_{-b} describe incident plus reflected radiation at top and bottom, while T_{+b} and T_{-b} describe radiated transmitted from top and bottom, respectively, to the other side of the atmosphere. All four operators are positive with respect to the cone K_p. This outlook of reflection and transmission has been chosen in Ref. 25 and 26.

A different landscape evolves when not incorporating the incident fluxes in the description, as found in Ref. 11.

Displaying the input-output mapping between the incident and the reflected plus transmitted radiation, one obtains the transfer (matrix) operator S_b, which satisfies

$$\begin{bmatrix} Q_+ \underset{\sim}{I}(b) \\ Q_- \underset{\sim}{I}(0) \end{bmatrix} = \begin{bmatrix} S_b^{++} & S_b^{+-} \\ S_b^{-+} & S_b^{--} \end{bmatrix} \begin{bmatrix} \underset{\sim}{J}_+ \\ \underset{\sim}{J}_- \end{bmatrix} . \qquad (19)$$

As a consequence, $S_b^{\pm\pm}$ and $S_b^{\mp\pm}$ are the restrictions to $Q_\pm[H_p]$ or $Q_\pm[H_{1,2}]$ of the operators $T_{\pm b}$ and $(R_{\pm b}-Q_\pm)$, respectively. All four entries of the transfer (matrix) operator are positive. The major gain from introducing this seemingly redundant transfer operator, however, becomes apparent when accounting for the transfer effects of two separate and adjacent optical layers. For the first such layers, $\tau\epsilon(0,b_1)$, we have the analog of Eq. (19), namely

$$Q_+ \underset{\sim}{I}(b_1) = S_{b_1}^{++} \underset{\sim}{J} + S_{b_1}^{+-} Q_- \underset{\sim}{I}(b_1)$$

$$Q_- \underset{\sim}{I}(0) = S_{b_1}^{-+} \underset{\sim}{J}_+ + S_{b_1}^{--} Q_- \underset{\sim}{I}(b_1) .$$

Similarly, we have the analog of Eq.(19) for the second layer, $\tau\epsilon(b_1,b)$ with $b=b_1+b_2$, written in the form

$$Q_+\underset{\sim}{I}(b) = S_{b_2}^{++}Q_+\underset{\sim}{I}(b_1)+S_{b_2}^{+-}\underset{\sim}{J}_-$$

$$Q_-\underset{\sim}{I}(b_1) = S_{b_2}^{-+}Q_+\underset{\sim}{I}(b_1)+S_{b_2}^{--}\underset{\sim}{J}.$$

By eliminating $Q_+\underset{\sim}{I}(b_1)$ and $Q_-I(b_1)$ from these equations and comparing the resulting equations with Eq.(19) we obtain the following adding equations:

$$S_b^{++} = S_{b_2}^{++}(1-S_{b_1}^{+-}S_{b_2}^{-+})^{-1}S_{b_1}^{++} \tag{20a}$$

$$S_b^{+-} = S_{b_2}^{+-}+ S_{b_2}^{++}(1-S_{b_1}^{+-}S_{b_2}^{-+})^{-1}S_{b_1}^{+-}S_{b_2}^{--} \tag{20b}$$

$$S_b^{-+} = S_{b_1}^{-+}+S_{b_1}^{--}(1-S_{b_2}^{-+}S_{b_1}^{+-})^{-1}S_{b_2}^{-+}S_{b_1}^{++} \tag{20c}$$

$$S_b^{--} = S_{b_1}^{--}(1-S_{b_2}^{-+}S_{b_1}^{+-})^{-1}S_{b_2}^{--} \tag{20d}$$

It can be proved that the inverses appearing in Eqs. (20) exist and can be obtained as the absolutely convergent series

$$(1-S_{b_1}^{+-}S_{b_2}^{-+})^{-1}= \sum_{n=0}^{\infty}(S_{b_1}^{+-}S_{b_2}^{-+})^n \;, \; (1-S_{b_2}^{-+}S_{b_1}^{+-})^{-1}= \sum_{n=0}^{\infty}(S_{b_2}^{-+}S_{b_1}^{+-})^n. \tag{21}$$

Using these each one of the adding equations can be written as an equation, where the right-hand side is an infinite series with each term modeling a reflection and/or transmission, followed by a finite number of double interface reflections, and then a final transmission. These so-called multiple interface reflection expansions appear in the adding method [6,7] for computing numerically the reflection and transmission properties of a combined layer from those of its constituent sublayers. The convergence of these expansions may thus be established. For details and ancillary results we refer to Ref. 11.

III. INHOMOGENEOUS MEDIA

On considering radiative transfer in vertically inhomogeneous plane-parallel atmospheres, one considers Eq.(1) with the albedo of single scattering a and the scattering matrix $\underset{\sim}{F}(\theta)$ depending on the optical depth τ. We shall develop in a concise way an existence and uniqueness theory and justify the adding method for such equations. The equation is given by

$$u \frac{\partial}{\partial\tau} \underset{\sim}{I}(\tau,u,\phi)+\underset{\sim}{I}(\tau,u,\phi) = \frac{1}{4\pi} \int_{-1}^{1} \int_{0}^{2\pi} \underset{\sim}{Z}(u,u',\phi-\phi';\tau)\underset{\sim}{I}(\tau,u',\phi')d\phi' du', \quad (22)$$

where

$$\underset{\sim}{Z}(u,u',\phi-\phi';\tau) = \underset{\sim}{L}(\pi-\sigma_1)\underset{\sim}{F}(\theta;\tau)\underset{\sim}{L}(-\sigma_2)$$

with the data $u,u',\phi,\phi',\sigma_1,\theta$ and σ_2 related as before. We assume that $\underset{\sim}{F}(\theta;\tau)$ is given by (20), where the nontrivial matrix elements also depend on τ. Since in Eq.(22) the albedo of single scattering, $a\in(0,1]$, is absent, we must replace condition (4) by

$$0< \int_{-1}^{1} a_1(\theta;\tau)d(\cos\theta)\le 2.$$

It is again assumed that $\underset{\sim}{F}(\theta;\tau)$ transforms vectors $\underset{\sim}{I} = (I,Q,U,V)$ satsfying (3) into vectors of the same type. In addition we make the following regularity assumption:

> The functions $c\in\{a_1,a_2,a_3,a_4,b_1,b_2\}$ satisfy the condition that, for some $r>1$, $\int_{-1}^{1}|c(\theta)|^r d\cos\theta$ is finite. Also, for each of such c we have $\tau \to c(\theta;\tau)$ continuous in $[0,b]$ if b is finite, and bounded and continuous on $[0,\infty)$ if $b=\infty$, in the L_r-norm of $[-1,1]$.

If no τ-dependence is present, this condition generalizes condition (16), since $|c|\leq a_1$ for every element c of the scattering matrix [28,2].

We may now define the operators $T,Q_{\pm},R,J$ and $H(\tau)$ on H_p, but the definitions of B and A must be modified to obtain

$$(B(\tau)\underset{\sim}{I})(u,\phi) = \frac{1}{4\pi}\int_{-1}^{1}\int_{0}^{2\pi} \underset{\sim}{Z}(u,u',\phi-\phi';\tau)\underset{\sim}{I}(u',\phi')d\phi'du', \quad A(\phi)=1-aB(\tau),$$

where we note the absence of $a\in(0,1]$ in the definition of $A(\tau)$. As a result, $B(\tau)$ generalizes aB rather than B. We may then write Eq. (22) in the abstract form

$$(TI)'(\tau) = -A(\tau)\underset{\sim}{I}(\tau) \ , \quad \tau\in(0,b), \tag{24}$$

endowed with the boundary conditions (10b) and (10c) if $b\in(0,\infty)$, (11b) and (11c) if $b=\infty$. As a consequence of the above regularity assumption one may write both boundary value problems in the form of the vector integral equations

$$\underset{\sim}{I}(\tau) - \int_0^b [H(\tau-\tau')+e^{(b-\tau)T^{-1}}Q_-J\,R\,H(b-\tau')]B(\tau')\underset{\sim}{I}(\tau')d\tau' =$$
$$= [e^{-\tau T^{-1}}+e^{(b-\tau)T^{-1}}Q_-J\,R\,e^{-bT^{-1}}]\underset{\sim}{J}_+ \ , \quad \tau\in(0,b), \tag{25}$$

if $b\in(0,\infty)$, and

$$\underset{\sim}{I}(\tau) - \int_0^\infty H(\tau-\tau')B(\tau')\underset{\sim}{I}(\tau')d\tau' = e^{-\tau T^{-1}}\underset{\sim}{J}_+ \ , \quad \tau\in(0,\infty), \tag{26}$$

if $b=\infty$. Equations (25) and (26) may be written in the form of Eqs. (14) and (15), respectively, for suitable choices of N_b, L_∞ and $\underset{\sim}{\omega}$. The functional space is $L_q(H_p)_0^b(1\leq q\leq\infty)$, provided R is bounded on H_p. We can repeat the arguments of the previous paragraph if H_p is replaced by $H_{1,2}$.

LEMMA - *Under the above regularity assumption, there is a matrix* $\underset{\sim}{G}(\theta)$ *of the form*

$$\underset{\sim}{G}(\theta) = \begin{bmatrix} \hat{a}_1(\theta) & \hat{b}_1(\theta) & 0 & 0 \\ \hat{b}_1(\theta) & \hat{a}_2(\theta) & 0 & 0 \\ 0 & 0 & \hat{a}_3(\theta) & \hat{b}_2(\theta) \\ 0 & 0 & -\hat{b}_2(\theta) & \hat{a}_4(\theta) \end{bmatrix}$$

having the following properties:

(i) *For every* $\tau\in[0,b]$ *if* b *is finite, or for every* $\tau\in[0,\infty)$ *if* $b=\infty$, *the vector*

$$\{\underset{\sim}{G}(\theta) - \underset{\sim}{F}(\theta;\tau)\}\underset{\sim}{I}$$

satisfies the condition (3) *if* $\underset{\sim}{I}$ *satisfies condition* (3).

(ii) *The elements* $\hat{c}(\theta)$ *of* $G(\theta)$ *are measurable functions of* θ *satisfying* $\int_{-1}^{1}|\hat{c}(\theta)|^r d\cos\theta<\infty$, *where* $r<1$ *is the constant appearing in the regularity assumption.*

(iii) *We have* $0<\int_{-1}^{1}\hat{a}_1(\theta)d\cos\theta\leq 2$.

Proof. Let E be a countable dense subset of [0,b] if b is finite or of $[0,\infty)$ if $b=\infty$. Then it is sufficient to construct $\underset{\sim}{G}(\theta)$ satisfying (i'),(ii) and (iii), where we have:

(i') *For every* $\tau\in E$ *the vector* $\{\underset{\sim}{G}(\theta)-\underset{\sim}{F}(\theta;\tau)\}\underset{\sim}{I}$ *satisfies condition* (3) *if* $\underset{\sim}{I}$ *satisfies condition* (3).

One may replace (i) by (i') because of the continuity clause in the regularity assumption.

Next, choose a constant N such that the integral $\int_{-1}^{1}|c(\theta;\tau)|^r d\cos\theta\leq N<\infty$ for every element $c(\theta;\tau)$ of $\underset{\sim}{F}(\theta;\tau)$. Since the cone K_p is closed with respect to the creation of finite suprema [29], we can construct, with respect to the order induced by the cone K_p,

$$\underset{\sim}{F}_k(\theta) = \sup_{1\le i\le k} \underset{\sim}{F}(\theta;\tau_i),$$

where $E = \{\tau_i\}_{i=1}^{\infty}$ is an enumeration of E. Since K_p is a normal cone [17], we have $\int_{-1}^{1} |c(\theta;\tau)|^r d\cos\theta \le M^r N < \infty$, where $c(\theta;\tau)$ is an element of $\underset{\sim}{F}_k(\theta)$ and the constant M (in fact, $M = \sqrt{3}$) does not depend on c nor k. It is now straightforward to see that $\underset{\sim}{G}(\theta) = \sup\{\underset{\sim}{F}_k(\theta) \mid k \in \mathbb{N}\}$, where the countable supremum taken with respect to the order induced by K_p exists, is the matrix function sought for. □

We now put

$$a = \frac{1}{2}\int_{-1}^{1} \hat{a}_1(\theta)\, d\cos\theta \ , \quad \underset{\sim}{F}(\theta) = (1/a)\underset{\sim}{G}(\theta),$$

using that $a \in (0,1]$. Then all elements c of $\underset{\sim}{F}(\theta)$, which has the form (2c), satisfy $\int_{-1}^{1} |c(\theta)|^r d\cos\theta < \infty$, while $\int_{-1}^{1} a_1(\theta)\, d\cos\theta =$ Thus the matrix $\underset{\sim}{F}(\theta)$ satisfies all the requirements for a scattering matrix, and therefore for this matrix the boundary value problems (10) and (11) are uniquely solvable in H_p and in $H_{1,2}$ (where we assume, of course, that R is bounded on H_p or $H_{1,2}$, and on H_2). Also, if R is bounded on H_p and H_2, we have $r(N_b) < 1$ and $r(L_\infty) = 1$ for the present homogeneous atmosphere problem, where these spectral radii do not depend on the functional formulation.

Let us assume that R is bounded on H_p, H_2 and $H_{1,2}$. Let us denote the operators N_b and L_∞ for the original inhomogeneous media problem by $N_b^{(inh)}$ and $L_\infty^{(inh)}$, and those for the dominating homogeneous media problem by

$N_b^{(hom)}$ and $L_\infty^{(hom)}$. Then for $\underset{\sim}{I}\in L_q(K_p)_o^b$, or for $\underset{\sim}{I}\in L_q(K_{1.2})_o^b$, we have

$$0\le N_b^{(inh)}\underset{\sim}{I}\le aN_b^{(hom)}\underset{\sim}{I} \quad , \quad 0\le L_\infty^{(inh)}\underset{\sim}{I}\le a\le L_\infty^{(hom)}\underset{\sim}{I}.$$

Using the strict monotonicity of the $H_{1.2}$ norm [17] we obtain

$$0\le\|N_b^{(inh)}\underset{\sim}{I}\|_{1.2} \le a\ \|N_b^{(hom)}\underset{\sim}{I}\|_{1.2} \quad <a.1 = a\le 1 \qquad (27)$$

and

$$0\le\|L^{(inh)}\underset{\sim}{I}\|_{1.2} \le a\ \|L_\infty^{(hom)}\underset{\sim}{I}\|_{1.2} \quad \le a.1=a, \qquad (28)$$

where we have exploited the results for homogeneous atmospheres. Again using the insensitivity of the spectral radii for the functional formulation, we obtain

THEOREM. *For* $a\in[0,1]$ *the finite atmosphere problem* (24)-(10b)-(10c) *is uniquely solvable in* H_p *(resp.*$H_{1.2}$*), if* R *is bounded on* H_p,$H_{1.2}$ *and* H_2 *(resp. on* $H_{1.2}$ *and* H_2*) and satisfies conditions* (6), (7) *and* (8). *For* $a\in[0,1]$ *the semiinfinite atmosphere problem* (24)-(11b)-(11c) *is uniquely solvable in* H_p *and* $H_{1.2}$.

It is straightforward to justify the adding method for inhomogeneous atmospheres. Using unique solvability one can define the reflection and transmission operators by Eq.(18) and the transfer (matrix) operator by Eq.(19), provided one replaces the subindex b (appropriate to a homogeneous layer) by a subindex linked to the layer under consideration. The adding equations (20) will not change at all. Again using the insensitivity of the spectral radii of $S_{b_1}^{+-}\,S_{b_2}^{-+}$ and $S_{b_2}^{-+}\,S_{b_1}^{+-}$ for the functional space, and exploiting the dominating homogeneous atmosphere problem and the fact that in the latter case these spectral radii are less than unity, we immediately see that these spectral radii are less than unity also for the inhomogeneous atmosphere problem. Hence, one may justify the series expansions appearing in the adding method for inhomogeneous atmospheres too.

It remains to prove the existence and uniqueness of the solutions of the inhomogeneous semifinite atmosphere problem for $a = 1$.

Footnotes

1) Talk given at the Ninth International conference on Transport at Montecatini, June 10-14, 1985.

2) The paper was written while the author was visiting the University of Florence in Summer 1985, supported by C.N.R. (Gruppo Nazionale per la Fisica-Matematica).

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4) The research was supported in part by N.S.F. under grant No. DMS-8501337 and by C.N.R. (Fisica-Matematica).

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13. The precise relationship is: $\omega = (x,y,z)$ where $x=(1-u^2)^{\frac{1}{2}} \cos\phi$, $y = (1-u^2)^{\frac{1}{2}} \sin\phi$ and $z = -u$.

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15. The vector function should be strongly measurable with respect to Lebesgue measure. The integrals in Eqs.(12) and (13) are Bochner integrals.

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17. For the concepts we refer to Ch.1 of the below Ref. 18. For convenience, K_p is closed with respect to addition, multiplication by a positive constant, and taking limits in H_p, while every $0 \neq \underset{\sim}{I} \in K_p$ satisfies $(-\underset{\sim}{I}) \notin K_p$. The real space H_p is the set of differences of pairs of vectors of K_p ("being reproducing"), and there exists a constant M_p, in fact $M = \sqrt{3}$, such that $0 \le \underset{\sim}{I}_1 \le \underset{\sim}{I}_2$ with respect to the partial order induced by K_p implies $\|\underset{\sim}{I}_1\|_p \le M \|\underset{\sim}{I}_2\|_p$. Analogous properties hold true for $H_{1,2}$ and $K_{1,2}$, but now one may take $M=1$; cf. Ref.8.

18. M.A. Krasnoselskii, "Positive Solutions of Operator Equations", Noordhoff, Groningen, 1964.

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20. The condition that R is bounded on H_2 is necessary, because Ref. 19 is written completely in a Hilbert space setting.

21. Those properties follow directly from the inequality $(2\ell+1-a\beta_\ell)(2\ell+1-a\alpha_\ell)-a^2\gamma_\ell^2 \ge 0$ from certain expansion coefficient α_ℓ, β_ℓ, and γ_ℓ. Their derivation due to Hovenier and van der Mee will appear in a forthcoming publication. The proof also appears in Section VI.2 of the below Ref. 22.

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29. In the terminology of Ref. 18, K_p is a minihedral cone.

Received: July 23, 1985