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Reflection and Transmission of Polarized Light: The Adding Method for Homogeneous Atmospheres

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A mathematical justification is given for the multiple interface reflection expansion arising when using the adding method for solving numerically the equation of transfer of polarized light in a homogeneous atmosphere. The ratio of convergence is investigated analytically as a function of the optical thicknesses of the layers. The proof is based on manipulations with the positivity properties of the reflection and transmission operators, whose existence may be based on recent existence and uniqueness results for the solution of the equation of polarized light transfer. © 1986 Academic Press, Inc.

1. INTRODUCTION

On neglecting vertical inhomogeneities and thermal emission, as well as reflection by the planetary surface, the equation of transfer of polarized light in a plane-parallel atmosphere of finite *optical thickness b* is the vector-valued integrodifferential equation

$$u \frac{d}{d\tau} \mathbf{I}(\tau, u, \varphi) + \mathbf{I}(\tau, u, \varphi) = \frac{a}{4\pi} \int_{-1}^{1} \int_{0}^{2\pi} \mathbf{Z}(u, u', \varphi - \varphi') \, \mathbf{I}(\tau, u', \varphi') \, d\varphi' \, du', \tag{1}$$

where $0 < \tau < b$, endowed with the boundary conditions

$$\mathbf{I}(0, u, \varphi) = \mathbf{J}(u, \varphi) \text{ for } u > 0, \qquad \mathbf{I}(b, u, \varphi) = \mathbf{J}(u, \varphi) \text{ for } u < 0. \tag{2}$$

Here $0 < a \leqslant 1$ is the *albedo of single scattering*, $\mathbf{Z}(u, u', \varphi - \varphi')$ the *phase matrix*, and $\mathbf{I}(\tau, u, \varphi)$ a four-vector depending on *optical depth* τ, direction cosine of propagation u, and azimuthal angle φ. The components I, Q, U, and V of the vector $\mathbf{I}$ are the Stokes parameters, which describe the intensity and state of polarization of the beam. The function $\mathbf{J}(u, \varphi)$ specifies the Stokes parameters of the light incident to the top of the atmosphere.

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Usually $\tau = 0$ is the top and $\tau = b$ is the bottom so that $\mathbf{J}(u, \varphi) = 0$ for $u < 0$, but for later convenience we choose to have the more general boundary conditions (2). A consistent treatment of polarized light transfer based on the (equivalent) conventions for polarization parameters of Chandrasekhar [2] and van de Hulst [17] is given in [15], on which we shall rely for the basic notations and physical background.

The phase matrix can be expressed as the product

$$\mathbf{Z}(u, u', \varphi - \varphi') = \mathbf{L}(\pi - \sigma_2)\, \mathbf{F}(\theta)\, \mathbf{L}(-\sigma_1) \tag{3}$$

of two rotation matrices of the form

$$\mathbf{L}(\alpha) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\alpha & \sin 2\alpha & 0 \\ 0 & -\sin 2\alpha & \cos 2\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and the *scattering matrix*

$$\mathbf{F}(\theta) = \begin{bmatrix} a_1(\theta) & b_1(\theta) & 0 & 0 \\ b_1(\theta) & a_2(\theta) & 0 & 0 \\ 0 & 0 & a_3(\theta) & b_2(\theta) \\ 0 & 0 & -b_2(\theta) & a_4(\theta) \end{bmatrix}. \tag{4}$$

The relationship between $u = -\cos v$, $u' = -\cos v'$, and θ $(0 \leqslant v, v', \theta < \pi)$ on the one hand and φ, φ', σ_1, and σ_2 on the other hand is given by the formulae

$$\cos \theta = \cos v \cos v' + \sin v \sin v' \cos(\varphi - \varphi') \tag{5}$$

$$\cos \sigma_1 = \frac{\cos v - \cos v' \cos \theta}{\sin v' \sin \theta}, \qquad \cos \sigma_2 = \frac{\cos v' - \cos v \cos \theta}{\sin v \sin \theta}, \tag{6}$$

where $\sin \sigma_1$ and $\sin \sigma_2$ have the same sign as $\sin(\varphi' - \varphi)$. When the denominator of any of the equations (6) vanishes, the appropriate limits have to be taken as to make the transformation continuous.

The existence and uniqueness theory for the boundary value problems (1)–(2) has been developed by van der Mee. In [24] he considered Eqs. (1)–(2) and corresponding equations for media of infinite optical thickness, while in [25] reflection by the planetary surface was incorporated. In both of these publications it is assumed that $\mathbf{F}(\theta)$ is a measurable matrix that leaves invariant the positive cone of vectors $\mathbf{I} = (I, Q, U, V)$ satisfying

$$I \geqslant \sqrt{Q^2 + U^2 + V^2} \geqslant 0, \tag{7}$$

while the *phase function* $a_1(\theta)$ is nonnegative and satisfies the conditions

$$\int_{-1}^{1} a_1(\theta)\, d(\cos\theta) = 2 \tag{8}$$

$$\exists r > 1: \qquad \int_{-1}^{1} a_1(\theta)^r\, d(\cos\theta) < \infty. \tag{9}$$

Condition (7) for a four-vector $\mathbf{I} = (I, Q, U, V)$ means physically that the degree of polarization of a beam of light with Stokes parameters I, Q, U, and V belongs to $(0, 1]$. The positive cone property (7) was first observed by Germogenova and Konovalov [5].

Let H_p denote the Banach space of measurable functions $\mathbf{I}: \Omega \to \mathbb{C}^4$, where Ω is the unit sphere in three-dimensional space, bounded with respect to the norm

$$\|\mathbf{I}\|_p = \left[\int_{-1}^{1} \int_0^{2\pi} \{|I(u, \varphi)|^p + |Q(u, \varphi)|^p + |U(u, \varphi)|^p + |V(u, \varphi)|^p\}\, d\varphi\, du\right]^{1/p}, \qquad 1 \leqslant p < \infty,$$

and let K_p denote the positive cone on H_p consisting of those functions whose values satisfy the condition (7) almost everywhere on Ω. On H_p we define the operators T, B, A, Q_+, and Q_- as follows:

$$(T\mathbf{I})(u, \varphi) = u\mathbf{I}(u, \varphi),$$

$$(B\mathbf{I})(u, \varphi) = \frac{1}{4\pi} \int_{-1}^{1} \int_0^{2\pi} \mathbf{Z}(u, u', \varphi - \varphi')\, \mathbf{I}(u', \varphi')\, d\varphi'\, du'$$

$$(A\mathbf{I})(u, \varphi) = \mathbf{I}(u, \varphi) - a(B\mathbf{I})(u, \varphi)$$

$$\begin{aligned} (Q_+\mathbf{I})(u, \varphi) &= \mathbf{I}(u, \varphi) \quad \text{for } u > 0; & (Q_-\mathbf{I})(u, \varphi) &= 0 \quad \text{for } u > 0 \\ &= 0 \quad \text{for } u < 0; & &= \mathbf{I}(u, \varphi) \quad \text{for } u < 0, \end{aligned}$$

where points $\omega \in \Omega$ are parametrized as (u, φ) with $u \in [-1, 1]$ and $\varphi \in [0, 2\pi)$. Then the boundary value problem (1)–(2) can be reformulated as

$$(T\mathbf{I})'(\tau) = -A\mathbf{I}(\tau), \qquad 0 < \tau < b \tag{10}$$

$$Q_+\mathbf{I}(0) = Q_+\mathbf{J}, \qquad Q_-\mathbf{I}(\tau) = Q_-\mathbf{J}. \tag{11}$$

For every $\mathbf{J} \in H_p$, $1 \leqslant p < \infty$, there exists a unique continuous function $\mathbf{I}: [0, b] \to H_p$ such that $T\mathbf{I}$ is strongly differentiable on $(0, b)$ and Eqs. (10)–(11) are fulfilled; moreover, if $\mathbf{J} \in K_p$, then $\mathbf{I}(\tau) \in K_p$ for every

$\tau \in [0, b]$ (cf. [24]). If we now define (unique) *reflection operators* R_{+b} and R_{-b} and *transmission operators* T_{+b} and T_{-b} by

$$\mathbf{I}(0) = (R_{+b} + T_{-b})\,\mathbf{J}, \qquad \mathbf{I}(b) = (T_{+b} + R_{-b})\,\mathbf{J}, \tag{12}$$

where

$$R_{\pm b} Q_{\pm} = R_{\pm b}, \qquad T_{\pm b} Q_{\pm} = T_{\pm b}, \tag{13}$$

then $(R_{\pm b} - Q_{\pm})\,\mathbf{J}$ denotes the Stokes vector of the light reflected by the surface at $\tau = 0$ (resp. $\tau = b$) and $T_{\pm\tau}\mathbf{J}$ accounts for the light transmitted from the surface at $\tau = 0$ (resp. $\tau = b$) to the opposite surface. If we next introduce the *transfer (matrix) operator* $\mathbf{S}_b$ by

$$S_b \mathbf{J} = Q_- \mathbf{I}(0) + Q_+ \mathbf{I}(b), \tag{14}$$

then with respect to the decomposition $H_p = Q_+[H_p] \oplus Q_-[H_p]$ we have

$$S_b = \begin{bmatrix} S_b^{++} & S_b^{+-} \\ S_b^{-+} & S_b^{--} \end{bmatrix}, \tag{15}$$

where $S_b^{\pm\pm} = T_{\pm b}$: $Q_{\pm}[H_p] \to Q_{\pm}[H_p]$ and $S_b^{\pm\mp} = R_{\mp b} - Q_{\mp}$: $Q_{\mp}[H_p] \to Q_{\pm}[H_p]$. For unpolarized light transfer reflection and transmission operators were introduced in [23] and studied further in [7]. There is a close relationship between these operators and the reflection and transmission matrices prevalent in the radiative transfer literature (for instance, [2, 18]). The operator S_b also appears in the study of stationary kinetic equations on finite layers with reflective boundary conditions at both surfaces (cf. [8, Chap. V]).

In this article we shall justify the series expansion arising when using the adding method. This method consists of computing the reflection and transmission properties of a medium of optical thickness $b = b_1 + b_2$ from the reflection and transmission properties of two constituent layers of optical thicknesses b_1 and b_2, using a series expansion in orders of multiple reflection by the interface of the two constituent layers. A comprehensive account of this method and of the doubling method (merely a repetitive application of the adding method using identical layers at each stage) has been given by Hansen and Travis [13] and van de Hulst [18], where many references can be found. A detailed description of the method for polarized light was recently given by de Haan [10], and has been supplemented by numerical results by de Haan, Bosma, and Hovenier [11]. A comprehensive account of the qualitative aspects of the adding of constituent media in neutron transport, displaying the philosophy of systems theory, was given by Ribarič [27].

Let us consider a layer of optical thickness $b = b_1 + b_2$ composed of two adjacent layers of optical thicknesses b_1 and b_2. Using (14) and (15) we obtain the pairs of formulae

$$Q_+ \mathbf{I}(b_1) = S_{b_1}^{++} Q_+ \mathbf{J} + S_{b_1}^{+-} Q_- \mathbf{I}(b_1) \tag{16}$$

$$Q_- \mathbf{I}(0) = S_{b_1}^{-+} Q_+ \mathbf{J} + S_{b_1}^{--} Q_- \mathbf{I}(b_1) \tag{17}$$

and

$$Q_+ \mathbf{I}(b_1) = S_{b_2}^{++} Q_+ \mathbf{I}(b_1) + S_{b_2}^{+-} Q_- \mathbf{J} \tag{18}$$

$$Q_- \mathbf{I}(b_1) = S_{b_2}^{-+} Q_+ \mathbf{I}(b_1) + S_{b_2}^{--} Q_- \mathbf{J}, \tag{19}$$

whence

$$Q_+ \mathbf{I}(b_1) = S_{b_1}^{++} Q_+ \mathbf{J} + S_{b_1}^{+-} [S_{b_2}^{-+} Q_+ \mathbf{I}(b_1) + S_{b_2}^{--} Q_- \mathbf{J}] \tag{20}$$

$$Q_- \mathbf{I}(b_1) = S_{b_2}^{--} Q_- \mathbf{J} + S_{b_2}^{-+} [S_{b_1}^{++} Q_+ \mathbf{J} + S_{b_1}^{+-} Q_- \mathbf{I}(b_1)]. \tag{21}$$

From the latter equations one easily derives

$$(\mathbb{1} - S_{b_1}^{+-} S_{b_2}^{-+})\, Q_+ \mathbf{I}(b_1) = S_{b_1}^{++} Q_+ \mathbf{J} + S_{b_1}^{+-} S_{b_2}^{--} Q_- \mathbf{J} \tag{22}$$

$$(\mathbb{1} - S_{b_2}^{-+} S_{b_1}^{+-})\, Q_- \mathbf{I}(b_1) = S_{b_2}^{--} Q_- \mathbf{J} + S_{b_2}^{-+} S_{b_1}^{++} Q_+ \mathbf{J}. \tag{23}$$

As we shall prove $(\mathbb{1} - S_{b_1}^{+-} S_{b_2}^{-+})$ and $(\mathbb{1} - S_{b_2}^{-+} S_{b_1}^{+-})$ invertible, we may write

$$S_b^{++} = S_{b_2}^{++} (\mathbb{1} - S_{b_1}^{+-} S_{b_2}^{-+})^{-1} S_{b_1}^{++} \tag{24}$$

$$S_b^{+-} = S_{b_2}^{+-} + S_{b_2}^{++} (\mathbb{1} - S_{b_1}^{+-} S_{b_2}^{-+})^{-1} S_{b_1}^{+-} S_{b_2}^{--} \tag{25}$$

$$S_b^{-+} = S_{b_1}^{-+} + S_{b_1}^{--} (\mathbb{1} - S_{b_2}^{-+} S_{b_1}^{+-})^{-1} S_{b_2}^{-+} S_{b_1}^{++} \tag{26}$$

$$S_b^{--} = S_{b_1}^{--} (\mathbb{1} - S_{b_2}^{-+} S_{b_1}^{+-})^{-1} S_{b_2}^{--}. \tag{27}$$

The adding method consists of calculating the operator S_b from the operators S_{b_1} and S_{b_2} using the series expansions

$$\begin{aligned} (\mathbb{1} - S_{b_1}^{+-} S_{b_2}^{-+})^{-1} &= \sum_{n=0}^{\infty} (S_{b_1}^{+-} S_{b_2}^{-+})^n, \\ (\mathbb{1} - S_{b_2}^{-+} S_{b_1}^{+-})^{-1} &= \sum_{n=0}^{\infty} (S_{b_2}^{-+} S_{b_1}^{+-})^n, \end{aligned} \tag{28}$$

which have the physical interpretation of giving the contributions of successive reflection by the interface between the two adjacent slabs. For instance, transmission from $\tau = 0$ to $\tau = b$ (incorporated in S_b^{++}) can be thought of as being composed of the contributions (for $n = 0, 1, 2, \ldots$) of first transmission from $\tau=0$ to $\tau=b_1$, then n consecutive pairs of a reflection by the surface $\tau=b_1$ of the layer at $b_1 \leqslant \tau \leqslant b$ followed by a reflection by the surface $\tau=b_1$ of the layer at $0 \leqslant \tau \leqslant b_1$, and finally transmission from $\tau=b_1$ to $\tau=b$. We shall justify the expansions (28) using a simple positivity argument.

Several authors have studied numerically the ratio of convergence of the series expansions (28). In particular, one has to study the eigenvalues, and foremost the spectral radii, of the operators $S_{b_1}^{+-}S_{b_2}^{-+}$ and $S_{b_2}^{-+}S_{b_1}^{+-}$. (In fact, we shall prove that both operators are compact on H_p, $1 \leqslant p < \infty$, and have the same eigenvalues.) Once an accurate value is obtained for the leading eigenvalue and possibly for the second largest eigenvalue in modulus, the remaining terms of the series can be approximately computed using extrapolation by a geometric series (cf. [12–14]). A comprehensive discussion of the eigenvalues involved in the rate of convergence of the series (28) has been given by van de Hulst [18]. In this article we shall study the behavior of the spectral radius of $S_{b_1}^{+-}S_{b_2}^{-+}$ (or $S_{b_2}^{-+}S_{b_1}^{+-}$) as a function of the optical thicknesses b_1 and b_2. In particular, we shall prove that the spectral radius is a continuous function of b_1 and b_2 (where $b_1, b_2 \in (0, \infty)$) which is strictly monotonically increasing in the albedo of single scattering a and in each of the optical thicknesses b_1 and b_2. It appears that the supremum of the spectral radius over all $b_1, b_2 \in (0, \infty)$ is strictly less than one if $a \in (0, 1)$ and equals one if $a=1$.

In Section 2 we shall establish the absolute convergence of the series expansions (28) and the monotonicity of the spectral radii of $S_{b_1}^{+-}S_{b_2}^{-+}$ and $S_{b_2}^{-+}S_{b_1}^{+-}$ in the optical thicknesses b_1 and b_2. In Section 3 we shall prove strict monotonicity and investigate the behavior of the spectral radii for large b_1 and b_2. Section 4 is devoted to a discussion of the results obtained and some possibilities of generalization to multigroup neutron transport.

2. Convergence of the Multiple Interface Reflection Expansion

In [24] we have proved that every continuous function $\mathbf{I}$: $[0, b] \to H_p$ such that $T\mathbf{I}$ is strongly differentiable on $(0, b)$ and Eqs. (10) and (11) hold true satisfies the vector-valued convolution equation

$$\mathbf{I}(\tau) - a \int_0^b \mathscr{H}(\tau-\tau')\, B\mathbf{I}(\tau')\, d\tau' = \boldsymbol{\omega}(\tau), \qquad 0 \leqslant \tau \leqslant b, \tag{29}$$

where

$$\begin{aligned} [\mathscr{H}(\tau)\,\mathbf{I}](u, \varphi) &= |u|^{-1}\, e^{-\tau/u}\mathbf{I}(u, \varphi) \qquad &&\text{for} \quad \tau u > 0 \\ &= 0 &&\text{for} \quad \tau u < 0 \end{aligned}$$

and

$$\begin{aligned} \boldsymbol{\omega}(\tau)(u,\varphi) &= e^{-\tau/u}\mathbf{I}(u,\varphi) \qquad \text{for} \quad u>0 \\ &= e^{(b-\tau)/u}\mathbf{I}(u,\varphi) \qquad \text{for} \quad u<0. \end{aligned} \tag{30}$$

Conversely, every bounded solution of Eq. (29) with right-hand side (30) is continuous on $[0, b]$, has $T\mathbf{I}$ strongly differentiable on $(0, b)$, and satisfies Eqs. (10) and (11). If we introduce the Banach space $C(H_p)_0^b$ of continuous functions $\mathbf{I}$: $[0, b] \to H_p$ with norm

$$\|\mathbf{I}\|_{C(H_p)_0^b} = \max_{0\leqslant\tau\leqslant b}\left[\int_{-1}^{1}\int_{0}^{2\pi}\{|I(u,\varphi)|^p + |Q(u,\varphi)|^p + |U(u,\varphi)|^p + V(u,\varphi)|^p\}\,d\varphi\,du\right]^{1/p}$$

then Eq. (29) can be written as

$$(\mathbb{1} - a\mathscr{L}_b)\,\mathbf{I} = \boldsymbol{\omega}, \tag{31}$$

where

$$(\mathscr{L}_b\mathbf{I})(\tau) = \int_0^b \mathscr{H}(\tau-\tau')\,B\mathbf{I}(\tau')\,d\tau', \qquad 0\leqslant\tau\leqslant b, \tag{32}$$

is a compact operator on $C(H_p)_0^b$. Writing $\boldsymbol{\omega} = \boldsymbol{\omega}_{\mathbf{J}}$ for (30), we obtain

$$S_b^{-+}\mathbf{J} = [\{(\mathbb{1} - a\mathscr{L}_b)^{-1} - \mathbb{1}\}\,\boldsymbol{\omega}_{Q_+\mathbf{J}}](0) \tag{33}$$

and

$$S_b^{+-}\mathbf{J} = [\{(\mathbb{1} - a\mathscr{L}_b)^{-1} - \mathbb{1}\}\,\boldsymbol{\omega}_{Q_-\mathbf{J}}](b), \tag{34}$$

whence S_b^{-+} and S_b^{+-} are compact operators on H_p, $1\leqslant p<\infty$. Since Eqs. (22) and (23) having more than one solution would contradict the unique solvability of Eqs. (1)–(2), the operators $(\mathbb{1} - S_{b_1}^{+-}S_{b_2}^{-+})$ and $(\mathbb{1} - S_{b_1}^{-+}S_{b_2}^{+-})$ are injective and, by the Fredholm alternative, are invertible on H_p. Moreover, the nonzero eigenvalues of the compact operators $S_{b_1}^{+-}S_{b_2}^{-+}$ and $S_{b_2}^{-+}S_{b_1}^{+-}$ coincide and their resolvents are related as follows:

$$(\mathbb{1} - cS_{b_1}^{+-}S_{b_2}^{-+})^{-1} = \mathbb{1} + cS_{b_1}^{+-}(\mathbb{1} - cS_{b_2}^{-+}S_{b_1}^{+-})^{-1}\,S_{b_2}^{-+} \tag{35}$$

and

$$(\mathbb{1} - cS_{b_2}^{-+}S_{b_1}^{+-})^{-1} = \mathbb{1} + cS_{b_2}^{-+}(\mathbb{1} - cS_{b_1}^{+-}S_{b_2}^{-+})^{-1}\,S_{b_1}^{+-}. \tag{36}$$

Moreover, in terms of the reflection operators R_{+b} and R_{-b} one has [cf. (12), (13), (14), and (15)]

$$(R_{+b_1}+R_{-b_1}-I)(R_{+b_2}+R_{-b_2}-I)=S_{b_1}^{+-}S_{b_2}^{-+}\oplus S_{b_1}^{-+}S_{b_2}^{+-} \tag{37}$$

and

$$(R_{+b_2}+R_{-b_2}-I)(R_{+b_1}+R_{-b_1}-I)=S_{b_2}^{+-}S_{b_1}^{-+}\oplus S_{b_2}^{-+}S_{b_1}^{+-}. \tag{38}$$

THEOREM 2.1. *The series expansions* (28) *are absolutely convergent in H_p-operator norm and the spectral radii of $S_{b_1}^{+-}S_{b_2}^{-+}$ and $S_{b_2}^{-+}S_{b_1}^{+-}$ coincide and are strictly less than one.*

Proof. Using an argument of Ribarič ([26, Lemma 12]) we first compute [cf. (22), (23)]

$$\sum_{n=0}^{L}(S_{b_1}^{+-}S_{b_2}^{-+})^n[S_{b_1}^{++}Q_+\mathbf{J}+S_{b_1}^{+-}S_{b_2}^{--}Q_-\mathbf{J}]$$
$$=Q_+\mathbf{I}(b_1)-(S_{b_1}^{+-}S_{b_2}^{-+})^{L+1}Q_+\mathbf{I}(b_1)\leqslant Q_+\mathbf{I}(b_1)$$

and

$$\sum_{n=0}^{L}(S_{b_2}^{-+}S_{b_1}^{+-})^n[S_{b_2}^{--}Q_-\mathbf{J}+S_{b_2}^{-+}S_{b_1}^{++}Q_+\mathbf{J}]$$
$$=Q_-\mathbf{I}(b_1)-(S_{b_2}^{-+}S_{b_1}^{+-})^{L+1}Q_-\mathbf{I}(b)\leqslant Q_-\mathbf{I}(b_1),$$

where (here and in the sequel) we have used the order derived from the positive cone K_p of H_p (i.e., $\mathbf{I}_1\leqslant I_2$ means $\mathbf{I}_2-\mathbf{I}_1\in K_p$) and $\mathbf{J}\in K_p$. [We have also used that $\mathbf{I}(b_1)\in K_p$ whenever $\mathbf{J}\in K_p$.] Since the partial sums of the iterated series deducible from Eqs. (22) and (23) are bounded above in H_p whenever $\mathbf{J}\in K_p$ and since K_p is a regular cone in H_p whenever $1\leqslant p<\infty$ (cf. [20] for the terminology), these iterated series converge for every $\mathbf{J}\in K_p$. Because K_p is a reproducing cone in H_p, these iterated series converge for every $\mathbf{J}\in H_p$.

It suffices to prove that the right-hand sides of (22) and (23) are dense sets in $Q_+[H_p]$ and $Q_-[H_p]$, since in combination with the invertibility of the operators $(\mathbb{1}-S_{b_1}^{+-}S_{b_2}^{-+})$ and $(\mathbb{1}-S_{b_2}^{-+}S_{b_1}^{+-})$ it would imply the positivity (relative to the cone K_p) of the inverses of these operators and thus $r(S_{b_1}^{+-}S_{b_2}^{-+})=r(S_{b_2}^{-+}S_{b_1}^{+-})<1$ for their spectral radii. Indeed, consider the operator $\mathscr{T}$ defined by

$$\mathscr{T}\mathbf{J}=S_{b_1}^{++}Q_+\mathbf{J}+S_{b_1}^{+-}S_{b_2}^{--}Q_-\mathbf{J}+S_{b_2}^{--}Q_-\mathbf{J}+S_{b_2}^{-+}S_{b_1}^{++}Q_+\mathbf{J},$$

which may be factorized as [cf. (12), (13), (14), (15)]

$$\mathscr{T} = (R_{+b_1} + R_{-b_1})\, \mathscr{S}_{b_1,b_2},$$

where

$$\begin{aligned}(\mathscr{S}_{b_1,b_2}\mathbf{J})(u, \varphi) &= e^{-b_1/u}\mathbf{J}(u, \varphi) \qquad \text{for} \quad u > 0\\ &= e^{b_2/u}\mathbf{J}(u, \varphi) \qquad \text{for} \quad u < 0.\end{aligned}$$

Clearly, $R_{+b_1} + R_{-b_1}$ is a compact perturbation of the identity satisfying

$$\begin{aligned}&(Q_+ - Q_-)(R_{+b_1} + R_{-b_1})(Q_+ - Q_-)\\ &\quad = (Q_+ - Q_-)\{(R_{+b_1} - Q_+) - (R_{-b_1} - Q_-) + (Q_+ - Q_-)\}\\ &\quad = -(R_{+b_1} - Q_+) - (R_{-b_1} - Q_-) + \mathbb{1}\\ &\quad = 2\,\mathbb{1} - (R_{+b_1} + R_{-b_1}).\end{aligned}$$

The invertibility of $(\mathbb{1} - S_{b_1}^{+-}S_{b_1}^{-+})$ and $(\mathbb{1} - S_{b_1}^{-+}S_{b_1}^{-+})$ in combination with Eqs. (37) and (38) (for $b_1 = b_2$) implies the invertibility of the operators $(R_{+b_1} + R_{-b_1})$ and $2\,\mathbb{1} - (R_{+b_1} + R_{-b_1})$. Since S_{b_1,b_2} has a dense range, we have a dense range for $\mathscr{T}$. ▌

It is immediate from (25) and (26) that $S_b^{+-} \geqslant S_{b_1}^{+-} \geqslant 0$ and $S_b^{-+} \geqslant S_{b_1}^{-+} \geqslant 0$, whence $S_{b_2}^{-+}S_{b_1}^{+-}$ and $S_{b_1}^{+-}S_{b_2}^{-+}$ are increasing if b_1 and b_2 are increasing. On writing

$$S_{b_1}^{\pm\pm} - S_b^{\pm\pm} = (\mathbb{1} - S_{b_2}^{\pm\pm})\, S_{b_2}^{\pm\pm} + \sum_{n=1}^{\infty} S_{b_2}^{\pm\pm}(S_{b_1}^{\pm\mp}S_{b_2}^{\mp\pm})^n\, S_{b_1}^{\pm\pm}$$

and

$$|T|(\mathbb{1} - S_{b_2})\,\mathbf{J} = |T|\,\mathbf{J} - |T|\,S_{b_2}\mathbf{J} = T\mathbf{I}(0) - T\mathbf{I}(b_2) = \int_0^{b_2} A\mathbf{I}(\tau)\,d\tau \geqslant 0$$

using Eqs. (10) and (11) with b replaced by b_2, we easily obtain $S_{b_1}^{\pm\pm} \geqslant S_b^{\pm\pm} \geqslant 0$, whence S_b^{++} and S_b^{--} are decreasing if b is increasing. Since K_p is a reproducing and normal cone in H_p (cf. [20] for the terminology) and therefore a renormalization of H_p is possible that turns H_p into a Banach space with strictly monotonic norm (with respect to K_p; i.e., $0 \leqslant \mathbf{I}_1 \leqslant \mathbf{I}_2$ implies $\|\mathbf{I}_1\| \leqslant \|\mathbf{I}_2\|$ after equivalent renormalization), we have monotonicity of the coinciding spectral radii of $S_{b_1}^{+-}S_{b_2}^{-+}$ and $S_{b_2}^{-+}S_{b_1}^{+-}$ with respect to b_1 and b_2, as well as monotonicity of the spectral radii of the four operators S_b^{++}, S_b^{+-}, S_b^{-+}, and S_b^{--} with respect to b.

3. Monotonicity and Smoothness Properties of the Convergence Rates

On transforming the convolution equation (29) to an equation for $\tau \in [0, 1]$ we may deduce that S_b is an analytic operator function of b in the open right half-plane. Indeed, on defining the operator

$$(\hat{\mathscr{L}}_b \mathbf{I})(\tau) = b \int_0^1 \mathscr{H}(b(\tau - \tau'))\, B\mathbf{I}(\tau')\, d\tau'$$

on $C(H_p)_0^1$ and inspecting closely local (in b) bounds on the strong derivative

$$\frac{d}{db} \{\mathscr{H}(b(\tau - \tau'))\, B\} = -|\tau - \tau'|\, |T|^{-2} \exp[-b\, |\tau - \tau'| \cdot |T|^{-1}]\, B,$$

we easily see (cf. [24], (2.2), using Eq. (9) from the present article) that

$$\int_0^1 \left\| \frac{d}{db} \{\mathscr{H}(b(\tau - \tau'))\, B\} \right\| d\tau' \leqslant M < \infty,$$

uniformly in $\tau \in [0, 1]$ and in b on compact subsets on the open right half-plane. Hence, $\hat{\mathscr{L}}_b$ is an analytic operator function in b, and so is $\mathscr{L}_b$. Using the formulae (33) and (34) we obtain the analyticity in b of the operators S_b^{-+} and S_b^{+-}. The analyticity of S_b^{++} and S_b^{--} follows analogously using the formulae

$$S_b^{++}\mathbf{J} = [(\mathbb{1} - a\mathscr{L}_b)^{-1}\, \boldsymbol{\omega}_{Q_+\mathbf{J}}](b)$$

and

$$S_b^{--}\mathbf{J} = [(\mathbb{1} - a\mathscr{L}_b)^{-1}\, \boldsymbol{\omega}_{Q_-\mathbf{J}}](0).$$

Next, let us discuss the Fourier decomposition of the radiative transfer problem in Eqs. (1)–(2). On writing

$$\mathbf{I}^{c0s}(\tau, u) = \left(\int_0^{2\pi} I(\tau, u, \varphi)\, d\varphi, \int_0^{2\pi} Q(\tau, u, \varphi)\, d\varphi, 0, 0 \right)$$

and

$$\mathbf{I}^{c0a}(\tau, u) = \left(0, 0, \int_0^{2\pi} U(\tau, u, \varphi)\, d\varphi, \int_0^{2\pi} V(\tau, u, \varphi)\, d\varphi \right),$$

where $\mathbf{I} = (I, Q, U, V)$ and s = symmetric and a = antisymmetric, we obtain the boundary value problems

$$u \frac{d}{d\tau} \mathbf{I}^{c0s}(\tau, u) + \mathbf{I}^{c0s}(\tau, u) = \frac{1}{2} a \int_{-1}^{1} \mathbf{Z}^{c0s}(u, u') \, \mathbf{I}^{c0s}(\tau, u') \, du' \tag{39}$$

$$\mathbf{I}^{c0s}(0, u) = \mathbf{J}^{c0s}(u) \text{ for } u > 0, \qquad \mathbf{I}^{c0s}(b, u) = \mathbf{J}^{c0s}(u) \text{ for } u < 0 \tag{40}$$

and

$$u \frac{d}{d\tau} \mathbf{I}^{c0a}(\tau, u) + \mathbf{I}^{c0a}(\tau, u) = \frac{1}{2} a \int_{-1}^{1} \mathbf{Z}^{c0a}(u, u') \, \mathbf{I}^{c0a}(\tau, u') \, du' \tag{41}$$

$$\mathbf{I}^{c0a}(0, u) = \mathbf{J}^{c0a}(u) \text{ for } u > 0, \qquad \mathbf{I}^{c0a}(b, u) = \mathbf{J}^{c0a}(u) \text{ for } u < 0, \tag{42}$$

where $0 < \tau < b$,

$$\mathbf{Z}^{c0s}(u, u') = \tilde{\mathbf{Z}}^{c0s}(u', u), \qquad \mathbf{Z}^{c0a}(u, u') = \mathbf{E}\tilde{\mathbf{Z}}^{c0a}(u', u) \, \mathbf{E}, \tag{43}$$

$\mathbf{E} = \mathrm{diag}(1, -1)$, and the tilde above a matrix symbol denotes transposition. Equations (39) to (43) were obtained from the full radiative transfer problem (1)–(2) by Kuščer and Ribarič [21]. Generalizations to azimuthally dependent real component equations are due to Siewert [29]. In terms of complex polarization parameters a Fourier decomposition was derived by Kuščer and Ribarič [21] and further studied by Germogenova and Konovalov [5]. It should be noticed that Eqs. (39)–(40) and Eqs. (41)–(42) can both be posed in the form of problem (10)–(11) on the Banach space $L_p[-1, 1] \oplus L_p[-1, 1]$ with corresponding operators B defined by

$$(B^{c0s}\mathbf{I}^{c0s})(u) = \frac{1}{2} \int_{-1}^{1} \mathbf{Z}^{c0s}(u, u') \, \mathbf{I}^{c0s}(u') \, du',$$

$$(B^{c0a}\mathbf{I}^{c0a})(u) = \frac{1}{2} \int_{-1}^{1} \mathbf{Z}^{c0a}(u, u') \, \mathbf{I}^{c0a}(u') \, du',$$

where B^{c0s} and $\mathbf{E}B^{c0a}$ are selfadjoint on $L_2[-1, 1] \oplus L_2[-1, 1]$. It appears that the operator $A^{c0s} = \mathbb{1} - aB^{c0s}$ is strictly positive selfadjoint if $a \in (0, 1)$, and positive selfadjoint with null space $\mathrm{span}\{(1, 0, 0, 0)\}$ if $a_1(\theta) \not\equiv a_4(\theta)$ and $\mathrm{span}\{(1, 0, 0, 0), (0, 0, 0, 1)\}$ if $a_1(\theta) \equiv a_4(\theta)$, if $a = 1$ (cf. Section VII.2 of [9]; the result is immediate from an inequality for expansion coefficients given in [16]).

LEMMA 3.1. *The operators $S_{b_1}^{+-} S_{b_2}^{-+}$ and $S_{b_2}^{-+} S_{b_1}^{+-}$ have the same positive spectral radius.*

Proof. Let us consider the component problem (39)–(40) and let us define the reflection and transmission operators $R_{\pm b}$, $T_{\pm b}$, and S_b for this problem in the same way as for the undecomposed problem (1)–(2), i.e., by restricting the operators related to the undecomposed problem to the subspace $H_p^{c0s} = \{\mathbf{I} = (I, Q, U, V) \in H_p / U = V = 0,\ I \text{ and } Q \text{ do not depend on } \varphi\}$. Introducing the Hilbert space $H_{2,T}^{c0s}$ of measurable functions $\mathbf{I} = (I, Q)$: $[-1, 1] \to \mathbb{C}^2$ which are bounded with respect to the norm

$$\|\mathbf{I}\|_{2,T} = \left[\int_{-1}^{1} |u| \{|I(u)|^2 + |Q(u)|^2\} \, du\right]^{1/2}$$

with corresponding inner product

$$(\mathbf{I}_1, \mathbf{I}_2)_T = \int_{-1}^{1} |u| \{I_1(u) \overline{I_2(u)} + Q_1(u) \overline{Q_2(u)}\} \, du,$$

one can exploit the positive selfadjointness of the operator A^{c0s} on H_2^{c0s} to prove the existence of a unique continuous function $\mathbf{I}$: $[0, b] \to H_{2,T}^{c0s}$ for every $\mathbf{J} \in H_{2,T}^{c0s}$ that is strongly differentiable on $(0, b)$ and satisfies the boundary value problem of the type (10)–(11) associated with Eqs. (39)–(40), as one may deduce from the abstract kinetic equations theory constructed by Beals [1]. As a result the reflection and transmission operators $R_{\pm b}$, $T_{\pm b}$, and S_b can be defined on $H_{2,T}^{c0s}$ and the addition formulae (12)–(23) can be formulated on $H_{2,T}^{c0s}$. As a consequence of Eqs. (37)–(38) and the selfadjointness on $H_{2,T}^{c0s}$ of the operators $(R_{+b_1} + R_{-b_1} - \mathbb{1})$ and $(R_{+b_2} + R_{-b_2} - \mathbb{1})$ with $H_{2,T}^{c0s}$-norm strictly less than one, the latter being a result of Greenberg and van der Mee [7], the invertibility of $(\mathbb{1} - S_{b_1}^{+-} S_{b_2}^{-+})$ and $(\mathbb{1} - S_{b_2}^{-+} S_{b_1}^{+-})$ on $H_{2,T}^{c0s}$, the addition formulae (24)–(27), and the absolute convergence in $H_{2,T}^{c0s}$ operator norm are clear, as also are Eqs. (35)–(36) extended to $H_{2,T}^{c0s}$. Since all operators $(R_{+b} + R_{-b})$, T_{+b}, and T_{-b} are selfadjoint on $H_{2,T}^{c0s}$ and satisfy identities of the type

$$|T| \, K = K^* \, |T|,$$

where $(T\mathbf{I})(u) = u\mathbf{I}(u)$, K is compact on H_2^{c0s}, and K^* is the adjoint of K with respect to H_2^{c0s}, the operators $(R_{+b} + R_{-b} - \mathbb{1})$, $(T_{+b} - e^{-bT^{-1}} Q_+)$, and $(T_{-b} - e^{bT^{-1}} Q_-)$ are compact on $H_{2,T}^{c0s}$ [6, Theorem V 3.4], where S_b^{+-}, S_b^{-+}, and $(S_b - e^{-b|T|^{-1}})$ are compact in $H_{2,T}^{c0s}$ topology. Using simple Fredholm arguments it can now be understood that the spectra of these compact operators do not depend on the choice of the space H_p^{c0s} $(1 \leqslant p < \infty)$ or $H_{2,T}^{c0s}$ of definition, and thus neither do their spectral radii. If any of the spectral radii of $S_{b_1}^{+-} S_{b_2}^{-+}$ or $S_{b_2}^{-+} S_{b_1}^{+-}$ would vanish, so would the spectral radii of $S_b^{+-} S_b^{-+}$ and $S_b^{-+} S_b^{+-}$ where $b_0 = \min(b_1, b_2)$ (because of monotonicity), and therefore $(R_{+b_0} + R_{-b_0} - \mathbb{1})$ would have zero spectral radius (cf. (37)–(38)). Since the latter operator is selfadjoint on $H_{2,T}^{c0s}$, we would have $(R_{+b}+R_{-b}-\mathbb{1})=0$ for $0<b\leqslant b_0$ and thus, by analytic continuation, $(R_{+b}+R_{-b})=\mathbb{1}$ for $0<b<+\infty$, which is a contradiction. Hence, $S_{b_1}^{+-}S_{b_2}^{-+}$ and $S_{b_2}^{-+}S_{b_1}^{+-}$ have positive spectral radii.

Hitherto in this proof we have dealt with the component problem (39)–(40), which is a restriction of the undecomposed problem (1)–(2). Hence, we also have the result for the latter. ∎

THEOREM 3.2. *The coinciding spectral radii of the operators* $S_{b_1}^{+-}S_{b_2}^{-+}$ *and* $S_{b_2}^{-+}S_{b_1}^{+-}$, $r(b_1, b_2)$, *are continuous functions on the open right half-plane which are strictly monotonically increasing in both variables* b_1 *and* b_2 *separately and satisfy the requirements*

$$\lim_{b_1\to 0} r(b_1, b_2)=\lim_{b_2\to 0} r(b_1, b_2)=0.$$

Proof. Fix $b_2\in(0,\infty)$. Then $S_{b_1}^{+-}S_{b_2}^{-+}$ is a compact operator on $Q_+[H_p]$ depending monotonically and analytically on b_1. As its spectral radius $r(b_1, b_2)$ is an eigenvalue, it depends analytically on b_1 except at a discrete set of algebraic branch points where it is still continuous in b_1 [19, Theorem VII 1.8]. If $r(b_1, b_2)$ would be constant and positive in a neighborhood of $b_1\in(0,\infty)$, then the nearest branch point of $r(b_1, b_2)$ as a function of b_1 cannot possibly be a branch point, and therefore $r(b_1, b_2)$ can be analytically continued in every $b_1\in(0,\infty)$, which implies the false statement $\lim_{b_1\to 0} r(b_1, b_2)>0$. Hence, $r(b_1, b_2)$ is strictly monotonically increasing and continuous in b_1. Similarly, one proves that $r(b_1, b_2)$ is strictly monotonically increasing and continuous in b_2. As a consequence of Eqs. (37) and (38) we have, using spr(C) for the spectral radius of C (here and consecutively),

$$\begin{aligned}\max\{r(b_1, b_2), r(b_2, b_1)\} &= \operatorname{spr}\{(R_{+b_1}+R_{-b_1}-\mathbb{1})(R_{+b_2}+R_{-b_2}-\mathbb{1})\}\\ &\leqslant \|R_{+b_1}+R_{-b_1}-\mathbb{1}\|\ \|R_{+b_2}+R_{-b_2}-\mathbb{1}\|\\ &\leqslant (\|S_{b_1}^{+-}\|+\|S_{b_1}^{-+}\|)(\|S_{b_2}^{+-}\|+\|S_{b_2}^{-+}\|),\end{aligned}$$

which vanishes if one of the variables b_1 or b_2 vanishes. The latter is immediate from (33) and (34). ∎

Next, we shall study the behavior of the spectral radius $r(b_1, b_2)$ if either b_1 or b_2 tends to infinity. In order to do so, we have to consider the accompanying radiative transfer equation on a semi-infinite medium, namely, the boundary value problem

$$\begin{aligned}&u\frac{d}{d\tau}\mathbf{I}(\tau, u, \varphi)+\mathbf{I}(\tau, u, \varphi)\\ &\quad=\frac{a}{4\pi}\int_{-1}^{1}\int_{0}^{2\pi}\mathbf{Z}(u, u', \varphi-\varphi')\,\mathbf{I}(\tau, u', \varphi')\,d\varphi'\,du' \qquad (44)\end{aligned}$$

$$\mathbf{I}(0, u, \varphi) = \mathbf{J}_+(u) \text{ for } u > 0,$$
$$\left[\int_{-1}^{1}\int_0^{2\pi} \|\mathbf{I}(\tau, u, \varphi)\|_p^p \, d\varphi \, du\right]^{1/p} = 0(1) \; (\tau \to \infty), \tag{45}$$

where $0 < \tau < \infty$. This problem appears to be uniquely solvable on H_p for every $\mathbf{J}_+ \in Q_+[H_p]$ (cf. [24]) and the solution at $\tau = 0$ can be expressed in terms of the Stokes vector of incident light $\mathbf{J}_+(u)$ using a reflection operator:

$$\mathbf{I}(0)(u, \varphi) = (R_{+\infty}\mathbf{J}_+)(u, \varphi). \tag{46}$$

On posing the problem in integral form we obtain the equivalent vector equation

$$(\mathbb{1} - a\mathscr{L}_\infty)\,\mathbf{I} = \boldsymbol{\omega}_{Q_+\mathbf{J}}$$

on the Banach space $C(H_p)_0^\infty$ of bounded continuous functions $\mathbf{I}\colon [0, \infty) \to H_p$ where $Q_+\mathbf{J} = \mathbf{J}_+$. For $a \in (0, 1)$ the operator $(\mathbb{1} - a\mathscr{L}_\infty)$ is invertible on $C(H_p)_0^\infty$, whereas invertibility breaks down for $a = 1$ (see [24]).

LEMMA 3.3. *We have for all $a \in (0, 1]$*

$$\lim_{b \to \infty} \|R_{+\infty} - R_{+b}\| = 0. \tag{47}$$

Proof. Since K_p is a reproducing and normal cone in H_p, there is a constant M (i.e., $M = \sqrt{3}$) such that $0 \leqslant \mathbf{I}_1 \leqslant \mathbf{I}_2$ on H_p implies $\|\mathbf{I}_1\| \leqslant M\|\mathbf{I}_2\|$ (cf. [20]). It should be observed that, if one indicates the dependence on $a \in (0, 1]$ by using a superscript (a), one has (i) $0 \leqslant R_{+b}^{(a)} \leqslant R_{+b}^{(1)}$ monotonically in a, as easily follows by iterating Eq. (31) for $\boldsymbol{\omega} = \boldsymbol{\omega}_{Q_+\mathbf{J}}$, and (ii) $0 \leqslant R_{+b}^{(a)} \leqslant R_{+\infty}^{(a)}$ monotonically in b and for $a \in (0, 1)$, as follows from the monotonicity in b of the reflection operator in combination with the "projection method" (cf. [4]; applicable since $\int_{-\infty}^{\infty} \|\mathscr{H}(\tau)\, B\| \, d\tau < \infty$ and $(\mathbb{1} - a\mathscr{L}_\infty)$ is invertible on $C(H_p)_0^\infty$ for $a \in (0, 1)$; see [24] for the latter ingredients). We have the following convergence properties:

(i) $\lim_{a \to 1} \|R_{+b}^{(1)} - R_{+b}^{(a)}\| = 0$, since $(\mathbb{1} - a\mathscr{L}_b)$ is invertible on $C(H_p)_0^b$ for all $a \in (0, 1]$;

(ii) $\lim_{b \to \infty} \|R_{+b}^{(a)} - R_{+\infty}^{(a)}\| = 0$ for $a \in (0, 1)$, as a direct consequence of the "projection method" [4];

(iii) $\lim_{a \to 1} \|R_{+\infty}^{(1)} - R_{+\infty}^{(a)}\| = 0$, as a result of Proposition 6.2 of [24].

The latter limit implies $0 \leqslant R_{+\infty}^{(a)} \leqslant R_{+\infty}^{(1)}$ monotonically in a.

Let us use the monotonicity of the L_p norm. First choose $b_0 \in (0, \infty)$ such that $\|R^{(1)}_{+\infty} - R^{(1)}_{+b_0}\| < (\varepsilon/3M^2)$; then

$$\|R^{(1)}_{+\infty} - R^{(1)}_{+b}\| < (\varepsilon/3M), \qquad b \in [b_0, \infty).$$

Next select $a_0 \in (0, 1)$ such that $\|R^{(1)}_{+b_0} - R^{(a)}_{+b_0}\| < (\varepsilon/3M)$ for $a \in [a_0, 1)$. Finally, choose $a_1 \in [a_0, 1)$ such that

$$\|R^{(1)}_{+b_0} - R^{(a)}_{+b_0}\| < (\varepsilon/3M), \qquad a \in [a_1, \infty).$$

Hence, on using these inequalities one obtains $\|R^{(1)}_{+\infty} - R^{(1)}_{+b_0}\| < \varepsilon/M$, whence by monotonicity

$$\|R^{(1)}_{+\infty} - R^{(1)}_{+b}\| < \varepsilon, \qquad b \in [b_0, \infty).$$

Then (47) is immediate. ▌

A similar result holds true for the reflection operators R_{-b}, which converge to the reflection operator $R_{-\infty}$ for a left semi-infinite medium ($\tau \in (-\infty, 0)$).

On using Eqs. (37) and (38) we obtain that $S^{+-}_{b_1} S^{-+}_{b_2}$ and $S^{-+}_{b_2} S^{+-}_{b_1}$ converge in the operator norms of $Q_+[H_p]$ and $Q_-[H_p]$, respectively, if either b_1 or b_2 tends to infinity.

THEOREM 3.4. *If either b_1 or b_2 tends to infinity while the other variable remains fixed and finite, the spectral radii of $S^{+-}_{b_1} S^{-+}_{b_2}$ and $S^{-+}_{b_2} S^{+-}_{b_1}$, $r(b_1, b_2)$, tend to a limit strictly less than one. Moreover,*

$$\sup\{r(b_1, b_2)/b_1, b_2 \in (0, \infty)\} = [\operatorname{spr}(R_{+\infty} + R_{-\infty} - \mathbb{1})]^2, \tag{48}$$

which is strictly less than one if $a \in (0, 1)$, and equals one if $a = 1$.

Proof. The result is a direct consequence of previous monotonicity and continuity properties and Eqs. (37)–(38). It should be observed that $(R_{+\infty} + R_{-\infty} - \mathbb{1})$ is a compact operator, since it is the limit of the compact operator $(R_{+b} + R_{-b} - \mathbb{1})$ in the operator norm on H_p as b tends to infinity. One should also observe that $(R_{+\infty} + R_{-\infty} - \mathbb{1})$ depends analytically on $a \in (0, 1)$ and continuously on $a \in [0, 1]$, tends to zero as $a \to 0$ and to a nonzero operator as $a \to 1$, and is monotonically increasing in a. Since certainly $\operatorname{spr}(R_{+\infty} + R_{-\infty} - \mathbb{1}) \geqslant \operatorname{spr}(R_{+b} + R_{-b} - \mathbb{1}) > 0$ for $a \in (0, 1)$, the spectral radius of $(R_{+\infty} + R_{-\infty} - \mathbb{1})$ is monotonically increasing in a from 0 (for $a = 0$) to some $l \in (0, 1]$. However, if $\mathbf{J} = (1, 0, 0, 0)$, then $A\mathbf{J} = 0$ for $a = 1$ and therefore $\mathbf{I}(\tau) \equiv \mathbf{J}$ satisfies the radiative transfer problem (44)–(45) on $\tau \in (0, \infty)$ with $\mathbf{J}_+ = Q_+\mathbf{J}$, while $\mathbf{I}(\tau) \equiv \mathbf{J}$ satisfies the analogous problem on $\tau \in (-\infty, 0)$ with $\mathbf{J}_- = Q_-\mathbf{J}$. Hence, $\mathbf{J} = (1, 0, 0, 0) \in \operatorname{Ker}(R_{+\infty} + R_{-\infty} - \mathbb{1})$ for $a = 1$ and therefore $l = 1$. ▌

The operator $R_{+\infty}+R_{-\infty}$ maps the Stokes vector of incident radiation $Q_+\mathbf{J}=\mathbf{J}_+$ onto the Stokes vector of incident plus reflected radiation $\mathbf{I}(0)$ for the right semi-infinite layer problem (44)–(45). On the other hand, $(R_{+\infty}+R_{-\infty}-\mathbb{1})$ transforms the Stokes vector of incident radiation $Q_-\mathbf{J}=\mathbf{J}_-$ for the corresponding left semi-infinite layer $(\tau\in(-\infty,0))$ problem into the Stokes vector of incident plus reflected radiation $\mathbf{I}(0)$. For $a\in(0,1)$ where $A=\mathbb{1}-aB$ has trivial null space, $R_{+\infty}+R_{-\infty}$ is the so-called albedo operator, which plays a crucial role in the existence and uniqueness theory for (abstract) kinetic equations in half-spaces (cf. [1, 22, 8, 9]). If $a=1$, $\operatorname{Ker} A\neq\{0\}$. (In fact, $\operatorname{Ker} A=\operatorname{span}\{(1,0,0,0),(0,0,0,1)\}$ if $a_1(\theta)\equiv a_4(\theta)$, and $\operatorname{Ker} A=\operatorname{span}\{(1,0,0,0)\}$ if $a_1(\theta)\not\equiv a_4(\theta)$.) In such situations one usually modifies the operator A and obtains an invertible finite-rank perturbation in order to deduce existence and uniqueness results. For "half-space" problems on $\tau\in(0,\infty)$ and $\tau\in(-\infty,0)$ one constructs separate albedo operators E_r and E_l such that E_r and E_l coincide on a subspace of finite co-dimension (4 or 2, depending on $\operatorname{Ker} A$) and $E_rQ_++E_lQ_-=R_{+\infty}+R_{-\infty}$.

4. Discussion

We have established the convergence of the multiple interface reflection expansion which is used when applying the adding method for polarized light transfer. We have also studied in detail the rate of convergence of the expansion in the form of results on the spectral radius $r(b_1,b_2)$ of certain operators. In fact, in the long term the multiple interface reflection expansion behaves like a geometric progression with ratio $r(b_1,b_2)$ $(\in(0,1))$, because $r(b_1,b_2)$ is the so-called dominant eigenvalue of the operators $S_{b_1}^{+-}S_{b_2}^{-+}$ (or $S_{b_2}^{-+}S_{b_1}^{+-}$) (cf. [20] for the existence of such an eigenvalue if one knows that the spectral radius of the compact positive operator $S_{b_1}^{+-}S_{b_2}^{-+}$, or $S_{b_2}^{-+}S_{b_1}^{+-}$, is strictly positive). It appears that the ratio of convergence $r(b_1,b_2)$ is a continuous and strictly monotonically increasing function of b_1 and b_2 with upper bounds $r(+\infty,b_2)$ for finite b_2 and $r(b_1,+\infty)$ for finite b_1 strictly less than one, which means that the multiple interface reflection expansion also converges on adding a finite layer to a semi-infinite layer. However, the supremum $r(+\infty,+\infty)$ over both optical thicknesses b_1 and b_2, corresponding to the ratio of convergence involved in the addition of two semi-infinite layers (a physically irrelevant situation), is less than one if $a\in(0,1)$, and one if $a=1$.

For more detailed information on the convergence rate for the above expansion one needs information about all (or the largest) eigenvalues of $S_{b_1}^{+-}S_{b_2}^{-+}$. It is known that there is a leading positive eigenvalue, $r(b_1,b_2)$, which exceeds or equals in modulus the remaining eigenvalues. In some cases it is known that all eigenvalues are real, namely, if polarization is neglected ($a_2(\theta) \equiv a_3(\theta) \equiv a_4(\theta) \equiv b_1(\theta) \equiv b_2(\theta) \equiv 0$), or if $b_2(\theta) \equiv 0$ (which occurs, for instance, for Rayleigh scattering with or without depolarization effects ([2, 17, 18], for instance)), or for the component problem (39)–(40). In all other cases, if $b_2(\theta) \not\equiv 0$ and for other Fourier component problems arising from Eqs. (1)–(2), the most that one could say is that the non-dominating eigenvalues occur in complex conjugate pairs of the same multiplicity, while the competing eigenvalues (those having modulus $r(b_1, b_2)$ but different from $r(b_1, b_2)$) are given by $\lambda = r(b_1, b_2) \exp[(2\pi k i)/n]$ where $k = 1, 2,..., n-1$, with multiplicity at most the multiplicity of the dominant eigenvalue $r(b_1, b_2)$. The property of having only real eigenvalues rests on the selfadjointness of the operator B, the property of having eigenvalues in complex conjugate pairs rests on the selfadjointness of $(R_{+b} + R_{-b} - \mathbb{1})$ in an indefinite inner product (cf. [24, Sect. 7], together with Eqs. (37)–(38)), while the property of competing eigenvalues is based on the positivity and compactness of $S_{b_1}^{+-} S_{b_2}^{-+}$ (cf. [28, Chap. V]). These results are in agreement with numerical results presented in [13, 18, 11].

With minor modifications the convergence proof and the analysis of the rate of convergence apply to multigroup neutron transport. Here the problem is to find the N-vector $\boldsymbol{\psi}(x, \mu, \varphi) = (\psi_i(x, \mu, \varphi))_{i=1}^N$ of neutron angular densities within N groups of neutrons with constant speed (cf. [3]). Here the equations read

$$\mu \frac{\partial \boldsymbol{\psi}}{\partial x}(x, \mu, \varphi) + \boldsymbol{\Sigma} \boldsymbol{\psi}(x, \mu, \varphi) = \frac{1}{4\pi} \int_{-1}^{1} \int_{0}^{2\pi} (\mathbf{C} \otimes \mathbf{P}(\omega \cdot \omega'))\, \boldsymbol{\psi}(x, \mu', \varphi')\, d\varphi'\, d\mu' \tag{49}$$

and

$$\boldsymbol{\psi}(0, \mu, \varphi) = \boldsymbol{\Phi}(\mu, \varphi) \text{ for } \mu > 0, \qquad \boldsymbol{\psi}(\tau, \mu, \varphi) = \boldsymbol{\Phi}(\mu, \varphi) \text{ for } \mu < 0, \tag{50}$$

where x is the distance to the surface of the finite-slab reactor medium with thickness τ, μ is the direction cosine of propagation, φ the azimuthal angle, and $\boldsymbol{\Sigma} = \operatorname{diag}(\sigma_1,..., \sigma_N)$ with $\sigma_1 \geqslant \cdots \geqslant \sigma_N = 1$ the diagonal matrix of neutron cross sections. Here $\mathbf{C}$ is a nonnegative $N \times N$ matrix, and $\mathbf{P}(\omega \cdot \omega')$ is an $N \times N$ matrix of nonnegative functions in $L_r[-1, 1]$ for some $r > 1$ which satisfy the conditions $\int_{-1}^{1} [\mathbf{P}(\cos\theta)]_{ij}\, d(\cos\theta) = 2$ ($1 \leqslant i, j \leqslant N$), $[\mathbf{C} \otimes \mathbf{P}]_{ij} = [\mathbf{C}]_{ij} [\mathbf{P}]_{ij}$, and $\omega \equiv (\mu, \varphi) \in \Omega$. On introducing as H_p the Banach space of measurable functions $\mathbf{h}\colon \Omega \to \mathbb{C}^N$ which are bounded with respect to the norm

$$\|\mathbf{h}\|_p = \left[\sum_{i=1}^{N} \sigma_i \int_{-1}^{1} \int_{0}^{2\pi} |h_i(\mu, \varphi)|^p\, d\varphi\, d\mu \right]^{1/p}$$

as well as the operators

$$(T\mathbf{h})(\mu, \varphi) = \mu\mathbf{\Sigma}^{-1}\mathbf{h}(\mu, \varphi),$$

$$(B\mathbf{h})(\mu, \varphi) = \frac{1}{4\pi}\int_{-1}^{1}\int_{0}^{2\pi} \mathbf{\Sigma}^{-1}(\mathbf{C}\otimes\mathbf{P}(\omega\cdot\omega'))\,\mathbf{h}(\mu, \varphi)\,d\varphi'\,d\mu'$$

$$(A\mathbf{h})(\mu, \varphi) = \mathbf{h}(\mu, \varphi) - (B\mathbf{h})(\mu, \varphi)$$

$$(Q_+\mathbf{h})(\mu, \varphi) = \mathbf{h}(\mu, \varphi) \text{ for } \mu > 0; \qquad (Q_-\mathbf{h})(\mu, \varphi) = 0 \qquad \text{for } \mu > 0$$
$$= 0 \qquad \text{for } \mu < 0; \qquad\qquad = \mathbf{h}(\mu, \varphi) \text{ for } \mu < 0,$$

we obtain the boundary value problem (10)–(11) (replacing (τ, b) by (x, τ) and $\mathbf{J}$ by $\mathbf{\Phi}$). Using unique solvability of the corresponding half-space problem if $\|B\| < 1$, and a monotonicity argument (cf. [8, Sect. IX.4], for details), we may derive unique solvability of Eqs. (49)–(50) for $\|B\| \leqslant 1$. (The spectral radius $\operatorname{spr}(B)$ does not depend on the space H_p, $1 \leqslant p < \infty$, since B is compact on all those spaces.) For $\|B\| \leqslant 1$ we may develop the formalism for the adding method and establish convergence with ratio less than one for the multiple interface reflection expansion. If $0 \leqslant \|B\| \leqslant 1$, the continuity properties and monotonicity of $r(b_1, b_2)$ can be proved in full; however, since taking the $b_1 \to +\infty$ and $b_2 \to +\infty$ limits requires existence and uniqueness for the corresponding half-space problem, these results only go through for $0 \leqslant \|B\| < 1$, or for the symmetric multigroup problem where $\mathbf{C}$ and $\mathbf{P}(\omega\cdot\omega')$ are real symmetric matrices (cf. [9] for the half-space result for the symmetric multigroup case). Strict monotonicity can only be shown in the previous way for the symmetric multigroup case with $\operatorname{spr}(B) \leqslant 1$. For general nonsymmetric cases one might have $\operatorname{spr}(B) = 0$ and perhaps $\operatorname{spr}(S_{\tau_1}^{+-}S_{\tau_2}^{-+}) = 0$. Intuitively one would expect $r(\tau_1, \tau_2) < 1$ in subcritical cases ($\operatorname{spr}(\mathscr{L}_\tau) < 1$ for $b = b_1 + b_2$; cf. (32)), but this cannot be proved by a complete repetition of the arguments used for polarized light transfer.

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