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Polarized Light Transfer Above a Reflecting Surface*)

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The existence and uniqueness are established for the solution of the equation of transfer of polarized light in a homogeneous atmosphere of finite optical thickness, assuming reflection by the planetary surface. A general L_p-space formulation is adopted. The boundary value problem is first written as a vector-valued integral equation. Using monotonicity properties of the spectral radii of the integral operators involved as well as recent half-range completeness results for kinetic equations with reflective boundary conditions, the present results follow as a corollary.

1 Introduction

If one neglects vertical inhomogeneities and thermal emission, the equation of transfer of polarized light in a plane-parallel atmosphere of finite *optical thickness b* is the vector-valued integro-differential equation

$$(1.1)\quad u\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{I}(\tau,u,\varphi) + \mathbf{I}(\tau,u,\varphi) = \frac{a}{4\pi}\int_{-1}^{1}\int_{0}^{2\pi}\mathbf{Z}(u,u',\varphi-\varphi')\mathbf{I}(\tau,u',\varphi')\mathrm{d}\varphi'\mathrm{d}u' ,$$

where $0 < \tau < b$. In this equation $0 < a \leqslant 1$ is the *albedo of single scattering,* $\mathbf{Z}(u,u',\varphi - \varphi')$ the phase matrix and $\mathbf{I}(\tau,u,\varphi)$ a four-vector depending on optical depth τ, direction cosine of propagation u and azimuthal angle φ. The components I, Q, U and V of the vector $\mathbf{I}$ describe the intensity and state of polarization of the beam. These so-called Stokes parameters always satisfy the inequalities

$$(1.2)\quad I \geqslant \sqrt{Q^2 + U^2 + V^2} \geqslant 0 ,$$

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which mean that the degree of polarization p of the beam satisfies $p \in [0,1]$. A consistent treatment of polarized light transfer based on the (equivalent) conventions for polarization parameters of Chandrasekhar [3] and Van de Hulst [13] is given in [12]. For notations we shall rely on this work as well as on the predecessor paper [25].

The *phase matrix* can be expressed as the product

$$(1.3)\quad \mathbf{Z}(u,u',\varphi-\varphi') = \mathbf{L}(\pi-\sigma_2)\mathbf{F}(\theta)\mathbf{L}(-\sigma_1)$$

of two *rotation matrices* of the type

$$(1.4)\quad \mathbf{L}(\alpha) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\alpha & \sin 2\alpha & 0 \\ 0 & -\sin 2\alpha & \cos 2\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and the *scattering matrix*

$$(1.5)\quad \mathbf{F}(\theta) = \begin{bmatrix} a_1(\theta) & b_1(\theta) & 0 & 0 \\ b_1(\theta) & a_2(\theta) & 0 & 0 \\ 0 & 0 & a_3(\theta) & b_2(\theta) \\ 0 & 0 & -b_2(\theta) & a_4(\theta) \end{bmatrix}.$$

The quantities $u = -\cos\vartheta$, $u' = -\cos\vartheta'$ and $\theta\,(0 \leqslant \vartheta,\vartheta',\theta < \pi)$ on the one hand and the angles φ, φ', σ_1 and σ_2 on the other hand are connected by the equations

$$(1.6)\quad \begin{aligned} &\cos\theta = \cos\vartheta\cos\vartheta' + \sin\vartheta\sin\vartheta'\cos(\varphi'-\varphi) \\ &\cos\sigma_1 = \frac{\cos\vartheta - \cos\vartheta'\cos\theta}{\sin\vartheta'\sin\theta}, \qquad \cos\sigma_2 = \frac{\cos\vartheta' - \cos\vartheta\cos\theta}{\sin\vartheta\sin\theta}, \end{aligned}$$

where $\sin\sigma_1$ and $\sin\sigma_2$ have the same sign as $\sin(\varphi'-\varphi)$. When the denominators of the equations (1.6) vanish, the appropriate limits must be taken. The *phase function* $a_1(\theta)$ must be nonnegative measurable with normalization

$$\int_{-1}^{1} a_1(\theta)\,\mathrm{d}(\cos\theta) = 2\,.$$

The elements of the scattering matrix are measurable functions and for almost every $\theta \in (0,\pi)$ the matrix $\mathbf{F}(\theta)$ transforms four-vectors satisfying (1.2) into four-vectors of the same type. This implies ([12], (82) – (85))

$$(1.7)\quad \left.\begin{aligned} &|b_1(\theta)| \leqslant \tfrac{1}{2}\{a_1(\theta) + a_2(\theta)\} \leqslant a_1(\theta) \\ &b_1(\theta)^2 + b_2(\theta)^2 + a_k(\theta)^2 \leqslant a_1(\theta)^2 \quad (k = 3,4) \end{aligned}\right\},$$

whence all entries of $\mathbf{F}(\theta)$ are real L_1-functions of $\cos\theta$. Moreover, if $a_1(\theta)$ is an L_r-function of $\cos\theta$ for some $r > 1$, so are the remaining elements of the scattering matrix.

The present article offers a complete existence and uniqueness theory for the solution of the equation of polarized light transfer endowed with the boundary conditions

$$\left.\begin{aligned} &\mathbf{I}(0,u,\varphi) = \mathbf{J}(u,\varphi) \\ &\mathbf{I}(b,-u,\varphi) = \frac{1}{\pi}\int_0^1\int_0^{2\pi} u'\mathbf{R}_g(u,u',\varphi-\varphi')\mathbf{I}(b,u',\varphi')\mathrm{d}\varphi'\mathrm{d}u' + \mathbf{J}(-u,\varphi) \end{aligned}\right\} \tag{1.8}$$

where $u > 0$. Here $\mathbf{J}(u,\varphi)$ specifies incident light and $\mathbf{R}_g(u,u',\varphi-\varphi')$ the reflection properties of the planetary surface. By physical necessity $\mathbf{J}(u,\varphi)$ is a four-vector of the type (1.2) and $\mathbf{R}_g(u,u',\varphi-\varphi')$ transforms four-vectors of the form (1.2) into vectors of the same type. We shall assume that the surface does not reflect more energy than it receives, i.e., that its *plane albedo* (cf. [3]) does not exceed unity:

$$0 \leqslant \frac{1}{\pi}\int_0^1\int_0^{2\pi} u'[\mathbf{R}_g(u,u',\varphi-\varphi')]_{11}\mathrm{d}(\varphi-\varphi')\mathrm{d}u' \leqslant 1\,. \tag{1.9}$$

We also assume that the ground surface displays reciprocity symmetry and mirror symmetry, i.e., we have (cf. [9, 10])

$$\mathbf{R}_g(u,u',\varphi-\varphi') = \mathbf{P}\tilde{\mathbf{R}}_g(u',u,\varphi'-\varphi)\mathbf{P} \tag{1.10}$$

and

$$\mathbf{R}_g(u,u',\varphi-\varphi') = \mathbf{D}\mathbf{R}_g(u,u',\varphi'-\varphi)\mathbf{D}\,, \tag{1.11}$$

respectively, where $\mathbf{P} = \mathrm{diag}(1,1,-1,1)$, $\mathbf{D} = \mathrm{diag}(1,1,-1,-1)$ and tilde above a matrix denotes transposition.

Let us introduce the functional formulation of the boundary value problem (1.1) and (1.8). Let H_p, $1 \leqslant p < \infty$, denote the direct sum of four copies of $L_p(\Omega)$, where Ω is the unit sphere in $\mathbf{R}^3$. The norm of a function $\mathbf{I}:\Omega \to \mathbb{C}^4$ is given by

$$\|\mathbf{I}\|_p = \left[\int_{-1}^1\int_0^{2\pi}\{|I(u,\varphi)|^p + |Q(u,\varphi)|^p + |U(u,\varphi)|^p + |V(u,\varphi)|^p\}\mathrm{d}\varphi\,\mathrm{d}u\right]^{1/p},$$

where $u = -\cos\vartheta$ and (ϑ,φ) are the polar coordinates of a point $\omega \in \Omega$. On H_p we define the bounded linear operators T, B, A, Q_+, Q_- and J by

$$(T\mathbf{I})(u,\varphi) = u\mathbf{I}(u,\varphi)\,, \qquad (A\mathbf{I})(u,\varphi) = \mathbf{I}(u,\varphi) - a(B\mathbf{I})(u,\varphi) \tag{1.12}$$

$$(B\mathbf{I})(u,\varphi) = (4\pi)^{-1}\int_{-1}^1\int_0^{2\pi}\mathbf{Z}(u,u',\varphi-\varphi')\mathbf{I}(u',\varphi')\mathrm{d}\varphi'\mathrm{d}u' \tag{1.13}$$

$$(Q_\pm\mathbf{I})(u,\varphi) = \begin{cases}\mathbf{I}(u,\varphi) & \text{for } u \gtrless 0\\ 0 & \text{for } u \lessgtr 0\end{cases}, \qquad (J\mathbf{I})(u,\varphi) = \mathbf{D}\mathbf{I}(-u,\pi-\varphi)\,, \tag{1.14}$$

and the H_p-vector $\mathbf{I}(\tau)$ by $\mathbf{I}(\tau)(u,\varphi) = \mathbf{I}(\tau,u,\varphi)$. If the surface reflection operator

$$(1.15)\quad (\mathscr{R}\mathbf{I})(u,\varphi) = \frac{1}{\pi}\int_0^1\int_0^{2\pi} u'\mathbf{R}_g(u,u',\varphi-\varphi')\mathbf{D}\mathbf{I}(u',\pi-\varphi')\mathrm{d}\varphi'\mathrm{d}u'$$

is bounded on H_p, then by a solution in H_p of the boundary value problem (1.1) and (1.8) we mean a vector-valued function $\mathbf{I}:(0,b)\to H_p$ such that $T\mathbf{I}$ is differentiable on $(0,b)$ in the strong sense and the following equations hold true:

$$(1.16)\quad (T\mathbf{I})'(\tau) = -A\mathbf{I}(\tau)(0<\tau<b)$$

$$(1.17)\quad \lim_{\tau\downarrow 0}\|Q_+\mathbf{I}(\tau) - Q_+\mathbf{J}\|_p = 0,$$

$$\lim_{\tau\uparrow b}\|Q_-\mathbf{I}(\tau) - J\mathscr{R}Q_+\mathbf{I}(\tau) - Q_-\mathbf{J}\|_p = 0.$$

It can be shown (see Section 2) that this boundary value problem has the same bounded solution as the vector-valued integral equation

$$(1.18)\quad \mathbf{I}(\tau) - a\int_0^b[\mathscr{H}(\tau-\tau') + \mathrm{e}^{(b-\tau)T^{-1}}Q_-J\mathscr{R}\mathscr{H}(b-\tau')]B\mathbf{I}(\tau')\mathrm{d}\tau'$$

$$= \mathrm{e}^{-\tau T^{-1}}Q_+\mathbf{J} + \mathrm{e}^{(b-\tau)T^{-1}}Q_-[\mathbb{I} + J\mathscr{R}\mathrm{e}^{-bT^{-1}}Q_+]\mathbf{J},$$

where $0<\tau<b$,

$$(1.19)\quad (\mathscr{H}(\sigma)\mathbf{I})(u,\varphi) = \begin{cases} |u|^{-1}\mathrm{e}^{-\sigma/u}\mathbf{I}(u,\varphi) & \text{for } \sigma u>0 \\ 0 & \text{for } \sigma u<0\end{cases}$$

defines the *propagator function* and

$$(\mathrm{e}^{\mp\tau T^{-1}}Q_+\mathbf{J})(u,\varphi) = \begin{cases}\mathrm{e}^{\mp b/u}\mathbf{J}(u,\varphi) & \text{for } u \gtrless 0\\ 0 & \text{for } u \lessgtr 0\end{cases}$$

defines two semigroups. On neglecting reflection, analogous results have appeared both for unpolarized light (cf. [22, 23]) and polarized light (cf. [25]). If $\mathscr{R}$ is bounded on H_p, we consider (1.18) on the Banach space $L_q(H_p)_b$ of strongly measurable functions $\mathbf{I}:(0,b)\to H_p$ which are bounded with respect to the L_q-norm. Since for phase functions $a_1(\theta)$ which are L_r-functions of $\cos\theta$ for some $r>1$ the norm estimate

$$\int_{-\infty}^{\infty}\|\mathscr{H}(\sigma)B\|_{H_p}\mathrm{d}\sigma<\infty$$

holds true, one may prove the boundedness on $L_q(H_p)_b$ of the operators

$$(1.20)\quad (L_b\mathbf{I})(\tau) = \int_0^b\mathscr{H}(\tau-\tau')B\mathbf{I}(\tau')\mathrm{d}\tau'$$

for all $1\leqslant q\leqslant\infty$, and the operator

$$(1.21)\quad (M_b\mathbf{I})(\tau) = \int_0^b\mathrm{e}^{(b-\tau)T^{-1}}Q_-J\mathscr{R}\mathscr{H}(b-\tau')B\mathbf{I}(\tau')\mathrm{d}\tau'$$

for sufficiently large q and rewrite (1.18) in the form

(1.22) $(\mathbb{I} - aN_b)\mathbf{I} = \omega$,

where $N_b = L_b + M_b$ and $\omega(\tau)$ denotes the right-hand side of (1.18). As observed by Germogenova and Konovalov [6], the solutions $\mathbf{I}(\tau)$ must belong to the cone

$$K_p = \{\mathbf{I} = (I, Q, U, V)/I \geqslant \sqrt{Q^2 + U^2 + V^2} \geqslant 0\}$$

on the real Banach space H_p, as a consequence of the physical requirement (1.2). In [6, 19, 20] the cone preservation methods of Krein and Rutman [17] and Krasnoselskii [16] have been applied to K_p to obtain information on the position and multiplicity of the zeros of the characteristic equation (which are the discrete eigenvalues of $T^{-1}A$) and the structure of the corresponding eigenfunctions, thereby generalizing the results of Maslennikov [21] for unpolarized light transfer. Related methods have been applied by Van der Mee [25] to the existence and uniqueness problem for the equation of transfer (1.1) with non-reflective boundary conditions (i.e., $\mathscr{R} = 0$), both for atmospheres of finite and infinite optical thickness. In this article we shall extend these methods to the equation (1.1) with reflection by the planetary surface taken into account. On the real Banach space $L_q(H_p)_b$ we define the cone

$$L_q(K_p)_b = \{\mathbf{I} \in L_q(H_p)_b/\mathbf{I}(\tau) \in K_p \text{ almost everywhere}\}.$$

Because physical assumptions cause B, Q_+, Q_-, J, $\mathscr{H}(\sigma)$ and $e^{\mp\sigma T^{-1}}Q_\pm$ to leave invariant the cone K_p and $\mathscr{R}$ to have the same property if it happens to be bounded on H_p, the operators L_b, M_b and N_b are positive with respect to the cone $L_q(K_p)_b$ for sufficiently large q. Using the compactness on $L_q(K_p)_b$ of the operators L_b, M_b and N_b and the u_0-positivity of L_b and N_b (in the sense of [16]) we may prove that the spectral radius $r(N_b)$ of N_b (which does not depend on q, if q is sufficiently large) is strictly monotonically increasing from zero to a finite positive value r_∞ as b increases from zero to infinity. (In the exceptional case $a_1(\theta) \equiv a_4(\theta)$ those operators fail to be u_0-positive, but the result is still true). The function $b \mapsto r(N_b)$ appears to be a C^∞-function. It remains to prove that $0 < r_\infty \leqslant 1$.

In order to establish the latter, we must assume that the ground reflection matrix is dissipative, i.e., that condition (1.9) is satisfied. By monotonicity it then suffices to prove $r_\infty \leqslant 1$ in all cases and to prove $r_\infty = 1$ for $\mathscr{R} = 0$. This is, in fact, done by identifying r_∞ as the spectral radius of a positive operator $\hat{N}_\infty$ on $L_p(H_p)_\infty$ (with q sufficiently large) and exploiting the equivalence of the vector-valued integral equation

$$(\mathbb{I} - a\hat{N}_\infty)\mathbf{I} = \omega$$

with suitable right-hand side ω to the half-space boundary value problem

$$(1.23)\quad \begin{cases} (T\mathbf{I})'(\tau) = -A\mathbf{I}(\tau) \quad (0 < \tau < \infty) \\ \lim_{\tau\downarrow 0} \|Q_+\mathbf{I}(\tau) - \mathscr{R}JQ_-\mathbf{I}(\tau) - Q_+\mathbf{J}\|_p = 0\,. \\ \|\mathbf{I}(\tau)\|_p = O(1) \quad (\tau \to \infty) \end{cases}$$

For $p = 2$ we may then apply recent results of Van der Mee and Protopopescu [26] and conclude that $0 < r_\infty \leqslant 1$. The boundedness of the surface reflection operator $\mathscr{R}$ on suitable spaces H_p and the dissipativity hypothesis (1.9) will enable us to extend these results to all spaces H_p, where $1 \leqslant p < \infty$. The existence and uniqueness results then follow as a corollary. Moreover, since apparently $r(N_b) < 1$, we may iterate the integral equation (1.18) and obtain the unique solution in the form

$$\mathbf{I} = \sum_{n=0}^{\infty} a^n (N_b)^n \omega ,$$

which belongs to the cone $L_q(K_p)_b$. In this way we shall obtain a mathematical justification for the method of expansion with respect to successive orders of multiple scattering (see [11] where reflection is neglected). All results can then be proved for the component equations obtained by Fourier decomposition and symmetry relations.

On neglecting polarization or for the simplest component equation the existence and uniqueness results in H_2 are immediate from results of Greenberg and Van der Mee [8]. However, in [8] the equations (1.1) and (1.8) are analyzed directly and cone preservation techniques do not play any role.

In Sec. 2 we discuss preliminaries and derive the integral equation (1.18). Section 3 is devoted to the monotonicity and continuity of the spectral radius of N_b as a function of b, while in Sec. 4 we introduce the half-space problem (1.23) and prove that $0 < r_\infty \leqslant 1$.

2 Preliminaries and Integral Formulations

Let us first compile some properties of the operator B (see [25], Prop. 2.1; part of it can be found in [6] as parts of Theorems 1 and 3).

Proposition 2.1. *For $1 \leqslant p < \infty$ the operator B is compact and has unit norm on H_p. If $a_1 \in L_r[-1,1]$ for some $r > 1$, then B is a bounded operator from H_p into H_{pr} and*

$$\int_{-\infty}^{\infty} \| \mathscr{H}(\sigma) B \|_{H_p} \mathrm{d}\sigma < \infty . \tag{2.1}$$

Moreover, in this case B acts as a compact operator from $H_{r/(r-1)}$ into the space $C^{(4)}(\Omega)$ of continuous functions $\mathbf{h}: \Omega \to \mathbf{C}^4$ with supremum norm.

As in the introduction, let $L_q(H_p)_b$ be the (real or complex) Banach space of strongly measurable functions $\mathbf{I}$ (0, B) $\to H_p$, which are finite with respect to the norm

$$\|\mathbf{I}\|_{L_q(H_p)_b} = \begin{cases} \left[\int_0^b \|\mathbf{I}(\tau)\|_{H_p}^q \mathrm{d}\tau \right]^{1/q}, & 1 \leqslant q < \infty \\ \operatorname*{ess\,sup}_{0<\tau<b} \|\mathbf{I}(\tau)\|_{H_p}, & q = \infty \end{cases} .$$

Strong measurability is defined in the sense of Sec. 31 of [28]. If $a_1 \in L_r[-1,1]$ for some $r > 1$, then (2.1) guarantees that the operator L_b defined by (1.20) is a bounded operator on $L_q(H_p)_b$, where $1 \leqslant p \leqslant \infty$. As B is uniformly approximable on $H_p(1 \leqslant p < \infty)$ by operators of finite rank, L_b is a compact operator also (cf. [7]; infinite-dimensional generalization of Lemma 1.1), where it is again assumed that $a_1 \in L_r[-1,1]$ for some $r > 1$. In order to analyze M_b, we use the proof of Proposition 2.1 of [25] to find the estimate

$$\|\mathscr{H}(\sigma)B\|_{H_p} = O(|\sigma|^{\alpha-1})(\sigma \to 0)$$

for every $0 < \alpha < (r-1)/pr$, whence

$$\text{(2.2)} \quad \int_{-\infty}^{\infty} \|\mathscr{H}(\sigma)B\|_{H_p}^{q'} \mathrm{d}\sigma < \infty$$

for all $1 \leqslant q' < (1-\alpha)^{-1}$ with $0 < \alpha < (r-1)/pr$. Thus $q = q'/(q'-1)$ must satisfy

$$\text{(2.3)} \quad pr/(r-1) < q \leqslant +\infty$$

in order that (2.2) is fulfilled. Since the convolution product of an $L_{q'}$-function and an L_q-function is bounded and continuous, it is clear that the operator M_b defined by (1.21) and $N_b = L_b + M_b$ are bounded on $L_q(H_p)_b$ whenever $\mathscr{R}$ is bounded on H_p and condition (2.3) is satisfied. One may, in fact, easily prove that under these hypotheses all three operators L_b, M_b and N_b are compact on $L_q(H_p)_b$.

Theorem 2.2. *Let $a_1 \in L_r[-1,1]$ for some $r > 1$, and let $\mathscr{R}$ be bounded on H_p. Then every solution of (1.18) in $L_\infty(H_p)_b$ is continuous on $[0,b]$ and satisfies the boundary value problem (1.16)–(1.17). Conversely, every solution $\mathbf{I}:(0,b) \to H_p$ of (1.16)–(1.17) that is continuous on $[0,b]$ satisfies the vector-valued integral equation (1.18).*

Proof. We shall use the equivalence theorem 2.2 of [25]. Let $\mathbf{I}:(0,b) \to H_p$ be a solution of (1.16)–(1.17) that is continuous on $[0,b]$ and satisfies the equation

$$\text{(2.4)} \quad \mathbf{I}(\tau) - a\int_0^b \mathscr{H}(\tau-\tau')B\mathbf{I}(\tau')\mathrm{d}\tau' = \mathrm{e}^{-\tau T^{-1}}Q_+\mathbf{I}(0) + \mathrm{e}^{(b-\tau)T^{-1}}Q_-\mathbf{I}(b),$$

where $0 \leqslant \tau \leqslant b$ and the boundary conditions

$$\text{(2.5)} \quad Q_+\mathbf{I}(0) = Q_+\mathbf{J}, \qquad Q_-\mathbf{I}(b) = J\mathscr{R}Q_+\mathbf{I}(b) + Q_-\mathbf{J}$$

are fulfilled. On substituting $\tau = b$ in the above integral equation and premultiplying by Q_+ we obtain

$$Q_+\mathbf{I}(b) - a\int_0^b \mathscr{H}(b-\tau')B\mathbf{I}(\tau')\mathrm{d}\tau' = \mathrm{e}^{-bT^{-1}}Q_+\mathbf{I}(0) = \mathrm{e}^{-bT^{-1}}Q_+\mathbf{J},$$

whence

$$\text{(2.6)} \quad Q_-\mathbf{I}(b) = aJ\mathscr{R}\int_0^b \mathscr{H}(b-\tau')B\mathbf{I}(\tau')\mathrm{d}\tau' + J\mathscr{R}\mathrm{e}^{-bT^{-1}}Q_+\mathbf{J} + Q_-\mathbf{J}.$$

Equation (1.18) then easily follows from (2.4) to (2.6).

Conversely, let $\mathbf{I} \in L_\infty(H_p)_b$ be a solution of (1.18). On rewriting (1.18) in the form

$$\mathbf{I}(\tau) - a\int_0^b \mathscr{H}(\tau - \tau')B\mathbf{I}(\tau')\mathrm{d}\tau' = \omega(\tau)\,, 0 < \tau < b\,,$$

where

$$\omega(\tau) = \mathrm{e}^{-\tau T^{-1}}Q_+\mathbf{J} + \mathrm{e}^{(b-\tau)T^{-1}}Q_-[\mathrm{I\!I} + J\mathscr{R}\mathrm{e}^{-bT^{-1}}Q_+]\mathbf{J}$$

is continuous on $[0,b]$ and $T\omega$ is differentiable on $(0,b)$, we may exploit the equivalence theorem 2.2 of [25] as well as the identities

$$(T\omega)'(\tau) + \omega(\tau) = 0(0 < \tau < b)$$
$$Q_+\omega(0) = Q_+\mathbf{J}\,, Q_-\omega(b) = J\mathscr{R}\mathrm{e}^{-bT^{-1}}Q_+\mathbf{J} + Q_-\mathbf{J}$$
$$Q_+\omega(b) = \mathrm{e}^{-bT^{-1}}Q_+\mathbf{J}$$

to obtain the equations (1.16) and (1.17). ∎

Let X be a real Banach space with a cone K. A positive operator L (with respect to K; i.e., $L[K] \subset K$) is said to be *u_0-bounded above* if for every $0 \neq f \in X$ there exist $n = n(f) \in \mathbf{N}$ and $\beta > 0$ satisfying $L^n f \leqslant \beta u_0$, and *$u_0$-bounded below* if for every $0 \neq g \in X$ there exist $m = m(g) \in \mathbf{N}$ and $\alpha > 0$ such that $L^m g \geqslant \alpha u_0$. If L is both u_0-bounded above and u_0-bounded below, it is said to be *u_0-positive* and in this case for every $0 \neq h \in X$ there exist $l = l(h) \in \mathbf{N}$ and $\alpha,\beta > 0$ such that $\alpha u_0 \leqslant L^l h \leqslant \beta u_0$. For more details on these notions we refer to Ch. 2 of [16].

Theorem 2.3. *Suppose that $a_1 \in L_r[-1,1]$ for some $r > 1$ and $a_1(\theta) \not\equiv a_4(\theta)$. Then the operators L_b and N_b are $\mathbf{v}_0$-positive on $L_q(H_p)_b$ where $\mathbf{v}_0(\tau,u,\varphi) = (1,0,0,0)$, provided $\mathscr{R}$ is bounded on H_t for $p \leqslant t \leqslant \infty$ and q satisfies condition* (2.3).

Proof. Under the assumptions of this theorem the integrability condition

$$\int_{-b}^{b} \| \mathscr{H}(\sigma)B\|_{H_p}^s \mathrm{d}\sigma < \infty$$

is fulfilled for $0 \leqslant s < pr/(1 + p(r-1))$ (cf. (2.3)). Using this integrability condition we may prove that L_b is a bounded operator from $L_q(H_p)_b$ into $L_{qs}(H_p)_b$ $(1 \leqslant q \leqslant \infty)$ and that M_b and N_b are bounded operators from $L_q(H_p)_b$ into $L_{qs}(H_p)_b(pr/(r-1) < q \leqslant \infty)$. Also, if $C(H_p)_b$ denotes the (real or complex) Banach space of continuous functions $\mathbf{I}: [0,b] \to H_p$ with supremum norm, the same integrability condition implies that L_b, M_b and N_b are bounded operators from $L_{s'}(H_p)_b$ into $C(H_p)_b$, where $s' = s/(s-1)$. Hence, if q satisfies condition (2.3) and $qs^{n-1} \geqslant s' = s/(s-1)$, then L_b^n and N_b^n are bounded operators from $L_q(H_p)_b$ into $C(H_p)_b$.

Let us return to the proof of Prop. 2.1 of [25]. It can be shown that for fixed $1 < t < r$ we have the integrability condition

$$\int_{-b}^{b} \| \mathscr{H}(\sigma)B\|_{H_p \to H_{pt}}^u \mathrm{d}\sigma\,,$$

where $0 \leqslant u < pt/(1 + p(t-1))$. Thus L_b, M_b and N_b are bounded operators from $C(H_p)_b$ into $C(H_{pt})_b$. We may also derive the estimate

$$\int_{-b}^{b} \|\mathscr{H}(\sigma)B\|_{H_{t/(t-1)}\to C}\,d\sigma < \infty\,,$$

where $C = C^{(4)}(\Omega)$ (see Prop. 2.1), and establish that L_b, M_b and N_b are bounded operators from $C(H_{t/(t-1)})_b$ into $C(C)_b$, where $C(C)_b$ is the (real or complex) Banach space of all continuous functions $\mathbf{I}:[0,b] \to C = C^{(4)}(\Omega)$ with supremum norm. Hence, if q satisfies (2.3), $1 < t < r$ is fixed in such a way that $q > pt/(t-1)$, and if we choose the integer m to satisfy the condition $pt^{m-1} \geqslant t' = t/(t-1)$, then L_b^m and N_b^m are bounded operators from $C(H_p)_b$ into $C(C)_b$. Thus L_b^{n+m} and N_b^{n+m} are bounded operators from $L_q(H_p)_b$ into $C(C)_b$ whenever condition (2.3) holds true. It is clear that $C(C)_b$ may be identified with the (real or complex) Banach space $C^{(4)}([0,b] \times \Omega)$ of continuous (real or complex) four-vector functions on $[0,b] \times \Omega$ with supremum norm.

It is straightforward that L_b and N_b are $\mathbf{v}_0$-bounded above as operators on $C([0,b] \times \Omega)$, where $\mathbf{v}_0(\tau)(u,\varphi) = (1,0,0,0)$. In order to prove that L_b and N_b are $\mathbf{v}_0$-bounded below on this space, we consider a function $\mathbf{I} \in C^{(4)}([0,b] \times \Omega)$ which does not vanish identically and whose values satisfy the positivity condition (1.2). Then there exist a subset E of $(0,b)$ of positive measure and a nonzero vector $\mathbf{I}_0$ satisfying (1.2) such that $\mathbf{I}(\tau) - \mathbf{I}_0$ satisfies condition (1.2) for all $\tau \in E$. Since $\mathscr{H}(\tau - \tau')$ is a scalar multiple of the identity matrix and $\int_E \mathscr{H}(\tau - \tau')\,d\tau' \geqslant \varepsilon \mathbb{I}$ for some fixed $\varepsilon > 0$, we easily obtain (in the partial order generated by the positive cone):

(2.6) $\quad (L_b^m \mathbf{I})(u) \geqslant (L_b^m(\chi_E \mathbf{I}_0))(u) \geqslant \varepsilon^m B^m \mathbf{I}_0\,,\ u \in (0,b)\,.$

Here $\chi_E(u) = 1$ for $u \in E$ and $\chi_E(u) = 0$ for $u \notin E$.

If $a_1(\theta) \not\equiv a_4(\theta)$, the operator B is $\mathbf{u}_0$-positive on $C^{(4)}(\Omega)$ (see [6]; the exception $a_1(\theta) \equiv a_4(\theta)$ was not considered there) where $\mathbf{u}_0 = (1,0,0,0)$. In combination with (2.6) we may conclude that L_b (and therefore N_b also) is $\mathbf{v}_0$-positive on $C^{(4)}([0,b] \times \Omega)$. Since some power of N_b acts as a bounded operator from $L_q(H_p)_b$ into $C^{(4)}([0,b] \times \Omega)$ whenever p and q satisfy (2.3), we find that under the hypothesis (2.3) the operators L_b and N_b are $\mathbf{v}_0$-positive on $L_q(H_p)_b$. ■

If $a_1(\theta) \equiv a_4(\theta)$, the operator B is not $\mathbf{u}_0$-positive on $C^{(4)}(\Omega)$. In view of (1.7) we then have

(2.7) $\quad b_1(\theta) \equiv b_2(\theta) \equiv 0\,, |a_2(\theta)| \leqslant a_1(\theta)\,, |a_3(\theta)| \leqslant a_1(\theta)\,,$

and B is a self-adjoint operator on H_2. In this exceptional case we shall derive our results by exploiting [8].

3 Monotonicity and Continuity Properties of the Spectral Radius

Throughout this section we assume $a_1(\theta) \not\equiv a_4(\theta)$. We shall apply the cone preservation arguments of Krasnoselskii ([16], Ch. 2) to prove the following result:

Theorem 3.1. *Let $a_1 \in L_r[-1,1]$ for some $r > 1$, and let q satisfy condition* (2.3). *If $\mathscr{R}$ is bounded on H_t for $p \leqslant t \leqslant \infty$, then the spectral radius $r(N_b)$ of N_b is a strictly monotonically increasing C^∞-function of b such that*

$$(3.1) \quad r_\infty = \lim_{b \to \infty} r(N_b)$$

exists and is finite.

The method of proof was used before by Van der Mee [24] and Borgioli et al. [2] for various transport models as far as the monotonicity and the finiteness of r_∞ are concerned. The C^∞-dependence of the spectral radius on the size parameter was proved for a specific model in [2] using our present technique.

Proof of Theorem 3.1. We remark that $L_q(K_p)_b$ is a reproducing and normal cone in $L_q(H_p)_b$ (cf. [6, 25]). This means (cf. [16]) that

(i) $L_q(H_p)_b = \{\mathbf{I}_1 - \mathbf{I}_2 | \mathbf{I}_1, \mathbf{I}_2 \in L_q(K_p)_b\}$, and
(ii) $\| \mathbf{I}_1 \| \leqslant M \| \mathbf{I}_2 \|$ in the norm of $L_q(H_p)_b$ whenever $0 \leqslant \mathbf{I}_1 \leqslant \mathbf{I}_2$ in the partial order generated by the cone.

[In fact, $M = \sqrt{3}$]. Since, for q satisfying (2.3) and $\mathscr{R}$ bounded on H_t for $p \leqslant t \leqslant \infty$, the operator N_b is compact and $\mathbf{v}_0$-positive with respect to $L_q(K_p)_b$, we may use Theorems 2.10, 2.12 and 2.13 of [16] to establish the following properties:

(i) if $r(N_b) > 0$, there exists an eigenfunction $\mathbf{I}_b \in L_q(K_p)_b$ of N_b satisfying $N_b \mathbf{I}_b = r(N_b)\mathbf{I}_b$; if $r(N_b) = 0$, no such eigenfunction exists,
(ii) the eigenvalue $r(N_b) > 0$ is algebraically simple,
(iii) there does not exist an eigenvalue different from $r(N_b)$ with the eigenfunction in $L_q(K_p)_b$, and
(iv) there are no non-positive eigenvalues of absolute value $r(N_b)$.

Now choose $0 < b' < b < \infty$ and write the equation $N_b \mathbf{I}_b = r(N_b)\mathbf{I}_b$, where $r(N_b) > 0$, in the form

$$(3.2) \quad (r(N_b)\mathbb{I} - N_{b'})\mathbf{I}_b = \int_{b'}^{b} \mathscr{H}(\tau - \tau')B\mathbf{I}_b(\tau')\mathrm{d}\tau' + \int_{b'}^{b} \mathrm{e}^{(b-\tau)T^{-1}} Q_- J \mathscr{R} \mathscr{H}(b - \tau')B\mathbf{I}_b(\tau')\mathrm{d}\tau' \,, \quad 0 < \tau < b' \,.$$

Since the $\mathbf{u}_0$-positivity of B excludes the possibility of a non-trivial null space of B, the right-hand side is non-trivial. As a consequence of Th. 2.16 of [16] we obtain

$$(3.3) \quad 0 \leqslant r(N_{b'}) < r(N_b) \,.$$

If $r(N_b) = 0$, (3.2) would read

$$-N_{b'}\mathbf{I}_b = \omega$$

for non-zero vectors $\mathbf{I}_b$ and ω in $L_q(K_p)_{b'}$, which is a contradiction. Hence, $r(N_b) > 0$ for all $b \in (0, \infty)$ and (3.3) holds true. We obviously have

$$0 < r(N_b) \leqslant \|N_b\| \to 0 \quad \text{as } b \downarrow 0 ,$$

and therefore $r(N_b)$ vanishes as $b \downarrow 0$.

The analyticity of the semigroups $e^{-\tau T^{-1}}Q_+$ and $e^{\tau T^{-1}}Q_-$ and their derivatives $\mathscr{H}(\tau)$ (see [15] for analytic semigroup theory) implies that the operators L_b and N_b allow analytic continuation from $(0, \infty)$ to the open right half-plane. As a result the algebraically simple eigenvalue $r(N_b)$ of N_b will also allow analytic continuation to an open neighbourhood of every subinterval $[b_1, b_2]$ of $(0, \infty)$ (see [15], Th. VII 1.8), whence $b \mapsto r(N_b)$ is a C^∞-function on $(0, \infty)$.

It remains to prove the existence and finiteness of the limit (3.1). Let us consider the isometric operator $\mathscr{S}$ on $L_q(H_p)_b$ defined by

$$(\mathscr{S}\mathbf{I})(\tau) = J\mathbf{I}(b - \tau) , \quad 0 < \tau < b .$$

Then $\mathscr{S}^2 = \mathbb{I}$ and $\mathscr{S}$ is positive with respect to $L_q(K_p)_b$. Because $BJ = JB$ ([25], (7.6)), we have

$$\mathscr{S}L_b = L_b\mathscr{S}$$

as well as

$$(\mathscr{S}M_b\mathscr{S}\mathbf{I})(\tau) = \int_0^b e^{-\tau T^{-1}}Q_+ \mathscr{R}J\mathscr{H}(-\tau')B\mathbf{I}(\tau')\mathrm{d}\tau' , \qquad 0 < \tau < b .$$

Since N_b and $\mathscr{S}N_b\mathscr{S}$ have the same spectral radius $r(N_b)$, we find (3.1), where

$$r_\infty = \lim_{b\to\infty} r(N_b) = \lim_{b\to\infty} r(\mathscr{S}N_b\mathscr{S}) = r(\hat{N}_\infty) \tag{3.4}$$

with

$$(\hat{N}_\infty\mathbf{I})(\tau) = \int_0^\infty \mathscr{H}(\tau - \tau')B\mathbf{I}(\tau')\mathrm{d}\tau' + \int_0^\infty e^{-\tau T^{-1}}Q_+ \mathscr{R}J\mathscr{H}(-\tau')B\mathbf{I}(\tau')\mathrm{d}\tau' , \quad 0 < \tau < b , \tag{3.5}$$

defined on $L_q(H_p)_\infty$.

Corollary 3.2. *Let $a_1 \in L_r[-1, 1]$ for some $r > 1$, and let q satisfy condition* (2.3). *If $\mathscr{R}$ is bounded on H_t for $p \leqslant t \leqslant \infty$, then the integral equation*

$$(\mathbb{I} - aN_b)\mathbf{I} = \omega \tag{3.6}$$

is uniquely solvable on $L_q(H_p)_b$ for $0 < a < r(N_b)^{-1}$ and the solution is given by the absolutely converging series

$$\mathbf{I} = \sum_{n=0}^{\infty} a^n N_b^n \omega . \tag{3.7}$$

Thus if $\omega \in L_q(K_p)_b$, we have $\mathbf{I} \in L_q(K_p)_b$.

As a consequence of the compactness of the operator N_b we remark that $r(N_b)$ does not depend on q and p, provided condition (2.3) is satisfied. As a result of this invariance property and (3.4), the spectral radius of $\hat{N}_\infty$ does not depend on q and p either, also under condition (2.3). We may also derive a full analog of Corollary 3.2 for $\hat{N}_\infty$. It takes some more trouble to prove that r_∞ belongs to the spectrum of N_∞. First, the cone $L_q(K_p)_\infty$ is reproducing and normal (cf. [6, 25]) and therefore the adjoint cone of bounded linear functionals on $L_q(H_p)_\infty$ leaving invariant $L_q(K_p)_\infty$ is reproducing (see [1], part 2 of Th. 1). Using Theorem 4 of [14] we derive that $r_\infty = r(\hat{N}_\infty)$ belongs to the spectrum of $\hat{N}_\infty$.

4 Existence and Uniqueness Theorems

In this section we shall prove that $r_\infty = 1$. In view of (3.3) and (3.1) we then get $r(N_b) < 1$, which implies the existence and uniqueness of the solution of (3.6) and the series expansion (3.7) for $0 < a \leqslant 1$. We shall assume that the *ground reflection matrix* $\mathbf{R}_g(u, u', \varphi - \varphi')$ transforms four-vectors of the form (1.2) into vectors of the same type and satisfies the dissipativity condition (1.9) and the symmetry relations (1.10) and (1.11). In analogy with the derivation in Section 4.1 of [12] we may use the series expansion

$$\mathbf{R}_g(u, u', \varphi - \varphi') = \mathbf{R}_g^{c0}(u, u') + 2 \sum_{j=1}^{\infty} [\mathbf{R}_g^{cj}(u, u') \cos\{j(\varphi - \varphi')\} + \mathbf{R}_g^{sj}(u, u') \sin\{j(\varphi - \varphi')\}],$$

a similar expansion (namely, eq. (107) of [12]) for the phase matrix $\mathbf{Z}(u, u', \varphi - \varphi')$ and the Fourier series

$$\mathbf{I}(\tau, u, \varphi) = \mathbf{I}^{c0}(\tau, u) + 2 \sum_{j=1}^{\infty} [\mathbf{I}^{cj}(\tau, u) \cos j\varphi + \mathbf{I}^{sj}(\tau, u) \sin j\varphi]$$

to decompose the boundary value problem (1.16) – (1.17) into a boundary value problem for $\mathbf{I}^{c0}(\tau, u)$ and boundary value problems coupling $\mathbf{I}^{cj}(\tau, u)$ and $\mathbf{I}^{sj}(\tau, u)$ for each $j \geqslant 1$. Using the symmetry relation (104) of [12] for the phase matrix and the symmetry relation (1.11) for the ground reflection matrix, a further decoupling can be obtained into equations involving real polarization parameters.

It has been known since the fundamental article of Kuščer and Ribarič [18] that the addition formula for the generalized spherical functions of Gelfand and Shapiro [5] may be exploited to obtain explicitly the Fourier component equations of the full equation of transfer (1.1). Usually the equations obtained involve complex polarization parameters (cf. [18, 4]). One may use various symmetry relations (developed in [9]) to obtain component equations involving real polarization parameters (see [27]; also [12]). The simplest component equation concerns the first two entries (I and Q) of the azimuthal-averaged Stokes vector

$$\mathbf{I}_0(\tau,u) = \int_0^{2\pi} \mathbf{I}(\tau,u,\varphi)\mathrm{d}\varphi ,$$

which may be written as the two-vector (with s = symmetric)

$$\mathbf{I}^s(\tau,u) = \left(\int_0^{2\pi} I(\tau,u,\varphi)\mathrm{d}\varphi , \int_0^{2\pi} Q(\tau,u,\varphi)\mathrm{d}\varphi\right) \tag{4.1}$$

and can be obtained by restricting the equation of radiative transfer to the "subspace" $H_p^s = \{(I,Q)\,|\,\mathbf{I} = (I,Q,U,V)$ does not depend on $\varphi\}$. The restricted equation of transfer can be written explicitly as

$$u\frac{\partial}{\partial\tau}\mathbf{I}^s(\tau,u) + \mathbf{I}^s(\tau,u) = \frac{1}{2}a\int_{-1}^{1} \mathbf{W}^s(u,u')\mathbf{I}^s(u')\mathrm{d}u' , \tag{4.2}$$

where $\mathbf{W}^s(u,u')$ is a real matrix function of order two satisfying

$$\mathbf{W}^s(u,u') = \tilde{\mathbf{W}}^s(u',u) \tag{4.3}$$

with tilde above a boldface matrix symbol denoting transposition. We now define T^s, B^s, A^s, Q_+^s, Q_-^s and J^s by

$$(T^s\mathbf{I}^s)(u) = u\mathbf{I}^s(u) , (A\mathbf{I}^s)(u) = \mathbf{I}^s(u) - a(B^s\mathbf{I}^s)(u)$$

$$(B\mathbf{I}^s)(u) = \frac{1}{2}\int_{-1}^{1} \mathbf{W}^s(u,u')\mathbf{I}^s(u')\mathrm{d}u'$$

$$(Q_+^s\mathbf{I}^s)(u) = \begin{cases} \mathbf{I}^s(u) & \text{for } u \gtrless 0 \\ 0 & \text{for } u \lessgtr 0 \end{cases}, \qquad (J^s\mathbf{I}^s)(u) = \mathbf{I}^s(-u) .$$

By its very form (1.15) the surface reflection operator $\mathscr{R}$ will leave invariant H_p^s; its restriction to H_p^s we will denote by $\mathscr{R}^s$. The cone K_p induces on H_p^s the cone

$$K_p^s = \{\mathbf{I}^s \in H_p^s / |Q(u)| \leqslant I(u) \text{ for almost every } u \in [-1,1]\} .$$

Using the Fourier decomposition and the symmetry relations we easily find

$$(\mathscr{R}^s\mathbf{I}^s)(u) = 2\int_0^1 u'\mathbf{R}_g^s(u,u')\mathbf{I}^s(u')\mathrm{d}u' , \tag{4.4}$$

where

$$\mathbf{R}_g^s(u,u') = \mathbf{R}_g^s(u',u) , 0 \leqslant 2\int_0^1 u'[\mathbf{R}_g^s(u,u')]_{11}\mathrm{d}u' \leqslant 1 . \tag{4.5}$$

We may now introduce the spaces $L_q(H_p^s)_b$ and the operators L_b^s and N_b^s as the restrictions of L_b and N_b to these spaces. We have

Proposition 4.1. *Let $a_1 \in L_r[-1,1]$ for some $r > 1$, and let q satisfy condition (2.3). Assume that the surface reflection operator $\mathscr{R}$ is bounded on H_t for $p \leqslant t \leqslant \infty$. Then the spectral radii of N_b and N_b^s coincide.*

Proof. Since N_b^s is a compact positive operator on $L_q(H_p^s)_b$ with respect to the cone $L_q(K_p^s)_b$ and the latter cone is reproducing (cf. [16] for the definition), there exists a non-trivial vector $\mathbf{I}_b^s \in L_q(K_p^s)_b$ and a constant $r_b \geqslant 0$ satisfying $N_b^s \mathbf{I}_b^s = r_b \mathbf{I}_b^s$. Putting

$$\mathbf{I}_b(\tau,u,\varphi) = (I_b^s(\tau,u), Q_b^s(\tau,u), 0, 0),$$

we obtain $N_b \mathbf{I}_b = r_b \mathbf{I}_b$ on $L_q(H_p)_b$ with $0 \neq \mathbf{I}_b \in L_q(K_p)_b$.

If $a_1(\theta) \not\equiv a_4(\theta)$, the operator N_b is $\mathbf{v}_0$-positive and therefore positive eigenfunctions (with respect to $L_q(K_p)_b$) are unique up to a positive constant factor. We may then conclude $r_b = r(N_b)$ and therefore $r(N_b^s) = r(N_b)$.

If $a_1(\theta) \not\equiv a_4(\theta)$, we use approximation of the scattering matrix $\mathbf{F}(\theta)$ by (physically relevant) scattering matrices such that $a_1(\theta) \not\equiv a_4(\theta)$. The conclusion will be the same. ■

A simple compactness argument implies that $r(N_b)$ does not depend on q and p, provided (2.3) holds true. We may therefore prove $r_\infty = 1$ for $p = 2$ and $q = \infty$, provided $\mathscr{R}$ is bounded on H_t for $\min(p,2) \leqslant t \leqslant \infty$. For this case the vector-valued integral equation

$$(4.6)\qquad (\mathbb{I} - aN_b^s)\mathbf{I}^s = \omega^s,$$

where

$$\omega^s(\tau,u) = \begin{cases} e^{-\tau/u}\mathbf{J}^s(u), & u > 0 \\ e^{(b-\tau)/u}[\mathbf{J}^s(u) + (\mathscr{R}^s e^{-b(T^s)^{-1}} Q_+^s \mathbf{J}^s)(-u)], & u < 0 \end{cases}$$

has the same bounded solutions as the boundary value problem

$$(4.7)\qquad \left.\begin{aligned} &(T^s\mathbf{I}^s)'(\tau) = -A^s\mathbf{I}^s(\tau), \ 0 < \tau < b \\ &\lim_{\tau\downarrow 0} \| Q_+^s \mathbf{I}^s(\tau) - Q_+^s \mathbf{J}^s \|_2 = 0 \\ &\lim_{\tau\uparrow b} \| Q_-^s \mathbf{I}^s(\tau) - J^s\mathscr{R}^s Q_+^s \mathbf{I}^s(\tau) - Q_-^s \mathbf{J}^s \|_2 = 0 \end{aligned}\right\}.$$

However, on defining the unitary operator $U\colon H_2^s \to H_2^s$ satisfying

$$(U\mathbf{I}^s)(\tau) = \frac{1}{\sqrt{}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\mathbf{I}^s(\tau), \ (U^{-1}\mathbf{I}^s)(\tau) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\mathbf{I}^s(\tau),$$

we see that $U\mathscr{R}^s U^{-1}$ leaves invariant the cone of H_2^s consisting of all vectors (I_x, I_y) with $I_x \geqslant 0$ and $I_y \geqslant 0$, while $\mathbf{R}_g^s(u,u')$ is transformed into the kernel

$$\mathbf{R}_g^U(u,u') = \frac{1}{2}\begin{bmatrix} R_{11} + R_{22} - (R_{12} + R_{21}) & R_{11} - R_{22} + (R_{12} - R_{21}) \\ R_{11} - R_{22} - (R_{12} - R_{21}) & R_{11} + R_{22} + (R_{12} + R_{21}) \end{bmatrix}$$

with $R_{ij} = R_{ij}(u,u') = [\mathbf{R}_g^s(u,u')]_{ij}$. Defining $\hat{\mathscr{R}}^s = U\mathscr{R}^s U^{-1}$ and using the positivity estimates

$$(4.8)\qquad |R_{12}(u,u') \pm R_{21}(u,u')| \leqslant R_{11}(u,u') \pm R_{22}(u,u'),$$

a straightforward estimation yields

$$\left|\int_0^1 u(\mathscr{R}^s\mathbf{I})(u)\cdot\overline{\mathbf{I}(u)}\,du\right| \leqslant \int_0^1\int_0^1 uu'\{(R_{11}(u,u')+R_{22}(u,u')) + |R_{12}(u,u')+R_{21}(u,u')| + (R_{11}(u,u') - R_{22}(u,u')) + |R_{12}(u,u')-R_{21}(u,u')|\} \times \sqrt{|I(u)|^2+|Q(u)|^2}\sqrt{|I(u')|^2+|Q(u')|^2}\,du\,du'.$$

On using the estimates (4.8), Schwartz's inequality and the dissipativity condition in the second part of (4.5), we obtain

$$\left|\int_0^1 u(\mathscr{R}^s\mathbf{I})(u)\cdot\overline{\mathbf{I}(u)}\,du\right| \leqslant \int_0^1 u\|\mathbf{I}(u)\|_2^2\,du\,.$$

Hence, by the unitarity of U,

$$0 \leqslant \int_0^1 u(\mathscr{R}^s\mathbf{I}^s)(u)\cdot\overline{\mathbf{I}^s(u)}\,du \leqslant \int_0^1 u\|\mathbf{I}^s(u)\|_2^2\,du\,. \tag{4.9}$$

Theorem 4.2. *We have* $r_\infty = 1$.

Proof. For $p = 2$ and $q = \infty$ the equation

$$(\mathbb{I} - a\hat{N}_\infty)\mathbf{I}^s = \omega^s,$$

where ω^s is a suitable right-hand side, has the same bounded solution as the boundary value problem

$$\begin{cases} (T^s\mathbf{I}^s)'(\tau) = -A^s\mathbf{I}^s(\tau)\,, 0<\tau<\infty \\ \lim_{\tau\downarrow 0}\|Q_+^s\mathbf{I}^s(\tau) - \mathscr{R}^sJ^sQ_-^s\mathbf{I}^s(\tau) - Q_+^s\mathbf{J}^s\|_2 = 0\,. \\ \|\mathbf{I}^s(\tau)\|_2 = O(1)(\tau\to\infty) \end{cases} \tag{4.10}$$

However, A^s is positive self-adjoint for $0 < a \leqslant 1$ and strictly positive self-adjoint on H_2^s for $0 < a < 1$. In view of (4.9), problem (4.10) satisfies the assumptions of the main result of [26] and as a consequence this problem is uniquely solvable for $0 < a < 1$. Because the above reasoning implies $r_\infty \leqslant 1$ and $r_\infty = 1$ is smallest for $\mathscr{R}^s = 0$, we have $r_\infty = 1$ in all cases. ∎

If $a_1(\theta) \not\equiv a_4(\theta)$, the $\mathbf{v}_0$-positivity of N_b on $L_q(H_p)_b$, Proposition 4.1, the identity (3.4) and $r_\infty = 1$ imply that

$$r(N_b) < 1\,. \tag{4.11}$$

In the exceptional case $a_1(\theta) \equiv a_4(\theta)$ we exploit the positive self-adjointness of A^s on H_2^s, the estimate (4.9) and the main result of [8] to obtain the unique solvability of problem (4.7) for $0 < a \leqslant 1$. Thus in this case we have $r(N_b^s) < 1$ and therefore (4.11) holds true (cf. Prop. 4.1).

Corollary 4.3. *Let* $a_1 \in L_r[-1,1]$ *for some* $r > 1$, *and let q satisfy condition* (2.3). *If* $\mathscr{R}$ *is bounded on* H_t *for* $\min(p,2) \leqslant t \leqslant \infty$, *then the vector equation* (3.6) *is uniquely solvable on* $L_q(H_p)_b$ *for* $0 < a \leqslant 1$ *and the solution is given by the absolutely convergent series* (3.7). *Thus if* $\omega \in L_q(K_p)_b$, *we have* $\mathbf{I} \in L_q(K_p)_b$.

This corollary justifies the method of expansion with respect to successive orders of multiple scattering. (For $\mathscr{R} = 0$ the method was explained in [11]).

We have obtained a complete existence and uniqueness theory for the equation of transfer of polarized light, which accounts for an extensive class of reflection laws. Equation (1.1) can be decomposed into separate Fourier component equations (see [27, 12, 25] for equations involving real polarization parameters). From the present results it is straightforward to establish the unique solvability of all component equations for atmospheres of finite optical thickness with and without reflection by the planetary surface.

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Literature

[1] Ando, T.: On fundamental properties of a Banach space with a cone. Pac. J. Math. **12** (1962) 1163 – 1169

[2] Borgioli, G.; Frosali, G.; van der Mee, C. V. M.: A stationary criticality problem in general L_p-space for energy-dependent neutron transport in cylindrical geometry. Z. Angew. Math. Phys. **35** (1984) 166 – 180

[3] Chandrasekhar, S.: Radiative transfer. London: Oxford University Press 1950 = New York: Dover Publ. 1960

[4] Domke, H.: The expansion of scattering matrices for an isotropic medium in generalized spherical functions. Astrophys. Space Science **29** (1974) 379 – 386

[5] Gelfand, I. M.; Shapiro, Z. Ya.: Representations of the group of rotations in three-dimensional space and their applications. A.M.S. Transl., Series 2, vol. **2** (1956) 207 – 316 = Uspehi Matem. Nauk **7** (1952) 3 – 117 [Russian]

[6] Germogenova, T. A.; Konovalov, N. V.: The spectrum of the characteristic equation in the theory of transfer with polarization taken into account. Moscow: Institute of Applied Mathematics, AN SSSR, preprint no. 62, 1978 [Russian]

[7] Gohberg, I. C.; Heinig, G.: On matrix integral operators on a finite interval with kernels depending on the difference of their arguments. Rev. Roum. Math. Pures et Appl. **20** (1975) 55 – 73 [Russian]

[8] Greenberg, W.; van der Mee, C. V. M.: An abstract model for radiative transfer in an atmosphere with reflection by the planetary surface. SIAM J. Math. Anal. **16** (1985) 695 – 702

[9] Hovenier, J. W.: Symmetry relations for scattering of polarized light in a slab of randomly oriented particles. J. Atmosph. Sci. **26** (1969) 488 – 499

[10] Hovenier, J. W.: Principles of symmetry for polarization studies of planets. Astron. Astrophys. **7** (1970) 86 – 90
[11] Hovenier, J. W.: Multiple scattering of polarized light in planetary atmospheres. Astron. Astrophys. **13** (1971) 7 – 29. Appendix
[12] Hovenier, J. W.; van der Mee, C. V. M.: Fundamental relationships relevant to the transfer of polarized light in a scattering atmosphere. Astron. Astrophys. **128** (1983) 1 – 16
[13] van de Hulst, H. C.: Light scattering by small particles. New York: Wiley 1957 = New York: Dover Publ. 1981
[14] Karlin, S.: Positive operators. J. Math. Mech. **8** (1959) 907 – 937
[15] Kato, T.: Perturbation theory for linear operators. Heidelberg: Springer 1966
[16] Krasnoselskii, M. A.: Positive solutions of operator equations. Groningen: Noordhoff 1964 = Moscow: Fizmatgiz 1962 [Russian]
[17] Krein, M. G.; Rutman, M. A.: Linear operators leaving invariant a cone in a Banach space. A.M.S. Transl., Series 1, **26** (1950) 199 – 325 = Uspehi Matem. Nauk **3** (1948) 3 – 95 [Russian]
[18] Kuščer, I.; Ribarič, M.: Matrix formalism in the theory of diffusion of light. Optica Acta **6** (1959) 42 – 51
[19] Kuzmina, M. G.: General functional properties of the transport equation of polarized radiation. Sov. Phys. Dokl. **23** (1978) 26 – 28 = Dokl. Akad. Nauk SSSR **238** (1978) 314 – 317 [Russian]
[20] Kuzmina, M. G.: The characteristic equation of transport theory of polarized radiation. Sov. Phys. Dokl. **24** (1979) 17 – 19 = Dokl. Akad. Nauk SSSR **244** (1979) 62 – 66 [Russian]
[21] Maslennikov, M. V.: The Milne problem with anisotropic scattering. Proc. Steklov Institute of Mathematics, vol. 97, 1969 = Trudy Matem. Instituta im. V. A. Steklova, AN SSSR, vol. 97, 1968 [Russian]
[22] van der Mee, C. V. M.: Semigroup and factorization methods in transport theory. Amsterdam, Ph. D. Thesis, Free University 1981
[23] van der Mee, C. V. M.: Transport theory in L_p-spaces. Int. Eqs. Oper. Theor. **6** (1983) 405 – 444
[24] van der Mee, C. V. M.: Positivity and monotonicity properties of transport equations with spatially dependent cross-sections. Transp. Theor. Stat. Phys. **11** (1982/83) 199 – 216
[25] van der Mee, C. V. M.: Half- and finite-range completeness for the equation of transfer of polarized light. Math. Meth. in the Appl. Sci. **6** (1984) 393 – 416
[26] van der Mee, C. V. M.; Protopopescu, V.: Kinetic equations with reflecting boundary conditions. SIAM J. Math. Anal., to appear
[27] Siewert, C. E.: On the equation of transfer relevant to the scattering of polarized light. Astrophys. J. **245** (1981) 1080 – 1086
[28] Zaanen, A. C.: Integration. Amsterdam: North-Holland 1967

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