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Half-Range Solutions of Indefinite Sturm–Liouville Problems

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For diffusion equations of indefinite Sturm–Liouville type, we develop two equivalent methods of constructing explicit representations for the solutions. The first method is based on an eigenfunction expansion whereas the second uses a Wiener–Hopf-type integral equation and factorization. Some illustrative examples are worked out. © 1987 Academic Press, Inc.

I. INTRODUCTION

In this article we shall obtain explicit representations for the solution of the boundary value problem

$$w(\mu)\frac{\partial\psi}{\partial x}(x,\mu)=\frac{\partial}{\partial\mu}\left(p(\mu)\frac{\partial\psi}{\partial\mu}(x,\mu)\right)-q(\mu)\,\psi(x,\mu)\qquad(0<x<\infty,\ \mu\in I),\tag{1}$$

$$\psi(0,\mu)=\varphi_+(\mu)\qquad\text{if}\quad\omega(\mu)>0,\tag{2}$$

$$\int_I|\psi(x,\mu)|^2\,|\omega(\mu)|\,d\mu=O(1)\qquad\text{or}\qquad o(1)\quad\text{as } x\to\infty,\tag{3}$$

$\psi(x,\mu)$ satisfies self-adjoint boundary conditions of the differential operator $h\mapsto-(ph')'+qh$. (4)

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Instead of (3) we may also consider the boundary condition

$$\int_I |\psi(x, \mu)|^2 \, |\omega(\mu)| \, d\mu = O(x) \qquad \text{as} \quad x \to \infty. \tag{5}$$

Here $I = (a, b)$ is a finite or infinite interval, $w(\mu)$ is an indefinite weight function with finitely many sign changes and $h \mapsto -(ph')' + qh$ is endowed with boundary conditions that make it into a positive self-adjoint Sturm–Liouville operator. It is assumed that this operator has its spectrum within the set $\{0\} \cup [\varepsilon, \infty)$ for some $\varepsilon > 0$.

Boundary value problems of this (forward–backward) type arise as various kinetic equations. Bothe's model for electron scattering leads to the partial differential equation [10, 5]

$$\mu \frac{\partial \psi}{\partial x}(x, \mu) = \frac{\partial}{\partial \mu}\left((1-\mu^2)\frac{\partial \psi}{\partial \mu}(x, \mu)\right) \qquad (x \in (0, \infty), \mu \in (-1, 1)), \tag{6}$$

where $\psi(x, \mu)$ has a finite limit as $\mu \to \pm 1$. A Fokker–Planck equation [31, 32, 9] has the form

$$v e^{-(1/2)v^2} \frac{\partial x}{\partial x}(x, v) = \frac{\partial}{\partial v}\left(e^{-(1/2)v^2} \frac{\partial \psi}{\partial v}(x, v)\right) \qquad (x \in (0, \infty), v \in (-\infty, \infty)), \tag{7}$$

where $h \mapsto -(e^{-(1/2)v^2} h')'$ is considered as an operator in $L_2((-\infty, \infty); e^{-(1/2)v^2}\, dv)$. Both problems involve Sturm–Liouville operators which are positive self-adjoint, have discrete spectrum and have a simple eigenvalue zero.

For boundary value problems of the type (1)–(4) a complete existence and uniqueness theory has been constructed by Beals [8] (also [7]; for Eq. (7) also [9]), who realized a synthesis of the variational approach of Baouendi and Grisvard [2] and the semigroup method used by Hangelbroek [20]. Under minor regularity assumptions on p, q, and w it has been proved that for strictly positive self-adjoint Sturm–Liouville operators there exists a unique continuous function $\psi: [0, \infty) \to H_T = L_2(I; |w(\mu)|\, d\mu)$ which is H_T-differentiable on $(0, \infty)$ and satisfies Eqs. (1)–(4). [Here we identify $\psi(x)(\mu)$ with $\psi(x, \mu)$]. If the Sturm–Liouville operator has an isolated (and simple) eigenvalue zero, the solution exists in the above sense and is unique. Some care should be taken with condition (3). If φ_0 denotes the zero eigenfunction, one has to read $O(1)$ in (3) if $\int_I w(\mu) |\varphi_0(\mu)|^2 \, d\mu \geqslant 0$ and $o(1)$ if $\int_I w(\mu) |\varphi_0(\mu)|^2 \, d\mu < 0$. [For the examples (6) and (7) the integral vanishes and $O(1)$ must be read in (3).] At present no explicit representations are known for the solution.

The formal similarity between Eqs. (1)–(4) and the half-space problems of neutron transport, radiative transfer and rarefied gas dynamics (cf. [11, 33, 12]) has given rise to the emergence of abstract kinetic equations theory (e.g., [6, 27, 19]). In this theory one deals with the issue of existence and uniqueness of solutions of the boundary value problem

$$T\psi'(x) = -A\psi(x) \qquad (0 < x < \infty), \tag{8}$$

$$Q_+\psi(0) = \varphi_+, \tag{9}$$

$$\|\psi(x)\| = O(1) \qquad \text{or} \qquad o(1) \quad \text{as} \quad x \to \infty \tag{10}$$

on the abstract Hilbert space H, where T is an injective self-adjoint operator, A is a (positive) self-adjoint Fredholm operator and $Q_\pm$ is the orthogonal projection onto the maximal positive/negative T-invariant subspace. [Thus A has a finite-dimensional kernel and closed range but may be unbounded.] If A is strictly positive, then $A^{-1}T$ is self-adjoint with respect to the inner product

$$(h, k)_A = (Ah, k); \tag{11}$$

on the completion H_A of the domain $D(A)$ of A we may then define the orthogonal projections $P_\pm$ onto the maximal positive/negative $A^{-1}T$-invariant subspace and introduce the operator

$$V = Q_+P_+ + Q_-P_-. \tag{12}$$

The solutions may then be written in the semigroup form

$$\psi(x) = e^{-xT^{-1}A}\psi(0), \qquad 0 \leqslant x < \infty,$$

where $\psi(0) = P_+\psi(0)$ satisfies $V\psi(0) = \varphi_+$. By establishing the invertibility of V and introducing the albedo operator $E = V^{-1}$ we obtain the unique solution

$$\psi(x) = e^{-xT^{-1}A}E\varphi_+ = \int_0^\infty e^{-x/\mu}\tau(d\mu)\, E\varphi_+, \tag{13}$$

where $\tau(\cdot)$ is the resolution of the identity of $A^{-1}T$. For positive operators A having a nonzero null space Ker A one has to modify A on the (finite-dimensional) zero root subspace of $T^{-1}A$ in order to reduce Eqs. (8)–(10) to the above set up. Depending on the structure of Ker A, Eqs. (8)–(10), either with $O(1)$ or $o(1)$ in condition (10), are uniquely or non-uniquely solvable. Similar results have been obtained under the boundary condition (5) (cf. [27, 9]).

The above functional formulation was first carried out by Hangelbroek for subcritical isotropic neutron transport [20] (also [22]) and developed further by several authors. There are two variants. One variant due to Beals involves general bounded A and Sturm–Liouville-type operators. In [6] he introduced the completion of $D(T)$ with respect to the inner product

$$(h, k)_T = (|T|h, k) \tag{14}$$

and for injective and certain noninjective A he established the unique solvability of Eqs. (8)–(10) on H_T. As a major tool for the invertibility proof of V on H_T, he proved the equivalence of (14) to the inner product

$$(h, k)_S = (|A^{-1}T|\, h, k)_A = (T(P_+ - P_-)\, h, k) \tag{15}$$

and hence the equivalence of their completions H_T and H_S. [For the non-injective A treated he introduced the above modification.] In [8] he obtained the same results for the Sturm–Liouville problems (1)–(4). Introducing, on $H = L_2(I; d\mu)$, the operators

$$(Th)(\mu) = w(\mu)\, h(\mu), \qquad (Q_+ h)(\mu) = \begin{cases} h(\mu), & w(\mu) > 0 \\ 0, & w(\mu) < 0, \end{cases}$$

and

$$(Q_- h)(\mu) = \begin{cases} h(\mu), & w(\mu) < 0 \\ 0, & w(\mu) > 0, \end{cases}$$

$$(Ah)(\mu) = -\frac{d}{d\mu}(p(\mu)\, h'(\mu)) + q(\mu)\, h(\mu) + \text{self-adjoint b.c.},$$

it is easily seen that (1)–(4) reduce to an example of (8)–(10) and the above existence and uniqueness result on $H_T = L_2(I; |w(\mu)|\, d\mu)$ can be proved along the same lines.

The second variant involves operators A which are compact perturbations of the identity and was developed by van der Mee [26, 27]. (In these works T is bounded but the result can be generalized to unbounded T; cf. [18]). For injective A the operator V is invertible and Eqs. (8)–(10) are uniquely solvable on H and $D(T)$. Although in this case the result can also be obtained in a somewhat weaker sense in H_T (using [6]), the major importance of the present variant is that the boundary value problem allows reformulation as a vector-valued Wiener–Hopf equation and therefore a solution by Wiener–Hopf factorization. In this way he generalized a plethora of concrete results in various applied fields of kinetic

theory and obtained an explicit expression for the albedo operator E of the form

$$E\varphi_+ = \varphi_+ + \int_{-\infty(\mu)}^{0} \int_{0(\nu)}^{\infty} \frac{\nu}{\nu-\mu} \sigma(d\mu)\, Bj\, H_l(-\mu)\, H_r(\nu)\, \pi\sigma(d\nu)\, \varphi_+ \quad (16)$$

(see [28]). Here $\sigma(\cdot)$ is the resolution of the identity of T, $B = I - A$ has finite rank and j: Ran $B \to H$ and π: $H \to$ Ran B are the natural embedding and orthogonal projection, respectively. The matrix functions H_l and H_r are Wiener–Hopf factors and can be computed using a generalization of Chandrasekhar's H-equation [13].

In this article we present two methods to compute in principle the albedo operator E for the Sturm–Liouville problem (1)–(4). The first method, when formulated for strictly positive A, is based on the vector equation

$$(I + K_-)\, g = K_+ \varphi_+, \quad (17)$$

where $K_- = -Q_- P_+ Q_-$: $Q_-[H_T] \to Q_-[H_T]$ and $K_+ = Q_- P_+ Q_+$: $Q_+[H_T] \to Q_-[H_T]$, which has $g = Q_- E\varphi_+$ as its unique solution. Using an explicit representation for the resolution of the identity of $T^{-1}A$, we obtain an equation to be solved for g in $Q_-[H_T]$, whence $E\varphi_+ = \varphi_+ + g$. For neutron transport and rarefied gas dynamics this procedure leads to singular integral equations as appearing in [11, 12].

The second method will yield a generalization of (16). The key step is the following well-known observation: We may add such self-adjoint boundary conditions at the sign changes $c_1, \ldots, c_N$ $(a < c_1 < \cdots < c_N < b)$ of the weight $w(\mu)$ that (i) the resulting Sturm–Liouville operators $\hat{A}_0, \hat{A}_1, \ldots, \hat{A}_N$ on the spaces $L_2(a, c_1)$, $L_2(c_1, c_2), \ldots, L_2(c_N, b)$ are strictly positive self-adjoint and (ii) the operator $A^{-1} - \hat{A}^{-1}$, where A is assumed strictly positive and $\hat{A} = \hat{A}_0 \oplus \cdots \oplus \hat{A}_N$, has finite rank N. By construction, $T_1 = \hat{A}^{-1}T$ will have $Q_\pm$ as its positive–negative spectral projections and Eqs. (8)–(10) (on H_T) will be equivalent to the boundary value problem

$$T_1 \psi'(x) = -A_1 \psi(x), \qquad (0 < x < \infty), \quad (18)$$

$$Q_+ \psi(0) = \varphi_+, \quad (19)$$

$$\|\psi(x)\|_T = O(1) \qquad \text{or} \qquad o(1) \quad \text{as } x \to \infty, \quad (20)$$

where $A_1 = \hat{A}^{-1}A$. Since also $A_1^{-1} T_1 = A^{-1}T$, the albedo operator E and the semigroups involved in the solution (13) coincide. As a result, Eqs. (18)–(20) are uniquely solvable on $H_T = L_2(I; |w(\mu)|\, d\mu)$. The finite rank of $C = A^{-1}T - \hat{A}^{-1}T$ is our major gain, since we may now reformulate the problem as an integral equation. However, since A_1 is unboun-

ded (in H_T), the reduction to a formula of the type (16) is more complicated than it is to be expected from [27, 28]. Introducing the function $\varphi(x) = -\psi'(x) = T^{-1}A\psi(x)$ we first derive the integral equation

$$\varphi(x) + \frac{d}{dx}\int_0^\infty \mathscr{H}_1(x-y)\, C\varphi(y)\, dy = \mathscr{H}_1(x)\, \varphi_+, \qquad 0 < x < \infty, \tag{21}$$

where, in terms of the resolution of the identity $\sigma_1(\cdot)$ of $T^{-1}\hat{A} = T_1^{-1}$,

$$\mathscr{H}_1(x) = \begin{cases} +\displaystyle\int_0^\infty \mu e^{-\mu x}\sigma_1(d\mu), & 0 < x < \infty \\ -\displaystyle\int_{-\infty}^0 \mu e^{-\mu x}\sigma_1(d\mu), & -\infty < x < 0. \end{cases} \tag{22}$$

If j and π are the natural embedding and orthogonal projection (with respect to (14)) satisfying $Cj\pi = C$, we shall then derive a (finite-dimensional) equation for $\zeta(x) = \pi\varphi(x)$, which we solve by Wiener–Hopf factorization. However, the existence proof for the Wiener–Hopf factorization will make essential use of the strict positivity of A and is, in fact, based on the geometric factorization principle of Bart *et al.* [4, 3]. For the latter case we shall obtain a complete analogue of (16). At present the second method does not work for noninjective A, such as needed for Eqs. (6) and (7).

The paper will be organized as follows. In Section 2 we explain in detail the first method. In Section 3 we construct $\hat{A}$ and arrive at the equivalent problem (18)–(20). In Section 4 we prove the equivalence of the latter problem to the integral equation (21) and its finite-dimensional reduced form. In Section 5 we establish in a formal sense the Wiener–Hopf factorization associated with the integral equation and construct the albedo operator explicitly. In Section 6 we shall specialize our results for indefinite weights with one sign change and work out some illustrative examples.

II. An Eigenfunction Expansion for Computing the Albedo Operator

The first method of computing the albedo operator has an analogue in neutron transport, radiative transfer and rarefied gas dynamics known as the method of singular eigenfunction expansion (cf. [11, 12]). We shall work out the details for the abstract problem (8)–(10), state our assumptions on the indefinite Sturm–Liouville problem and then specialize the method for these problems.

Let T be an injective self-adjoint and A a positive self-adjoint Fredholm operator on a Hilbert space H, and let $Q_\pm$ denote the orthogonal projec-

tion of H onto the maximal positive/negative T-invariant subspace. We denote by H_T the completion of $D(T)$ with respect to the inner product (14), and put $H_A = D(A^{1/2})$. Then H_A is densely and continuously embedded in H. Let us *assume* that $H_A \subset D(T)$ and T is a bounded operator from H_A into H. (In [8] this was, in fact, assumed.) Then we have the decomposition

$$H_A = Z_0 \oplus Z_1,$$

where $Z_0 = \operatorname{Ker}(T^{-1}A)^2 \supset \operatorname{Ker} A$; this decompsition reduces the operator $T^{-1}A$, and $S_1 = (T^{-1}A|_{Z_1})^{-1}$ is bounded and self-adjoint on Z_1 with respect to the inner product (11) (cf. [19] for the most general result; first result of this type in [25]). If P_0 denotes the projection of H_A onto Z_0 along Z_1 and $\{P_0, P_+, P_-\}$ is the triple of complementary projections of H_A such that $P_\pm|_{Z_1}$ is the $(.,.)_A$-orthogonal projection of Z_1 onto the maximal $(.,.)_A$-positive–negative S_1-invariant subspace, then H_S will denote the completion of $Z_0 \oplus D(S_1)(\subset H_A)$ with respect to the inner product

$$\begin{aligned}(h, k)_S &= \{P_0 h, P_0 k\} + (|S_1|(I - P_0)\, h, (I - P_0)\, k)_A \\ &= \{P_0 h, P_0 k\} + (T(P_+ - P_-)\, h, k).\end{aligned}$$

Here $\{.,.\}$ is an arbitrary inner product of Z_0. We shall *assume* that $(.,.)_T$ and $(.,.)_S$ are equivalent and therefore H_T and H_S may be identified. This assumption is known to be satisfied if A is bounded (cf. [6]) or in the case of a Sturm–Liouville problem of the type (1)–(4) (cf. [8]). If $\operatorname{Ker} A = 0$, then $P_0 = 0$ and $(.,.)_S$ is given by (15). Finally, the projections $Q_\pm$, P_0, and $P_\pm$ allow continuous extension to $H_T \cong H_S$.

THEOREM 2.1 (cf. [6, 19, 8]). *If* $\operatorname{Ker} A = \{0\}$ *or if* $\operatorname{Ker} A = \operatorname{span}\{\varphi_0\}$ *with* $(T\varphi_0, \varphi_0) \geqslant 0$, *there exists a unique solution of the differential equation* (8) *with boundary conditions* (9) *and* $\|\psi(x)\|_T = O(1)(x \to \infty)$. *This solution satisfies*

$$\exists M, \qquad r > 0: \quad \|\psi(x) - \psi_\infty\|_T \leqslant M e^{-rx} \tag{23}$$

for some $\psi_\infty \in \operatorname{Ker} A$. *On the other hand, if* $\operatorname{Ker} A = \{0\}$ *or if* $\operatorname{Ker} A = \operatorname{span}\{\varphi_0\}$ *with* $(T\varphi_0, \varphi_0) < 0$, *there exists a unique solution of* Eq. (8) *with boundary conditions* (9) *and* $\|\psi(x)\|_T = o(1)(x \to \infty)$. *This solutions satisfies*

$$\exists M, \qquad r > 0: \quad \|\psi(x)\|_T \leqslant M e^{-rx}. \tag{24}$$

In all cases one may write $\psi(0) = E_+ \varphi_+$ where E_+ is the (H_T-bounded) projection along $Q_-[H_T]$ onto $P_+[H_S] \oplus \operatorname{Ker} A$ in the former situation, and onto $P_+[H_S]$ in the latter situation. For injective A we have $E_+ = EQ_+$ where $E = V^{-1}$. We define on H_S:

$$\tilde{P}_+ = \begin{cases} P_+ & \text{if} \quad \operatorname{Ker} A = \{0\} \text{ or } (T\varphi_0, \varphi_0) < 0 \\ P_+ + P_0 & \text{if} \quad (T\varphi_0, \varphi_0) > 0 \\ P_+ + \dfrac{(\cdot, T\psi_0)}{(\varphi_0, T\psi_0)} \varphi_0 & \text{if} \quad (T\varphi_0, \varphi_0) = 0. \end{cases} \tag{25}$$

[In the latter case there exists ψ_0 such that $A\psi_0 = T\psi_0$. We then have $(\varphi_0, T\psi_0) = (A\psi_0, \psi_0) > 0$ and we may select ψ_0 in such a way that $(T\psi_0, \psi_0) = 0$. As a result we find $\tilde{P}_+ \varphi_0 = \varphi_0$ and $\tilde{P}_+ \psi_0 = 0$.] Next, put

$$K_\pm = \pm Q_- \tilde{P}_+ Q_\pm : \quad Q_\pm[H_T] \to Q_-[H_T].$$

LEMMA 2.2. *The vector $g = Q_- E_+ \varphi_+$ is the unique solution in $Q_-[H_T]$ of the equation*

$$(I + K_-) g = K_+ \varphi_+ . \tag{26}$$

Proof. One easily calculates

$$\begin{aligned} K_+ \varphi_+ - K_- g &= Q_- \tilde{P}_+ (\varphi_+ + g) = Q_- \tilde{P}_+ (Q_+ E_+ \varphi_+ + Q_- E_+ \varphi_+) \\ &= Q_- \tilde{P}_+ E_+ \varphi_+ = Q_- E_+ \varphi_+ = g, \end{aligned}$$

using $\tilde{P}_+ E_+ = E_+$. Conversely, if g is a solution of (26) in $Q_-[H_T]$, we have

$$Q_-(g - \tilde{P}_+(\varphi_+ + g)) = g - K_+ \varphi_+ + K_- g = 0,$$

and therefore there exists $\varphi_+ \in Q_+[H_T]$ satisfying

$$g - \tilde{P}_+(\varphi_+ + g) = -\varphi_+ + \psi_+ .$$

The latter implies

$$\psi_+ \in Q_+[H_T] \cap (I - \tilde{P}_+)[H_S] = \{0\},$$

whence $\varphi_+ + g \in \tilde{P}_+[H_S]$. As a result we necessarily have $g = Q_- E_+ \varphi_+$. ∎

We may compute E_+ in principle by solving (26), putting

$$E_+ \varphi_+ = \varphi_+ + (I + K_-)^{-1} K_+ \varphi_+ \tag{27}$$

and applying the method to Sturm–Liouville diffusion problems. Let us consider the Sturm–Liouville differential operator

$$(Ah)(\mu) = -\frac{d}{d\mu}\left(p(\mu) \frac{dh}{d\mu} \right) + q(\mu)\, h(\mu)$$

defined on a domain where it is self-adjoint and positive with $\sigma(A) \subset \{0\} \cup [\varepsilon, \infty)$ for some $\varepsilon > 0$. We shall make the usual assumption that p is locally absolutely continuous and positive on I and g is continuous and positive on I. [In fact, we may require less, e.g., $\{p^{-1}, q\} \subset L_1^{\mathrm{loc}}(I)$.] A more specific description of the differential operator will follow in the next section. Next let us assume that the indefinite weight function $w(\mu)$ has the following properties:

(i) $w(\mu)$ is continuous and nonzero on I, except possible at $c_1, \ldots, c_N$;

(ii) for each $j = 1, 2, \ldots, N$ there is neighborhood U_j of c_j and a number $\alpha_j > -\frac{1}{2}$ satisfying $w(\mu) = \pm \operatorname{sgn}(\mu - c_j) \cdot |\mu - c_j|^{\alpha_j} v(\mu)$ for $\mu \in U_j \setminus \{c_j\}$ with $v(c_j) \neq 0$ and $v(\mu)$ continuously differentiable on U_j, and

(iii) the operator

$$(Th)(\mu) = w(\mu)\, h(\mu) (\mu \in I)$$

satisfies the condition

$$\int_I |w(\mu)\, h(\mu)|^2\, d\mu \leqslant C^2 \left\{ \int_I |(A^{1/2}h)(\mu)|^2\, d\mu + \int_I |(\Pi_A h)(\mu)|^2\, d\mu \right\},$$

where C is some fixed constant and Π_A is the orthogonal projection of $L_2(I)$ onto Ker A.

As a result, if we define H_A as the direct sum of the finite-dimensional subspace Ker A, with some inner product, and $\{h \in D(A^{1/2}) \mid (h, g) = 0$ for all $g \in \operatorname{Ker} A\}$, the latter endowed with the (complete) inner product

$$(h, k)_A = (A^{1/2}h, A^{1/2}k) = \int_I (A^{1/2}h)(\mu) \overline{(A^{1/2}k)(\mu)}\, d\mu,$$

the third hypothesis means that $H_A \subset D(T)$ and T is a bounded operator from H_A into H. Moreover, T and A meet the assumptions at the beginning of this section and Theorem 2.1 holds true in full, where in this case

$$(h, k)_T = \int_I |w(\mu)| \cdot h(\mu) \overline{k(\mu)}\, d\mu.$$

We shall consider the more elementary situation where $T^{-1}A$ has a compact resolvent. Let λ_n be the (simple) eigenvalues of $T^{-1}A$ with corresponding eigenfunctions φ_n, numbered according to the following convention:

(i) if Ker $A = \{0\}$, the index n runs through the nonzero integers and

$$(\varphi_n, \varphi_m)_A = \delta_{nm}, \qquad n\lambda_n \geqslant 0; \tag{28}$$

(ii) if Ker $A \neq \{0\}$ and Ker $A = \operatorname{span}\{\varphi_0\}$, the index n runs through all integers and (28) is valid for $0 \neq n, m \in \mathbb{Z}$;

(iii) if (and only if) $(T\varphi_0, \varphi_0) = 0$, there is a generalized eigenvector ψ_0 which can be chosen uniquely by requiring $(T\psi_0, \psi_0) = 0$. We then have

$$\tilde{P}_+ = P_{1,+} + \tilde{P}_0, \tag{29}$$

where

$$P_{1,+} h = \sum_{n>0} (h, |\lambda_n|^{1/2} \varphi_n)_s \, |\lambda_n|^{1/2} \varphi_n$$

and

$$\tilde{P}_0 h = \begin{cases} 0 & \text{if} \quad \operatorname{Ker} A = \{0\} \text{ or } (T\varphi_0, \varphi_0) < 0 \\ \dfrac{(h, T\varphi_0)}{(\varphi_0, T\varphi_0)} \varphi_0 & \text{if} \quad (T\varphi_0, \varphi_0) > 0 \\ \dfrac{(h, T\psi_0)}{(\varphi_0, T\psi_0)} \varphi_0 & \text{if} \quad (T\varphi_0, \varphi_0) = 0. \end{cases} \tag{30}$$

We shall now use the identity [21, Eq. (2.8); 19, Sect. 3]

$$(f, g)_S = (f, (2V - I) g)_T = ((Q_+ - Q_-) f, (P_{1,+} - P_{1,-}) g)_T,$$

where $\{f, g\} \subset Z_1 \subset H_S \simeq H_T$. As a result we obtain for Ker $A = \{0\}$:

$$\begin{aligned} K_- h &= -\sum_{n>0} \lambda_n ((Q_+ - Q_-) Q_- h, (P_+ - P_-) \varphi_n)_T \\ &= \sum_{n>0} \lambda_n (h, Q_- \varphi_n)_T Q_- \varphi_n, \\ K_+ h &= +\sum_{n>0} \lambda_n ((Q_+ - Q_-) Q_+ h, (P_+ - P_-) \varphi_n)_T \\ &= \sum_{n>0} \lambda_n (h, Q_+ \varphi_n)_T Q_- \varphi_n. \end{aligned}$$

For nontrivial Ker A we obtain the same formulas if $(T\varphi_0, \varphi_0) < 0$, and a correction term if $(T\varphi_0, \varphi_0 \geqslant 0$, which is easily derived from (29) and (30). The equation (26) for $g = Q_- E Q_+ \varphi_+$ now has the following form:

(i) $g + \sum_{n>0} \lambda_n (g, Q_- \varphi_n)_T Q_- \varphi_n = \sum_{n>0} \lambda_n (\varphi_+, Q_+ \varphi_n)_T Q_- \varphi_n$ if Ker $A = \{0\}$ or $(T\varphi_0, \varphi_0) < 0$,

(i)

$$g+\sum_{n>0}\lambda_n(g, Q_-\varphi_n)\,Q_-\varphi_n+\frac{(g, TQ_-\varphi_0)}{(\varphi_0, T\varphi_0)}Q_-\varphi_0$$
$$=\sum_{n>0}\lambda_n(\varphi_+, Q_+\varphi_n)_T\,Q_-\varphi_n+\frac{(\varphi_+, TQ_+\varphi_0)}{(\varphi_0, T\varphi_0)}Q_-\varphi_0$$

if $(T\varphi_0, \varphi_0)>0$, and

(iii)

$$g+\sum_{n>0}\lambda_n(g, Q_-\varphi_n)_T\,Q_-\varphi_n+\frac{(g, TQ_-\psi_0)}{(\varphi_0, T\psi_0)}Q_-\varphi_0$$
$$=\sum_{n>0}\lambda_n(\varphi_+, Q_+\varphi_n)_T\,Q_-\varphi_n+\frac{(\varphi_+, TQ_+\psi_0)}{(\varphi_0, T\psi_0)}Q_-\varphi_0,$$

if $(T\varphi_0, \varphi_0)=0$. Finally, if $T^{-1}A$ has continuous spectrum on parts of $(0, \infty)$ a similar integral equation for g can be derived involving integrations over a continuous rather than a discrete measure.

As an example we consider the Fokker–Planck equation (7) for which the eigenvalues and eigenvectors of $T^{-1}A$ have been computed by Pagani [32],

$$\lambda_{\pm n}=\pm\sqrt{n}, \qquad \lambda_0=0,$$
$$\varphi_0(v)=1,$$
$$\varphi_n(v)=\varphi_{-n}(-v)=(n^{1/4}\sqrt{n!(2\pi n)})^{-1}\,e^{v^2/4}D_n(v-2\sqrt{n}),$$

where $n=1, 2, 3,\ldots$, and D_n are the Weber functions. As in this case $(T\varphi_0, \varphi_0)=0$, we use the above formula (iii) for the albedo operator and obtain the solution $\psi(0, v)$, where $\psi(0, v)=\varphi_+(v)$ for $v>0$, $\psi(0, v)=g(v)$ for $v<0$ and $g(v)$ has to be computed from the integral equation

$$g(v)+\sum_{n>0}\sqrt{n}\int_0^\infty v'e^{-(1/2)(v')^2}\varphi_n(v)\,\varphi_n(-v')\,g(-v')\,dv'$$
$$-(1/\sqrt{2\pi})\int_0^\infty (v')^2\,e^{-(1/2)(v')^2}g(-v')\,dv'$$
$$=\sum_{n>0}\sqrt{n}\int_0^\infty v'e^{-(1/2)(v')^2}\varphi_n(v)\,\varphi_n(v')\,\varphi_+(v')\,dv'$$
$$+(1/\sqrt{2\pi})\int_0^\infty (v')^2\,e^{-(1/2)(v')^2}\varphi_+(v')\,dv'.$$

III. First Reduction: The Modified Sturm–Liouville Problem

Let us consider the Sturm–Liouville differential equation

$$-(pg')' + q\varphi = \lambda g + f \tag{32}$$

on the interval $I = (a, b)$ (cf. [14, 15, 1, 23, 16] for the general theory), where it is assumed that p is locally absolutely continuous and positive on I, q is continuous on I and $f \in L_2(I; d\mu)$. The endpoint a (or b) is called *regular* if it is finite, p and q extend continuously to $[a, b)$ (resp. $(a, b]$) and $p(a) > 0$ (resp. $p(b) > 0$). Otherwise it is called *singular*. If both endpoints are regular, we may turn the Sturm–Liouville operator $g \mapsto -(pg')' + qg$ into a self-adjoint operator on $L_2(I; d\mu)$ by imposing the boundary conditions

$$\cos \alpha g(a) - p(a) \sin \alpha g'(a) = 0, \tag{33}$$

$$\cos \beta g(b) - p(b) \sin \beta g'(b) = 0, \tag{34}$$

for certain $\alpha, \beta \in [0, \pi)$. If a is a regular and b is a singular endpoint, we have to distinguish between the *limit-circle* (at b) case where for some (and hence all) complex λ all solutions of the differential equation

$$-\frac{d}{d\mu}\left(p(\mu)\frac{d}{d\mu} g(\mu, \lambda)\right) + q(\mu)\, g(\mu, \lambda) = \lambda g(\mu, \lambda) \tag{35}$$

are square integrable near b, and the *limit-point* case where for some (and hence all) nonreal λ there is only one linearly independent solution of Eq. (35) that is square-integrable near b. In the limit-point case a self-adjoint boundary value problem arises by only imposing condition (33), while in the limit-circle case one needs condition (33) as well as a condition for $\mu \to b$ in order to ensure self-adjointness. If a is a singular and b is a regular endpoint, we have a similar distinction as before with a and b interchanged. If both endpoints a and b are singular, we choose $c \in (a, b)$, impose the boundary condition

$$\cos \gamma g(c) - p(c) \sin \gamma g'(c) = 0 \tag{36}$$

and consider the limit-circle versus limit-point classification at each of the endpoints a and b. Depending on the four possibilities as to each of the dichotomic classifications, we impose self-adjoint boundary conditions for Eq. (35) on I. The classification is independent of the particular choice of c and γ. In particular, if the differential equation (32) meets the limit-point condition at both endpoints a and b, no boundary conditions are to be

imposed to guarantee self-adjointness. If A denotes the self-adjoint Sturm–Liouville operator (with boundary conditions), then the resolvent $(A-\lambda)^{-1}$ is given as $g(\mu, \lambda) = [(A-\lambda)^{-1} f](\mu)$, where

$$g(\mu, \lambda) = \frac{1}{W(\lambda)} \left\{ \psi(\mu, \lambda) \int_a^\mu \varphi(v, \lambda) f(v)\, dv + \varphi(\mu, \lambda) \int_\mu^b \psi(v, \lambda) f(v)\, dv \right\}. \tag{37}$$

Here

$$W(\lambda) = p(\mu) \{ \varphi'(\mu, \lambda)\, \psi(\mu, \lambda) - \psi'(\mu, \lambda)\, \varphi(\mu, \lambda) \} \tag{38}$$

is a constant relating to the Wronskian and $\varphi(\mu, \lambda)$ and $\psi(\mu, \lambda)$ are nontrivial solutions of Eq. (35) satisfying the boundary conditions at a and b, respectively. [In the limit-point case the condition is square integrability near the endpoint.] The isolated eigenvalues of A correspond to isolated zeros of $W(\lambda)$ and are simple. The resolvent set of A consists of the open upper and lower half-planes and that part of the real line to which $W(\lambda)$ has the same analytic continuation from above and below leading to a nonzero value. The remaining part of the real line is continuous spectrum. Next let us see what happens if one decomposes the self-adjoint Sturm–Liouville problem (32) on $I = (a, b)$ into a problem on (a,c) and a problem on (c, b). We shall give a detailed discussion of basically well-known material, since the precise details are required for the subsequent construction of the albedo operator. Let $\chi(\mu, \lambda)$ be a nontrivial solution of Eq. (35) on (a, b) that satisfies the additional boundary condition (36). The resolvents of the Sturm–Liouville problems on (a, c) and (c, b) have the respective forms

$$g_l(\mu, \lambda) = \frac{1}{W_l(\lambda)} \left\{ \chi(\mu, \lambda) \int_a^\mu \varphi(v, \lambda) f(v)\, dv + \varphi(\mu, \lambda) \int_\mu^c \chi(v, \lambda) f(v)\, dv \right\}, \tag{39}$$

where

$$W_l(\lambda) = p(\mu) \{ \varphi'(\mu, \lambda)\, \chi(\mu, \lambda) - \chi'(\mu, \lambda)\, \varphi(\mu, \lambda) \}, \tag{40}$$

and

$$g_r(\mu, \lambda) = \frac{1}{W_r(\lambda)} \left\{ \psi(\mu, \lambda) \int_c^\mu \chi(v, \lambda) f(v)\, dv + \chi(\mu, \lambda) \int_\mu^b \psi(v, \lambda) f(v)\, dv \right\}, \tag{41}$$

where

$$W_r(\lambda) = p(\mu)\,\{(\chi'(\mu, \lambda)\,\psi(\mu, \lambda) - \psi'(\mu, \lambda)\,\chi(\mu, \lambda)\}. \tag{42}$$

Writing

$$\chi(\mu, \lambda) = c_l(\lambda)\,\varphi(\mu, \lambda) + c_r(\lambda)\,\psi(\mu, \lambda) \tag{43}$$

and using the formulas

$$c_l(\lambda) = W_r(\lambda)/W(\lambda), \qquad c_r(\lambda) = W_l(\lambda)/W(\lambda) \tag{44}$$

obtained by substituting (43) into (40) and (42), we finally get

$$g(\mu, \lambda) - g_l(\mu, \lambda) = -\frac{W_r(\lambda)}{W_l(\lambda)\,W(\lambda)}\,\varphi(\mu, \lambda)\int_a^c \varphi(v, \lambda) f(v)\,dv, \tag{45}$$

where $\mu \in (a, c)$ and $f \in L_2(a, c)$, and

$$g(\mu, \lambda) - g_r(\mu, \lambda) = -\frac{W_l(\lambda)}{W_r(\lambda)\,W(\lambda)}\,\psi(\mu, \lambda)\int_c^b \psi(v, \lambda) f(v)\,dv, \tag{46}$$

where $\mu \in (c, b)$ and $f \in L_2(c, b)$. Thus if λ belongs to the resolvent set of all three Sturm–Liouville operators and $\hat{A}$ denotes the direct sum of the Sturm–Liouville operators on $L_2(a, c)$ and $L_2(c, b)$ (viewed as an operator on $L_2(a, b)$), we have for $f \in L_2(a, b)$

$$\begin{aligned}&([(A-\lambda)^{-1} - (\hat{A}-\lambda)^{-1}]f)(\mu)\\ &\quad = -\frac{W_l(\lambda)\,W_r(\lambda)}{W(\lambda)}\,\kappa(\mu, \lambda)\int_a^b \kappa(v, \lambda) f(v)\,dv,\end{aligned} \tag{47}$$

where

$$\kappa(\mu, \lambda) = \begin{cases} W_l(\lambda)^{-1}\,\varphi(\mu, \lambda), & \mu \in (a, c) \\ W_r(\lambda)^{-1}\,\psi(\mu, \lambda), & \mu \in (c, b) \end{cases}. \tag{48}$$

Hence, the difference of the resolvents is an operator of rank one.

If A is positive self-adjoint with spectrum $\sigma(A) \subset \{0\} \cup [\varepsilon, \infty)$ for some $\varepsilon > 0$, it is possible to choose the constant γ in condition (36) in such a way that the resulting operator $\hat{A}$ is strictly positive self-adjoint. Let us normalize $\chi(\mu, \lambda)$ by requiring $\chi(c, \lambda) = p(c)\sin\gamma$ and $\chi'(c, \lambda) = \cos\gamma$. We then easily derive the identities

$$W_l(\lambda) = +p(c)\{p(c)\sin\gamma\varphi'(c, \lambda) - \cos\gamma\varphi(c, \lambda)\},$$

$$W_r(\lambda) = -p(c)\{p(c)\sin\gamma\psi'(c, \lambda) - \cos\gamma\psi(c, \lambda)\},$$

from (43) and (44) and the derivative of (43) with respect to μ (for λ real and $W(\lambda) \neq 0$) and observe that $W_l(\lambda) = W_r(\lambda) = 0$ implies $W(\lambda) = 0$. The latter implies the existence of unique and distinct $\gamma_l = \gamma_l(\lambda)$ and $\gamma_r = \gamma_r(\lambda)$ in $[0, \pi)$ satisfying $W_l(\lambda) = 0$ for $\gamma = \gamma_l(\lambda)$ and $W_r(\lambda) = 0$ for $\gamma = \gamma_r(\lambda)$. This in turn gives the existence of an interval of values of γ where $W_l(\lambda)\, W_r(\lambda)\, W(\lambda)^{-1} > 0$ and an interval of values of γ where $W_l(\lambda)\, W_r(\lambda)\, W(\lambda)^{-1} < 0$. On inspecting (47) it appears that we may choose γ in such a way that $(\hat{A} - \lambda)^{-1} \geqslant (A - \lambda)^{-1}$ as self-adjoint operators. If $\sigma(A) \subset (0, \infty)$ and $\lambda = 0$, we get $\hat{A}^{-1} \geqslant A^{-1} \geqslant 0$ for an interval of values of γ. If A is positive with isolated (simple) zero eigenvalue, then $W(0) = 0$, $W_l(0) \neq 0$ and $W_r(0) \neq 0$, except for the unique $\gamma = \gamma_0$ satisfying the equality

$$p(c) \sin \gamma_0 \varphi'(c, 0) = \cos \gamma_0 \varphi(c, 0)$$

with $\varphi(\cdot, 0) = d\psi(\cdot, 0)$ the zero eigenfunction of A. Because of the usual oscillation theorems we have $\varphi(\mu, 0) \neq 0$ for $\mu \in I$ and therefore $\gamma_0 \in (0, \pi)$. However, for every $\lambda < 0$ we have an interval of values of γ where $(\hat{A} - \lambda)^{-1} \geqslant (A - \lambda)^{-1} \geqslant 0$. Thus if we exclude γ_0 from this interval, we find an interval of γ (the same or a smaller one) where $\sigma(\hat{A}) \subset (0, \infty)$. Hence, the constant γ in condition (36) can be chosen as to make $\hat{A}$ strictly positive.

THEOREM 3.1. *Let us consider a self-adjoint Sturm–Liouville operator A and a weight function $w(\mu)$ on (a, b), both of which satisfy the previous assumptions. Let A be either strictly positive or positive with an isolated zero eigenvalue. At the sign changes $c_1, \ldots, c_N$ of the weight function we may add boundary conditions of the type*

$$\cos \gamma_j g(c_j) - p(c_j) \sin \gamma_j g'(c_j) = 0, \qquad j = 1, 2, \ldots, N,$$

and obtain strictly positive Sturm–Liouville operators $\hat{A}_0, \hat{A}_1, \ldots, \hat{A}_N$ on the respective subintervals (a, c_1), $(c_1, c_2), \ldots, (c_N, b)$. Then the direct sum

$$\hat{A} = \hat{A}_0 \oplus \hat{A}_1 \oplus \cdots \oplus \hat{A}_N \tag{49}$$

is strictly positive self-adjoint on $L_2(I)$. If the original Sturm–Liouville operator A is strictly positive, then $C = A^{-1}T - \hat{A}^{-1}T$ is a bounded operator on $H_T = L_2(I; |w(\mu)|\, d\mu)$ of rank N.

The proof follows easily by induction on N if the weight function is bounded. First we construct $\hat{A}_0$ by using Eq. (47) for $c = c_1$. On (c_1, b) we further split up the Sturm–Liouville operator obtained at $c = c_2$ and apply (47) again. After N steps we have constructed $\hat{A}_0, \ldots, \hat{A}_N$, and $(A - \lambda)^{-1} - (\hat{A} - \lambda)^{-1}$ is an operator of rank N for $\lambda \notin \sigma(A) \cup \sigma(\hat{A}) = \sigma(A) \cup \sigma(\hat{A}_0) \cup \cdots \cup \sigma(\hat{A}_N)$. If the weight function $w(\mu)$ is bounded, then $C(\lambda) =$

$\{(A-\lambda)^{-1}-(\hat{A}-\lambda)^{-1}\}\,T$ $(\lambda \notin \sigma(A)\cup\sigma(\hat{A}))$ is bounded on H_T. If the weight function $w(\mu)$ is unbounded, it must be unbounded near either a, $c_1,\ldots,c_N$ or b. Let $\varphi(\mu)$ be a positive C^∞-function on I with compact support within $I\backslash\{c_1,\ldots,c_N\}$. For each of the functions $\kappa_j(\mu,\lambda)(\lambda\notin\sigma(A)\cup\sigma(\hat{A}))$ constructed in (48) the function $\varphi(\mu)\,\kappa_j(\mu,\lambda)$ is continuous and has a compact support on which $w(\mu)$ is bounded; hence, $\int_I |w(\mu)|^2\,|\varphi(\mu)\,\kappa_j(\mu,\lambda)|^2\,d\mu<\infty$. The function $(1-\varphi(\mu))\,\kappa_j(\mu,\lambda)$ is square integrable on I with weight $|w(\mu)|$ for the following reasons:

(i) $(1-\varphi(\mu))\,\kappa_j(\mu,\lambda)$ is continuous on (a,b) and $w(\mu)=0(|\mu-c_j|^{\alpha_j})$ for some $\alpha_j>-\frac{1}{2}$,

(ii) $(1-\varphi(\mu))\,\kappa_j(\mu,\lambda)$ satisfies the boundary conditions (or local square integrability conditions) at a and b, and

(iii) $D(A)\subset D(T)$. Hence, $C(\lambda)=\{(A-\lambda)^{-1}-(\hat{A}-\lambda)^{-1}\}\,T$ is a bounded operator on H_T and has rank N if $\lambda\notin\sigma(A)\cup\sigma(\hat{A})$.

As a result of the above theorem, the operator $T_1=\hat{A}^{-1}T$ is bounded on $L_2(I)$ and compact if A has a compact resolvent. Also, T_1 commutes with the projections $Q_\pm$ defined above (following Eq. (15)). If we denote by $H_{\hat{A}}=D(\hat{A}^{1/2})$ the completion of $D(\hat{A})$ with respect to $(.,.)_{\hat{A}}=(\hat{A}.,.)$, then T_1 is self-adjoint on $H_{\hat{A}}$, Q_+ and Q_- are the $H_{\hat{A}}$-orthogonal projections onto the maximal $H_{\hat{A}}$-positive and negative T_1-invariant subspaces and $|T_1|=\hat{A}^{-1}\,|T|$ is the $H_{\hat{A}}$-positive absolute value of T_1. Therefore, by analogy with H_T we have

$$(h,k)_{T_1}=(|T_1|h,k)_{\hat{A}}=(\hat{A}^{-1}\,|T|\,h,k)_{\hat{A}}=(h,k)_T. \tag{50}$$

Also, defining $A_1=\hat{A}^{-1}A$ as an $H_{\hat{A}}$-positive operator, we have

$$T_1^{-1}A_1=T^{-1}A \tag{51}$$

and, if Ker $A=\{0\}$, and $S_1=A_1^{-1}T_1$,

$$(h,k)_{S_1}=(A_1\,|A_1^{-1}T_1|\,h,k)_{\hat{A}}=(A_1^{-1}T_1(P_+-P_-)\,h,k)_A=(h,k)_S. \tag{52}$$

Hence, the boundary value problem (8)–(10) on $H_T\simeq H_S$ has precisely the same solutions as the modified problem

$$T_1\psi'(x)=-A_1\psi(x) \qquad (0<x<\infty), \tag{53}$$

$$Q_+\psi(0)=\varphi_+, \tag{54}$$

$$\|\psi(x)\|_T=O(1) \qquad \text{or} \qquad o(1) \quad (x\to\infty), \tag{55}$$

and the existence and uniqueness properties of the latter problem can be described by Theorem 2.1. We have obtained the problem (53)–(55) of the

same type and with the same solutions as (8)–(10), but now the resolvents of $T^{-1}A$ and $T_1^{-1} = T^{-1}\hat{A}$ have a rank N difference. As we shall see, it is exactly this rank condition which guarantees a further reduction to a matrix integral equaton and factorization.

IV. Second Reduction: Wiener-Hopf Equation and Factorization

In this section we shall reduce the (modified) boundary value problem (53)–(55) further. As a result, we shall obtain a Wiener–Hopf integral equation which can be solved by factorization. Throughout we assume A strictly positive self-adjoint. (At the end of this section we shall indicate how to relax these assumptions.) We shall need the following technical condition:

$$\exists 0 < \alpha < 1: \qquad \operatorname{Ran} C \subset |T_1|^{\alpha}[H_T] \cap |A^{-1}T|^{\alpha}[H_T], \tag{56}$$

where $|T_1| = A^{-1}|T|$ and $|A^{-1}T|$ are the absolute values with respect to the inner products (14) and (15). In the Appendix we show that condition (56), respectively, the equivalent conditions (80) to be defined below, are valid for a large class of operators A with one sign change in the weight. We believe that the extension to more than one sign change can be done by the same methods. We shall make use of the estimates

$$\begin{aligned} \| \, |T_1|^{\alpha} \mathscr{H}_1(x) \|_{H_T} &= O(|x|^{\alpha-1}), \\ \| \, |A^{-1}T|^{\alpha-1} e^{-xT^{-1}A} P_+ \|_{H_T} &= O(|x|^{\alpha-1}) \end{aligned} \tag{57}$$

as $x \to 0$. Here $\mathscr{H}_1(x)$ is the *propagator function* defined by (22).

THEOREM 4.1. *Let* $\varphi_+ \in Q_+[H_T]$. *Then the vector function* $\varphi(x) = T^{-1}Ae^{-xT^{-1}A}E\varphi_+$ *is the unique solution of the integral equation*

$$\varphi(x) + \frac{d}{dx}\int_0^{\infty} \mathscr{H}_1(x-y)\, C\varphi(y)\, dy = \mathscr{H}_1(x)\, \varphi_+, \qquad 0 < x < \infty, \tag{58}$$

satisfying $\int_{\varepsilon}^{\infty} e^{rx} \, \|\varphi(x)\|_T \, dx < \infty$ *for some* $r > 0$, *where* $C = A^{-1}T - \hat{A}^{-1}T$.

Proof. Using (56) and (57) one easily proves that for every $\varepsilon \leqslant x < \infty$,

$$\int_{\varepsilon}^{\infty} \| \mathscr{H}_1(x-y)\, C\varphi(y) \|_T \, dy \leqslant M(\varepsilon) < \infty, \tag{59}$$

so that $\int_0^\infty \mathscr{H}_1(x-y)\, C\varphi(y)\, dy$ is an absolutely convergent Bochner integral for $0<x<\infty$. We compute

$$\begin{aligned} T_1 \int_0^\infty & \mathscr{H}_1(x-y)\, C\varphi(y)\, dy \\ &= \int_0^x \frac{d}{dy} \{e^{-(x-y)T_1^{-1}} Q_+ e^{-yT^{-1}A} E\varphi_+\}\, dy \\ &\quad - \int_x^\infty \frac{d}{dy} \{e^{-(x-y)T_1^{-1}} Q_- e^{-yT^{-1}A} E\varphi_+\}\, dy \\ &= \psi(x) - e^{-xT_1^{-1}} \varphi_+, \qquad 0<x<\infty, \end{aligned}$$

where we used the identity $Q_+ E Q_+ = Q_+$. By differentiation and using $\psi'(x) = -\varphi(x)$, we get (58).

Conversely, putting $\psi(x) = \int_x^\infty \varphi(y)\, dy$ (well defined since $\int_1^\infty e^{rx} \|\varphi(x)\|_T\, dx < \infty$ for some $r>0$) we may integrate (58) and obtain

$$\psi(x) - \int_0^\infty \mathscr{H}_1(x-y)\, C\varphi(y)\, dy = e^{-xT_1^{-1}} \varphi_+, \qquad 0<x<\infty, \tag{60}$$

where the integral term is strongly differentiable for $x \in (0, \infty)$. Let us write

$$\begin{aligned} T_1^2 \psi(x) - \int_0^x T_1 e^{-(x-y)T_1^{-1}} Q_+ C\varphi(y)\, dy + \int_x^\infty T_1 e^{-(x-y)T_1^{-1}} Q_- C\varphi(y)\, dy \\ = T_1^2 e^{-xT_1^{-1}} \varphi_+. \end{aligned}$$

Because of the estimate (59) and by dominated convergence (for Bochner integrals; cf. [34, Sect. 31, p. 30]) we may differentiate this equation in the following manner:

$$\begin{aligned} T_1^2 \psi'(x) &+ \int_0^x e^{-(x-y)T_1^{-1}} Q_+ C\varphi(y)\, dy - T_1 Q_+ C\varphi(x) \\ &- \int_x^\infty e^{-(x-y)T_1^{-1}} Q_- C\varphi(y)\, dy - T_1 Q_- C\varphi(x) \\ &= -T_1 e^{-xT_1^{-1}} \varphi_+, \end{aligned}$$

whence

$$T_1 \psi'(x) - C\varphi(x) + \int_0^\infty \mathscr{H}_1(x-y)\, C\varphi(y)\, dy = -e^{-xT_1^{-1}} \varphi_+, \qquad 0<x<\infty. \tag{61}$$

On adding (60) and (61) we obtain

$$T_1\psi'(x) - C\varphi(x) + \psi(x) = 0, \qquad 0 < x < \infty.$$

By virtue of $\psi'(x) = -\varphi(x)$ and $T_1 + C = A^{-1}T$, this in turn implies (53). Using (60) we conclude

$$Q_+\psi(x) = e^{-xT_1^{-1}}\varphi_+ + \int_0^x \mathscr{H}_1(x-y)\, C\varphi(y)\, dy,$$

whence (54). ∎

Let $C^\dagger$ denote the H_T-adjoint of C, i.e.,

$$C^\dagger = (Q_+ - Q_-)\, C(Q_+ - Q_-). \tag{62}$$

Let j denote the natural imbedding of Ran $C^\dagger$ into H_T, and π the H_T-orthogonal projection of H_T onto Ran $C^\dagger$ (as an operator π: $H_T \to$ Ran $C^\dagger$). Then j and π are adjoints and

$$Cj\pi = C. \tag{63}$$

It is then clear that $\zeta(x) = \pi\varphi(x)$ satisfies the finite-dimensional convolution equation

$$\zeta(x) + \frac{d}{dx}\int_0^\infty \pi\mathscr{H}_1(x-y)\, Cj\zeta(y)\, dy = \pi\mathscr{H}_1(x)\, \varphi_+, \qquad 0 < x < \infty. \tag{64}$$

The function $\varphi(x)$ may be computed from $\zeta(x)$ by putting

$$\varphi(x) = \mathscr{H}_1(x)\, \varphi_+ - \frac{d}{dx}\int_0^\infty \mathscr{H}_1(x-y)\, Cj\zeta(y)\, dy. \tag{65}$$

Clearly, $\int_\varepsilon^\infty e^{rx}\, \|\zeta(x)\|\, dx < \infty$ for some $r > 0$. Conversely, if $\zeta(x)$ is a solution of Eq. (64) satisfying $\int_\varepsilon^\infty e^{rx}\, \|\zeta(x)\|\, dx < \infty$ for some $r > 0$ and $\varphi(x)$ is given by (65), then we may multiply (65) by π from the left and subtract the resulting equation from (64). As a result, we get $\zeta(x) = \pi\varphi(x)$. Substituting the latter into (65) and utilizing (63) we find (58). Hence, Eq. (58) is equivalent to the pair of equations (64) and (65). The latter pair of equations, however, is defined on the finite-dimensional space Ran $C^\dagger$.

Let us transform (64) into a Riemann–Hilbert problem, as arising from the application of the classical Wiener–Hopf method. Using (59) it is not difficult to establish that $\int_0^\infty \pi\mathscr{H}_1(x-y)\, Cj\zeta(y)\, dy$ is (strongly) differentiable for $x \in (-\infty, 0)$. Let us put

$$\zeta(x) = -\frac{d}{dx}\int_0^\infty \pi\mathscr{H}_1(x-y)\, Cj\zeta(y)\, dy, \qquad x \in (-\infty, 0). \tag{66}$$

Since $\zeta(x) = \pi T^{-1} A e^{-xT^{-1}} E\varphi_+$ (see Theorem 4.1 in combination with the equivalence of Eq. (58) and Eqs. (64), (65)) and (56) holds true, we have $\int_0^\infty e^{rx} \|\zeta(x)\| \, dx < \infty$ and therefore

$$\int_{-\infty}^{\infty} e^{-s|x|} \left\| \int_0^\infty \pi \mathscr{H}_1(x-y) \, Cj\zeta(y) \, dy \right\| dx < \infty$$

for some $s > 0$. Let us define the Laplace transforms

$$\hat{\zeta}_\pm(\lambda) = \pm \int_0^{\pm\infty} e^{\lambda x} \zeta(x) \, dx,$$

$$\hat{K}(\lambda) = \int_{-\infty}^{\infty} e^{\lambda x} \pi \mathscr{H}_1(x) \, Cj \, dx, \qquad \hat{\omega}(\lambda) = \int_0^\infty e^{\lambda x} \pi \mathscr{H}_1(x) \, \varphi_+ \, dx,$$

where $\operatorname{Re} \lambda = 0$. Using the formula

$$\int_{-\infty}^{\infty} e^{\lambda x} f'(x) \, dx = [e^{\lambda x} f(x)]_{x=-\infty}^{0-} + [e^{\lambda x} f(x)]_{x=0}^{\infty} - \lambda \int_{-\infty}^{\infty} e^{\lambda x} f(x) \, dx,$$

we obtain the Riemann–Hilbert problem

$$[I - \lambda \hat{K}(\lambda)] \, \hat{\zeta}_+(\lambda) + \hat{\zeta}_-(\lambda) = \hat{\omega}(\lambda), \qquad \operatorname{Re} \lambda = 0. \tag{67}$$

The right-hand side is easily computed and reads

$$\hat{\omega}(\lambda) = \pi(I - \lambda T_1)^{-1} \, T_1 \varphi_+, \qquad \operatorname{Re} \lambda = 0. \tag{68}$$

Equation (67) has to be solved by factorizing the so-called *dispersion function*

$$\Lambda(\lambda) = I - \lambda \hat{K}(\lambda) = I - \lambda \pi (I - \lambda T_1)^{-1} \, Cj, \qquad \operatorname{Re} \lambda = 0,$$

into matrix functions which are analytic and invertible on appropriate half-planes.

Finally, let us consider the case when A is positive self-adjoint with an isolated (simple) eigenvalue zero. Recalling the definitions of P_0 and S_1 in Section 2, we define

$$C = (S_1 - \hat{A}^{-1} T)(I - P_0),$$

which is an operator of rank at most N. We rephrase the regularity assumption (56) as

$$\exists 0 < \alpha < 1: \quad \operatorname{Ran} C \subset |T_1|^\alpha [H_T], \qquad \operatorname{Ran} (I - P_0) \, C \subset |S_1|^\alpha [H_T].$$

By construction, and for E_+ defined in Section 2, the function

$$\varphi(x) = T^{-1} A e^{-xT^{-1}A} E_+ \varphi_+, \qquad 0 < x < \infty,$$

has its values in $(I - P_0)[H_T] \oplus \operatorname{Ker} A$. We may then repeat the proof of Theorem 4.1 and the derivation of (64) and (65) with the sole adaptation that

$$\psi(x) - \psi(\infty) = \int_x^\infty \varphi(y)\, dy$$

(and similarly for $\zeta(x)$). We then obtain the Riemann–Hilbert problem (67) as a result. Unfortunately, as will become clear shortly, the existence proof of the factorization of $\Lambda(\lambda)$ will break down if A has a zero eigenvalue.

V. Wiener–Hopf Factorization and Albedo Operator

Let us again assume that A is strictly positive. It is then easily seen that

$$\Lambda(1/\xi) = I - \pi(\xi - T_1)^{-1}\, Cj \tag{69}$$

has the form of a transfer function of linear systems theory. For such functions a factorization principle has been developed by Bart, Gohberg, Kaashoek and Van Dooren [4; 3, Chap. 1]. In the terminology of [3] we have Ran $C^\dagger$ as the input–output space, H_T as the state space, the identity as the external operator, T_1 as the main operator and $T_1 + Cj\pi = A^{-1}T$ as the associate operator. Let us consider EQ_+ where E is the albedo operator. Then $E = V^{-1}$ with V defined by (12) and satisfying $Q_\pm V = VP_\pm$ implies that EQ_+ maps H_T onto $P_+[H_T]$. The boundary condition $Q_+\psi(0) = \varphi_+$ with $\psi(0) = E\varphi_+$ implies $Q_+ EQ_+ = Q_+$ and therefore $(EQ_+)^2 = EQ_+$, $Q_+(I - EQ_+) = 0$ and $Q_- = (I - EQ_+)\, V(P_- - P_+)$; hence $(I - EQ_+)[H_T] = Q_-[H_T]$. It is then clear that EQ_+ is a bounded projection on H_T with range $P_+[H_T]$ (which is invariant under the associate operator $A^{-1}T$) and kernel $Q_-[H_T]$ (which is invariant under the main operator T_1), whence, in the terminology of [3], EQ_+ is a supporting projection. As a result we immediately obtain the factorization formula

$$\Lambda(1/\xi)^{-1} = H_l(-1/\xi)\, H_r(1/\xi), \tag{70}$$

where

$$H_r(1/\xi)^{-1} = I - \pi(\xi - T_1)^{-1}\, (I - EQ_+)\, Cj, \tag{71}$$

$$H_l(-1/\xi)^{-1} = I - \pi EQ_+(\xi - T_1)^{-1}\, Cj, \tag{72}$$

$$H_l(-1/\xi) = I + \pi(\xi - A^{-1}T)^{-1} EQ_+ Cj, \tag{73}$$

$$H_r(1/\xi) = I + \pi(I - EQ_+)(\xi - A^{-1}T)^{-1} Cj, \tag{74}$$

$$\Lambda(1/\xi)^{-1} = I + \pi(\xi - A^{-1}T)^{-1} Cj. \tag{75}$$

The factors obtained have the following properties:

(i) $H_l(z)$, $H_l(z)^{-1}$, $H_r(z)$, and $H_r(z)^{-1}$ are continuous in the closed right half-plane (except possibly at infinity) and analytic in the open right half-plane.

(ii) $H_l(0^+)$ and $H_r(0^+)$ are the identity operators where the limits to $z = 0$ must be taken from the closed right half-plane.

(iii) At infinity we have the estimates

$$\| H_l(z) \| = o(z) \quad \text{and} \quad \| H_l(z)^{-1} \| = o(z) \quad (z \to \infty, \operatorname{Re} z \geqslant 0),$$

$$\| H_r(z) \| = o(z) \quad \text{and} \quad \| H_r(z)^{-1} \| = o(z) \quad (z \to \infty, \operatorname{Re} z \geqslant 0).$$

Property (iii) follows from the following fact: If S is a strictly positive self-adjoint operator on a Hilbert space H, then for every $h \in H$ we have

$$\lim_{\xi \to 0,\ \operatorname{Re} \xi \leqslant 0} \| \xi(\xi - S)^{-1} h \| = 0.$$

The dispersion function itself has the property

$$\| \Lambda(z) \| = o(z) \quad \text{and} \quad \| \Lambda(z)^{-1} \| = o(z) \quad \left(z \to \infty, \left| \frac{\pi}{2} - \arg z \right| \leqslant \omega \right)$$

for all $\omega \in (0, \frac{1}{2}\pi)$. This property is based on the fact that if S is an injective self-adjoint operator on a Hilbert space H, then for every $h \in H$ one has

$$\lim_{\xi \to 0,\ |1/2\pi - \arg \xi| \leqslant \omega} \| \xi(\xi - S)^{-1} h \| = 0, \qquad \omega \in (0, \tfrac{1}{2}\pi).$$

LEMMA 5.1. *If the dispersion function has two factorizations of the type* (70) *with the factors satisfying properties* (i)–(iii), *then the factorizations are connected by the formulas*

$$H_l^{(2)}(z) = H_l^{(1)}(z)(I + zA), \qquad H_r^{(2)}(z) = (I + zA)\, H_r^{(1)}(z),$$

where $A^2 = 0$ *and for* $i = 1, 2$ *the expressions* $\| H_l^{(i)}(z)\, A \|$, $\| H_l^{(i)}(z)^{-1} A \|$, $\| H_r^{(i)}(z)\, A \|$ *and* $\| H_r^{(i)}(z)^{-1} A \|$ *converge to zero as* $z \to \infty$ *from the closed right half-plane.*

Proof. We easily obtain

$$F(z) = H_l^{(2)}(-z)^{-1}\, H_l^{(1)}(-z) = H_r^{(2)}(z)\, H_r^{(1)}(z)^{-1}, \qquad \operatorname{Re} z = 0.$$

Using Liouville's theorem it appears that $F(z)$ is an entire function satisfying

$$\|F(z)\| = o(z^2), \qquad \|F(z)^{-1}\| = o(z^2) \quad (z \to \infty, \operatorname{Re} z \leqslant 0),$$
$$\|F(z)\| = o(z^2), \qquad \|F(z)^{-1}\| = o(z^2) \quad (z \to \infty, \operatorname{Re} z \geqslant 0),$$

whence $F(z) = I + zA$ and $F(z)^{-1} = I + zB$. We easily find $B = -A$ and $A^2 = 0$. ∎

As we shall see shortly, the fact that in the case when the weight $w(\mu)$ has more than one sign change the factorization (70) with factors having properties (i), (ii), and (iii) may be non-unique, does not affect the formula for the albedo operator E. The non-uniqueness may affect a computational algorithm to obtain the factors.

Using a factorization of the type (70) we easily reduce the algebraic equation (67) to the Riemann–Hilbert problem

$$H_l(-\lambda)^{-1}\,\hat{\zeta}_+(\lambda) + H_r(\lambda)\,\hat{\zeta}_-(\lambda) = H_r(\lambda)\,\hat{\omega}(\lambda), \qquad \operatorname{Re}\lambda = 0. \tag{76}$$

LEMMA 5.2. *The Riemann–Hilbert problem* (76) *has precisely one solution of the following type*:

(i) $\hat{\zeta}_\pm(\lambda)$ *is analytic in the open left–right half-plane and continuous on the closed left–right half-plane*;

(ii) $\lim_{\lambda\to\infty} \lambda\hat{\zeta}_+(\lambda)$ *exists, as* λ *approaches infinity from the closed left half-plane*;

(iii) $\lim_{\lambda\to\infty} \lambda^c\hat{\zeta}_-(\lambda)$ *exists for all* $0 < c < 1$, *as* λ *approaches infinity from the closed right half-plane.*

This solution leads to the unique solution of the boundary value problem (53)–(55).

Proof. Put $h_\pm = \lim_{\lambda\to\infty} \lambda\hat{\zeta}_\pm(\lambda)$ with the limit computed as in the formulation of the lemma. Then the conditions on H_l and H_r imply

$$\lim_{\lambda\to\infty,\,\operatorname{Re}\lambda\leqslant 0} H_l(-\lambda)^{-1}\,\hat{\zeta}_+(\lambda) = 0, \qquad \lim_{\lambda\to\infty,\,\operatorname{Re}\lambda\geqslant 0} H_r(\lambda)\,\hat{\zeta}_-(\lambda) = 0.$$

From (68) we also have

$$\lim_{\lambda\to\infty,\,\operatorname{Re}\lambda=0} \lambda\hat{\omega}(\lambda) = -\varphi_+, \qquad \lim_{\lambda\to\infty,\,\operatorname{Re}\lambda=0} H_r(\lambda)\,\hat{\omega}(\lambda) = 0.$$

Given a Hölder continuous function $\hat{h}(\lambda)$ on the extended imaginary line satisfying $\hat{h}(\pm i\infty) = 0$, we can find unique functions $\hat{h}_\pm(\lambda)$ that are analytic

in the open left–right half-plane, continuous in the closed left–right half-plane and satisfying $\hat{h}_{\pm}(i\infty)=0$ (when approached from the appropriate half-plane), such that

$$\hat{h}(\lambda)=\hat{h}_{+}(\lambda)+\hat{h}_{-}(\lambda), \qquad \operatorname{Re}\lambda=0$$

(cf. [28]). We therefore obtain

$$\hat{\zeta}_{+}(\lambda)=H_{\mathrm{l}}(-\lambda)(H_{\mathrm{r}}\hat{\omega})_{+}(\lambda), \qquad \hat{\zeta}_{-}(\lambda)=H_{\mathrm{r}}(\lambda)^{-1}(H_{\mathrm{r}}\hat{\omega})_{-}(\lambda).$$

As a consequence of Lemma 5.1, these formulas do not depend on the particular choice of H_{l} and H_{r}.

Finally, the relevant solution of the problem (76) (for which we are now going to prove (i), (ii), and (iii)) is given by

$$\hat{\zeta}_{+}(\lambda)=\int_{0}^{\infty} e^{\lambda x}\pi e^{-xT^{-1}A}E\varphi_{+}\,dx=\pi(I-\lambda A^{-1}T)^{-1}A^{-1}TE\varphi_{+}, \quad \operatorname{Re}\lambda=0,$$

and

$$\hat{\zeta}_{-}(\lambda)=\int_{-\infty}^{0} e^{\lambda x}\zeta(x)\,dx=-\pi(I-\lambda T_{\mathrm{l}})^{-1}Q_{-}E\varphi_{+}, \qquad \operatorname{Re}\lambda=0.$$

In the latter formula we have employed the expression

$$\zeta(x)=\frac{-d}{dx}\int_{0}^{\infty}\pi\mathscr{H}_{\mathrm{l}}(x-y)\,C\varphi(y)\,dy=\frac{d}{dx}\left[\pi e^{-(x-y)T_{\mathrm{l}}^{-1}}Q_{-}e^{-yT^{-1}A}E\varphi_{+}\right]_{y=0}^{\infty}$$

$$=-\frac{d}{dx}\pi e^{-xT_{\mathrm{l}}^{-1}}Q_{-}E\varphi_{+}=-\pi\mathscr{H}_{\mathrm{l}}(x)\,E\varphi_{+}, \qquad -\infty<x<0.$$

From these expressions the properties (i), (ii), and (iii) are clear, because $\|H_{\mathrm{r}}(z)\|=0(z^{1-\alpha})(z\to\infty,\ \operatorname{Re}z\geqslant 0)$. ∎

THEOREM 5.3. *The albedo operator is given by*

$$E\varphi_{+}=\varphi_{+}+\int_{\substack{-\infty\\(\mu)}}^{0}\int_{\substack{0\\(\nu)}}^{\infty}\frac{\mu\nu}{\mu-\nu}\,\sigma_{1}(d\mu)\,CjH_{\mathrm{l}}(-\mu)\,H_{\mathrm{r}}(\nu)\,\pi\sigma_{1}(d\nu)\,\varphi_{+}, \tag{77}$$

where $\varphi_{+}\in Q_{+}[H_{T}]$ and $\sigma_{1}(\cdot)$ is the resolution of the identity of $T^{-1}\hat{A}$.

Proof. Choose $\varphi_{+}\in Q_{+}[H_{T}]$ and some factorization of the type (70). Then

$$
\begin{aligned}
E\varphi_+ = \psi(0) &= \int_0^\infty \varphi(x)\,dx \\
&= \int_0^\infty \left\{ \mathscr{H}_1(x)\,\varphi_+ - \frac{d}{dx}\int_0^\infty \mathscr{H}_1(x-y)\,Cj\zeta(y)\,dy \right\} dx \\
&= \varphi_+ + \int_0^\infty \mathscr{H}_1(-y)\,Cj\zeta(y)\,dy \\
&= \varphi_+ + \int_0^\infty \int_{\substack{-\infty \\ (\mu)}}^0 (-\mu)\,e^{\mu y}\sigma_1(d\mu)\,Cj\zeta(y)\,dy \\
&= \varphi_+ + \int_{-\infty}^0 (-\mu)\,\sigma_1(d\mu)\,Cj\hat{\zeta}_+(\mu). \qquad (78)
\end{aligned}
$$

The right-hand side of (76), with $\lambda = \mu$, can be written as

$$
\begin{aligned}
H_r(\mu)\,\hat{\omega}(\mu) &= H_r(\mu)\int_0^\infty e^{\mu x}\int_{\substack{0 \\ (\nu)}}^\infty \nu e^{-\nu x}\pi\sigma_1(d\nu)\,\varphi_+ \\
&= H_r(\mu)\int_{\substack{0 \\ (\nu)}}^\infty \frac{\nu}{\nu-\mu}\,\pi\sigma_1(d\nu)\,\varphi_+ \\
&= \int_{\substack{0 \\ (\nu)}}^\infty \nu\left\{\frac{H_r(\nu)}{\nu-\mu} - \frac{H_r(\nu)-H_r(\mu)}{\nu-\mu}\right\}\pi\sigma_1(d\nu)\,\varphi_+,
\end{aligned}
$$

whence

$$
H_1(-\mu)^{-1}\hat{\zeta}_+(\mu) = \int_{\substack{0 \\ (\nu)}}^\infty \nu\,\frac{H_r(\nu)}{\nu-\mu}\,\pi\sigma_1(d\nu)\,\varphi_+. \qquad (79)
$$

The identity (77) now immediately follows from (78) and (79). ∎

We remark that the factorization formulas (71)–(74) are not valid if A has a zero eigenvalue. Therefore, it is not clear whether for this case the albedo oprator can be written in the form (77).

VI. Albedo Operators for Models with One Sign Change

In addition to previous hypotheses, we shall assume that (i) the weight $w(\mu)$ has one sign change only (say at $c \in (a, b)$, where $\omega(\mu) < 0$ for

$\mu \in (a, c)$ and $w(\mu) > 0$ for $\mu \in (c, b)$), and (ii) $T^{-1}A$ has discrete spectrum. If we now add a boundary condition at c as to make the resulting direct sum $\hat{A}$ of Sturm–Liouville operators strictly positive, the rank one difference of the resolvents of $T^{-1}A$ and $T^{-1}\hat{A}$ implies that $T^{-1}\hat{A}$ has discrete spectrum. Let $(\lambda_n)_{0 \neq n \in \mathbb{Z}}$ be the nonzero (simple) eigenvalues of $T^{-1}A$ with corresponding real eigenfunctions $(\varphi_n)_{0 \neq n \in \mathbb{Z}}$ satisfying

$$\cdots < \lambda_{-2} < \lambda_{-1} < 0 < \lambda_1 < \lambda_2 < \cdots \qquad (A\varphi_n, \varphi_m) = \delta_{nm}.$$

Let $(\zeta_n)_{0 \neq n \in \mathbb{Z}}$ be the (nonzero, simple) eigenvalues of $T^{-1}\hat{A}$ with corresponding real eigenfuctions $(\psi_n)_{0 \neq n \in \mathbb{Z}}$ satisfying

$$\cdots < \zeta_{-2} < \zeta_{-1} < 0 < \zeta_1 < \zeta_2 < \cdots, \qquad (\hat{A}\psi_n, \psi_m) = \delta_{nm}.$$

Then $\psi_n(\mu) = 0$ if $n(\mu - c) < 0$. It is readily seen that

$$(\psi_n, \psi_m)_T = (|\hat{A}^{-1}T|\psi_n, \psi_m)_{\hat{A}} = |\zeta_n|^{-1}\delta_{nm},$$

and therefore $(|\zeta_n|^{1/2}\psi_n)_{0 \neq n \in \mathbb{Z}}$ is an orthonormal basis in $H_T = L_2(I; |w(\mu)|\, d\mu)$. In the same way we see that $(|\lambda_n|^{1/2}\varphi_n)_{0 \neq n \in \mathbb{Z}}$ is an orthonormal basis of $(I - P_0)[H_S]$.

Let us define $\kappa(\mu) = \kappa(\mu, 0)$ by (48), and let us compute its expansion coefficients

$$\kappa_n = (\kappa, |\zeta_n|^{1/2}\psi_n)_T, \qquad \tilde{\kappa}_n = (\kappa, |\lambda_n|^{1/2}\varphi_n)_S,$$

where $0 \neq n \in \mathbb{Z}$. Then $\kappa \in |T_1|^{\alpha}[H_T]$ and $(I - P_0)\kappa \in |S_1|^{\alpha}[H_S]$ if and only if

$$\sum_{0 \neq n \in \mathbb{Z}} |\zeta_n|^{2\alpha}|\kappa_n|^2 < \infty, \qquad \sum_{0 \neq n \in \mathbb{Z}} |\lambda_n|^{2\alpha}|\tilde{\kappa}_n|^2 < \infty. \tag{80}$$

We shall assume the existence of $0 < \alpha < 1$ satisfying (8).

THEOREM 6.1. *Let A be strictly positive self-adjoint. Then the albedo operator E is given by the formula*

$$(E\varphi_+)(\mu) = \begin{cases} \varphi_+(\mu), \ \mu \in (c, b) \\ (\operatorname{sgn} K) \displaystyle\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{\zeta_{-m}\zeta_n}{\zeta_{-m} - \zeta_n}\kappa_{-m}\kappa_n g_m^{\mathrm{l}} g_n^{\mathrm{r}} \varphi_{+n} |\zeta_{-m}|^{1/2}\psi_{-m}(\mu), \end{cases}$$

$$\mu \in (a, c).$$

Here $\varphi_{+n} = (\varphi_+, |\zeta_n|^{1/2}\psi_n)_T$. *Moreover,* $g_n^{\mathrm{l}} = \tilde{H}_{\mathrm{l}}(-\zeta_{-n})$ *and* $g_n^{\mathrm{r}} = \tilde{H}_{\mathrm{r}}(\zeta_n)$, *where* $\tilde{H}_{\mathrm{l}}(z)$ *and* $\tilde{H}_{\mathrm{r}}(z)$ *are the unique functions appearing in the factorization*

$$\tilde{H}_{\mathrm{l}}(-z)\,\tilde{H}_{\mathrm{r}}(z) = (\operatorname{sgn} K)\left[\frac{1}{K} + z\sum_{n=1}^{\infty}\left\{\frac{\zeta_n}{\zeta_n - z}|\kappa_n|^2 - \frac{\zeta_{-n}}{\zeta_{-n} - z}|\kappa_{-n}|^2\right\}\right]^{-1}, \tag{81}$$

where $K = (K_0(0)\,K_{\mathrm{r}}(0))/K(0)$, *and having the following properties*:

(i) $\tilde{H}_{\mathrm{l}}(z)$, $\tilde{H}_{\mathrm{l}}(z)^{-1}$, $\tilde{H}_{\mathrm{r}}(z)$, *and* $\tilde{H}_{\mathrm{r}}(z)^{-1}$ *are continuous in the closed right half-plane (except at infinity) and analytic in the open right half-plane.*

(ii) $\tilde{H}_{\mathrm{l}}(0^+) = \tilde{H}_{\mathrm{r}}(0^+) = 1/\sqrt{|K|}$.

(iii) *At infinity all four functions* $\tilde{H}_{\mathrm{l}}(z)$, $\tilde{H}_{\mathrm{l}}(z)^{-1}$, $\tilde{H}_{\mathrm{r}}(z)$, *and* $\tilde{H}_{\mathrm{r}}(z)^{-1}$ *are* $o(z)$ *as* $z \to \infty$ *from the closed right half-plane.*

Proof. The uniqueness of the factorization (81) is a direct consequence of Lemma 5.1 if there is only one sign change. We then easily compute

$$\begin{aligned}\Lambda(z)\,\tilde{\kappa} &= \tilde{\kappa} - \frac{z}{\|\tilde{\kappa}\|_T^2}\frac{K_{\mathrm{l}}(0)\,K_{\mathrm{r}}(0)}{-K(0)}\sum_{n\neq 0}\left(\frac{\zeta_n}{\zeta_n - z}\kappa_n\,\|\kappa\|_T^2\,|\zeta_n|^{1/2}\psi_n, \tilde{\kappa}\right)_T \\ &= \left[1 + z\frac{K_{\mathrm{l}}(0)\,K_{\mathrm{r}}(0)}{K(0)}\sum_{n=1}^{\infty}\left\{\frac{\zeta_n}{\zeta_n - z}|\kappa_n|^2 - \frac{\zeta_{-n}}{\zeta_{-n} - z}|\kappa_{-n}|^2\right\}\right],\end{aligned}$$

and we may therefore put $\tilde{H}_{\mathrm{l}} = |K|^{1/2}\,\mathrm{H}_{\mathrm{l}}$ and $\tilde{H}_{\mathrm{r}} = |K|^{1/2}\,H_{\mathrm{r}}$. We now easily obtain

$$\begin{aligned}Q_- E\varphi_+ = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\zeta_{-m}\zeta_n}{\zeta_{-m} - \zeta_n}&\left(\frac{-K_{\mathrm{l}}(0)\,K_{\mathrm{r}}(0)}{K(0)}\tilde{\kappa}, |\zeta_{-m}|^{1/2}\psi_{-m}\right)_T \\ &\cdot |K|^{-1}\,g_m^{\mathrm{l}}\,g_n^{\mathrm{r}}\,\varphi_{+n}\,|\zeta_{-m}|^{1/2}\psi_{-m},\end{aligned}$$

which reduces to the stated expression for E. ∎

Let us consider the special case where $\varphi_+(\mu) = |\zeta_l|^{1/2}\psi_{\mathrm{l}}(\mu)$ for some $\mathrm{l} \in \mathbb{N}$. Then $\varphi_{+n} = \delta_{nl}$, as a result of the orthonormality of the functions $\{|\zeta_n|^{1/2}\psi_n\}_{0\neq n\in\mathbb{Z}}$ in H_T. We then obtain for $\mu \in (a, c)$,

$$(E\varphi_+)(\mu) = (\operatorname{sgn} K)\sum_{m=1}^{\infty}\frac{\zeta_{-m}\zeta_l}{\zeta_{-m} - \zeta_l}\kappa_{-m}\kappa_l\,g_m^{\mathrm{l}}\,g_l^{\mathrm{r}}\,|\zeta_{-m}|^{1/2}\psi_{-m}(\mu).$$

Clearly, we must have

$$C_l^2 = \sum_{m=1}^{\infty}\left|\frac{\zeta_{-m}}{\zeta_{-m} - \zeta_l}\kappa_{-m}\,g_m^{\mathrm{l}}\right|^2 = \frac{-1}{|\zeta_l\kappa_l\,g_l^{\mathrm{r}}|^2}\int_a^c w(\mu)|(E\varphi_+)(\mu)|^2\,d\mu < \infty,$$

and therefore

$$\zeta_l \kappa_l g_l^{\mathrm{r}} C_l = O(1) \qquad (l \to +\infty).$$

Next, let us consider the special case (arising from most applications) where $I = (-N, N)$ is a (finite or infinite) interval symmetric about the origin and the weight and Sturm–Liouville operator satisfy the conditions

(i) $w(-\mu) = w(\mu)$,

(ii) $(Af)(\mu) = (A \operatorname{sgn}(\mu) f)(-\mu)$.

The eigenvalues and eigenfunctions will then satisfy $\zeta_{-n} = \zeta_n$ and $\psi_{-n}(\mu) = \psi_n(-\mu)$, whence $\kappa(-\mu) = -\kappa(\mu)$ and $g_n^{\mathrm{l}} = g_n^{\mathrm{r}} = g_n$. We obtain the simplified expressions

$$\frac{1}{K} A(z) = \frac{1}{K} + 2z^2 \sum_{n=1}^{\infty} \frac{\zeta_n}{\zeta_n^2 - z^2} \kappa_n^2 \tag{82}$$

and

$$(E\varphi_+)(\mu) = (\operatorname{sgn} K) \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\zeta_m \zeta_n}{\zeta_m + \zeta_n} \times \kappa_m \kappa_n g_m g_n \varphi_{+n}(\zeta_m)^{1/2} \psi_m(-\mu), \qquad \mu \in (-N, 0). \tag{83}$$

Here we have the summability condition (where $g_n > 0$)

$$\sup_{n \in \mathbb{N}} \zeta_n |\kappa_n| g_n \left[\sum_{m=1}^{\infty} \left(\frac{\zeta_m}{\zeta_m + \zeta_n} \kappa_m g_m \right)^2 \right]^{1/2} < \infty. \tag{84}$$

EXAMPLE 6.2. For $\alpha \in [0, \frac{1}{2}\pi]$ we consider the operator $Ah = -h''$ on $I = (-1, 1)$ with boundary conditions

$$\cos \alpha h(\pm 1) \pm \sin \alpha h'(\pm 1) = 0$$

and indefinite weight $w(\mu) = \operatorname{sgn} \mu$. Then A is strictly positive for $\alpha \in [0, \frac{1}{2}\pi)$ and positive with simple zero eigenvalue for $\alpha = \frac{1}{2}\pi$. To construct $\hat{A}$ we impose the additional boundary condition $h(0) = 0$; we obtain a strictly positive $\hat{A}$. As special solutions of the equation $-h'' = 0$ we have

$$\varphi(\mu, 0) = \mu + 1 + \tan \alpha, \quad \mu \in I_- = (-1, 0), \quad \cos \alpha \varphi(-1, 0) = \sin \alpha \varphi'(-1, 0),$$

$$\psi(\mu, 0) = \mu - 1 - \tan \alpha, \quad \mu \in I_+ = (0, 1), \quad \cos \alpha \psi(1, 0) = -\sin \alpha \psi'(1, 0),$$

$$\chi(\mu, 0) = \mu, \quad \mu \in I = (-1, 1), \quad \chi(0, 0) = 0.$$

As $p(\mu) \equiv 1$, we find

$$K(0) = -2(1+\tan\alpha), \quad L_l(0) = K_r(0) = -(1+\tan\alpha), \quad K = -\tfrac{1}{2}(1+\tan\alpha),$$

whence (cf. (48))

$$\kappa(\mu) = (\operatorname{sgn}\mu)\{1-(1+\tan\alpha)^{-1}\,|\mu|\}.$$

For the $H_{\hat{A}}$-normalized eigenfunctions ψ_n, $n \geqslant 1$, we obtain

$$\psi_n(\mu) = \left[\frac{2}{\zeta_n}\left(1-(\sin 2\sqrt{\zeta_n}/2\sqrt{\zeta_n})\right)^{-1}\right]^{1/2} \sin(\mu\sqrt{\zeta_n}), \qquad \mu > 0, \quad (85)$$

where $(\tan\sqrt{\zeta_n})/\sqrt{\zeta_n} = -\tan\alpha$ and $(n-\frac{1}{2})\,\pi \leqslant \sqrt{\zeta_n} \leqslant n\pi$. For $n \geqslant 1$ it then follows that

$$\kappa_n = \left[\frac{2}{\zeta_n}(1-(\sin 2\sqrt{\zeta_n}/2\sqrt{\zeta_n}))^{-1}\right]^{1/2}.$$

As conditions (80) are satisfied (cf. Appendix), we may immediately write down the expression

$$\Lambda(z) = 1 - 2z^2(1+\tan\alpha) \sum_{n=1}^{\infty} (1-(\sin 2\sqrt{\zeta_n}/2\sqrt{\zeta_n}))^{-1} \frac{1}{\zeta_n^2 - z^2},$$

and compute $|K| = 2(1+\tan\alpha)$ and $\operatorname{sgn} K = -1$. The estimate (84) simplifies to

$$\sup_{n\in\mathbb{N}} n g_n \left[\sum_{m=1}^{\infty}\left(\frac{m g_m}{m^2+n^2}\right)^2\right]^{1/2} < \infty,$$

and the albedo operator can be written down using (83). For $\alpha = 0$ we have $\zeta_n = n^2\pi^2$ $(n \in \mathbb{N})$ and therefore (cf. [17, Eqs. 1.421(3), (4)])

$$\frac{1}{|K|}\Lambda(z) = \frac{1}{4}\sqrt{z}\,(\cot\sqrt{z} + \coth\sqrt{z}).$$

For $\alpha = \frac{1}{2}\pi$ we have $\zeta_n = (n-\frac{1}{2})^2\pi^2 = \lim \zeta_n(\alpha)$ as $\alpha \downarrow \frac{1}{2}\pi$. We then find

$$\lim_{\alpha\downarrow(1/2)\pi} \frac{1}{|K|}\Lambda(z) = \frac{1}{2}\sqrt{z}\,(-\tan\sqrt{z} + \tanh\sqrt{z}).$$

For the $\alpha = 0$ case the example was presented by Kaper *et al.* [24] as a way to illustrate the existence and uniqueness theory of Sturm–Liouville diffusion equations. For this example they proved the existence of E in a way different from [8]. Here we have been concerned with an explicit construction of E.

VII. Conclusions

We have presented two different methods of constructing explicit representations of the solution (albedo) operator for a class of indefinite Sturm–Liouville problems, culminating in the formulas (27) and (77), respectively. Problems of this type have been shown to have a unique solution [8], which ensures automatically the equivalence of the two representations whenever they can be constructed. So far, the second method has only been developed for Ker $A = \{0\}$, since the existence proof for the factorization (70) breaks down for Ker $A \neq \{0\}$. The first method works for both Ker $A = \{0\}$ and Ker $A \neq \{0\}$. The practical advantage of having two different methods appears when trying to solve concrete problems, when one method may be considerably more expedient than the other. If the original Sturm–Liouville problem $(T^{-1}A)$ is easier to handle than the modified one $(T^{-1}\hat{A})$, the first method should be preferred and viceversa. The situation is acutally illustrated by the examples given in Sections 2 and 6, respectively.

Appendix

Verification of condition (56) (resp. conditions (80)). We consider the eigenvalue problem

$$Ah = \lambda wh, \tag{A1}$$

where $Ah = -h'' + q(\mu)\, h$ with boundary conditions

$$h(-a) = h(b) = 0 \qquad (a, b > 0) \tag{A2}$$

and

$$w(\mu) = \operatorname{sgn} \mu \, |\mu|^{\sigma} \, v(\mu), \qquad \sigma > -\tfrac{1}{2}. \tag{A3}$$

The boundary conditions (A2) are used for simplicity only. Other boundary conditions can easily be accomodated. For $\mu \in [-a, b]$ suppose that q and v'' are continuous, and $v > 0$. Moreover, let q be such that A is positive with Ker $A = \{0\}$. Of the two conditions (80) we first consider

$$\sum_{0 \neq n \in \mathbb{Z}} |\lambda_n|^{2\alpha} \, |\tilde{\kappa}_n|^2 < \infty \qquad \text{for some } 0 < \alpha < 1, \tag{A4}$$

where λ_n denotes the eigenvalues of problem (A1). It suffices to prove (A4) for $n > 0$. We have

$$\tilde{\kappa}_n = \frac{(|A^{-1}T|\,\kappa, \varphi_n)_A}{\|\varphi_n\|_A} = \frac{(\kappa, w\varphi_n)}{(\varphi_n, w\varphi_n)^{1/2}} \tag{A5}$$

with

$$\kappa = \kappa(\mu) = \begin{cases} \dfrac{\mu + a}{-a} & -a \leqslant \mu \leqslant 0 \\ \dfrac{\mu - b}{-b}, & 0 \leqslant \mu \leqslant b. \end{cases} \tag{A6}$$

On introducing the variables

$$\zeta(\mu) = -\left\{\int_{\mu}^{0} |w(t)|^{1/2}\, dt\right\}^{1/\beta}, \qquad -a \leqslant \mu \leqslant 0, \tag{A7}$$

$$\zeta(\mu) = \left\{\int_{0}^{\mu} w^{1/2}(t)\, dt\right\}^{1/\beta}, \qquad 0 \leqslant \mu \leqslant b, \tag{A8}$$

where $\beta = (\sigma + 2)/2$, and letting

$$\hat{v}(\mu) = \frac{|w(\mu)|}{|\zeta|^{\sigma}}, \tag{A9}$$

$$H(\zeta) = \hat{v}^{1/4}(\mu)\, h(\mu), \tag{A10}$$

Eq. (A1) can be rewritten as

$$\frac{d^2 H}{d\zeta^2} = \left\{\pm \frac{\lambda}{4} (\sigma + 2)^2\, |\zeta|^{\sigma} + R(\zeta)\right\} H \tag{A11}$$

with

$$R(\zeta) = \frac{(\sigma + 2)^2\, q(\mu)}{4\hat{v}(\mu)} + \hat{v}^{-1/4}(\mu) \frac{d^2}{d\zeta^2} \{\hat{v}^{1/4}(\mu)\}. \tag{A12}$$

The upper (lower) sign in (A11) pertains to the case $-a \leqslant \mu \leqslant 0$ $(0 \leqslant \mu \leqslant b)$. Equation (A11) has two linearly independent solutions of the form

$$H_1(\zeta) = |\zeta|^{1/2} \{I_\nu(\lambda^{1/2} |\zeta|^{\beta}) + \varepsilon_1(\lambda, \zeta)\}, \tag{A13}$$

$$H_2(\zeta) = |\zeta|^{1/2} \{K_\nu(\lambda^{1/2} |\zeta|^{\beta}) + \varepsilon_2(\lambda, \zeta)\}, \tag{A14}$$

for $-a \leqslant \mu \leqslant 0$, and

$$H_3(\zeta) = \zeta^{1/2} \{J_\nu(\lambda^{1/2} \zeta^{\beta}) + \varepsilon_3(\lambda, \zeta)\}, \tag{A15}$$

$$H_4(\zeta) = \zeta^{1/2} \{J_{-\nu}(\lambda^{1/2} \zeta^{\beta}) + \varepsilon_4(\lambda, \zeta)\}, \tag{A16}$$

for $0 \leqslant \mu < b$; $\nu = (2+\sigma)^{-1}$. For the construction of these solutions and a discussion of the magnitude of the remainders $\varepsilon_i(\lambda, \zeta)$ relative to the corresponding Bessel-functions we refer to Olver [30].

We put $\Phi_n(\zeta) = \hat{v}^{1/4}(\mu)\, \varphi_n(\mu)$ and impose the initial conditions

$$\Phi_n(\zeta(-a)) = 0, \qquad \Phi_n'(\zeta(-a)) = 1. \tag{A17}$$

If we then write

$$\Phi(\zeta) = c_1 H_1(\zeta) + c_2 H_2(\zeta), \qquad -\zeta(-a) \leqslant \zeta \leqslant 0, \tag{A18}$$

$$\Phi_n(\zeta) = c_3 H_3(\zeta) + c_4 H_4(\zeta), \qquad 0 \leqslant \zeta \leqslant \zeta(b), \tag{A19}$$

the conditions (A17) and the requirement that Φ_n and Φ_n' be continuous across $\zeta = 0$ determine the coefficients c_i uniquely. It follows from the work of Olver cited above that the leading behavior as $\lambda \to \infty$ of these coefficients is not affected by the remainders $\varepsilon_i(\lambda, \zeta)$ (see [30, pp. 233, 235, 238]). This is not surprising since one expects the potential q to become negligible as $\lambda \to \infty$. By using the familiar asymptotics of Bessel functions we find

$$c_1 \sim -\frac{|\zeta(-a)|^{1/2}}{\beta} \left(\frac{\pi}{2x_n}\right)^{1/2} e^{-x_n}, \tag{A20}$$

$$c_2 \sim \frac{|\zeta(-a)|^{1/2}}{\beta} \frac{e^{x_n}}{(2\pi x_n)^{1/2}}, \tag{A21}$$

$$c_3 \sim \frac{|\zeta(-a)|^{1/2}}{2\beta} \Gamma(1-\nu) \frac{e^{x_n}}{(2\pi x_n)}, \tag{A22}$$

$$c_4 \sim \frac{|\zeta(-a)|^{1/2}\, \Gamma(\nu)\, \Gamma(1-\nu)}{2\beta} \frac{e^{x_n}}{(2\pi x_n)^{1/2}}, \tag{A23}$$

where $x_n = \lambda_n |\zeta(-a)|^{\beta}$. Hence

$$\Phi_n(0) \sim \frac{|\zeta(-a)|^{1/2}\, \Gamma(\nu)\, 2^{\nu-1}}{\beta} \frac{e^{x_n}}{(2\pi x_n)^{1/2}} \lambda_n^{-\nu/2}, \tag{A24}$$

$$\Phi_n'(0) \sim -\frac{|\zeta(-a)|^{1/2}\, \Gamma(-\nu)\, 2^{-\nu-1}}{\beta \Gamma(\nu)} \frac{e^{x_n}}{(2\pi x_n)^{1/2}} \lambda_n^{\nu/2}, \tag{A25}$$

and

$$\Phi_n(\zeta(b)) \sim A_n \cos\left(y_n - \frac{\pi}{4}\right), \tag{A26}$$

$$\Phi_n'(\zeta(b)) \sim B_n \sin\left(y_n - \frac{\pi}{4}\right), \tag{A27}$$

where

$$A_n = \frac{|\zeta(-a)|^{1/2} (\zeta(b))^{1/2} e^{x_n}}{x_n^{1/2} y_n^{1/2} 2\beta \sin(\pi/2\nu)} \tag{A28}$$

$$B_n = -\frac{\beta x_n}{\zeta(b)} A_n, \tag{A29}$$

$$y_n = \lambda_n^{1/2} (\zeta(b))^{1/2}. \tag{A30}$$

Solving the equation $\Phi_n(\zeta(b)) = 0$ yields

$$\lambda_n \sim \frac{n^2\pi^2}{\int_0^b (w^{1/2}(t)\, dt)^{1/\beta}}, \qquad n \to \infty. \tag{A31}$$

To proceed with our investigation of $\tilde{\kappa}_n$ we write

$$(\kappa, w\varphi_n) = \lambda_n^{-1}(\kappa, q\varphi_n) + 2\lambda_n^{-1}\varphi_n'(0) + \lambda_n^{-1}\left(\frac{1}{b} - \frac{1}{a}\right)\varphi_n(0), \tag{A32}$$

where we have used (A1), (A2), and an integration by parts. Since $\max_{-a \leqslant x \leqslant b} |\varphi_n| = O(e^{x_n})$ it follows that the second term on the right dominates the others as $n \to \infty$ and thus

$$(\kappa, w\varphi_n) = O(\lambda_n^{\nu/2 - 5/4} e^{x_n}). \tag{A33}$$

Next we turn to the denominator in (A5). We note that

$$(\varphi_n, w\varphi_n) = -\beta \int_{\zeta(-a)}^{0} \Phi_n^2(\zeta) |\zeta|^\sigma\, d\zeta + \beta \int_0^{\zeta(b)} \Phi_n^2(\zeta)\, \zeta^\sigma\, d\zeta. \tag{A34}$$

To find the large-n behavior of these integrals we employ the identity

$$\frac{d}{d\zeta}\{\beta^2\lambda_n |\zeta|^{\sigma+1} \Phi_n^2 + \zeta\Phi_n'^2 - \Phi_n\Phi_n'\} = \pm\lambda_n\beta^2(\sigma+2)|\zeta|^\sigma \Phi_n^2, \tag{A35}$$

where the $+$ $(-)$ sign is taken if $\zeta > 0 (\zeta < 0)$. Therefore, by using (A17), we obtain

$$\beta(\sigma+2)(\varphi_n, w\varphi_n) = \lambda_n^{-1}(\zeta(b)\, \phi_n'(\zeta(b))^2 + \zeta(-a)) = O(\lambda_n^{-1} e^{2x_n}) \tag{A36}$$

and hence by (A33) and (A36) we see that

$$\tilde{\kappa}_n \sim \text{const. } \lambda_n^{-1/2} \frac{\Phi_n'(0)}{\Phi_n(\zeta(b))}, \tag{A37}$$

i.e.,

$$\tilde{\kappa}_n = O(\lambda_n^{\nu/2 - 3/4}). \tag{A38}$$

Thus (A4) holds for $0<\alpha<(1+\sigma)/2(2+\sigma)$. Checking the other condition in (80) is now easy if we note that

$$\kappa_n=\frac{(\kappa\,|\zeta_n|^{1/2}\,\psi_n)_T}{\|\,|\zeta_n|^{1/2}\,\psi_n\|_T}=\frac{(\kappa,\,|w|\,\psi_n)}{(\psi_n,\,|w|\,\psi_n)^{1/2}}. \tag{A39}$$

Considering $n>0$ we know that $\psi_n(\mu)=0$ if $\mu\leqslant 0$. If we let $\Psi_n(\zeta)=\hat{v}^{1/4}\psi_n(\mu)$, then Ψ_n is just a multiple of $H_3(\zeta)$. The analogs of (A32) and (A36) are readily worked out and it is found that (A38) remains the same. A similar argument works if $n<0$.

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