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STABILITY OF SOLUTIONS TO STURM-LIOUVILLE DIFFUSION EQUATIONS

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ABSTRACT

Stability of solutions of abstract half-space problems of the type $T\psi'(x) = -A\psi(x)(0 < x < \infty)$ is established under perturbations of the resolvent of the (unbounded) positive self-adjoint operator A. Applications are given to Sturm-Liouville type diffusion equations.

I. INTRODUCTION

In this article we investigate the stability of solutions of the boundary value problem

$$T\psi'(x) = -A\psi(x) \quad (0 < x < \infty) \tag{1.1}$$

$$Q_+\psi(0) = \phi_+ \tag{1.2}$$

$$||\psi(x)|| = O(1) \quad (x \to \infty) \tag{1.3}$$

under perturbations of A. Throughout T and A are defined on the Hilbert space H, T is bounded, self-adjoint and injective, Q_+ and Q_-

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are the orthogonal projections of H onto the maximal positive and negative T-invariant subspaces and A is positive self-adjoint and Fredholm. (Thus A has closed range and finite-dimensional null space but may be unbounded). Special instances occur in neutron transport[1], radiative transfer[2] and Fokker-Planck models[3] under steady state conditions with boundary conditions appropriate to incoming flux specification. We shall prove the stability of solutions to this problem under resolvent perturbations of A and give new applications to Sturm-Liouville type diffusion equations.

Stability results of this type were derived by Van der Mee[4] for the case when A is a compact perturbation of the identity and is either strictly positive or has a one-dimensional null space. The proof used stability properties of an equivalent vector-valued Wiener-Hopf equation and was inspired by related results of Feldman[5]. The results were generalized by Ran and Rodman[6] to arbitrary A of this type. In Refs. [4] and [6] the stability was obtained under perturbations of A in the operator norm. Recently, Hangelbroek[7] proved a stability result for arbitrary bounded and strictly positive self-adjoint A using the functional formulation and existence and uniqueness results of Beals[8]. We shall prove stability results under perturbations of $(A+K)^{-1}$ in the norm, using Beals' functional formulation.

In Section 2 we prove the basic stability result. Section 3 is devoted to Sturm-Liouville type applications. In Section 4 we make some remarks on related stability problems.

2. THE BASIC STABILITY THEOREM

Let H_T denote the completion of H with respect to the inner product[8]

$$(h,K)_T = (|T|h,K). \qquad (2.1)$$

Let H_A denote the direct sum of the null space Ker A and the completion of the domain D(A) of A with respect to the inner product[9]

$$(h,K)_A = (Ah,K). \qquad (2.2)$$

Then $H_A = D(A^{1/2}) \subset H \subset H_T$ and H_A allows the decomposition[10,11,12]

$$H_A = Z_0 \oplus Z_1$$

with the following properties: (i) $Z_0 = \text{Ker}(T^{-1}A)^2$ has finite dimension, contains Ker A and is invariant under $T^{-1}A$, (ii) Z_1 is an H_A-closed $T^{-1}A$-invariant subspace and $S_1 = (T^{-1}A|_{Z_1})^{-1}$ is bounded and $(.,.)_A$-selfadjoint on Z_1, and (iii) $(Th,K) = 0$ for all $h \varepsilon Z_0$ and $K \varepsilon Z_1$. Using the Spectral Theorem we may construct three bounded and complementary projections P_0, P_+^1 and P_-^1 on H_A satisfying: (i) Ran $P_0 = Z_0$ and Ran $P_+^1 \oplus$ Ran $P_-^1 = Z_1$, and (ii) $(\pm S_1 h,h)_A \geq 0$ for all $h \varepsilon$ Ran $P_\pm^1$. We may then define H_S as the direct sum of Z_0 and the completion of H_A with respect to the inner product[8]

$$(h,K)_{S_1} = (|S_1|h,K)_A = (T(P_+^1 - P_-^1)h,K). \qquad (2.3)$$

If $\{.,.\}$ is an arbitary inner product on the finite-dimensional space Z_0, we may identify H_T and H_S if (and only if) for some constants $c_1, c_2 > 0$

$$c_1||h||_T^2 \leq \{P_0h,P_0h\} + ((I-P_0)h,(I-P_0)h)_{S_1} \leq c_2||h||_T^2, \quad h \varepsilon H_A . \qquad (2.4)$$

For all our models we shall <u>assume</u> the norm equivalence (2.4). This equivalence is valid if either A is bounded[8], or T is a multiplication by an indefinite weight and A is a differential operator of Sturm-Liouville type[13]. If Ker A = {0}, then $Z_0 = \{0\}$ and $P_0 = 0$; we shall then drop the subscript 1 of P_+^1, P_-^1 and S_1.

LEMMA 2.1. *For all non-real* λ *the operators* $(A-\lambda T)^{-1}$, $(T^{-1}A-\lambda)^{-1}$ *and* $(AT^{-1}-\lambda)^{-1}$ *are defined as bounded operators on* H.

Proof. Let $(h_n)_n$ be an H-bounded sequence in D(A) and let λ be a non-real number satisfying $\|(A-\lambda T)h_n\| \to 0$. As $|\text{Im}\lambda| \neq 0$, we find $(Ah_n,h_n) \to 0$ and $(Th_n,h_n) \to 0$. Thus if A is strictly positive, we have $\|h_n\| \to 0$ and $(A-\lambda T)^{-1}$ bounded on its closed domain. However, since

$$[\text{Ran}(A-\lambda T)]^{\perp} = \text{Ker}(A^*-\bar{\lambda}T^*) = \text{Ker}(A-\bar{\lambda}T) = \{0\},$$

we have $(A-\lambda T)^{-1}$ bounded on H. On the other hand, if A has a non-zero null space, we change A into a strictly positive operator A_β satisfying $A|_{Z_1} = A_\beta|_{Z_1}$ (see Refs. [11,12]), observe that $(A-\lambda T)|_{Z_0}$ has a bounded inverse on $T[Z_0]$, exploit $(A_\beta-\lambda T)|_{Z_1} = (A-\lambda T)|_{Z_1}$ and obtain the same result. Finally, we notice the equalities

$$(T^{-1}A-\lambda)^{-1} = (A-\lambda T)^{-1}T, \; (AT^{-1}-\lambda)^{-1} = T(A-\lambda T)^{-1},$$

which imply the boundedness of $(T^{-1}A-\lambda)^{-1}$ and $(AT^{-1}-\lambda)^{-1}$. ∎

LEMMA 2.2. *Let us assume that* $(A_n)_{n=1}^{\infty}$ *is a sequence of positive self-adjoint Fredholm operators on* H *satisfying*

$$\exists K > 0 : \lim_{n\to\infty} \|(A_n+K)^{-1}-(A+K)^{-1}\| = 0. \tag{2.5}$$

Then for every λ *in the resolvent set* $\rho(T^{-1}A)$ *of* $T^{-1}A$ *there exists* $m = m(\lambda,K)$ *such that* $\lambda \in \rho(T^{-1}A_n)$ *for* $n \geq m$ *and*

$$\lim_{n\to\infty} ||(T^{-1}A_n-\lambda)^{-1} - (T^{-1}A-\lambda)^{-1}|| = 0 \qquad (2.6)$$

$$\lim_{n\to\infty} ||(A_nT^{-1}-\lambda)^{-1} - (AT^{-1}-\lambda)^{-1}|| = 0. \qquad (2.7)$$

Proof. It is easy to prove the following: If $\zeta \in \rho(A)$, then $\zeta \in \rho(A_n)$ for sufficiently large n and uniformly on compact subsets of $\rho(A)$

$$\lim_{n\to\infty} ||(A_n-\zeta)^{-1} - (A-\zeta)^{-1}|| = 0.$$

Indeed, for $B_n = (A_n+K)^{-1}$ and $B = (A+K)^{-1}$ we may write

$$(A_n-\zeta)^{-1} = B_n\{I-(K+\zeta)B_n\}^{-1}, \quad (A-\zeta)^{-1} = B\{I-(K+\lambda)B\}^{-1}.$$

Since $||B_n-B|| \to 0$ as $n\to\infty$, the above statement, including the uniformity part, is clear.

Also, if $\zeta \in \rho(T^{-1}A)$, then $\zeta \in \rho(T^{-1}A_n)$ for sufficiently large n, while Eqs. (2.6) and (2.7) with λ replaced by ζ would imply (2.6) and (2.7) themselves. Thus it suffices to prove the identities

$$\lim_{n\to\infty} ||(T^{-1}A_n-\zeta)^{-1} - (T^{-1}A-\zeta)^{-1}|| = 0$$

$$\lim_{n\to\infty} ||(A_nT^{-1}-\zeta)^{-1} - (AT^{-1}-\zeta)^{-1}|| = 0$$

for some $\zeta \in \rho(A) \cap \rho(T^{-1}A)$, uniformly in ζ on compact subsets of $\rho(A) \cap \rho(T^{-1}A)$. Indeed, let us first compute

$$(A_n-\zeta)^{-1}(A_n-\zeta T) = I + \zeta(A_n-\zeta)^{-1}(I-T) \to (A-\zeta)^{-1}(A-\zeta T)$$

as $n \to \infty$. We then have

$$(A_n-\zeta T)^{-1} = [I+\zeta(A_n-\zeta)^{-1}(I-T)]^{-1}(A_n-\zeta)^{-1} \to (A-\zeta T)^{-1}$$
$$(T^{-1}A_n-\zeta)^{-1} = (A_n-\zeta T)^{-1}T \to (T^{-1}A-\zeta)^{-1}$$
$$(A_nT^{-1}-\zeta)^{-1} = T(A_n-\zeta T)^{-1} \to (AT^{-1}-\zeta)^{-1}$$

as $n \to \infty$, uniformly in ζ on compact subsets of $\rho(T^{-1}A)$. All limits have been taken in the operator norm on H. ∎

Put

$$K_n = (T^{-1}A_n-\zeta)^{-1}-(T^{-1}A-\zeta)^{-1}, \quad K_n^+ = (A_nT^{-1}-\zeta)^{-1}-(AT^{-1}-\zeta)^{-1}.$$

Then K_n and K_n^+ are bounded operators on H satisfying $TK_n = K_n^+T$. We then easily obtain that K_n and $\tilde{K}_n = (Q_+ - Q_-)K_n^+(Q_+ - Q_0)$ are bounded operators on H having the property

$$(K_nf,g)_T = (f,\tilde{K}_ng)_T$$

for f, g ε H. Hence[14], K_n estends to a bounded operator on H_T satisfying

$$||K_n||_{H_T} \leq \max(||K_n||_H, ||\tilde{K}_n||_H) = \max(||K_n||_H, ||K_n^+||_H),$$

whence

$$\lim_{n\to\infty} ||(T^{-1}A_n-\zeta)^{-1}-(T^{-1}A-\zeta)^{-1}||_{H_T} = 0, \quad \zeta \in \rho(T^{-1}A), \tag{2.8}$$

uniformly in ζ on compact subsets of $\rho(T^{-1}A)$. As a consequence of (2.8) and the zero eigenvalue of $T^{-1}A$ (when present) being isolated, there exists an interval $[-L,L]$ and $m_1 \in \mathbb{N}$ such that (i) $\Sigma = \sigma(T^{-1}A) \cap [-L,L] \subset \{0\}$ and (ii), for $n \geq m_1$, $\Sigma_n = \sigma(T^{-1}A_n) \cap [-L,L]$ consists of finitely many eigenvalues whose algebraic multiplicities add up to the algebraic multiplicity of the zero eigenvalue of $T^{-1}A$. Moreover, if Γ is the positively oriented rectangle with vertices $\pm L \pm i$, then the spectral projections $Q_{0,n}$ satisfy $\lim_{n\to\infty} ||Q_{0,n}-P_0||_{H_T} = 0$, where

$$P_0 = \frac{-1}{2\pi i}\int_\Gamma (T^{-1}A-\zeta)^{-1}d\lambda.$$

PROPOSITION 2.3. For $n \geq m_1$, let $Q_{+,n}$ and $Q_{-,n}$ be the spectral projections of $T^{-1}A_n$ corresponding to the parts of the spectrum on (L,∞) and $(-\infty,L)$, respectively. We then have

$$\lim_{n\to\infty} ||[Q_{+,n}-P_+^1]h||_T = 0, \quad \lim_{n\to\infty}||[Q_{-,n}-P_-^1]h||_T = 0, \qquad (2.9)$$

where $h \in H_T$ is arbitrary.

Proof. Each of the subspaces $Q_{0,n}[H_T]$ and $P_0[H_T]$ is a nondegenerate indefinite inner product space[11,12] with respect to the sesquilinear form

$$[h,K] = (Th,K). \qquad (2.10)$$

Defining $W_{0,n}$ by

$$W_{0,n} = Q_{0,n}P_0 + (I-Q_{0,n})(I-P_0),$$

we have

$$\lim_{n\to\infty} ||W_{0,n}-I||_{H_T} = 0,$$

and therefore $W_{0,n}$ is invertible on H_T for all $n \geq m_2$ (where $m_2 \geq m_1$). Thus the restriction $S_{0,n}$ of $W_{0,n}^{-1}T^{-1}A_nW_{0,n}$ to $P_0[H_T]$ is a self-adjoint operator on $P_0[H_T]$ with respect to the indefinite inner product (2.10) which converges to the restriction S_0 of $T^{-1}A$ to $P_0[H_T]$. If we choose a fixed "fundamental" decomposition of $P_0[H_T]$, i.e.,

$$M_+ \oplus M_- = P_0[H_T],$$

where (i) $\pm[h,h] > 0$ for all $0 \neq h \in M_\pm$ and (ii) $[h,K] = 0$ for all $h \in M_+$ and $K \in M_-$, and define $P_{0,+}$ and $P_{0,-}$ as the projections of H_T onto M_+ and M_- along $(I-P_0)[H_T] \oplus M_0$ and $(I-P_0)[H_T] \oplus M_+$, respectively, then we may define[11,12]

$$A_{\beta,n} = A_n(I-Q_{0,n}) + TW_{0,n}(P_{0,+} - P_{0,-})W_{0,n}^{-1}$$

and obtain $A_{\beta,n}^{-1}T$ as a bounded self-adjoint operator on H_T (with respect to a suitable positive definite inner product) satisfying

$$\lim_{n\to\infty} ||A_{\beta,n}^{-1}T-A_\beta^{-1}T||_{H_T} = 0. \qquad (2.11)$$

Here

$$A_\beta = A(I-P_0) + T(P_{0,+} - P_{0,-}). \qquad (2.12)$$

It should be observed that $T^{-1}A_{\beta,n}$ and $T^{-1}A_\beta$ have restrictions to Ker $Q_{0,n}$ and Ker P_0 that coincide with the restrictions of $T^{-1}A_n$ and $T^{-1}A$ to the subspaces.

Using the identity

$$(\ell A_{\beta,n}-\zeta)^{-1}(\ell A_{\beta,n}-\zeta\ell T) = I+\zeta(\ell A_{\beta,n}-\zeta)^{-1}(I-\ell T)$$

where $\ell > 0$, we obtain

$$||(\ell A_{\beta,n}-\zeta)^{-1}(\ell A_{\beta,n}-\zeta\ell T)-I|| \le \frac{|\zeta|(1+\ell||T||)}{\sqrt{(|\zeta|^2+\ell^2\delta^2)}}, \text{ Re } \zeta = 0, \qquad (2.13)$$

where $\sigma(A_{\beta,n})\subset[\delta,\infty)$ for some $\delta > 0$ and $n \ge n_0$. If this upper bound is less than one, then for Re $\zeta = 0$

$$||(\ell A_{\beta,n}-\zeta\ell T)^{-1}|| \le [1 - \frac{|\zeta|(1+\ell||T||)}{\sqrt{(|\tau|^2+\ell^2\delta^2)}}]^{-1} \quad \frac{1}{\sqrt{(|\zeta|^2+\ell^2\delta^2)}},$$

whence

$$||(T^{-1}A_{\beta,n}-\zeta)^{-1}|| \le [1 - \frac{|\zeta|(1+\ell||T||)}{\sqrt{(|\zeta|^2+\ell^2\delta^2)}}]^{-1} \quad \frac{\ell||T||}{\sqrt{(|\zeta|^2+\ell^2\delta^2)}}, \quad \text{Re } \zeta = 0,$$

and $||(A_{\beta,n}T^{-1}-\zeta)^{-1}||$ has the same upper bound. Thus for Re $\zeta = 0$

$$||\zeta(T^{-1}A_{\beta,n}-\zeta)^{-1}||_{H_T} \le [1 - \frac{|\zeta|(1+\ell||T||)}{(\sqrt{|\zeta|^2+\ell^2\delta^2})}]^{-1} \quad \frac{\ell|\zeta|||T||}{\sqrt{|\zeta|^2+\ell^2\delta^2}}, \qquad (2.14)$$

provided the right-hand side of (2.13) is less than one. The latter is satisfied if

$$|\zeta| < \delta||T||^{-1} \text{ and } 1 + \frac{2}{\ell||T||} < \left(\frac{\delta}{|\zeta|||T||}\right)^2. \tag{2.15}$$

If we now choose $\ell = \frac{1}{4}||T||^{-1}$ and $|\zeta| \le \frac{1}{4}\delta\,||T||^{-1}$, then the conditions (2.15) are fulfilled and the right-hand side of (2.14) does not exceed $[1-(5\sqrt{2})/8]^{-1} \simeq 8.6120$, whence, for Re $\lambda = 0$, $|\lambda| > 4||T||/\delta$ and a constant M not depending on n,

$$||\lambda(\lambda-A_{\beta,n}^{-1}T)^{-1}||_{H_T} \le ||A_{\beta,n}^{-1}T||_{H_T}[1-(5\sqrt{2})/8]^{-1} \le M < \infty.$$

On the other hand, in order to prove the uniform H_T-boundedness of $\lambda(\lambda-A_{\beta,n}^{-1}T)^{-1}$ for imaginary λ with $|\lambda| \le 4||T||/\delta$ and $n \ge n_0$, we use Eq. (2.11), which implies that

$$\lim_{n\to\infty} ||\zeta(T^{-1}A_{\beta,n}-\zeta)^{-1} - \zeta(T^{-1}A_\beta-\zeta)^{-1}||_{H_T} = 0$$

uniformly in ζ for $\pm i\zeta \in (\frac{1}{4}\delta||T||^{-1},\infty)$ [cf. the second half of the proof of Lemma 2.2]. Hence, for $n \ge n_0$ and Re $\lambda = 0$ we have

$$||\lambda(\lambda-A_{\beta,n}^{-1}T)^{-1}||_{H_T} \le N < \infty,$$

where N does not depend on λ and n. We may now repeat the argument of the proof of Theorem VIII 1.15 of Ref. [15], where we exploit the estimate

$$||H_n(H_n^2+\eta^2)^{-1}H_n(H_n^2+I)^{-1}||_{H_T} \le N^2(1+N^2)\cdot\min(1,\eta^{-2})$$

for $\eta \in (0,\infty)$, $H_n = A_{\beta,n}^{-1}T$ and $n \ge n_0$ and the Spectral Theorem, and obtain the equalities (2.9). ∎

If Ker A = {0}, we choose 1/2L as the spectral radius of $A^{-1}T$. As a result we obtain

$$\lim_{n\to\infty} ||P_{+,n}h-P_+h||_T = 0, \quad \lim_{n\to\infty}||P_{-,n}h-P_-h|| = 0$$

for all $h \in H_T$, where $P_{+,n}$ and $P_{-,n}$ are the spectral projections of $T^{-1}A_n$ for the positive and negative parts of the spectrum. Since the unique solutions $\psi(x)$ of Eqs. (1.1)-(1.3) and $\psi_n(x)$ of the analogous problem with A replaced by A_n have the form

$$\psi(x) = e^{-xT^{-1}A}E\phi_+, \quad \psi_n(x) = e^{-xT^{-1}A_n}E_n\phi_+$$

where $E = V^{-1}$ with $V = Q_+P_+ + Q_-P_-$ and $E_n = V_n^{-1}$ with $V_n = Q_+P_{+,n} + Q_-P_{-,n}$, and since[12]

$$||E-I||_{H_T} < 1, \qquad ||E_n-I||_{H_T} < 1,$$

the identity

$$\lim_{n\to\infty} ||V_nh-Vh||_T = 0, \quad h \in H_T,$$

immediately gives

$$\lim_{n\to\infty} ||\psi(0) - \psi_n(0)||_T = 0. \tag{2.16}$$

Also, we may prove

$$\lim_{n\to\infty}||e^{-xT^{-1}A}P_+^1 h-e^{-xT^{-1}A_n}(I-Q_{0,n})P_{+,n}^1h||_T = 0, \quad h \in H_T.$$

In combination with (2.11) we then get

$$\lim_{n\to\infty} ||\psi(x) - \psi_n(x)||_T = 0,$$

which means that for Ker A = {0} the unique solution of Eqs. (1.1) - (1.3) is stable under perturbations of A.

Let us consider the more complicated case Ker A ≠ {0}. We first analyze the auxiliary problem

$$\begin{cases} T\hat{\psi}'(x) = -A_\beta\hat{\psi}(x) \quad (0 < x < \infty) & (2.17) \\ Q_+\hat{\psi}(0) = \phi_+ & (2.18) \\ ||\hat{\psi}(x)|| = O(1) \quad (x \to \infty), & (2.19) \end{cases}$$

where A_β is defined by (2.12). It is then easy to prove the identity

$$\lim_{n\to\infty} ||\hat{\psi}(x) - \hat{\psi}_n(x)||_T = 0,$$

where $\hat{\psi}_n(x)$ is the unique solution of the analogous problem with A_β replaced by $A_{\beta,n}$. In combination with $\lim_{n\to\infty} ||Q_{0,n}-P_0||_{H_T} = 0$ we find

$$\lim_{n\to\infty} ||(I-P_0)\hat{\psi}(x) - (I-Q_{0,n})\hat{\psi}_n(x)||_T = 0.$$

THEOREM 2.4. If Ker $A = \{0\}$ or if Ker $A = \text{span}\{\phi_0\}$ with $(T\phi_0,\phi_0) \geq 0$, and if

$$\exists K > 0 : \lim_{n\to\infty}||(A_n+K)^{-1} - (A+K)^{-1}|| = 0 \qquad (2.20)$$

for a sequence $(A_n)_{n=1}^{\infty}$ of positive self-adjoint Fredholm operators, then the unique solution in H_T of Eqs. (1.1)-(1.3) is approximated in the norm of H_T by the unique solution (for sufficiently large n) in H_T of Eqs. (1.1)-(1.3) with A replaced by A_n. If Ker $A = \{0\}$ or if Ker $A = \text{span}\{\phi_0\}$ with $(T\phi_0,\phi_0) < 0$, and if there exists $K > 0$ satisfying (2.20) for a sequence $(A_n)_{n=1}^{\infty}$ of positive self-adjoint Fredholm operators, then the unique solution in H_T of the boundary value problem

$$\begin{cases} T\psi'(x) = -A\psi(x) \quad (0 < x < \infty) & (2.21) \\ Q_+\psi(0) = \phi_+ & (2.22) \\ \lim_{n\to\infty} ||\psi(x)||_T = 0 & (2.23) \end{cases}$$

is approximated in the norm of H_T by the unique solution (for sufficiently large n) of the analogous problem with A replaced by A_n.

For the case when A is a compact perturbation of the identity, Ran and Rodman[6] have removed all restrictions on the structure of Ker A and have obtained stability as well as non-stability results. Their method also hinges on the auxiliary problem (2.17)-(2.19) in combination with a finite-dimensional stability problem and for this reason their results allow complete extension to the present situation.

Proof of Theorem 2.4. Since we have the identities

$$\lim_{n\to\infty} ||Q_{0,n}-P_0||_{H_T} = 0, \lim_{n\to\infty}||W_{0,n}-I||_{H_T} = 0$$

and

$$\lim_{n\to\infty} ||S_{0,n}-S_0||_{P_0[H_T]} = 0,$$

the theorem depends decisively on stability properties of certain invariant subspaces of S_0. If Ker A = {0} or if Ker A = span$\{\phi_0\}$ with $(T\phi_0,\phi_0) < 0$, then the boundary value problem (2.21) - (2.23) is uniquely solvable[15] and the stability result is immediate. If Ker A = span$\{\phi_0\}$ with $(T\phi_0,\phi_0) > 0$, then Z_0 = Ker A has dimension one and the stability result is again immediate from the unique solvability[12] of Eqs. (1.1)-(1.3) in H_T. If Ker A = span$\{\phi_0\}$ with $(T\phi_0,\phi_0) = 0$, then $T\phi_0 \in (\text{Ker } A)^{\perp}$ = Ran A implies the existence of $\phi_1 \in Z_0$ satisfying $T^{-1}A\phi_1 = \phi_0$, whence $Z_0 = \text{span}\{\phi_0,\phi_1\}$. Notice that the S_0-invariant subspace Ker A is a stable invariant subspace of S_0 in the sense of Ref. [17]. Using Corollary 8.3 of Ref. [17] the above stability result is immediate. ∎

3. Applications to Sturm-Liouville type diffusion equations

Let us consider the formal differential operator[18,19]

$$(A_{formal}h)(\mu) = - \frac{d}{d\mu} (p(\mu)h'(\mu)) + q(\mu)h(\mu) \qquad (3.1)$$

on $I = (a,b)$, where $p(\mu) > 0$ is locally absolutely continuous and $q(\mu)$ is bounded, continuous and nonnegative. If $-\infty < a < b < \infty$, $p(\mu)$ is continuous and strictly positive and $q(\mu)$ is continuous on $[a,b]$, the endpoints a and b are called regular and (3.1) can be turned into a self-adjoint operator on $L_2(I)$ by imposing the boundary conditions

$$\cos\alpha\, h(a) - p(a)\sin\alpha\, h'(a) = 0 \tag{3.2}$$

$$\cos\beta\, h(b) - p(b)\sin\beta\, h'(b) = 0. \tag{3.3}$$

The identity

$$\int_I (Ah)(\mu)\overline{h(\mu)}\, d\mu = \int_I \{p|h'|^2+q|h|^2\}d\mu+\text{cotan}\alpha|h(a)|^2-\text{cotan}\beta|h(b)|^2$$

indicates that A will be positive in $0 \leq \alpha \leq \frac{1}{2}\pi$ and $\frac{1}{2}\pi \leq \beta \leq \pi$. (For $\alpha = 0$ or $\beta = \pi$ the corresponding term is absent). For regular boundary value problems A will have a compact resolvent and zero will be a point of the resolvent of A or a simple isolated eigenvalue. Choosing a bounded continuous weight function $w(\mu)$ which changes its sign at the points $a < c_1 < c_2 < \ldots < c_n < b$ and satisfies the condition

$$w(\mu) = \pm|\mu - c_j|^{\alpha_j} v_j(\mu)$$

[for some $\alpha_j \geq 0$ and continuously differentiable function $v_j(\mu)$ such that $v_j(c_j) \neq 0$] in some neighborhood of c_j, the boundary value problem

$$w(\mu)\frac{\partial\psi}{\partial x}(x,\mu) = -(A\psi(x))(\mu) \qquad (0 < x < \infty,\quad \mu \in I) \tag{3.4}$$

$$\psi(0,\mu) = \phi_+(\mu) \quad \text{if } w(\mu) > 0 \tag{3.5}$$

$$\int_I |w(\mu)| \cdot |\phi(x,\mu)|^2 d\mu = O(1) \qquad (x \to \infty) \tag{3.6}$$

is uniquely solvable[13] if Ker A = {0} or if Ker A = span$\{\phi_0\}$ with

$\int_I w(\mu)|\phi_0(\mu)|^2 d\mu \geq 0$. If Ker A = {0} or if Ker A = span$\{\phi_0\}$ with $\int_I w(\mu)|\phi_0(\mu)|^2 d\mu < 0$, then the boundary value problem (3.4)-(3.6) where (3.6) is replaced by the condition

$$\lim_{x\to\infty} \int_I |w(\mu)| \cdot |\psi(x,\mu)|^2 d\mu = 0$$

is uniquely solvable[13,16] also. These results remain valid if the interval I of definition of the differential operator has one or two singular endpoints and the corresponding self-adjoint differential operator A is either strictly positive or is positive with an isolated simple zero eigenvalue.

Special cases of this result of Beals[13] are the following:

(I) Electron scattering[20]. Here Eqs. (3.4)-(3.6) have the form

$$\mu \frac{\partial\psi}{\partial x}(x,\mu) = \frac{\partial}{\partial\mu}((1-\mu^2)\frac{\partial\psi}{\partial\mu}) \quad (0 < x < \infty, \quad \mu \in (-1,1))$$

$$\psi(x, \pm 1) \text{ bounded}$$

$$\psi(0,\mu) = \phi_+(\mu) \quad (\mu \in (0,1))$$

$$\int_{-1}^{1} |\mu| \cdot |\psi(x,\mu)|^2 d\mu = O(1) \quad (x \to \infty).$$

In this problem A is positive self-adjoint[21] with an isolated simple zero eigenvalue and eigenfunction $\phi_0(\mu) \equiv 1$. Since $w(\mu) = \mu$ and $\int_{-1}^{1} \mu|\phi_0(\mu)|^2 d\mu = 0$, the corresponding half-space problem is uniquely solvable[22].

(II) Fokker-Planck equation[3]. Equations (3.4)-(3.6) can be written in the form

$$ve^{-v^2/2} \frac{\partial\psi}{\partial x}(x,v) = \frac{\partial}{\partial v}(e^{-v^2/2} \frac{\partial\psi}{\partial v}) \quad (0 < x < \infty, \quad v \in (-\infty,\infty))$$

$$\psi(0,v) = \phi_+(v) \quad (v \in (0,\infty))$$

$$\int_{-\infty}^{\infty} |v|e^{-v^2/2} |\psi(x,v)|^2 dv = O(1) \quad (x \to \infty).$$

Again A is positive self-adjoint with an isolated simple zero eigenvalue and eigenfunction $\phi_0(v) \equiv 1$ when defined on the L_2-space with weight $e^{-v^2/2}$. Since $w(v) = ve^{-v^2/2}$ and $\int_{-\infty}^{\infty} ve^{-v^2/2} |\phi_0(v)|^2 dv = 0$, the corresponding half-space problem is uniquely solvable.[23]

(III) Example with regular endpoints[24]. Equations (3.4)-(3.6) have the form

$$\mathrm{sgn}(\mu) \cdot \frac{\partial\psi}{\partial x}(x,\mu) = \frac{\partial^2\psi}{\partial\mu^2} \quad (0 < x < \infty,\ \mu \in (-1,1)) \tag{3.7}$$

$$\psi(x, \pm 1) = 0 \tag{3.8}$$

$$\psi(0,\mu) = \phi_+(\mu) \qquad (\mu \in (0,1)) \tag{3.9}$$

$$\int_{-1}^{1} |\psi(x,\mu)|^2 d\mu = O(1) \qquad (x \to \infty). \tag{3.10}$$

Here A is strictly positive self-adjoint and the half-space problem is uniquely solvable.[24] The same result remains true[24] if $\mathrm{sgn}(\mu)$ is replaced by μ and $\int_{-1}^{1} |\psi(x,\mu)|^2 d\mu$ by $\int_{-1}^{1} |\mu| \cdot |\psi(x,\mu)|^2 d\mu$.

It should be noted that the differential operators of examples (I) and (II) have two singular endpoints on their intervals of definition.

For Sturm-Liouville problems of the above type the stability results of Section 2 are available. Two types of perturbation of the operator A are of particular interest. First, we take $A_n = A + \varepsilon_n I$ for some sequence $\varepsilon_1 > \varepsilon_2 > \ldots > 0$ with zero limit, i.e., we replace (3.1) by

$$(A_{n,\mathrm{formal}} h)(\mu) = - \frac{d}{d\mu} (p(\mu)h'(\mu)) + \{q(\mu) + \varepsilon_n\}h(\mu)$$

while keeping invariant the boundary conditions. If A is strictly positive or if A is positive with an isolated simple zero eigenvalue and eigenfunction $\phi_0(\mu)$ satisfying

$$\int_I |w(\mu)| \cdot |\phi_0(\mu)|^2 d\mu \geq 0,$$

then the unique solution $\psi(x)$ of Eqs. (3.4)-(3.6) is approximated by the unique solution of the equation

$$w(\mu)\,\frac{\partial\psi_n}{\partial x}\,(x,\mu) = -(A_n\psi_n(x))(\mu) \qquad (0 < x < \infty, \quad \mu \in I)$$

with boundary conditions (3.5) and (3.6):

$$\psi_n(0,\mu) = \phi_+(\mu) \quad \text{if } w(\mu) > 0$$

$$\lim_{n\to\infty} \int_I |w(\mu)| \cdot |\psi(x,\mu) - \psi_n(x,\mu)|^2 d\mu = 0.$$

This situation applies, in particular, to the above first and third examples. A second type of perturbation is obtained by keeping invariant the formal differential operator (3.1) and by perturbing continuously the boundary conditions to the differential operator. We would then find the same stability result as for the first type of perturbation. For example, if in Eqs. (3.7)-(3.10) one replaces the Dirichlet boundary condition (3.8) by the Neumann boundary condition

$$\psi'(x, \pm 1) = 0, \tag{3.11}$$

one may consider a sequence $0 \leq \alpha_1 < \alpha_2 < \alpha_3 < \ldots < \frac{1}{2}\pi$ of angles converging to $\frac{1}{2}\pi$ and a concomitant sequence of boundary conditions of the type (3.2) and (3.3) with $\alpha = \alpha_n$ and $\beta = \pi - \alpha_n$ and prove that the unique solution $\psi_n(x)$ of Eqs. (3.7), (3.9) and (3.10) with the latter boundary conditions converges to the unique solution of Eqs. (3.7), (3.11), (3.9) and (3.10) as $n \to \infty$:

$$\lim_{n\to\infty} \int_{-1}^{1} |\psi(x,\mu) - \psi_n(x,\mu)|^2 d\mu = 0.$$

4. Concluding remarks

We have developed a stability theory for solutions of abstract half-space problems that is sufficiently general to apply to Sturm-Liouville type diffusion equations. As for these problems Ker A has at most dimension one, a generalization to more general structures of Ker A is not required, as far as Sturm-Liouville diffusion is concerned. However, one may use recent results of Ran and Rodman[6] to remove all restrictions on the structure of Ker A and to arrive at stability as well as non-stability results. In particular, if all vectors $\phi \in$ Ker A satisfy $(T\phi,\phi) \geq 0$, the unique solution of Eqs. (1.1)-(1.3) is stable under suitable perturbations of A.

Analogous stability results can be developed also for the finite-slab boundary value problem

$$T\psi'(x) = -A\psi(x) \qquad (0 < x < \tau) \tag{4.1}$$

$$Q_+\psi(0) = \phi_+, \quad Q_-\psi(\tau) = \phi_-. \tag{4.2}$$

More precisely, if $(A_n)_{n=1}^{\infty}$ is a sequence of strictly positive self-adjoint operators or positive self-adjoint operators with isolated zero eigenvlaue, satisfying (2.20) for some $K > 0$, then the unique solution $\psi(x)$ of Eqs. (4.1)-(4.2) can be approximated in H_T by the unique solution of the boundary value problem

$$T\psi_n'(x) = -A_n\psi_n(x) \qquad (0 < x < \tau)$$

$$Q_+\psi_n(0) = \phi_+, \quad Q_-\psi_n(\tau) = \phi_- .$$

Here all of the operators T, A_n and A should be chosen in such a way that the space H_S and the analogous spaces H_{S_n} connected with T and A_n can all be identified with H_T. We shall omit the details. We observe that in this way we may generalize recent finite-slab stability results of Hangelbroek[7] for bounded A and A_n.

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