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COMPARISON OF THE CRITICAL EIGENVALUES FOR INTEGRAL NEUTRON TRANSPORT EQUATIONS IN DIFFERENT GEOMETRIES

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ABSTRACT

The critical eigenvalues of the monoenergetic time-independent neutron transport equation are compared for spheres, cylinders and slabs. Strict monotonicity properties are derived and the generalization to the energy-dependent case is discussed.

I. Introduction

Neutron transport equations have been studied intensively both in integro-differential and in integral form. A large amount of literature on this

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field has been devoted to the cases when the particular symmetry induced by the geometry of the medium leads to a one-dimensional transport equation.

The criticality problem connected with the one-dimensional transport equation in slab, spherical and cylindrical geometry has been successfully investigated in different functional spaces, both in one-speed as well as multigroup approximation and in energy-dependent cases.

For the integral equation in the one-speed approximation it has been proved that the first (critical) eigenvalue of the transport operator in spherical geometry is the second eigenvalue of the transport operator in slab geometry [1]. Hence, the critical eigenvalue for spheres is smaller than the critical eigenvalue for slabs as one would expect from physical considerations.

There is substantial numerical evidence that the critical eigenvalue of a sphere of radius R is less than the critical eigenvalue of an infinite cylinder of radius R and that the latter is smaller than the critical eigenvalue of a slab of thickness 2R. Tabulating some of the results (obtained by Busoni et al. (1973)[2] for spheres and slabs, and by Sanchez and Ganapol (1982)[3] for cylinders) we get

R	sphere	cylinder	slab
0.5	0.308910	0.417853	0.619049
1	0.502919	0.618935	0.783022
3	0.821042	0.883141	0.944914
7	0.949160	0.969182	0.986466
10	0.972620	0.983652	0.992914

These results as well as similar results by Carlvik (1968)[4], Syros and Theocharopoulos (1977)[5], Dahl and Sjöstrand (1979)[6], Pomraning (1980)[7] and Premuda et al. (1982)[8] all support the above strict order relation between the critical eigenvalues of sphere, cylinder and slab. Such support also comes, more indirectly, from values for critical slab and sphere sizes (rather than critical eigenvalues) obtained numerically by Kaper, Lindeman and Leaf(1974)[9]. While studying the behavior of the neutron flux for vanishing small media, all three one-dimensional geometries were considered simultaneously before by Pomraning [7].

In this article we shall prove this strict order relation analytically. This problem is substantially complicated by the completely different nature of the

kernels of the transport operators in one-dimensional sphere, cylinder and slab geometry.

This trouble leads us to return to three-dimensional formulations of the transport operator. This allows us to deal with integral operators defined on different domains but having the same kernel. Resorting to a common three-dimensional form of the transport equation has suggested us to extend the investigation on the comparison of the first critical eigenvalues to some more general classes of domains.

With this aim we examine in Sect. 2 the problems connected with the transport equation in arbitrary domains (possibly unbounded). We give a weak order relation for the spectral radii of the transport operators acting on two domains of $\mathbb{R}^3$, $D_i \subset D_j$. In Sect. 3 we study the eigenvalue problem for the transport operator in the class of bounded domains. We prove the existence of a dominant eigenvalue with corresponding positive eigenfunction. Then we establish a strict order relation between the dominant eigenvalues for a couple of domains $D_i \subset D_j$. In Sect. 4 we obtain the same results for domains of cylindrical type. In Sect. 5 we discuss the eigenvalue problem for sphere, cylinder and slab and prove a strict order relation among the respective dominant eigenvalues. In Sect. 6 we extend the results obtained in the preceding sections to the energy-dependent case.

The functional space, which we shall choose in order to formulate the transport equation as a vector equation, is the space of uniformly continuous bounded functions defined on a domain D, i.e. UCB(D). For unbounded domains this Banach space is the natural extension of the space of continuous functions defined on compact domains. We shall use a C-setting when dealing with bounded domains. Because of its physical relevance we shall also study the eigenvalue problem for the transport operator in an L_p-setting. In fact, in Sects. 3 and 4 we establish the independence of the spectrum of the integral transport operator of the specific C- or L_p-setting, $1\le p<\infty$.

2. The integral transport equation

Let us write down the stationary integral equation for monoenergetic neutron transport in a homogeneous multiplying medium occupying a convex (possibly unbounded) region $D\subset\mathbb{R}^3$ embedded in vacuum or in a purely absorbing medium. Under the assumption of isotropic scattering, the equation reads as follows:

$$\phi(\underline{r})=\frac{c\Sigma}{4\pi}\int_D \frac{\exp(-\Sigma|\underline{r}-\underline{r}'|)}{|\underline{r}-\underline{r}'|^2}\,\phi(\underline{r}')\,d\underline{r}', \tag{1}$$

where $\phi(\underline{r})$ is the total neutron flux at position $\underline{r}\in D$, c is the average number of secondaries per collision and Σ is the total cross-section.

We shall analyze Eq. (1) as a vector equation in the Banach space of uniformly continuous bounded (UCB) functions defined on the domain D:

$$X=X(D):=UCB(D)$$

with norm $\|f\|=\|f;X\|:=\sup_{\underline{r}\in D}|f(\underline{r})|$.

The abstract formulation of Eq. (1) in UCB(D) then reads as follows:

$$\phi=cK_D\phi, \tag{2}$$

where K_D is the integral operator defined by Eq. (1). Eq. (2) represents a simplified version of the well-known Peierls equation, with the total cross-section Σ constant and in absence of sources [10]. Usually the functional-analytic and numerical study of Eq. (2) is restricted to transport in simple geometries, as, for instance, slab, sphere and cylinder. In these cases the specific symmetry of the geometry reduces the transport equation to a one-dimensional equation which has been analyzed in great detail. As a matter of fact, the study of Eq. (2) in a more general domain D is often complicated by the absence of suitable properties of K_D. On the one hand, for any, even an unbounded, domain D, the operator K_D is continuous on X(D) with norm $\|K_D;B(X(D))\| \leq 1$. Moreover, since K_D has a weakly singular kernel, it is a completely continuous operator on C(D) if $D\subset\mathbb{R}^3$ is a compact set. On the other hand, if D is unbounded, K_D need not be compact. For instance, in a slab of thickness 2a, K, defined by (1) as acting on $UCB([-a,a]\times\mathbb{R}^2)$, is neither a compact operator nor a power-compact operator. Direct application of the theory of positive cones[11] is impossible. First of all, K maps a positive function with compact support into a function vanishing at infinity in the direction of the y and z coordinates, which is not an interior element of the positive cone. Secondly, K^n maps a function vanishing at infinity into another function of the same kind, for any n. Hence, K is not a strongly positive operator on the positive cone. One can also verify that K does not even have the weaker property of u_0-positivity (as defined in Ref. [11], Sec. 2.3).

In order to study the stationary criticality problem, we associate to Eq. (2) the eigenvalue equation for the operator K_D:

$$\lambda\phi=K_D\phi. \tag{3}$$

It is well-known [12] that the spectral radius of a positive compact operator acting on a Banach lattice is

also an eigenvalue with corresponding positive eigenfunction. The inverse of the spectral radius then gives the average number of secondaries per collision which keeps critical the system. For unbounded domains D the eigenvalue problem (3) cannot be successfully analyzed directly. Only by taking account of the possible symmetries of the medium, a more manageable equation can be obtained.

The principal aim of this paper is to compare the critical eigenvalues (i.e., the spectral radii of the transport operators) for different geometries. In this paper we shall compare the spectral radii of two transport operators for two different domains, disregarding for the moment whether the corresponding eigenvalue problems admit solutions.

First of all, we remark that $K_D\in B(X)$, i.e. K_D is a bounded operator on the complex Banach space X, for any $D\subset\mathbb{R}^3$ (possibly unbounded). Thus, by definition, the spectral radius $r(K_D)$ is given by

$$r(K_D)=\lim_{n\to\infty}\|K_D^n;B(X)\|^{1/n}.$$

Let $k(\underline{r},\underline{r}')$ be the kernel defining the operator K_D:

$$k(\underline{r},\underline{r}')=\frac{\Sigma\exp(-\Sigma|\underline{r}-\underline{r}'|)}{4\pi|\underline{r}-\underline{r}'|^2}. \tag{4}$$

As regards the operator norm of K_D, it is easy to see that

$$\|K_D;B(X)\| = \sup_{\|f;X\|=1} \sup_{\underline{r}\in D} \left|\int_D k(\underline{r},\underline{r}')f(\underline{r}')d\underline{r}'\right| =$$

$$= \sup_{\underline{r}\in D} \int_D k(\underline{r},\underline{r}')1_D d\underline{r}',$$

where 1_D is the characteristic function of D. Hence $\|K_D;B(X)\| = \|K_D 1_D;X\|$ and similarly

$$\|K_D^n;B(X)\| = \|K_D^n 1_D;X\|.$$

Consider now D_i and $D_j \subset \mathbb{R}^3$ such that $D_i \subset D_j$; from the above considerations it follows that

$$\|K_{D_i}^n;B(X_i)\| \le \|K_{D_j}^n;B(X_j)\|,$$

where $X_i(X_j)$ is the UCB-space on $D_i(D_j)$. Finally, one has the following order relation between the spectral radii:

$$r(D_i) \le r(D_j).$$

3. Bounded domains

In this section we shall pay attention to the eigenvalue problem (3) where the domain D is bounded and closed. In this case the Banach space X is reduced to the space of continuous functions defined on the compact set D, C(D). Since the eigenvalue problem can be successfully studied in C(D) as well as in $L_p(D)$, $1\le p<\infty$, we shall derive some propositions in this direction and write K instead of K_D.

PROPOSITION 1. K is compact as an operator acting on $L_p(D)$, $1\le p<\infty$, and C(D).

Proof. K is an integral operator with a weakly singular kernel; hence, K is compact both on C(D) and $L_2(D)$ [13]. To prove the compactness of K on L_1 we can follow the route of Ref. [14], but, as in the one-dimensional case, we may also use the following simple argument.

Let us define a sequence of continuous kernels

$$k_n(\underline{r},\underline{r}') = \begin{cases} \frac{\Sigma n^2}{4\pi}\exp(-\frac{\Sigma}{n}), & \text{if } |\underline{r}-\underline{r}'| \le \frac{1}{n} \\ k(\underline{r},\underline{r}'), & \text{if } |\underline{r}-\underline{r}'| > \frac{1}{n}. \end{cases}$$

For each n, the approximating operator $(K_n\phi)(\underline{r}) = \int_D k_n(\underline{r},\underline{r}')\phi(\underline{r}')d\underline{r}'$, with continuous kernel, is compact. In fact,

$$|(K_n\phi)(\underline{r})| \le \frac{\Sigma n^2}{4\pi}\exp(-\frac{\Sigma}{n})\|\phi;L_1(D)\|, \text{ and}$$

$$|(K_n\phi)(\underline{r})-(K_n\phi)(\underline{r}')| \le \max_{\underline{r}''\in D}|k_n(\underline{r},\underline{r}'')-k_n(\underline{r}',\underline{r}'')|\,\|\phi;L_1(D)\|.$$

Hence, by using Ascoli's theorem, K_n is compact as an operator from $L_1(D)$ into C(D); moreover, since a compact set in C(D) is always compact in $L_1(D)$, the operator K_n is compact as acting on $L_1(D)$. Because the operators K_n converge to K in the operator norm of $L_1(D)$, K is also compact as acting on $L_1(D)$. Moreover,

K is bounded on $L_\infty(D)$. This permits us to interpolate the compactness property of K and to prove that K is compact on $L_p(D)$, $1\le p<\infty$[15].

PROPOSITION 2. The spectra of the operator K acting on $L_p(D)$, $1\le p<\infty$, and $C(D)$ are the same.

Proof. As regards the dependence of the spectrum of the specific L_p-setting, we follow the procedure of Ref. [16]. We denote here by T_L and T_C an operator T as acting on $L_1(D)$ and $C(D)$, respectively. Since D is a compact set, $C(D)\subset L_1(D)$ and hence, for each λ,

$$\mathrm{Ker}(I-\lambda K)_L \supseteq \mathrm{Ker}(I-\lambda K)_C$$

$$\mathrm{Im}(I-\lambda K)_L \supseteq \mathrm{Im}(I-\lambda K)_C.$$

Recalling the definition of nullity $n(T)$ and deficiency $d(T)$ of an operator T, we have

$$n(I-\lambda K)_L \ge n(I-\lambda K)_C$$

$$d(I-\lambda K)_L \le d(I-\lambda K)_C. \qquad (5)$$

It is well-known that, for a compact operator T and for every λ, $I-\lambda T$ is a Fredholm operator of index 0 [17]. This means that, using the definition of the index $\mathrm{ind}(T) = n(T) - d(T)$,

$$0 = \mathrm{ind}(I-\lambda K)_L = n(I-\lambda K)_L - d(I-\lambda K)_L,$$

$$0 = \mathrm{ind}(I-\lambda K)_C = n(I-\lambda K)_C - d(I-\lambda K)_C.$$

Thus, in (5) the equality signs hold. Hence, $\lambda^{-1}\notin\sigma(K_L)$ if and only if $n(I-\lambda K)_L=d(I-\lambda K)_L=0$, and also $\lambda^{-1}\notin\sigma(K_C)$ if and only if $n(I-\lambda K)_C=d(I-\lambda K)_C=0$. Thus from the equivalence of $n(I-\lambda K)=d(I-\lambda K)$ in both L_1 and C, it follows that $\sigma(K)$ is the same on L_1 and C.

The above propositions lead us to work in C, where it is possible to study the stationary problem in the simplest way. In fact, the property of C of having a positive cone C^+ with non-empty interior allows an easier use of the theory of positive operators in Banach lattices.

Before stating our results on the solution of the eigenvalue problem (3), we recall some definitions. A linear operator A is called strongly positive with respect to C^+ if for each $f>0$ (positive function not identically zero) there exists a natural number $n=n(f)$ such that A^nf is a function in the interior of C^+. The points of the spectrum with modulus equal to the spectral radius are said to form the peripheral spectrum. A positive eigenvalue greater than the moduli of all other eigenvalues is said to be dominant.

Owing to the strict positivity of the kernel $k(\underline{r},\underline{r}')$ on $D\times D$ (which in this case is bounded) the operator K is strongly positive. Thus we can formulate the following proposition, whose proof directly follows from the theory of positive operators[11].

PROPOSITION 3. i) The operator K has a unique eigenvalue with a positive eigenfunction interior to C^+; the eigenvalue is equal to the spectral radius r(K).

ii) The peripheral point spectrum of K consist solely of the dominant eigenvalue r(K).

iii) The eigenspace corresponding to r(K) is one-dimensional; moreover, r(K) is algebraically simple.

We conclude this section with the comparison of the dominant eigenvalues for two different bounded domains D_1, D_2, the former included in the latter: $D_1 \subset D_2$. Let us denote by λ_i and ϕ_i the dominant eigenvalue, equal to $r(K_{D_i})$, and the corresponding positive eigenfunction of the operator K_{D_i}, i=1,2. We can rewrite the eigenvalue equation $\lambda_2\phi_2 = K_{D_2}\phi_2$ as follows:

$$\lambda_2\phi_2 - K_{D_1}\phi_2 = \hat{K}\phi_2$$

where $\hat{K}$ is the integral operator

$$(\hat{K}\phi)(\underline{r}) = \int_{D_2 \setminus D_1} \hat{k}(\underline{r},\underline{r}')\phi(\underline{r}')d\underline{r}'$$

and $\hat{k}(\underline{r},\underline{r}')$ is defined as in (4) with $\underline{r}' \in D_2 \setminus D_1$ and $r \in D_2$. Obviously, when K_{D_1} is applied to a function ϕ, this function must be restricted to D_1. Thus, the non-homogeneous equation

$$\lambda_2 f - K_{D_1} f = g$$

with $g = \hat{K}\phi_2$ restricted to D_1 is solvable in $C^+(D_1)$, while g>0. Hence, λ_2 must be greater than the spectral radius of K_{D_1}, i.e. $\lambda_2 > \lambda_1$ (Ref. [11], Theorem 2.16).

4. Domains of cylindrical type

Let us study now the eigenvalue problem (3) in the family of domains which are unbounded in the z-direction only and have z-translational symmetry:

$$D_c = \{(x,y,z) \in \mathbb{R}^3 : (x,y) \in E \subset \mathbb{R}^2,\ E \text{ bounded}\};$$

throughout this section we shall concisely call D_c a domain of cylindrical type.

We define the following operator on C(E):

$$(L_E\, g)(\underline{x}) = \int_E \ell(\underline{x},\underline{x}')g(\underline{x}')d\underline{x}'$$

with $\underline{x} = (x,y)$, $\underline{x}' = (x',y') \in E$, and

$$\ell(\underline{x},\underline{x}') = \frac{\Sigma}{4\pi}\int_{-\infty}^{+\infty} \frac{\exp[-\Sigma((x-x')^2+(y-y')^2+(z-z')^2)^{\frac{1}{2}}]}{(x-x')^2+(y-y')^2+(z-z')^2}dz'.$$

Moreover, if K_0 denotes the modified Bessel function of zero order, then

$$\ell(\underline{x},\underline{x}') = \frac{\Sigma}{4\pi}\int_{-\infty}^{+\infty} \frac{\exp[-\Sigma(|\underline{x}-\underline{x}'|^2+s^2)^{\frac{1}{2}}]}{|\underline{x}-\underline{x}'|^2+s^2}ds=$$

$$= \frac{\Sigma}{2\pi|\underline{x}-\underline{x}'|} \int_{\Sigma|\underline{x}-\underline{x}'|}^{\infty} K_0(t)dt, \tag{6}$$

which behaves as $1/4|\underline{x}-\underline{x}'|$ for $|\underline{x}-\underline{x}'|\to 0$ (for the equality (6) see Ref. [19], p. 483, 11.2.8 and 11.2.10). Hence, $\ell(\underline{x},\underline{x}')$ is a weakly singular kernel and L_E is a compact operator on $C(E)$[13]. It is also easy to prove that L_E is a strongly positive operator on the cone of positive functions of $C(E)$. In fact, let $d=\sup_{\underline{x},\underline{x}'\in E}|\underline{x}-\underline{x}'|$; since E is bounded, $d<\infty$. Now, directly from (6) we have

$$\ell(\underline{x},\underline{x}') \geq \frac{\Sigma}{2\pi d}\int_{\Sigma d}^{+\infty} K_0(t)dt \geq \varepsilon > 0, \tag{7}$$

which proves the strict positivity of $\ell(\underline{x},\underline{x}')$ and hence our statement. From classical results on the peripheral point spectrum of positive operators [18] we have the existence of a positive dominant eigenvalue of L_E, $\lambda_E=r(L_E)$, and of a corresponding strictly positive eigenfunction $\phi_E\in C^+(E)$. It is obvious that by inserting ϕ_E in the eigenvalue problem (3) we find that ϕ_E is also a solution of (3), with λ_E the corresponding eigenvalue. Moreover, $\lambda_E=r(K_{D_c})$ if D_c is a domain of cylindrical type. In fact, if we denote $X_c=UCB(D_c)$, we have

$$\|K_{D_c};B(X_c)\| = \sup_{\|f;X_c\|=1}\ \sup_{\underline{r}\in D_c}\left|\int_D k(\underline{r},\underline{r}')f(\underline{r}')d\underline{r}'\right| =$$

$$= \sup_{\underline{r}\in D_c}\int_D k(\underline{r},\underline{r}')1_{D_c}d\underline{r}'$$

Now, we obtain

$$\|K_{D_c};B(X_c)\| = \sup_{\underline{x}\in E}\int_E \ell(\underline{x},\underline{x}')1_E d\underline{x}' = \|L_E;B(C(E))\|.$$

By induction we have

$$\|K_{D_c}^n;B(X_c)\| = \|L_E^n;B(C(E))\|,$$

and hence

$$r(K_{D_c}) = r(L_E) = \lambda_E.$$

Thus, the dominant eigenvalue which solves the two-dimensional eigenvalue problem

$$\lambda\phi = L_E\phi \tag{8}$$

is also an eigenvalue of the three-dimensional problem (2) and is equal to $r(K_{D_c})$.

We mention here concisely some results on the spectrum of L_E. Since L_E is an operator with a weakly singular kernel, it is compact on both $L_1(E)$ and $L_2(E)$ [13]. Moreover, L_E is bounded on $L_\infty(E)$ and hence [15] compact on each $L_p(E)$, $1\leq p<\infty$. Condition (7) implies that L_E is strongly positive, i.e. for each $f>0$, $L_E f$ is quasi-interior to the positive cone $L_p^+(E)$ [18]. Thus, the eigenvalue problem (8) admits a dominant eigenvalue with corresponding strictly positive eigenfunction in each $L_p(E)$. By considerations similar to the ones developed in the case of bounded domains, we obtain the equivalence of the spectra of L_E on C and L_p, $1\leq p<\infty$, i.e. these (point) spectra coincide as sets and corresponding multiplicities are the same.

We now prove a strict order relation between the dominant eigenvalues corresponding to different domains of cylindrical type. Let $E\subset E'$; we can write the following identity which follows from the solution $\phi_{E'}$ of (8) for $L_{E'}$:

$$\lambda_{E'}\phi_{E'}(\underline{x})=\int_{E'} \ell(\underline{x},\underline{x}')\phi_{E'}(\underline{x}')d\underline{x}'.$$

Rearranging the above identity we have

$$\lambda_{E'}\phi_{E'}(\underline{x})-\int_{E}\ell(\underline{x},\underline{x}')\phi_{E'}(\underline{x}')d\underline{x}'=\int_{E'\setminus E}\ell(\underline{x},\underline{x}')\phi_{E'}(\underline{x}')d\underline{x}',$$

which can be written in the shorter form

$$\lambda_{E'}\phi_{E'}-L_E\phi_{E'}=g.$$

As in Sect. 3, $g>0$ because the eigenfunction $\phi_{E'}$ is positive; therefore $\lambda_{E'}>\lambda_E$ (Ref. [11], Theorem 2.16). Extending this result to the spectral radii of the corresponding K_{D_c}, we can state a strict order relation, even if the domains D_c involved are unbounded: if $D_c\subset D_c'$, $D_c\neq D_c'$, then $r(K_{D_c})<r(K_{D_c'})$.

Summarizing the results of the present and the previous section on the comparison of dominant eigenvalues we obtain

THEOREM 1. i) If $D_i\subset D_j$ ($D_i\neq D_j$) are two bounded domains in $\mathbb{R}^3$, then the dominant eigenvalues of Eq. (3) satisfy $\lambda_i<\lambda_j$.

ii) If $D_c\subset D_c'$ ($D_c\neq D_c'$) are two domains of cylindrical type contained in $\mathbb{R}^3$, then the dominant eigenvalues of Eq. (3) satisfy

$$\lambda_c<\lambda_c'.$$

Let us comment on the physical meaning of Theorem 1. Recall that the dominant eigenvalue is equal to the inverse of c, the number of secondaries per collision keeping critical the medium. Then the number of secondaries necessary to keep critical a domain containing another one is smaller for the larger domain. This agrees with the physical fact that the relative loss of neutrons by escape is larger for the smaller domain.

5. Sphere, cylinder and slab

Let us denote by D_i, $i=1,2,3$, the following sets:

$$D_1=\{(x,y,z)\in\mathbb{R}^3:\ x^2+y^2+z^2\leq R^2\}$$

$$D_2=\{(x,y,z)\in\mathbb{R}^3:\ x^2+y^2\leq R^2\}$$

$$D_3=\{(x,y,z)\in\mathbb{R}^3:\ -R\leq x\leq R\}.$$

D_1 is the sphere of radius R, D_2 is the infinite circular cylinder of radius R, D_3 is the infinite slab of thickness 2R: $D_1 \subset D_2 \subset D_3$. In this section we analyze the eigenvalue problem for each operator K_{D_i} in X_i: $\lambda\phi = K_{D_i}\phi$, $i=1,2,3$ (here X_i denotes $UCB(D_i)$). For each domain D_i we shall prove the existence of a positive eigenvalue $\lambda_i = r(K_i)$ that corresponds to a one-dimensional positive eigenfunction ϕ_i. At the end of the section we shall give our main result, which is the strict order relation $\lambda_1 < \lambda_2 < \lambda_3$.

Let us consider the cases of slab, cylindrical and spherical geometry separately.

a) <u>The slab</u>. It is well-known that the one-dimensional transport equation in a slab of thickness 2R reads as follows:

$$\psi(x) = c\int_{-R}^{R} k_3(x,x')\psi(x')dx' \tag{9}$$

with $k_3(x,x') = \frac{\Sigma}{2}E_1(\Sigma|x-x'|)$,

where E_1 is the exponential integral [19]. If we formulate Eq. (9) as an abstract eigenvalue problem in $C[-R,R]$ and put $\lambda = 1/c$, we have

$$\lambda\psi = K_3\psi. \tag{10}$$

It is well-known that Eq. (10) admits a positive dominant eigenvalue λ_3, which is equal to the spectral radius of K_3 and to which corresponds a unique positive eigenfunction $\phi_3 \in C[-R,R]$ [20, 21]. On the other hand,

we can identify $\phi_3 = \phi_3(x)$ as an element of X_3. Thus in order to study the eigenvalue equation $\lambda\phi = K_{D_3}\phi$, we put $\phi(\underline{r}) = \phi_3(x)$; by means of straightforward computation [22], one obtains

$$\lambda\phi_3(x) = \int_{-R}^{R}dx' \int_{-\infty}^{+\infty}dy' \int_{-\infty}^{+\infty}dz'\,k(\underline{r},\underline{r}')\phi_3(x') = \\ = \frac{\Sigma}{2}\int_{-R}^{R}E_1(|x-x'|)\phi_3(x')dx'. \tag{11}$$

From the preceding considerations it is manifest that Eq. (11) can be solved for $\lambda = \lambda_3$. It follows also that λ_3 is a positive eigenvalue of K_{D_3} corresponding to the positive eigenfunction $\phi_3(x) \in X_3$.

Let us now prove that the spectral radius of K_{D_3} is equal to the spectral radius of K_3:

$$r(K_{D_3}) = r(K_3). \tag{12}$$

Let Y_3 denote the Banach space of all bounded linear operators on X_3 and Z_3 the Banach space of all bounded linear operators on $C[-R,R]$. Let us prove (12) by first establishing the following equality:

$$\|K_{D_3}^n; Y_3\| = \|K_3^n; Z_3\|, \quad n \geq 1.$$

In fact, $\|K_{D_3}; Y_3\| = \sup_{\underline{r} \in D_3} \int_{D_3} k(\underline{r},\underline{r}')1_{D_3}d\underline{r}'$, where 1_{D_3} is the constant unit function of X_3. It follows immediately that

$$\|K_{D_3}; Y_3\| = \sup_{x \in [-R,R]} \frac{\Sigma}{2}\int_{-R}^{R} E_1(\Sigma|x-x'|)dx' = \|K_3; Z_3\|.$$

By giving a similar argument for the n-th powers of K_{D_3} and K_3 and taking the limit as $n\to\infty$ we may conclude that

$$\lambda_3=r(K_3)=r(K_{D_3}).$$

b) The cylinder. The one-dimensional transport equation in a cylinder of radius R and infinite height reads as follows:

$$\psi(x)=c\int_0^R k_2(x,x')\psi(x')dx', \qquad (13)$$

where

$$k_2(x,x') = \frac{\Sigma}{4\pi}\int_{-\infty}^{+\infty}dz \int_0^{2\pi}d\theta\ \frac{x'\exp(-\Sigma(x^2+x'^2+z^2-2xx'\cos\theta)^{\frac{1}{2}})}{x^2+x'^2+z^2-2xx'\cos\theta}.$$

Formulating Eq. (13) as an abstract eigenvalue problem in $C[0,R]$ and putting $\lambda=1/c$, we have

$$\lambda\psi=K_2\psi. \qquad (14)$$

Eq. (14) admits a positive dominant eigenvalue λ_2, $\lambda_2=r(K_2)$, to which corresponds a unique positive eigenfunction $\phi_2(x)\in C[0,R]$ [23,24] By complete analogy to the slab problem one first identifies $\phi_2=\phi_2(x)$ as a vector in X_2 which satisfies Eq. (3) for $D=D_2$. Similar arguments then lead to the conclusion

$$\lambda_2 = r(K_2) = r(K_{D_2}).$$

c) The sphere. Let us consider the one-dimensional transport equation in a sphere of radius R:

$$\psi(x)=c\int_0^R k_1(x,x')\psi(x')dx', \qquad (15)$$

where

$$k_1(x,x')=\frac{\Sigma}{2}\{E_1(\Sigma|x-x'|)-E_1(\Sigma|x+x'|)\}\frac{x'}{x}.$$

The abstract eigenvalue problem $\lambda\psi=K_1\psi$ in $C[0,R]$ admits a positive dominant eigenvalue λ_1, $\lambda_1=r(K_1)$, corresponding to a unique positive eigenfunction $\phi_1(x)\in C[0,R]$. Moreover, it has been proved that λ_1 is the second eigenvalue of K_3, the integral transport operator in one-dimensional slab geometry [1]. Hence, $\lambda_3>\lambda_1$. Analogously to the preceding cases we may derive that

$$\lambda_1=r(K_1)=r(K_{D_1}).$$

Now, if we recall the relationship between the spectral radii of K_D proved at the end of Sect. 2, we conclude

$$\lambda_1\le\lambda_2\le\lambda_3.$$

The remaining part of this section will be devoted to proving a strict order relation between λ_1, λ_2 and λ_3. Let $D_4=\{(x,y,z)\in\mathbb{R}^3:\ x^2+y^2\le R^2,\ -R\le z\le R\}$ be the finite cylinder containing the sphere of radius R. From our results on finite domains we have the existence of a dominant eigenvalue of K_{D_4},

$\lambda_4=r(K_{D_4})>\lambda_1$. Since $r(K_{D_4})\le r(K_{D_2})=\lambda_2$, we conclude

$$\lambda_1<\lambda_4\le\lambda_2.$$

Next, let $E_5=\{(x,y)\in\mathbb{R}^2: -R\le x\le R,\ -R\le y\le R\}$ and $D_5=\{(x,y,z)\in\mathbb{R}^3: (x,y)\in E_5\}$. We have $D_2\subset D_5\subset D_3$; hence,

$$\lambda_2=r(K_{D_2})\le r(K_{D_5})\le r(K_{D_3})=\lambda_3.$$

But D_5 is a domain of cylindrical type, as well as D_2, with $E_2=\{(x,y)\in\mathbb{R}^2: x^2+y^2\le R^2\}\subset E_5$. We have proved that a dominant eigenvalue exists for the corresponding "two-dimensional" operators L_{E_2} and L_{E_5} (see the previous section) satisfying

$$\lambda_2=r(L_{E_2})=r(K_{D_2})<\lambda_5=r(L_{E_5})=r(K_{D_5}),$$

and from this we also have

$$\lambda_2<\lambda_5\le\lambda_3.$$

Thus, we summarize the above results by the following theorem:

<u>THEOREM</u> 2. Let D_1, D_2 and D_3 be the previous spherical, cylindrical and slab domains. The dominant eigenvalues of the corresponding Eqs. (3) satisfy the strict order relation

$$\lambda_1<\lambda_2<\lambda_3.$$

6. The energy- dependent case

Throughout this paper we have developed our results in a UCB-setting, since we had to consider a functional space containing functions non-zero and constant at infinity. At the same time, whenever we treated bounded domains (or bounded "projections" of infinite domains), we used a C-setting. We showed the independence of the spectral properties of the monoenergetic transport operator of the specific UCB-, C- and even L_p-setting. Now, if we want to generalize our results to the energy-dependent case, we must write Eq. (1) in a different form, introducing an energy-dependent total cross-section $\Sigma(E)$ and a so-called energy transfer function $S(E,E')$. Keeping the assumptions of homogeneity of the multiplying medium and of isotropic scattering the transport equation reads as follows:

$$\phi(\underline{r},E)=$$
$$=\frac{c}{4\pi}\int_D\int_U \frac{\exp(-\Sigma(E)|\underline{r}-\underline{r}'|)}{|\underline{r}-\underline{r}'|^2}\,S(E,E')\phi(\underline{r}',E')\,dE'd\underline{r}' \quad (16)$$

where $U=[E_m,E_M]$, $0<E_m<E_M<+\infty$, is the energy interval.

The choice of the functional space on which to study Eq. (16) requires suitable assumptions on the functions $\Sigma(E)$ and $S(E,E')$. We shall employ the following set of assumptions appropriate to a C-setting [25, 26]:

A.1 $\Sigma:E \mapsto \Sigma(E)$ is a continuous and strictly positive function of $E \in U$:

$$0<\Sigma_m \le \Sigma(E) \le \Sigma_M < +\infty, \quad \forall E \in U.$$

The energy transfer function $S(E,E')$ is given by $(E'/E)^{1/2}\Sigma(E')f(E,E')$, where $f(E,E')$ is the probability density that a neutron packet with energy in $(E',E'+dE')$ emerges with energy in $(E,E+dE)$ after collision; $f(E,E')$ is normalized by [22]

$$\int_U f(E,E')dE=1, \quad \forall E' \in U.$$

We give the following conditions on $S(E,E')$:

A.2 $S(E,E')$ is a nonnegative measurable function such that

$$\int_U S(E,E')dE'<M, \quad \forall E \in U;$$

$$\lim_{E''\to E}\int_U |S(E'',E')-S(E,E')|dE'=0, \quad \forall E \in U.$$

Let $S:C(U)\to C(U)$, $(Sg)(E)=\int_U S(E,E')g(E')dE'$, be the corresponding operator acting on $C(U)$. We also require:

A.3 The n-th iterated kernel $S^{(n)}(E,E')$ of S^n, for some n and some constant ε, satisfies

$$S^{(n)}(E,E')\ge\varepsilon>0, \quad \forall E,\ E' \in U.$$

A.2 and A.3 guarantee that S is a compact and strongly positive operator on $C(U)$.

For every D we define the integral transport operator K by Eq. (16) as acting on $UCB(D\times U)$. By analogy to the monoenergetic case, we prove the following proposition:

PROPOSITION 4. For any bounded and closed D, K is a compact and strongly positive operator acting on $C(D\times U)$.

Proof. The compactness follows from A.1 and A.2 and arguments of Ascoli-Arzelà type. To exploit the positivity of the transport operator, let $k(\underline{r},\underline{r}',E,E')$ be the kernel defining K. Now, $k(\underline{r},\underline{r}',E,E')\ge S(E,E')\exp(-\Sigma_M d)/4\pi d^2$, where $\sup_{\underline{r},\underline{r}'\in D} |\underline{r}-\underline{r}'|=d<\infty$. By iteration, from A.3 we obtain directly the strong positivity of K, which completes the proof.

As in Sect. 3 we conclude with the following proposition:

PROPOSITION 5. i) The operator K has a unique eigenvalue with a positive eigenfunction interior to $C^+(D\times U)$; the eigenvalue is equal to the spectral radius $r(K)$.

ii) The peripheral point spectrum of K consists solely of the dominant eigenvalue r(K).

iii) The eigenspace corresponding to r(K) is one-dimensional; moreover, r(K) is algebraically simple.

We compare now the dominant eigenvalues for two bounded geometries $D_1 \subset D_2$ $(D_1 \neq D_2)$. Denoting by λ_i and ϕ_i, i=1,2, the dominant eigenvalue and the corresponding positive eigenfunction of the operator K_{D_i}, let us rearrange the eigenvalue equation $\lambda_2\phi_2=K_{D_2}\phi_2$ in the following way:

$$\lambda_2\phi_2-K_{D_1}\phi_2=\hat{K}\phi_2,$$

where $\hat{K}$ is the integral operator given by

$$(\hat{K}\phi)(\underline{r},E)=$$
$$=\int_{D_2\setminus D_1}\int_U \hat{k}(\underline{r},\underline{r}',E,E')\phi(\underline{r}',E')dE'd\underline{r}'.$$

$\hat{k}(\underline{r},\underline{r}',E,E')$ is defined in (16) with $\underline{r}'\in D_2\setminus D_1$ and $\underline{r}\in D_2$. Thus, by perfect analogy to the monoenergetic case, it follows that λ_2 is larger than $r(K_{D_1})$, i.e. $\lambda_2>\lambda_1$ [11].

The preceding results for the case of bounded domains are the same, under assumptions A.1-A.3, as the ones obtained in Sect. 3 for the monoenergetic transport operator. We omit the extension of the results of Sects. 4 and 5 to the energy-dependent case. However, on the basis of the analogy to the preceding arguments and under assumptions A.1-A.3, we may prove that in the energy-dependent model the strict order relation for the critical eigenvalues holds both for domains of cylindrical type and for spherical, cylindrical and slab geometry.

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