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Transport Equations with Boundary Conditions of Reverse Reflection Type

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A study is performed of transport equations on arbitrary three-dimensional domains with boundary conditions of reverse reflection type. The existence of the dominant eigenvalue of the criticality problem is proved and its independence of the functional setting and its continuous dependence on a variety of data are established. The corresponding time-dependent problem is shown to be well-posed, also for a conservative boundary. The relationship between the criticality and the time-dependent problem is given explicitly.

1. Introduction

In this paper we consider the (closely related) problems of criticality and time-evolution for the one-speed transport process in bounded geometry with boundary conditions of reverse reflection type. For purely absorbing walls, these two problems have been thoroughly investigated by Vladimirov[15] and Jörgens,[7] respectively. Purely absorbing boundary conditions are typical for neutron transport, from which these studies, as well as many others that followed, received impetus and motivation. Yet, when trying to understand other transport processes (for neutral molecules), more complicated boundary conditions have to be considered. One of them is (partial) reverse reflection, describing a scattering process at the surface wall in which the molecules are scattered back in the same direction they came from with—possibly—a modification of the magnitude of the carried current. Reverse reflection is far less far-fetched than it may at first seem. Together with perfectly specular reflection, perfect reverse reflection is the only isotropic, planar and conservative reflection law that does not decrease the entropy of the system. Moreover, perfect reverse reflection leads to the important no-slip boundary condition in fluid dynamics (whereas the perfectly specular boundary condition yields zero tangential stress). For an extensive discussion of these aspects, we refer the reader to Reference 14. Both reverse and specular reflection (perfect or partial) can be treated to some extent within a general formalism including many other types of boundary condition.[2,5,16]

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However, in the present article we deal with two specific aspects. First, for the purely conservative case (i.e. no absorption at the boundary), the existence of solutions of the time evolution problem is not automatically covered by the general theory (unless one allows for a weakening of the notion of solution) and requires a separate study.[2] Secondly, for reverse specular reflection many computations can be performed analytically, which simplifies the approach to a great extent and makes the results more refined and transparent. Specular reflection in a general geometry does not enjoy this property. Its great importance for entropy-related billiard schemes and boundary conditions in fluid dynamics entitles it to a separate study, which is deferred to a forthcoming article.

The plan of the paper is the following. In Section 2 we introduce the notation, state the criticality and time-dependent problems and establish a type of equivalence between the two. This equivalence is realized via the associated integral operators. To some extent it therefore suffices to study thoroughly the integral operators in one version (e.g. criticality) and to transfer the results afterwards.

Section 3 is devoted to deriving some basic properties (mostly compactness-related) of the integral operator, and in Section 4 their positivity aspects and the related spectral properties are discussed.

Some features not obtainable from the criticality analysis and specific to the time-dependent problem are tackled in Section 5. In particular, it is proved that the transport operator on a domain of functions satisfying reverse reflection boundary conditions generates the strongly continuous contraction semigroup of evolution. For more general boundary conditions such a result usually does not go through if the reflection preserves the current[16]. We conclude this article with a discussion. The compactness properties of the integral operator relevant to reverse reflection will be derived in the Appendix.

2. Statement of the problem. The equivalence

Let us consider a bounded convex body $V \subset \mathbf{R}^3$ with sufficiently smooth boundary ∂V. Since for traditional and physical reasons we want to express the boundary condition in terms of the normal vector at the surface, we shall actually suppose that ∂V is piecewise C^1. We denote the unit inner normal at $\mathbf{y} \in \partial V$ by $\mathbf{n}(\mathbf{y})$.

We shall consider two integrodifferential equations in V, one for the (stationary) criticality problem and one for the time-dependent problem. They read

$$0 = -\boldsymbol{\Omega}\cdot\nabla u(\mathbf{x}, \boldsymbol{\Omega}) - \sigma u(\mathbf{x}, \boldsymbol{\Omega}) + \frac{\gamma}{4\pi}\int_S u(\mathbf{x}, \boldsymbol{\Omega}')\mathrm{d}\boldsymbol{\Omega}' = (Qu)(\mathbf{x}, \boldsymbol{\Omega}) \tag{1}$$

and

$$\frac{\partial u}{\partial t}(\mathbf{x}, \boldsymbol{\Omega}, t) = -\boldsymbol{\Omega}\cdot\nabla u(\mathbf{x}, \boldsymbol{\Omega}, t) - \sigma u(\mathbf{x}, \boldsymbol{\Omega}, t) + \frac{\gamma}{4\pi}\int_S u(\mathbf{x}, \boldsymbol{\Omega}', t)\mathrm{d}\boldsymbol{\Omega}' = (Qu)(\mathbf{x}, \boldsymbol{\Omega}, t), \tag{2}$$

respectively, where σ is the cross-section or collision frequency, γ is the albedo of single scattering, the position vector $\mathbf{x}$ belongs to V, the velocity $\boldsymbol{\Omega}$ runs over the unit sphere $S \subset \mathbf{R}^3$ and the time t appearing in (2) is in $\mathbf{R}_+$. We refer to (1) as the criticality equation and to (2) as the time-dependent equation. The transport operator Q can be conveniently written as the sum of the streaming operator $T = -\mathbf{\Omega}\cdot\nabla - \sigma I$ and the bounded perturbation $P = (4\pi)^{-1}\int_S \cdot \mathrm{d}\mathbf{\Omega}'$ multiplied by γ.

The two equations for the angular particle density u will be considered in a variety of spaces, namely C and L_p $(1 \le p < \infty)$, as explained in the sequel, and will be supplemented with the boundary condition of reverse specular type:

$$u(\mathbf{y}, \mathbf{\Omega}) = \alpha u(\mathbf{y}, -\mathbf{\Omega}), \quad \mathbf{y} \in \partial V,\ \mathbf{\Omega}\cdot\mathbf{n}(\mathbf{y}) \ge 0, \tag{3}$$

which specifies the reflection of particles by the boundary. The coefficient $\alpha \in [0, 1]$ is the accommodation coefficient and accounts for the fraction of the current scattered back by the surface wall. In the time-dependent setting we have to supplement the data with the initial condition

$$\lim_{t\to 0+} u(\mathbf{x}, \mathbf{\Omega}, t) = u_0(\mathbf{x}, \mathbf{\Omega}). \tag{4}$$

Throughout the collision frequency σ and the albedo of single scattering γ will be taken as constants. One of the main problems related to (1) and (2) is solving the corresponding eigenvalue equations:

$$Tu + \gamma Pu = 0, \tag{5}$$

$$\lambda u - Tu - \gamma Pu = 0, \tag{6}$$

for the eigenvalues γ and λ, respectively. One can see easily that the two eigenvalue problems are equivalent in the following sense: for any real $\lambda > -\sigma$ which satisfies (6), we define $\sigma' = \sigma + \lambda$; then (6) becomes the criticality problem for a system with the same dimensions and a new collision frequency. We may proceed differently, by rescaling in equation (6) the spatial dimensions as $\mathbf{x}' = \mathbf{x}(\sigma+\lambda)/\sigma$, by which we obtain the criticality problem for a system with scaled spatial dimensions but with the same collision frequency. The eigenvalues γ are themselves rescaled by $\gamma' = \gamma\sigma/(\sigma+\lambda)$. These relationships, which reflect the invariance of the transport equation with respect to the transformations $t \to t/\delta$, $\mathbf{x} \to \mathbf{x}/\delta$, $\sigma \to \delta\sigma$ and $\gamma \to \delta\gamma$, allow one to transfer information between the two problems. The equivalence is even more evident and easier to exploit when the eigenvalue equations (5) and (6) are written in integral form: we shall come back to this point later after having defined the associated integral operators. We mention here that the validity of the integral equation makes the equivalence valid for every λ in $\mathrm{Re}\lambda > -\sigma$. However, owing to the reality of $\mathbf{x}$ and σ the equivalence between (5) and (6) is meaningful and useful only for real $\lambda > -\sigma$ and the full spectral description of the transport operator, also for complex λ, cannot be inferred from a study of the criticality problem, using the above equivalence.

3. Integral formulation and preliminaries

Let V be a bounded convex region in $\mathbf{R}^3$ with piecewise C^1-boundary ∂V, and let $\mathbf{n}(\mathbf{y})$ denote the unit inner normal at $\mathbf{y} \in \partial V$. For $(\mathbf{x}, \mathbf{\Omega}) \in V \times S$, put (cf. Fig. 1) $r(\mathbf{x}, \pm\mathbf{\Omega}) = \inf\{t > 0/\mathbf{x} \mp t\mathbf{\Omega} \notin V\}$, $d(\mathbf{x}, \mathbf{\Omega}) = r(\mathbf{x}, \mathbf{\Omega}) + r(\mathbf{x}, -\mathbf{\Omega})$.

Lemma 1. *The unique solution of the differential equation*

$$-\mathbf{\Omega}\cdot\nabla u(\mathbf{x}, \mathbf{\Omega}) - \sigma u(\mathbf{x}, \mathbf{\Omega}) + f(\mathbf{x}) = 0, \quad (\mathbf{x}, \mathbf{\Omega}) \in V \times S, \tag{7}$$

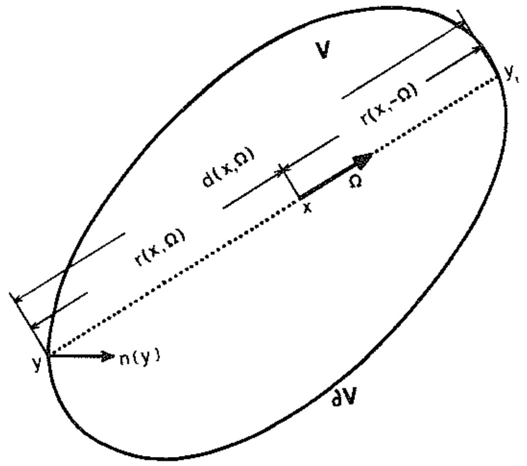

Fig. 1. When subject to reverse specular reflection, the particle moves on a fixed, dotted, line. Given $(\mathbf{x}, \mathbf{\Omega})$, $r(\mathbf{x}, \mathbf{\Omega})$ and $r(\mathbf{x}, -\mathbf{\Omega})$ denote the distances from $\mathbf{x}$ to ∂V in the directions of $-\mathbf{\Omega}$ and $\mathbf{\Omega}$, respectively. The total length of the dotted line is denoted by $d(\mathbf{x}, \mathbf{\Omega})$

with boundary condition (3) *is given by*

$$u(\mathbf{x},\mathbf{\Omega}) = \int_0^{r(\mathbf{x},\mathbf{\Omega})} e^{-\sigma t} f(\mathbf{x}-t\mathbf{\Omega})\mathrm{d}t + \int_0^{d(\mathbf{x},\mathbf{\Omega})} \frac{\alpha \exp(-\sigma r(\mathbf{x},\mathbf{\Omega}))}{1-\alpha^2\exp(-2\sigma d(\mathbf{x},\mathbf{\Omega}))} e^{-\sigma t}$$

$$\times f(\mathbf{x}-r(\mathbf{x},\mathbf{\Omega})\mathbf{\Omega}+t\mathbf{\Omega})\mathrm{d}t$$

$$+\int_0^{d(\mathbf{x},\mathbf{\Omega})} \frac{\alpha^2\exp(-\sigma r(\mathbf{x},\mathbf{\Omega}))\exp(-\sigma d(\mathbf{x},\mathbf{\Omega}))}{1-\alpha^2\exp(-2\sigma d(\mathbf{x},\mathbf{\Omega}))} e^{-\sigma t}$$

$$\times f(\mathbf{x}+r(\mathbf{x},-\mathbf{\Omega})\mathbf{\Omega}-t\mathbf{\Omega})\mathrm{d}t. \tag{8}$$

Proof. We first observe that the motion takes place on the segment with direction $\mathbf{\Omega}$ through $\mathbf{x}$. Let us introduce the parameter r measuring this segment, satisfying $r = 0$ at $(\mathbf{x}-r(\mathbf{x},\mathbf{\Omega})\mathbf{\Omega})$ and $r = d(\mathbf{x},\mathbf{\Omega})$ at $(\mathbf{x}+r(\mathbf{x},-\mathbf{\Omega})\mathbf{\Omega})$.

Putting $u(\mathbf{x},\mathbf{\Omega}) = v(r)$ and $f(\mathbf{x}) = g(r)$ where $r = r(\mathbf{x},\mathbf{\Omega})$, one may convert equation (7) into the equation

$$v'(r)+\sigma v(r) = g(r), \quad 0 < r < d(\mathbf{x},\mathbf{\Omega}) = d,$$

whence

$$v(r) = e^{-\sigma r}\, v(0) + \int_0^r e^{-\sigma(r-r')}\, g(r')\mathrm{d}r'. \tag{9}$$

Similarly, $w(r) = u(\mathbf{x},-\mathbf{\Omega})$ with $r = r(\mathbf{x},\mathbf{\Omega})$ satisfies the equation

$$w'(r)-\sigma w(r) = -g(r), \quad 0 < r < d(\mathbf{x},\mathbf{\Omega}) = d,$$

whence

$$w(r) = e^{\sigma r} w(0) - \int_0^r e^{\sigma(r-r')}\, g(r')\mathrm{d}r'. \tag{10}$$

The boundary condition (3) applied for $\mathbf{y}_0 = \mathbf{x} - r(\mathbf{x}, \boldsymbol{\Omega})\boldsymbol{\Omega}$ where $\boldsymbol{\Omega} \cdot \mathbf{n}(\mathbf{y}_0) > 0$, and $\mathbf{y}_d = \mathbf{x} + r(\mathbf{x}, -\boldsymbol{\Omega})\boldsymbol{\Omega}$ where $(-\boldsymbol{\Omega}) \cdot \mathbf{n}(\mathbf{y}_d) > 0$ yields the respective identities

$$v(0) = \alpha w(0), \quad w(d) = \alpha v(d).$$

Substitution of these identities into (9) and (10) gives

$$\begin{aligned} v(0) &= \frac{\mathrm{e}^{-\sigma d}}{1-\alpha^2\mathrm{e}^{-2\sigma d}}\left\{\alpha^2 \int_0^d \mathrm{e}^{-\sigma(d-r')} g(r')\mathrm{d}r' + \alpha \int_0^d \mathrm{e}^{\sigma(d-r')} g(r')\mathrm{d}r'\right\} \\ &= \frac{1}{1-\alpha^2\mathrm{e}^{-2\sigma d}}\left\{\alpha \int_0^d \mathrm{e}^{-\sigma r'} g(r')\mathrm{d}r' + \alpha^2 \int_0^d \mathrm{e}^{-\sigma(2d-r')} g(r')\mathrm{d}r'\right\}, \end{aligned}$$

whence

$$\begin{aligned} v(r) = &\int_0^r \mathrm{e}^{-\sigma r'} g(r-r')\mathrm{d}r' + \int_0^d \frac{\alpha \mathrm{e}^{-\sigma(r+r')}}{1-\alpha^2\mathrm{e}^{-2\sigma d}} g(r')\mathrm{d}r' \\ &+ \int_0^d \frac{\alpha^2\mathrm{e}^{-\sigma(r+d+r')}}{1-\alpha^2\mathrm{e}^{-2\sigma d}} g(d-r')\mathrm{d}r'. \end{aligned}$$

Substituting $r = r(\mathbf{x}, \boldsymbol{\Omega})$, $d = d(\mathbf{x}, \boldsymbol{\Omega})$ and $r' = t$ one obtains (8).

On applying Lemma 1 for $f(x) = (\gamma/4\pi)\int_S u(\mathbf{x}, \boldsymbol{\Omega}')\mathrm{d}\boldsymbol{\Omega}'$, one obtains the equation

$$u = \gamma B_\alpha u, \tag{11}$$

or written explicitly

$$\begin{aligned} u(\mathbf{x}, \boldsymbol{\Omega}) = &\frac{\gamma}{4\pi}\int_0^{r(\mathbf{x},\boldsymbol{\Omega})} \mathrm{e}^{-\sigma t}\int_S u(\mathbf{x}-t\boldsymbol{\Omega}, \boldsymbol{\Omega}')\mathrm{d}\boldsymbol{\Omega}'\mathrm{d}t + \frac{\gamma}{4\pi} \\ &\times \int_0^{d(\mathbf{x},\boldsymbol{\Omega})} \frac{\alpha\exp(-\sigma r(\mathbf{x}, \boldsymbol{\Omega}))}{1-\alpha^2\exp(-2\sigma d(\mathbf{x}, \boldsymbol{\Omega}))}\mathrm{e}^{-\sigma t}\int_S u(\mathbf{x}-r(\mathbf{x}, \boldsymbol{\Omega})\boldsymbol{\Omega}+t\boldsymbol{\Omega}, \boldsymbol{\Omega}')\mathrm{d}\boldsymbol{\Omega}'\mathrm{d}t \\ &+ \frac{\gamma}{4\pi}\int_0^{d(\mathbf{x},\boldsymbol{\Omega})} \frac{\alpha^2\exp(-\sigma r(\mathbf{x}, \boldsymbol{\Omega}))\exp(-\sigma d(\mathbf{x}, \boldsymbol{\Omega}))}{1-\alpha^2\exp(-2\sigma d(\mathbf{x}, \boldsymbol{\Omega}))}\mathrm{e}^{-\sigma t} \\ &\times \int_S u(\mathbf{x}+r(\mathbf{x}, -\boldsymbol{\Omega})\boldsymbol{\Omega}-t\boldsymbol{\Omega}, \boldsymbol{\Omega}')\mathrm{d}\boldsymbol{\Omega}'\mathrm{d}t, \end{aligned}$$

which is the integral formulation for the angular particle density $u(\mathbf{x}, \boldsymbol{\Omega})$. Put

$$\phi(\mathbf{x}) = \int_S u(\mathbf{x}, \boldsymbol{\Omega}')\mathrm{d}\boldsymbol{\Omega}',$$

and introduce the quantities (cf. Fig. 2)

$$r^{\pm}(\mathbf{x}, \mathbf{x}') = r\left(\mathbf{x}, \mp\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}\right), \quad d(\mathbf{x}, \mathbf{x}') = r^+(\mathbf{x}, \mathbf{x}') + r^-(\mathbf{x}, \mathbf{x}').$$

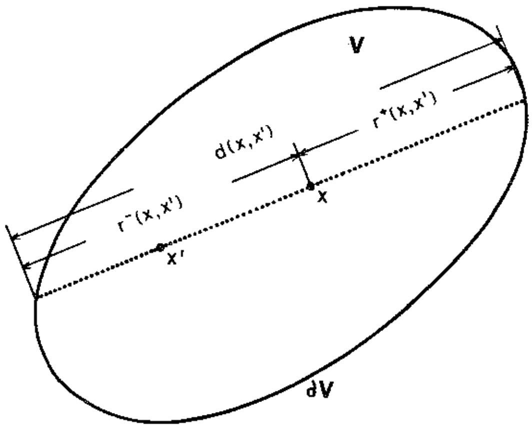

Fig. 2. $r^-(\mathbf{x},\mathbf{x}')$ and $r^+(\mathbf{x},\mathbf{x}')$ are the distances from x to the boundary ∂V, in the direction of $\mathbf{x}'$ and in the opposite direction, respectively. Their sum, the 'diameter' of V along the line through x and $\mathbf{x}'$, is $d(\mathbf{x},\mathbf{x}')$. Moreover, we have $r^-(\mathbf{x},\mathbf{x}')=r^+(\mathbf{x}',\mathbf{x})+|\mathbf{x}-\mathbf{x}'|$

We easily obtain

$$\begin{aligned}\phi(\mathbf{x}) = \frac{\gamma}{4\pi}\int_S \mathrm{d}\Omega\Bigg[&\int_0^{r(\mathbf{x},\boldsymbol{\Omega})} \mathrm{e}^{-\sigma t}\phi(\mathbf{x}-t\boldsymbol{\Omega})\\ &+\int_0^{r(\mathbf{x},\boldsymbol{\Omega})} \frac{\alpha\exp(-2\sigma r(\mathbf{x},\boldsymbol{\Omega}))}{1-\alpha^2\exp(-2\sigma d(\mathbf{x},\boldsymbol{\Omega}))}\mathrm{e}^{\sigma t}\phi(\mathbf{x}-t\boldsymbol{\Omega})\\ &+\int_0^{r(\mathbf{x},-\boldsymbol{\Omega})} \frac{\alpha\exp(-2\sigma r(\mathbf{x},\boldsymbol{\Omega}))}{1-\alpha^2\exp(-2\sigma d(\mathbf{x},\boldsymbol{\Omega}))}\mathrm{e}^{-\sigma t}\phi(\mathbf{x}+t\boldsymbol{\Omega})\\ &+\int_0^{r(\mathbf{x},-\boldsymbol{\Omega})} \frac{\alpha^2\exp(-2\sigma d(\mathbf{x},\boldsymbol{\Omega}))}{1-\alpha^2\exp(-2\sigma d(\mathbf{x},\boldsymbol{\Omega}))}\mathrm{e}^{\sigma t}\phi(\mathbf{x}+t\boldsymbol{\Omega})\\ &+\int_0^{r(\mathbf{x},\boldsymbol{\Omega})} \frac{\alpha^2\exp(-2\sigma d(\mathbf{x},\boldsymbol{\Omega}))}{1-\alpha^2\exp(-2\sigma d(\mathbf{x},\boldsymbol{\Omega}))}\mathrm{e}^{-\sigma t}\phi(\mathbf{x}-t\boldsymbol{\Omega})\Bigg]\mathrm{d}t.\end{aligned}$$

On writing $t=|\mathbf{x}-\mathbf{x}'|$ and $\mathrm{d}\mathbf{x}'=|\mathbf{x}-\mathbf{x}'|^2\mathrm{d}\boldsymbol{\Omega}\mathrm{d}t$, we finally get the integral formulation for the total particle density $\phi(\mathbf{x})$,

$$\begin{aligned}\phi(\mathbf{x}) &= \frac{\gamma}{4\pi}\int_V \frac{\exp(-\sigma|\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}-\mathbf{x}'|^2}\phi(\mathbf{x}')\mathrm{d}\mathbf{x}'\\ &+\alpha\frac{\gamma}{4\pi}\int_V \frac{\exp(-2\sigma r^+(\mathbf{x},\mathbf{x}'))+\exp(-2\sigma r^+(\mathbf{x}',\mathbf{x}))}{1-\alpha^2\exp(-2\sigma d(\mathbf{x},\mathbf{x}'))}\\ &\times\frac{\exp(-\sigma|\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}-\mathbf{x}'|^2}\phi(\mathbf{x}')\mathrm{d}\mathbf{x}'+\alpha^2\frac{\gamma}{4\pi}\\ &\times\int_V \frac{\exp(-\sigma|\mathbf{x}-\mathbf{x}'|)+\exp(\sigma|\mathbf{x}-\mathbf{x}'|)}{1-\alpha^2\exp(-2\sigma d(\mathbf{x},\mathbf{x}'))}\frac{\exp(-2\sigma d(\mathbf{x},\mathbf{x}'))}{|\mathbf{x}-\mathbf{x}'|^2}\phi(\mathbf{x}')\mathrm{d}\mathbf{x}'.\end{aligned}$$

The latter can be written in vector form as

$$\phi = \gamma A_\alpha \phi. \tag{12}$$

We shall study the equations (11) and (12) on suitable functional spaces. By $L_p(V)$ and $L_p(V \times S)$, $1 \leq p \leq \infty$, we denote the usual L_p-spaces with respect to Lebesgue measure; $C(\bar{V})$ and $C(\bar{V} \times S)$ will be the Banach spaces of complex continuous functions on the compact sets $\bar{V}$ and $\bar{V} \times S$, respectively, endowed with the maximum norm. At this point we wish to strengthen our assumption on ∂V. For every $\mathbf{y} \in \partial V$ we require the existence of a 'tangent sphere' with radius $R(\mathbf{y}) > 0$ such that

$$\{\mathbf{z} \in V / |\mathbf{z} - (\mathbf{y} + R(\mathbf{y})\mathbf{n}(\mathbf{y}))| < R(\mathbf{y})\} \subset V, \tag{13}$$

whereas $R(\mathbf{y}) \geq R > 0$ for all $\mathbf{y} \in \partial V$ and some fixed R. In particular, ∂V is everywhere C^1, and the additional assumption is satisfied if ∂V is everywhere C^2, since in this case the curvature depends continuously on $\mathbf{y} \in \partial V$.

Lemma 2. *Suppose condition* (13) *is satisfied. Then for* $0 \leq \alpha \leq 1$ *the operator* A_α *is compact on the functional spaces* $L_p(V)$ $(1 \leq p < \infty)$ *and* $C(\bar{V})$.

For the proof of this proposition, essentially due to Belleni-Morante[3] for $L_2(V)$, we refer to the Appendix. We remark that condition (13) is not needed if $0 \leq \alpha < 1$ (see Appendix).

In order to investigate in detail the connection between the spectral properties of A_α and those of B_α, we introduce some additional operators. As before P is the operator from $L_p(V \times S)$ into $L_p(V)$, or from $C(\bar{V} \times S)$ into $C(\bar{V})$, defined by

$$(Pu)(\mathbf{x}) = (4\pi)^{-1} \int_S u(\mathbf{x}, \mathbf{\Omega}')\mathrm{d}\mathbf{\Omega}'.$$

Let J be the natural imbedding of $L_p(V)$ into $L_p(V \times S)$, or of $C(\bar{V})$ into $C(\bar{V} \times S)$. Then PJ is the identity operator on $L_p(V)$ or $C(\bar{V})$, and JP is the projection (as an operator defined on $L_p(V \times S)$ or $C(\bar{V} \times S)$) onto the functions depending only on position. Then

$$A_\alpha = PB_\alpha J.$$

If we define $K_\alpha = B_\alpha J$, then

$$A_\alpha = PK_\alpha, \quad B_\alpha = K_\alpha P. \tag{14}$$

Since B_α is bounded on $L_p(V \times S)$ and $C(\bar{V} \times S)$, so will be K_α.

Lemma 3. *For* $0 \leq \alpha \leq 1$ *the operators* A_α *and* B_α *have the same spectra, consisting of a countably infinite sequence of eigenvalues of finite algebraic multiplicity converging to zero. The partial multiplicities of the eigenvalues of* A_α *and* B_α *coincide pairwise.*

Proof. If $\lambda \neq 0$ is in the resolvent set of A_α, $\rho(A_\alpha)$, then $\lambda \in \rho(B_\alpha)$ and

$$(\lambda - B_\alpha)^{-1} = \frac{1}{\lambda}\{I + K_\alpha(\lambda - A_\alpha)^{-1}P\}.$$

Conversely, if $\lambda \neq 0$ belongs to $\rho(B_\alpha)$, then $\lambda \in \rho(A_\alpha)$ and

$$(\lambda - A_\alpha)^{-1} = \frac{1}{\lambda}\{I + P(\lambda - B_\alpha)^{-1}K_\alpha\}.$$

Both resolvent identities follow easily from (14), and therefore non-zero parts of the spectra of A_α and B_α coincide. Since A_α is compact, the non-zero part of $\sigma(B_\alpha)$ consists solely of eigenvalues of finite algebraic multiplicity.

Suppose $\phi_0 \neq 0$ is a generalized eigenvector of A_α at the non-zero eigenvalue λ_0, i.e. $(A_\alpha - \lambda_0)\phi_0 = \phi_1, (A_\alpha - \lambda_0)\phi_1 = \phi_2, \ldots, (A_\alpha - \lambda_0)\phi_k = 0$ where $\phi_k \neq 0$. Using the first one of the intertwining relations (cf. (14))

$$K_\alpha A_\alpha = B_\alpha K_\alpha, \quad A_\alpha P = PB_\alpha \tag{15}$$

we have $(B_\alpha - \lambda_0)K_\alpha\phi_0 = K_\alpha\phi_1, (B_\alpha - \lambda_0)K_\alpha\phi_1 = K_\alpha\phi_2, \ldots, (B_\alpha - \lambda_0)K_\alpha\phi_k = 0$.

If one would have $K_\alpha\phi_k = 0$, then $\phi_k = \lambda_0^{-1}A_\alpha\phi_k = \lambda_0^{-1}PK_\alpha\phi_k = 0$, which is a contradiction. Thus $K_\alpha\phi_k \neq 0$, $K_\alpha\phi_0 \neq 0$ and therefore $K_\alpha\phi_0$ is a generalized eigenvector of B_α of (precisely) rank k. Conversely, if $u_0 \neq 0$ is a generalized eigenvector at λ_0, i.e. $(B_\alpha - \lambda_0)u_0 = u_1, (B_\alpha - \lambda_0)u_1 = u_2, \ldots, (B_\alpha - \lambda_0)u_k = 0$ where $u_k \neq 0$, then on using (15) one obtains $(A_\alpha - \lambda_0)Pu_0 = Pu_1, (A_\alpha - \lambda_0)Pu_1 = Pu_2, \ldots, (A_\alpha - \lambda_0)Pu_k = 0$. If one would have $Pu_k = 0$, then $u_k = \lambda_0^{-1}B_\alpha u_k = \lambda_0^{-1}K_\alpha Pu_k = 0$, which is a contradiction. Thus $Pu_k \neq 0$, $Pu_0 \neq 0$ and therefore Pu_0 is a generalized eigenvector of A_α of (precisely) rank k.

We may thus conclude that the partial multiplicities of the non-zero eigenvalues of A_α and B_α coincide pairwise. The infinitude of the number of eigenvalues will be proved after Theorem 4.

Theorem 4. *The spectra of A_α and B_α, including the partial multiplicities of the non-zero eigenvalues, do not depend on the functional space, which may either be $L_p(V)$ $(1 \leq p < \infty)$ or $C(\bar{V})$ for A_α, or $L_p(V \times S)$ $(1 \leq p < \infty)$ or $C(\bar{V} \times S)$ for B_α.*

Proof. As a consequence of Lemma 3, it suffices to consider A_α only. We recall that A_α is defined on $L_p(V)$ $(1 \leq p < \infty)$ and $C(\bar{V})$, and observe that these spaces are linearly ordered with respect to set inclusion with the natural imbedding from the smaller into the larger space being continuous with dense range. For every such space E we denote by $n_k^E(\lambda_0)$ and $d_k^E(\lambda_0)$ the following numbers:

$$n_k^E(\lambda_0) = \dim \{\phi \in E / (A_\alpha - \lambda_0)^k\phi = 0\},$$

$$d_k^E(\lambda_0) = \operatorname{codim} \{(A_\alpha - \lambda_0)^k\phi / \phi \in E\}.$$

If $\lambda_0 \neq 0$, we have

$$n_k^E(\lambda_0) = d_k^E(\lambda_0) < \infty. \tag{16}$$

Now consider two spaces E and F from the chain of spaces $L_p(V)$ $(1 \leq p < \infty)$ and $C(\bar{V})$, and suppose $E \subset F$. Then E is continuously and densely imbedded in F and consequently

$$n_k^E(\lambda_0) = \dim\{\phi \in E / (A_\alpha - \lambda_0)^k\phi = 0\} \leq \dim\{\phi \in F / (A_\alpha - \lambda_0)^k\phi = 0\} = n_k^F(\lambda_0)$$

$$d_k^E(\lambda_0) = \operatorname{codim} \overline{\{(A_\alpha - \lambda_0)^k\phi / \phi \in E\}}^{(E)}$$

$$\geq \operatorname{codim} \overline{\{(A_\alpha - \lambda_0)^k\phi / \phi \in F\}}^{(F)} = d_k^F(\lambda_0).$$

In combination with (16) (for E and F) these two identities imply

$$n_k^E(\lambda_0) = n_k^F(\lambda_0) = d_k^E(\lambda_0) = d_k^F(\lambda_0),$$

which proves the theorem.

As a result we actually find $\{\phi \in L_p(V)/(A_\alpha - \lambda_0)^k \phi = 0\} \subset C(\bar{V})$ for every $\lambda_0 \neq 0$. In order to establish the inclusion

$$\{u \in L_p(V \times S)/(B_\alpha - \lambda_0)^k u = 0\} \subset C(\bar{V} \times S) \tag{17}$$

for every $\lambda_0 \neq 0$, we take $u \in L_p(V \times S)$ with $(B_\alpha - \lambda_0)^k u = 0$. Then $(A_\alpha - \lambda_0)^k Pu = P(B_\alpha - \lambda_0)^k u = 0$, and hence $Pu \in C(\bar{V})$. But then $B_\alpha u = K_\alpha Pu \in C(\bar{V} \times S)$. Since $(B_\alpha - \lambda_0)^k u = 0$, $\{B_\alpha u, B_\alpha^2 u, \ldots, B_\alpha^k u\} \subset C(\bar{V} \times S)$ and $\lambda_0 \neq 0$, we easily obtain $u \in C(\bar{V} \times S)$, which proves (17).

Since the kernel of A_α is symmetric and non-degenerate, A_α is self-adjoint on $L_2(V)$ with infinitely many eigenvalues, all of which are real. Moreover, the algebraic and geometric multiplicities coincide for each of them. As a corollary of Theorem 4, these properties of A_α on $L_2(V)$ are shared by A_α on $L_p(V)$ $(1 \leq p < \infty)$, A_α on $C(\bar{V})$, B_α on $L_p(V \times S)$ $(1 \leq p < \infty)$ and B_α on $C(\bar{V} \times S)$. We note however that these operators may differ considerably with regard to the zero point of their spectra.

4. Positivity and criticality analysis

In the previous section we have proved that, irrespective of the functional setting and whether one deals with A_α (total particle density) or B_α (angular density), the non-zero spectrum is the same and consists of a denumerably infinite sequence of real eigenvalues with finite algebraic multiplicity, without generalized eigenvectors. It is a trivial matter at this point to observe that A_α is a positive operator, in the sense that A_α maps non-negative functions u into non-negative functions. In the natural partial order, we observe easily that

$$0 \leq A_0 \leq A_{\alpha_1} \leq A_{\alpha_2} \leq A_1, \quad 0 \leq B_0 \leq B_{\alpha_1} \leq B_{\alpha_2} \leq B_1,$$

where $0 \leq \alpha_1 \leq \alpha_2 \leq 1$, while the kernel of A_0, $(4\pi)^{-1}|\mathbf{x}-\mathbf{x}'|^{-2} \exp(-\sigma|\mathbf{x}-\mathbf{x}'|) \geq \delta(V) > 0$. Here $\delta(V) = (4\pi)^{-1} D^{-2} \exp(-\sigma D)$ where D is the diameter of V.

The latter strict positivity property of all operators A_α and the mere fact that the analysis of the spectral properties of A_α and B_α may be reduced to the spectral analysis of A_α on $C(\bar{V})$ (at least if we do not consider the zero spectrum), enable a simple application of the Krein–Rutman theory.[12]

Theorem 5. *The largest eigenvalue of A_α, its positive spectral radius, r_α, is algebraically simple and, irrespective of the functional setting, the corresponding eigenfunction, $\phi_\alpha^{\text{crit}}$, belongs to $C(\bar{V})$ and is strictly positive. Similarly, the eigenfunction u_α^{crit} corresponding to the eigenvalue r_α of B_α belongs to $C(\bar{V} \times S)$ and is strictly positive. Moreover,*

$$u_\alpha^{\text{crit}} = \frac{1}{r_\alpha} K_\alpha \phi_\alpha^{\text{crit}}, \quad \phi_\alpha^{\text{crit}} = P u_\alpha^{\text{crit}} \tag{18}$$

Proof. Basically, everything follows directly from the above results in combination with Reference 12, except (18) and the strict positivity of u_α^{crit}. Since $A_\alpha \phi_\alpha^{\text{crit}} = r_\alpha \phi_\alpha^{\text{crit}}$ and $B_\alpha u_\alpha^{\text{crit}} = r_\alpha u_\alpha^{\text{crit}}$, while (14) and (15) hold true and the eigenvalue r_α is geometrically simple, we may relate the eigenfunctions as in (18). As K_α is an integral operator with strictly positive kernel and $\phi_\alpha^{\text{crit}}$ is strictly positive, u_α^{crit} is strictly positive as well.

Using Reference 12 it is straightforward to prove that r_α is the only eigenvalue of A_α or B_α where the corresponding eigenfunction is non-negative. In order to establish this result, it again suffices to restrict oneself to A_α on $C(\bar{V})$. Owing to the strict positivity of the kernel of A_α, r_α is the only eigenvalue of A_α where the eigenfunction is non-negative (Reference 10, Theorems 2.10, 2.11 and 2.13, noting that A_α is u_0-positive on $C(\bar{V})$ for $u_0 \equiv 1$). If $B_\alpha u = \lambda_0 u$ with $\lambda_0 \neq 0$ for some $u \geq 0$ and $u \not\equiv 0$, then $A_\alpha(Pu) = \lambda_0 Pu$ (cf. (15)). However, if Pu were to vanish, then $u = \lambda_0^{-1} B_\alpha u = \lambda_0^{-1} K_\alpha Pu$ would vanish identically, which contradicts $u \not\equiv 0$. Hence, we must have $\lambda_0 = r_\alpha$ and $u = cu_\alpha^{\text{crit}}$ for some $c > 0$; thus r_α is the only eigenvalue of B_α where the eigenfunction is non-negative. For other non-zero eigenvalues λ_0 the eigenfunctions of A_α and B_α are related by two identities of the type (18), namely $u = (1/\lambda_0)K_\alpha\phi$ and $\phi = Pu$.

Now that we have established the basic compactness, positivity and spectral properties of A_α and B_α, we shall study them as a function of the spatial domain V, the cross-section σ and the accommodation coefficient α. To begin with the latter we have

Theorem 6. *The critical eigenvalue r_α is a monotonically increasing and continuous function of α, which satisfies $r_0 > 0$ and $r_1 = 1/\sigma$.*

Proof. Let us write, for $|\alpha| \leq 1$,

$$(A_\alpha\phi)(\mathbf{x}) = \frac{1}{4\pi}\int_V \frac{\exp(-\sigma|\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}-\mathbf{x}'|^2} l_\alpha(\mathbf{x},\mathbf{x}')\phi(\mathbf{x}')\mathrm{d}\mathbf{x}' + \frac{1}{4\pi}\int_V \frac{\exp(\sigma|\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}-\mathbf{x}'|^2} m_\alpha(\mathbf{x},\mathbf{x}')\phi(\mathbf{x}')\mathrm{d}\mathbf{x}', \tag{19}$$

where

$$l_\alpha(\mathbf{x},\mathbf{x}') = 1 + \frac{\alpha\exp(-2\sigma r^+(\mathbf{x},\mathbf{x}')) + \alpha\exp(-2\sigma r^+(\mathbf{x}',\mathbf{x})) + \alpha^2\exp(-2\sigma d(\mathbf{x},\mathbf{x}'))}{1-\alpha^2\exp(-2\sigma d(\mathbf{x},\mathbf{x}'))} \tag{20}$$

and

$$m_\alpha(\mathbf{x},\mathbf{x}') = \frac{\alpha^2\exp(-2\sigma d(\mathbf{x},\mathbf{x}'))}{1-\alpha^2\exp(-2\sigma d(\mathbf{x},\mathbf{x}'))}.$$

Then

$$\frac{\partial}{\partial\alpha} l_\alpha(\mathbf{x},\mathbf{x}') = \frac{\{\exp(-2\sigma r^+(\mathbf{x},\mathbf{x}')) + \exp(-2\sigma r^+(\mathbf{x}',\mathbf{x}))\}\,\{1+\alpha^2\exp(-2\sigma d(\mathbf{x},\mathbf{x}'))\}}{[1-\alpha^2\exp(-2\sigma d(\mathbf{x},\mathbf{x}'))]^2} + \frac{2\alpha\exp(-2\sigma d(\mathbf{x},\mathbf{x}'))}{[1-\alpha^2\exp(-2\sigma d(\mathbf{x},\mathbf{x}'))]^2}$$

and

$$\frac{\partial}{\partial\alpha} m_\alpha(\mathbf{x},\mathbf{x}') = \frac{2\alpha\exp(-2\sigma d(\mathbf{x},\mathbf{x}'))}{[1-\alpha^2\exp(-2\sigma d(\mathbf{x},\mathbf{x}'))]^2}.$$

We easily estimate, for $|\alpha| < 1$,

$$\left|\frac{\partial}{\partial\alpha} l_\alpha(\mathbf{x},\mathbf{x}')\right| \leq \frac{2(1+|\alpha|^2)+2|\alpha|}{(1-|\alpha|^2)^2} \leq \frac{2}{(1-|\alpha|)^2}$$

and

$$\left|\frac{\partial}{\partial\alpha} m_\alpha(\mathbf{x}, \mathbf{x}')\right| \leq \frac{2|\alpha|}{(1-|\alpha|^2)^2}.$$

Therefore, A_α can be differentiated with respect to α in the operator norm if $|\alpha| < 1$, and the result is an integral operator with weakly singular kernel. Hence, irrespectively of the functional space, $L_p(V)$ $(1 \leq p < \infty)$ or $C(\bar{V})$, $\alpha \mapsto A_\alpha$ is an analytic function on $|\alpha| < 1$ with respect to the operator norm topology. In combination with the compactness of A_α (proved for $0 \leq \alpha \leq 1$, but extended to $|\alpha| < 1$ by analyticity) and the simplicity of r_α in the algebraic sense if $0 \leq \alpha \leq 1$, it follows that the scalar function $\alpha \to r_\alpha$ on $[0, 1)$ has an analytic continuation to a neighbourhood of $[0, 1)$ in the open unit disk (cf. Reference 9, Theorem VII 1.8).

It is trivial to see that $r_0 = \lim_{\alpha\to 0} r_\alpha > 0$. It remains to consider the behaviour as $\alpha \uparrow 1$. We may restrict ourselves to considering A_α on $L_p(V)$ where $1 \leq p < \infty$, and choose a non-negative $\phi \in L_p(V)$. Then $\{A_\alpha\phi\}_{\alpha\uparrow 1}$ is a monotonically increasing net in $L_p(V)$ dominated by $A_1\phi$; hence, there exists a strong limit of the net of operators $\{A_\alpha\}_{\alpha\uparrow 1}$, C say. Since $\lim_{\alpha\uparrow 1} l_\alpha(\mathbf{x}, \mathbf{x}') = l_1(\mathbf{x}, \mathbf{x}')$ and $\lim_{\alpha\uparrow 1} m_\alpha(\mathbf{x}, \mathbf{x}') = m_1(\mathbf{x}, \mathbf{x}')$, we have $C = A_1$ and A_1 is the strong limit in $L_p(V)$ of the net $\{A_\alpha\}_{\alpha\uparrow 1}$. In particular, if one normalizes ϕ_1^{crit} by $\|\phi_1^{\text{crit}}\|_2 = 1$, where from now on $\|\cdot\|_p = \|\cdot; L_p\|$ denotes the norm in L_p, then one has simultaneously

$$\|A_\alpha\phi_1^{\text{crit}}\|_2 \leq r_\alpha \leq \|A_1\phi_1^{\text{crit}}\|_2 = r_1$$

and

$$\lim_{\alpha\uparrow 1} \|A_\alpha\phi_1^{\text{crit}}\|_2 = \|A_1\phi_1^{\text{crit}}\|_2,$$

whence $\lim_{\alpha\uparrow 1} r_\alpha = r_1$. Finally, we notice that $u(\mathbf{x}, \mathbf{\Omega}) \equiv 1$ satisfies equation (1) if $\alpha = 1$ and $\gamma = \sigma$.

So $\phi(\mathbf{x}) = (4\pi)^{-1} \int_S u(\mathbf{x}, \mathbf{\Omega}')\mathrm{d}\mathbf{\Omega}' \equiv 1$ satisfies $\sigma A_\alpha\phi = \phi$, whence $\phi_1^{\text{crit}}(\mathbf{x}) \equiv (\text{mes } V)^{-\frac{1}{2}}$ and $r_1 = 1/\sigma$.

Next, let us investigate in detail the dependence of r_α on the spatial domain V. Having a family of bounded convex domains with piecewise C^1-boundary depending on a parameter b, $V = V_b$, we say that the family $\{V_b\}_b$ is *continuous* if (i) there exists a bounded set V_u, and therefore a bounded convex region V_u with piecewise C^2-boundary, containing all closures $\bar{V}_b$, and (ii) the characteristic function χ_b, $\chi_b(\mathbf{x}) = 1$ for $\mathbf{x} \in V_b$ and $\chi_b(\mathbf{x}) = 0$ for $\mathbf{x} \notin V_b$, depends continuously on b in the norm of $L_p(V_u)$ where $1 \leq p < \infty$. One should observe that the definition does not depend on the choice of $p \in [1, \infty)$ or on the choice of V_u. To every continuous family and every corresponding 'universe' V_u, we associate a family of operators $\{\Pi_b\}_b$ defined by

$$(\Pi_b\phi)(\mathbf{x}) = \begin{cases} \phi(\mathbf{x}), & \mathbf{x} \in V_b \\ 0, & \mathbf{x} \in V_u \setminus V_b; \end{cases}$$

then Π_b is a bounded projection on $L_p(V_u)$ for every $1 \leq p < \infty$. Also, $b \mapsto \Pi_b$ is continuous in the strong operator topology of $L_p(V_u)$, provided $1 \leq p < \infty$.

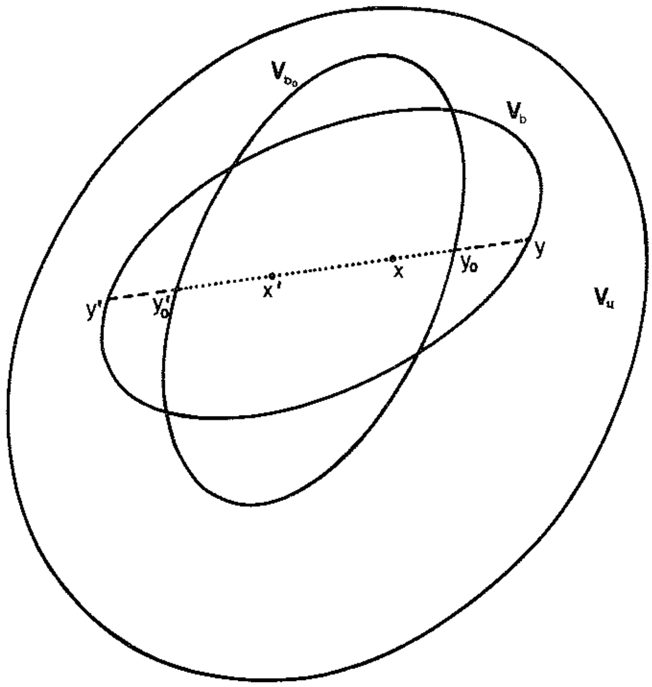

Fig. 3. Two regions, V_{b_0} and V_b, from the continuous family $\{V_b\}_b$. $r_{b_0}^-(\mathbf{x}, \mathbf{x}')$ equals $\mathbf{y}_0'\mathbf{x}$, $r_{b_0}^+(\mathbf{x}, \mathbf{x}')$ equals $\mathbf{x}\mathbf{y}_0$, $r_b^-(\mathbf{x}, \mathbf{x}')$ equals $\mathbf{y}'\mathbf{x}$ and $r_b^+(\mathbf{x}, \mathbf{x}')$ equals $\mathbf{x}\mathbf{y}$

Theorem 7. *Let $\{V_b\}_b$ be a continuous family of spatial domains. Then for every $0 \le \alpha \le 1$ the critical eigenvalues $r_\alpha = r_\alpha(b)$ depend continuously on b.*

Proof. Since $r_\alpha = 1/\sigma$ if $\alpha = 1$, it suffices to consider the case $0 \le \alpha < 1$. First we observe that the functions $l_\alpha(\mathbf{x}, \mathbf{x}')$ and $m_\alpha(\mathbf{x}, \mathbf{x}')$ in (20) are bounded from above, i.e.

$$0 \le l_\alpha(\mathbf{x}, \mathbf{x}') \le 1 + \frac{2\alpha + \alpha^2}{1-\alpha^2} \le \frac{2}{1-\alpha},$$

and

$$0 \le m_\alpha(\mathbf{x}, \mathbf{x}') \le \frac{\alpha^2}{1-\alpha^2} \le \frac{1}{1-\alpha^2}.$$

Adopting a 'universe' V_u for the family $\{V_b\}$, we associate with each operator $A_\alpha = A_{\alpha, b}$ on $L_p(V_b)$ the operator $C_{\alpha, b}$ on $L_p(V_u)$ defined by

$$(C_{\alpha, b}\phi)(\mathbf{x}) = \frac{1}{4\pi}\int_{V_u} \frac{\exp(-\sigma|\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}-\mathbf{x}'|^2} l_{\alpha, b}(\mathbf{x}, \mathbf{x}')\phi(\mathbf{x}')d\mathbf{x}'$$
$$+\frac{1}{4\pi}\int_{V_u} \frac{\exp(\sigma|\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}-\mathbf{x}'|^2} m_{\alpha, b}(\mathbf{x}, \mathbf{x}')\phi(\mathbf{x}')d\mathbf{x}',$$

where $l_{\alpha, b}(\mathbf{x}, \mathbf{x}') = m_{\alpha, b}(\mathbf{x}, \mathbf{x}') = 0$ if either $\mathbf{x}$ or $\mathbf{x}'$ does not belong to V_b, and $l_{\alpha, b}(\mathbf{x}, \mathbf{x}')$ and $m_{\alpha, b}(\mathbf{x}, \mathbf{x}')$ are defined by (20) with $r^+(\mathbf{x}, \mathbf{x}')$ and $d(\mathbf{x}, \mathbf{x}')$ written in terms of V_b (cf. Fig. 3) if $\mathbf{x}, \mathbf{x}' \in V_b$. This means that with respect to the decomposition

$$L_p(V_u) = L_p(V_b) \oplus L_p(V_u \setminus V_b)$$

one can decompose $C_{\alpha,b}$ as

$$C_{\alpha,b} = A_{\alpha,b} \oplus \hat{A}_{0,b},$$

where

$$(\hat{A}_{0,b}\phi)(\mathbf{x}) = \frac{1}{4\pi}\int_{V_u \setminus V_b} \frac{\exp(-\sigma|\mathbf{x}-\mathbf{x}'|)+\exp(\sigma|\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}-\mathbf{x}'|^2}\phi(\mathbf{x}')d\mathbf{x}',$$

$\mathbf{x} \in V_u \setminus V_b$. Thus $\hat{A}_{0,b}$ is the operator A_0 corresponding to $V_u \setminus V_b$. Choose b_0. Select a neighbourhood U_1 of b_0 such that for the operators A_0 corresponding to V_b and V_{b_0} one has

$$\| A_{0,b}|_{L_2(V_b \setminus V_{b_0})};\, L_2(V_b)\| < \frac{1}{8}(1-\alpha)\{1+\exp(\sigma D)\}\,\varepsilon$$

and

$$\| A_{0,b_0}|_{L_2(V_{b_0} \setminus V_b)};\, L_2(V_{b_0})\| < \frac{1}{8}(1-\alpha)\{1+\exp(\sigma D)\}\,\varepsilon,$$

whenever $b \in U_1$. Also select a neighbourhood U_2 of b_0 such that

$$\sup_{\mathbf{x},\mathbf{x}' \in V_b \cap V_{b_0}} \{|l_{\alpha,b}(\mathbf{x},\mathbf{x}')-l_{\alpha,b_0}(\mathbf{x},\mathbf{x}')|+|m_{\alpha,b}(\mathbf{x},\mathbf{x}')-m_{\alpha,b_0}(\mathbf{x},\mathbf{x}')|\}$$

$$< \frac{\varepsilon}{2M}\{1+\exp(\sigma D)\}^{-1},$$

where M is the L_2-norm of the operator A_0 corresponding to $V_b \cap V_{b_0}$. Then for every $b \in U_1 \cap U_2$ we have, using self-adjointness,

$$\begin{aligned}|r_{\alpha,b}-r_{\alpha,b_0}| &\le |r(C_{\alpha,b})-r(C_{\alpha,b_0})|\\ &= |\,\|C_{\alpha,b}\|_2-\|C_{\alpha,b_0}\|_2| \le \|C_{\alpha,b}-C_{\alpha,b_0}\|_2\\ &< \frac{2}{1-\alpha}\frac{1}{8}(1-\alpha)\varepsilon+\frac{2}{1-\alpha}\frac{1}{8}(1-\alpha)\varepsilon+M\frac{\varepsilon}{2M}=\varepsilon,\end{aligned}$$

which completes the proof.

Finally, the dependence of r_α on the cross-section σ can be studied as an application of Theorem 7. First we notice that none of the spectral, positivity, and criticality properties of A_α and B_α will change if V is replaced by the region $\tau[V]$ where τ is a similarity transformation of $\mathbf{R}^3$. Let us consider a homeomorphic C^1-transformation from $\mathbf{R}^3$ onto itself, denoted τ_σ, which stretches up all distances by a factor σ, i.e. $|\tau_\sigma(\mathbf{x})-\tau_\sigma(\mathbf{x}')| = \sigma|\mathbf{x}-\mathbf{x}'|$, and let $V_\tau = \tau_\sigma[V]$. Let us consider the invertible operators U_τ from $L_p(V)$ onto $L_p(V_\sigma)$, and W_τ from $L_p(V \times S)$ onto $L_p(V_\sigma \times S)$, defined by

$$(U_\tau\phi)(\mathbf{x}) = \phi(\tau_\sigma^{-1}(\mathbf{x})), \quad (W_\tau u)(\mathbf{x},\mathbf{\Omega}) = u(\tau_\sigma^{-1}(\mathbf{x}),\mathbf{\Omega}).$$

Put

$$A_\alpha^{(\tau)} = U_\tau A_\alpha U_\tau^{-1}, \quad B_\alpha^{(\tau)} = W_\tau B_\alpha W_\tau^{-1};$$

then

$$(A_\alpha^{(\tau)}\phi)(\mathbf{x}) = \frac{\sigma}{4\pi}\int_{V_\tau}\left[\frac{\exp(-|\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}-\mathbf{x}'|^2}\right.$$

$$\times\left\{1+\frac{\alpha\exp(-2r^+(\mathbf{x},\mathbf{x}'))+\alpha\exp(-2r^+(\mathbf{x}',\mathbf{x}))+\alpha^2\exp(-2d(\mathbf{x},\mathbf{x}'))}{1-\alpha^2\exp(-2d(\mathbf{x},\mathbf{x}'))}\right\}$$

$$\left.+\frac{\exp|\mathbf{x}-\mathbf{x}'|}{|\mathbf{x}-\mathbf{x}'|^2}\frac{\alpha^2\exp(-2d(\mathbf{x},\mathbf{x}'))}{1-\alpha^2\exp(-2d(\mathbf{x},\mathbf{x}'))}\right]\phi(\mathbf{x}')\mathrm{d}\mathbf{x}',$$

and similarly for $B_\alpha^{(\tau)}$. Hence,

$$r_\alpha(V,\sigma)=\sigma r_\alpha(V_{\tau_\sigma},1). \tag{21}$$

Thus we may find a description of the dependence of r_α on σ in terms of a description of the dependence of r_α on V by combining (21) with Theorem 7.

A second route to the same result is based on the notion of collective compactness introduced by Anselone (Reference 1, Chapter 4). It can be proved that $\{A_\alpha(\sigma_n)\}_{n=1}^\infty$ is a collectively compact family of operators if $(\sigma_n)_{n=1}^\infty$ is a convergent sequence of cross-sections, and this suffices to prove the continuity of $r_\alpha(\sigma)$ in σ.

5. Time dependent problems

As we have pointed out in the Introduction, the existence, uniqueness and positivity of the solution of (2) for $0\le\alpha<1$ have been settled in a rather general framework.[2] The case $\alpha=1$ may lead to some pathologies,[16] and has to be treated on a case-by-case basis unless one weakens the notion of a solution. In the present context we include the case $\alpha=1$, investigating the time-dependent problem directly.

Let

$$\begin{cases} C_\alpha u=-\boldsymbol{\Omega}\cdot\nabla u\\ D(C_\alpha)=\{f\in L_1(V\times S)/\boldsymbol{\Omega}\cdot\nabla f\in L_1(V\times S),\\ \qquad f^\pm\in L_1(\partial V\times S^\pm),\ \ f^+(\mathbf{y},\boldsymbol{\Omega})=\alpha f^-(\mathbf{y},-\boldsymbol{\Omega})\}\end{cases}$$

where $\boldsymbol{\Omega}\cdot\nabla f$ denotes the directional derivative of f taken in distributional sense, $S^-(\mathbf{y})=\{\boldsymbol{\Omega}\in S/\boldsymbol{\Omega}\cdot\mathbf{n}(\mathbf{y})<0\}$, $S^+(\mathbf{y})=\{\boldsymbol{\Omega}\in S/\boldsymbol{\Omega}\cdot\mathbf{n}(\mathbf{y})\ge 0\}$ and $f^\pm$ denote the traces (i.e. the restrictions) of f on $\partial V\times S^\pm$.

Lemma 8. *For $\alpha<1$ the streaming operator $C_\alpha-\sigma I$ is the infinitesimal generator of a strongly continuous contraction semigroup $\mathrm{e}^{-\sigma t}V_\alpha(t)$, i.e. $C_\alpha-\sigma I\in G(1,-\sigma;L_1(V\times S))$.*

Proof. Let $g=(\lambda I-C_\alpha)f$, $\forall f\in D(C_\alpha)$. Suppose, for the moment, that the function f belongs to the positive cone $L_1^+(V\times S)$. Then, for $\lambda>0$,

$$\|g;L_1(V\times S)\|\ge\left|\int_V\mathrm{d}\mathbf{x}\int_S\mathrm{d}\boldsymbol{\Omega}[\lambda f(\mathbf{x},\boldsymbol{\Omega})+\boldsymbol{\Omega}\cdot\nabla f(\mathbf{x},\boldsymbol{\Omega})]\right|$$

$$=|\lambda|\|f;L_1(V\times S)\|+\int_{\partial V}\mathrm{d}\Sigma_{\mathbf{y}}\int_{S^-(\mathbf{y})}\mathrm{d}\boldsymbol{\Omega}|\boldsymbol{\Omega}\cdot\mathbf{n}(\mathbf{y})|f^-(\mathbf{y},\boldsymbol{\Omega})$$

$$-\int_{\partial V}\mathrm{d}\Sigma_{\mathbf{y}}\int_{S^+(\mathbf{y})}\mathrm{d}\boldsymbol{\Omega}(\boldsymbol{\Omega}\cdot\mathbf{n}(\mathbf{y}))f^+(\mathbf{y},\boldsymbol{\Omega})|$$

$$=\lambda\|f;L_1(V\times S)\|+(1-\alpha)\|f^-;L_1(\partial V\times S^-)\|.$$

We have computed the resolvent $(\sigma I - C_\alpha)^{-1}$ explicitly for $\sigma > 0$ in (8), where one must replace $f(\mathbf{x})$ by $f(\mathbf{x}, \mathbf{\Omega})$. From this expression it is immediate that $(\sigma I - C_\alpha)^{-1}$ is bounded on $L_1(V \times S)$ for $\sigma > 0$ and $\alpha < 1$. Thus

$$\|f; L_1(V \times S)\| = \|(\lambda I - C_\alpha)^{-1} g; L_1(V \times S)\|$$

$$\leq \frac{1}{\lambda} \|g; L_1(V \times S)\| \ \forall \lambda > 0, \quad \forall g \in L_1 \text{ and } \forall \alpha \in [0, 1). \tag{22}$$

Moreover, $(\lambda I - C_\alpha)^{-1}$ bounded implies C_α closed for any $\alpha \in [0, 1)$, while $D(C_\alpha)$ is dense in $L_1(V \times S)$, because it contains the functions with compact support and infinitely differentiable.

From the Hille–Yosida theorem and classical semigroup theory,[4] the result follows immediately.

For $\alpha = 1$ we proceed differently. For $g \geq 0$ in $L_1(V \times S)$ we observe that $\{(\lambda I - C_\alpha)^{-1} g\}_{\alpha < 1}$ is a monotonically increasing net in $L_1(V \times S)$ having the (α-independent) L_1-bound (22), whenever $\lambda > 0$. Hence, for $\lambda > 0$ there exists a bounded operator $D(\lambda)$ on $L_1(V \times S)$ that is the strong limit of the net $\{(\lambda I - C_\alpha)^{-1}\}_{\alpha < 1}$ for $\alpha \to 1$. On using the resolvent identity for C_α and $\lambda, \mu > 0$ and taking the strong limit as $\alpha \to 1$, we find

$$D(\lambda) - D(\mu) = (\mu - \lambda) D(\lambda) D(\mu) = (\mu - \lambda) D(\mu) D(\lambda). \tag{23}$$

Since $\|D(\lambda)\| \leq \lambda^{-1}$ (cf. (22)), we obtain that $D(\lambda)$ depends continuously on $\lambda \in (0, \infty)$ in the uniform sense. Using (23) we see that Ran $D(\lambda)$ does not depend on λ, and hence there is a closed linear operator $\tilde{C}_1$ on $L_1(V \times S)$ such that $D(\tilde{C}_1) = \text{Ran } D(\lambda)$, $(0, \infty) \subset \rho(\tilde{C}_1)$ and $(\lambda I - \tilde{C}_1)^{-1} = D(\lambda)$.

Next, $\{V_\alpha(t) g\}_{\alpha < 1}$ is a monotonically increasing net in $L_1(V \times S)$ with L_1-bound $\|g\|_1$ if $g \geq 0$. As a result, there exists a bounded linear operator $V_1(t)$ on $L_1(V \times S)$ arising as a strong limit of the net $\{V_\alpha(t)\}_{\alpha < 1}$ as $\alpha \to 1$. This operator is strongly measurable in t on $[0, \infty)$ and satisfies

$$(\lambda I - \tilde{C}_1)^{-1} g = \int_0^\infty e^{-\lambda t} V_1(t) g \, dt, \quad g \in L_1(V \times S), \quad \lambda > 0. \tag{24}$$

On choosing $g \geq 0$ and Re $\lambda > 0$ we obtain

$$\left\| \int_0^\infty e^{-\lambda t} V_1(t) g \, dt \right\|_1 \leq \int_0^\infty e^{-t \text{Re}\lambda} \| V_1(t) g \|_1 \, dt$$

$$= \left\| \int_0^\infty e^{-t \text{Re}\lambda} V_1(t) g \, dt \right\|_1 = \| (\text{Re}\lambda I - \tilde{C}_1)^{-1} g \|_1 \leq (\text{Re}\lambda)^{-1} \|g\|_1 .$$

Hence, $(\lambda I - \tilde{C}_1)^{-1}$ has an analytic continuation to the open right half-plane, $\sigma(\tilde{C}_1) \subseteq \{\lambda \in \mathbf{C} : \text{Re}\lambda \leq 0\}$ and $\|(\lambda I - \tilde{C}_1)^{-1}\| \leq (\text{Re}\lambda)^{-1}$. As a result of the latter estimate, $\tilde{C}_1$ generates a contraction semigroup on $L_1(V \times S)$. From (24) it is clear that this C_0-semigroup coincides with $\{V_1(t)\}_{t \geq 0}$.

Now the physical meaning of the problem suggests the form of the semigroup $U_\alpha(t)$ describing the free evolution of the system. Let $\alpha \in [0, 1]$. We introduce the following flow:

$$\tau(t)\,(\mathbf{x},\mathbf{\Omega}) = \begin{cases} (\mathbf{x}-t\mathbf{\Omega},\mathbf{\Omega}) & \text{if } 0 \le t \le t_0(\mathbf{x},\mathbf{\Omega}) \\ (\mathbf{x}-r(\mathbf{x},\mathbf{\Omega})\mathbf{\Omega}+(t-t_{2n}(\mathbf{x},\mathbf{\Omega}))\mathbf{\Omega},-\mathbf{\Omega}) & \\ & \text{if } t_{2n}(\mathbf{x},\mathbf{\Omega}) < t \le t_{2n+1}(\mathbf{x},\mathbf{\Omega}) \\ (\mathbf{x}+r(\mathbf{x}-\mathbf{\Omega})\mathbf{\Omega}-(t-t_{2n+1}(\mathbf{x},\mathbf{\Omega}))\mathbf{\Omega},\mathbf{\Omega}) & \\ & \text{if } t_{2n+1}(\mathbf{x},\mathbf{\Omega}) < t \le t_{2(n+1)}(\mathbf{x},\mathbf{\Omega}), \end{cases} \tag{25}$$

where $n = 0, 1, \ldots$ and $t_n(\mathbf{x},\mathbf{\Omega}) = r(\mathbf{x},\mathbf{\Omega}) + nd(\mathbf{x},\mathbf{\Omega})$.

In terms of this flow we define the family of operators

$$(U_\alpha(t)f)\,(\mathbf{x},\mathbf{\Omega}) = \alpha^n f(\tau(t)\,(\mathbf{x},\mathbf{\Omega})), \quad t_{n-1}(\mathbf{x},\mathbf{\Omega}) < t \le t_n(\mathbf{x},\mathbf{\Omega}),$$

where $n = 0, 1, \ldots$ and $t_{-1}(\mathbf{x},\mathbf{\Omega}) = 0$. Then this family of operators is a strongly continuous function in t, which satisfies the semigroup property. If G_α is the infinitesimal generator of this C_0-semigroup, then G_α operates as a distributional directional derivative and $G_\alpha \supset C_\alpha$; for the proof, see the analogous proof in Reference 6.

Lemma 9. *The semigroup $\{U_\alpha(t): t \ge 0\}$ coincides with the semigroup generated by C_α, for any $\alpha \in [0, 1)$, and with the semigroup generated by $\tilde{C}_1$ for $\alpha = 1$; more precisely, $V_\alpha(t) \equiv U_\alpha(t)$, $t \ge 0$, and $C_\alpha = G_\alpha$, $\alpha \in [0, 1)$, and $\tilde{C}_1 = G_1$.*

Proof. As C_α and G_α generate the C_0-semigroups $V_\alpha(t)$ and $U_\alpha(t)$, respectively, there exists some ω such that the half plane $\{\mathrm{Re}\lambda > \omega\}$ is contained in $\rho(C_\alpha) \cap \rho(G_\alpha)$. Because $C_\alpha \subset G_\alpha$, we have $(\lambda I - C_\alpha)f = (\lambda I - G_\alpha)f$, for $f \in D(C_\alpha) \subset D(G_\alpha)$ and $\mathrm{Re}\,\lambda > \omega$. Thus for $\mathrm{Re}\,\lambda > \omega$, $R(\lambda, C_\alpha)$ and $R(\lambda, G_\alpha)$ coincide on the dense subspace $(\lambda I - C_\alpha)[\mathrm{D}(C_\alpha)]$, which actually is the complete space. Consequently $R(\lambda, C_\alpha) = R(\lambda, G_\alpha)$, for $\mathrm{Re}\lambda > \omega$. Using the exponential formula, we obtain

$$\begin{aligned} V_\alpha(t) &= s-\lim_{n\to\infty}\left[\frac{n}{t}R\left(\frac{n}{t}, C_\alpha\right)\right]^n = s-\lim_{n\to\infty}\left[\frac{n}{t}R\left(\frac{n}{t}, G_\alpha\right)\right]^n \\ &= U_\alpha(t). \end{aligned}$$

Finally, since a semigroup has precisely one generator, we have $C_\alpha = G_\alpha$. For $\alpha = 1$ one reasons likewise with C_α and G_α replaced by $\tilde{C}_1$ and G_1, respectively.

Now, for $\mathrm{Re}\,\lambda > -\sigma$, one can easily compute the resolvent of $C_\alpha - \sigma I$ by

$$((\lambda - C_\alpha + \sigma I)^{-1}f)\,(\mathbf{x},\mathbf{\Omega}) = \int_0^{+\infty} \mathrm{e}^{-(\lambda+\sigma)t}\,(U_\alpha(t)f)\,(\mathbf{x},\mathbf{\Omega})\mathrm{d}t \tag{26}$$

if $[0, 1)$; for $\alpha = 1$ one must replace C_1 by $\tilde{C}_1$.

This formula, with $\lambda = 0$ and $f = \gamma Pu$, enables us to put equation (5) in the integral form (11).

We refrain here from writing down explicitly the resolvent operator of $C_\alpha - \sigma I$; we only remark that it is bounded whenever $1 - \alpha^2\exp(-2(\lambda+\sigma)d(\mathbf{x},\mathbf{\Omega})) \ne 0$, (compare with the denominator in (8)), i.e. for

$$\lambda \ne -\sigma + \frac{\log\alpha}{d} + \frac{n\pi\mathrm{i}}{d}$$

where $d = d(\mathbf{x}, \mathbf{\Omega})$ varies between 0 and the diameter D of V.

Since for Re $\lambda > -\sigma + (\log\alpha/D)$ the resolvent operator is well defined, one sees that in that half plane the original eigenvalue problem (6) is equivalent (see the discussion in Section 2) to

$$u = \gamma \bar{B}_\alpha u$$

where the operator $\bar{B}_\alpha$ is obtained from B_α of Section 3 by replacing σ with $\sigma + \lambda$. Furthermore, for the total particle density ϕ, we have

$$\phi = \gamma \bar{A}_\alpha \phi$$

where $\bar{A}_\alpha = P \bar{B}_\alpha J$.

We conclude with the following general theorem:

Theorem 10. *λ is an eigenvalue of Q, if $1/\gamma$ is an eigenvalue of $\bar{A}_\alpha$, and vice versa. P establishes a one-to-one correspondence between the eigenfunctions corresponding to $\lambda \in \sigma_p(Q)$ and the ones corresponding to $1/\gamma \in \sigma_p(\bar{A}_\alpha)$. The first eigenvalue of $\bar{A}_\alpha$, for $\lambda = 0$, is the critical one; see* Theorem 5.

6. Concluding remarks

For reverse specular reflection we have given a rather complete account of the criticality and time-dependent theory, also for $\alpha = 1$, with special attention to their close relationship. Our study was greatly facilitated by the explicit knowledge of the particle flow $\tau(t)$ for all $t \geq 0$ under reverse specular reflection, which allowed us to write down every relevant operator explicitly and establish their relevant spectral properties. For the sake of simplicity we have restricted ourselves to constant σ and γ, although our study could have been extended to the case of non-constant σ and γ, but at the expense of complicated integral operator kernels. Having analysed reverse specular reflection, it seems natural to generalize the present study to deterministic boundary conditions where the position of each particule in phase space at time $t \geq 0$ is given in terms of the initial position in phase space $(\mathbf{x}, \mathbf{\Omega})$ by $\tau(t)\,(\mathbf{x}, \mathbf{\Omega})$. Here $\tau(t)$ is a flow that is measure preserving in phase space. The most important example is the billiard transformation which describes perfectly specular reflection. However, such a study is severely complicated by the non-explicit nature of general $\tau(t)$ and has therefore been deferred to a future publication.

Appendix

In this appendix we shall prove Lemma 2, which states that A_α is a compact operator on $L_p(V)$ $(1 \leq p < \infty)$ and $C(\bar{V})$ for $0 \leq \alpha \leq 1$ if condition (13) is satisfied. For $L_2(V)$ the proof has been given essentially by Belleni-Morante[3] and we shall use his method to some extent. We shall concentrate our efforts on providing the following three statements: (1) A_α is bounded on $L_p(V)$ $(1 \leq p \leq \infty)$, (2) A_α is compact on $L_p(1 \leq p < \infty)$ and $C(\bar{V})$ if $0 \leq \alpha < 1$, (3) A_α is compact on $L_p(V)$ $(1 \leq p < \infty)$ and $C(\bar{V})$ if condition (13) is satisfied.

(1) *A_α is bounded on $L_p(V)$* $(1 \le p \le \infty)$. On the phase space, $V \times S$, we consider the flow $\tau(t)$ defined by (25). Then, for every $t \ge 0$, $\tau(t)$ is defined almost everywhere on $V \times S$ and $\tau(t)\,[M]$ has the same Lebesgue measure as M if $M \subset V \times S$ is Lebesgue measurable. Writing

$$\tau(t)\,(\mathbf{x}, \mathbf{\Omega}) = (\tau_1(t)\,(\mathbf{x}, \mathbf{\Omega}), \tau_2(t)\,(\mathbf{x}, \mathbf{\Omega})),$$

we may cast A_1 in the form

$$(A_1\phi)\,(\mathbf{x}) = \frac{1}{4\pi}\int_S \int_0^\infty \mathrm{e}^{-\sigma t}\phi(\tau_1(t)\,(\mathbf{x}, \mathbf{\Omega}))\mathrm{d}t\mathrm{d}\mathbf{\Omega}.$$

As a result of the measure preservation property we obtain, for $\phi \ge 0$ in $L_1(V)$,

$$\begin{aligned}\|A_1\phi\|_1 &= \frac{1}{4\pi}\int_V \int_S \int_0^\infty \mathrm{e}^{-\sigma t}\phi(\tau_1(t)\,(\mathbf{x}, \mathbf{\Omega}))\mathrm{d}t\mathrm{d}\mathbf{\Omega}\mathrm{d}\mathbf{x} \\ &= \int_0^\infty \mathrm{e}^{-\sigma t}\left\{\frac{1}{4\pi}\int_S \int_V \phi(\tau_1(t)\,(\mathbf{x}, \mathbf{\Omega}))\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{\Omega}\right\}\mathrm{d}t \\ &= \int_0^\infty \mathrm{e}^{-\sigma t}\,\mathrm{d}t\,\|\phi\|_1 = \frac{1}{\sigma}\|\phi\|_1,\end{aligned}$$

whence A_1 is bounded on $L_1(V)$. One trivially finds the same estimate for $\phi \ge 0$ in $L_\infty(V)$. By interpolation (Reference 11, Theorem I 2.4) and using $0 \le A_\alpha \le A_1$, A_α is bounded on $L_p(V)$ if $0 \le \alpha \le 1$ and $1 \le p \le \infty$.

(2) *A_α is compact on $L_p(V)$ $(1 \le p < \infty)$ and $C(\bar{V})$ if $0 \le \alpha < 1$.* For $0 \le \alpha < 1$ we have

$$(A_\alpha\phi)\,(\mathbf{x}) = \int_V h_\alpha(\mathbf{x}, \mathbf{x}')\phi(\mathbf{x}')\mathrm{d}\mathbf{x}', \quad \mathbf{x} \in V,$$

where $|\mathbf{x}-\mathbf{x}'|^{2+\varepsilon}h_\alpha(\mathbf{x}, \mathbf{x}')$ is continuous on $\bar{V} \times \bar{V}$, while $V \subset \mathbf{R}^3$ is a bounded set. Thus A_α has a weakly singular kernel and therefore A_α is compact on $L_2(V)$ (Reference 13, Theorem 7.3.2) and $C(\bar{V})$ (Reference 13, Theorem 7.4.1) and maps $L_\infty(V)$ into $C(\bar{V})$ (Reference 13, Section 7.4). Since $h_\alpha(\mathbf{x}, \mathbf{x}')$ is symmetric, A_α will be compact on the space $M(V)$ of regular Borel measures on V, bounded on its subspace $L_1(V)$ and hence compact on $L_1(V)$. By compact interpolation (Reference 11, Theorem I 3.10), A_α is compact on $L_p(V)$, $1 \le p < \infty$.

(3) *A_α is compact on $L_p(V)$ $(1 \le p < \infty)$ and $C(\bar{V})$ if $0 \le \alpha \le 1$ and condition* (13) *is satisfied.*

It is sufficient to consider the case $\alpha = 1$. Writing

$$\begin{aligned}l_1^0(\mathbf{x}, \mathbf{x}') &= \exp(-2\sigma r^+(\mathbf{x}, \mathbf{x}')) + \exp(-2\sigma r^+(\mathbf{x}', \mathbf{x})) + \exp(-2\sigma d(\mathbf{x}, \mathbf{x}')), \\ m_1^0(\mathbf{x}, \mathbf{x}') &= \exp(-2\sigma d(\mathbf{x}, \mathbf{x}')),\end{aligned}$$

we may write A_1 in the form (19) (for $\alpha = 1$), where

$$l_1(\mathbf{x}, \mathbf{x}') = l_1^0(\mathbf{x}, \mathbf{x}') \sum_{i=0}^{M-1} \exp(-2\sigma i d(\mathbf{x}, \mathbf{x}')) + l_1^M(\mathbf{x}, \mathbf{x}')$$

$$m_1(\mathbf{x}, \mathbf{x}') = m_1^0(\mathbf{x}, \mathbf{x}') \sum_{i=0}^{M-1} \exp(-2\sigma i d(\mathbf{x}, \mathbf{x}')) + m_1^M(\mathbf{x}, \mathbf{x}') \qquad (27)$$

and

$$l_1^M(\mathbf{x}, \mathbf{x}') = l_1^0(\mathbf{x}, \mathbf{x}') \exp(-2\sigma M d(\mathbf{x}, \mathbf{x}')) [1 - \exp(-2\sigma d(\mathbf{x}, \mathbf{x}'))]^{-1}$$

$$m_1^M(\mathbf{x}, \mathbf{x}') = m_1^0(\mathbf{x}, \mathbf{x}') \exp(-2\sigma M d(\mathbf{x}, \mathbf{x}')) [1 - \exp(-2\sigma d(\mathbf{x}, \mathbf{x}'))]^{-1}.$$

Thus we may write

$$A_1 = A_1^0 + A_1^M$$

in accordance with the additive decomposition (27) of l_1 and m_1. Using the weak singularity of the kernel and compact interpolation as in part (2) of the Appendix, we obtain that A_1^0 is compact on $L_p(V)$ $(1 \le p < \infty)$ and $C(\bar{V})$. Using the estimate

$$\int_V \frac{\exp[-\sigma|\mathbf{x}-\mathbf{x}'| - 2\sigma M d(\mathbf{x}, \mathbf{x}')]}{|\mathbf{x}-\mathbf{x}'|^2 [1 - \exp(-2\sigma d(\mathbf{x}, \mathbf{x}'))]} d\mathbf{x}' = O\left(\frac{1}{M}\right), \text{ as } M \to \infty, \tag{28}$$

and the analogous one with + instead of − in front of $\sigma|\mathbf{x}-\mathbf{x}'|$, with the order constant independent of $\mathbf{x}$ (the proof of (28) will be given below), we derive the following:

(i) For $\phi \in C(\bar{V})$ we have

$$\begin{aligned} |A_1^M \phi(\mathbf{x})| &\le 3\|\phi\|_C \int_V \frac{\exp[-\sigma|\mathbf{x}-\mathbf{x}'| - 2\sigma M d(\mathbf{x}, \mathbf{x}')]}{|\mathbf{x}-\mathbf{x}'|^2 [1 - \exp(-2\sigma d(\mathbf{x}, \mathbf{x}'))]} d\mathbf{x}' \\ &\quad + \|\phi\|_C \int_V \frac{\exp[\sigma|\mathbf{x}-\mathbf{x}'| - 2\sigma M d(\mathbf{x}, \mathbf{x}')]}{|\mathbf{x}-\mathbf{x}'|^2 [1 - \exp(-2\sigma d(\mathbf{x}, \mathbf{x}'))]} d\mathbf{x}' \\ &\le \frac{\text{const}_1}{M} \|\phi\|_C. \end{aligned}$$

A similar estimate holds for $\phi \in L_\infty(V)$.

(ii) For $\phi \in L_1(V)$ we have, using $D = \sup\{|\mathbf{x}-\mathbf{x}'|/\mathbf{x}, \mathbf{x}' \in V\} < \infty$,

$$\begin{aligned} \|A_1^M \phi\|_1 &\le 4 \int_V d\mathbf{x} \int_V \frac{\exp\{\sigma|\mathbf{x}-\mathbf{x}'| - 2\sigma M d(\mathbf{x}, \mathbf{x}')\}}{|\mathbf{x}-\mathbf{x}'|^2 \{1 - \exp(-2\sigma d(\mathbf{x}, \mathbf{x}'))\}} |\phi(\mathbf{x}')| d\mathbf{x}' \\ &\le \frac{4\text{const}_2}{M} \int_V |\phi(\mathbf{x}')| d\mathbf{x}' = \frac{4\text{const}_2}{M} \|\phi\|_1. \end{aligned}$$

By interpolation (Reference 11, Theorem I 3.10), we see that A_1^M is bounded on $L_p(V)$ $(1 \le p < \infty)$, where for $q = p/(p-1)$

$$\|A_1^M\|_{L_p(V)} \le \|A_1^M\|_{L_1(V)}^{1/p} \|A_1^M\|_{L_\infty(V)}^{1/q} \le \frac{4\text{const}_3}{M}.$$

Hence,

$$\lim_{M \to \infty} \|A_1 - A_1^0\| = \lim_{M \to \infty} \|A_1^M\| = 0$$

on the spaces $L_p(V)$ $(1 \le p < \infty)$ and $C(\bar{V})$, and therefore A_1 is a compact operator on these spaces.

It remains to prove the estimate (28). Consider the subsets V_i and V_e of V defined by

$$V_i = \{\mathbf{x} \in V / \inf_{\mathbf{y} \in \partial V} |\mathbf{x} - \mathbf{y}| \ge R\}, \quad V_e = V \backslash V_i,$$

where R is the constant in condition (13). For $x \in V_i$ we have $d(\mathbf{x}, \mathbf{x}') \geq 2R$, whence

$$\int_V \frac{\exp\{-\sigma|\mathbf{x}-\mathbf{x}'|-2\sigma M d(\mathbf{x}, \mathbf{x}')\}}{|\mathbf{x}-\mathbf{x}'|^2\{1-\exp(-2\sigma d(\mathbf{x}, \mathbf{x}'))\}} \mathrm{d}\mathbf{x}'$$

$$= \int_S \mathrm{d}r \int_0^{r(\mathbf{x},\mathbf{\Omega})} \frac{\exp\{-\sigma s - 2\sigma M d(\mathbf{x}, \mathbf{\Omega})\}}{1-\exp(-2\sigma d(\mathbf{x}, \mathbf{\Omega}))} \mathrm{d}s$$

$$\leq \frac{1}{\sigma}\exp(-4\sigma MR) \int_S \frac{1-\exp(-\sigma r(\mathbf{x}, \mathbf{\Omega}))}{1-\exp(-2\sigma d(\mathbf{x}, \mathbf{\Omega}))} \mathrm{d}\mathbf{\Omega} \leq \frac{4\pi}{\sigma}\exp(-4\sigma MR).$$

Next, let $\mathbf{x} \in V_e$. If $\cos\alpha = |\mathbf{\Omega}\cdot\mathbf{n}(\mathbf{z})|$, where $\mathbf{z} \in \partial V$ and $|\mathbf{z}-\mathbf{x}|$ is minimal, then

$$d(\mathbf{x}, \mathbf{\Omega}) \geq 2R \cos\alpha,$$

as a consequence of condition (13). Hence,

$$\int_V \frac{\exp\{-\sigma|\mathbf{x}-\mathbf{x}'|-2\sigma M d(\mathbf{x}, \mathbf{x}')\}}{|\mathbf{x}-\mathbf{x}'|^2\{1-\exp(-2\sigma d(\mathbf{x}, \mathbf{x}'))\}} \mathrm{d}\mathbf{x}'$$

$$\leq \int_S \mathrm{d}\mathbf{\Omega}\exp(-4\sigma MR\cos\alpha) \int_0^{r(\mathbf{x},\mathbf{\Omega})} \frac{\exp(-\sigma s)\mathrm{d}s}{1-\exp(-2\sigma d(\mathbf{x}, \mathbf{\Omega}))}$$

$$\leq \int_S \mathrm{d}\mathbf{\Omega}\exp(-4\sigma MR\cos\alpha) \int_0^{\infty} \frac{\exp(-\sigma s)\mathrm{d}s}{1-\exp(-2\sigma D)} = \frac{1}{\tau}\int_S \exp(-4\sigma MR\cos\alpha)\mathrm{d}\mathbf{\Omega}$$

$$= \frac{2}{\tau}\int_0^{2\pi} \mathrm{d}\alpha' \int_0^{\pi/2} \exp(-4\sigma MR\cos\alpha)\sin\alpha \mathrm{d}\alpha = \frac{\pi}{\sigma\tau MR}\{1-\exp(-4\sigma MR)\}$$

$$\leq \frac{\pi}{\sigma\tau R}\frac{1}{M},$$

where D is the diameter of V and $\tau = \sigma\{1-\exp(-2\sigma D)\}$. This completes the proof of the estimate (28).

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