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# PROBLEMS IN THE ABSTRACT THEORY OF STATIONARY ONE DIMENSIONAL TRANSPORT*

by

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ABSTRACT

Eight unsolved problems, related to the abstract stationary one-dimensional linear transport equation, are presented.

Nearly 25 years ago, K. M. Case introduced the method of singular eigenfunctions to solve one-dimensional stationary transport problems. His papers on linear Vlasov theory [1] and neutron transport [2] began an industry which is still thriving. Numerous equations in neutron transport, radiative transfer, gas kinetics, phonon and electron transport and plasma theory have been studied on a case-by-case basis. Several years ago, R. Beals [3], expanding on the seminal contribution

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of R. J. Hangelbroek [4], initiated the study of the abstract transport equation

$$T \frac{\partial f}{\partial x} = -A f(x) \tag{1}$$

on a half space $0 \le x < \infty$, with boundary conditions appropriate to incoming flux problems. Here, T and A are operators on a suitable linear space. The abstract equation includes, as special cases, virtually all of the stationary one-dimensional equations previously studied.

There is now available a complete existence and uniqueness theory [3,5,6,7] for Eq. (1), where T and A are (possibly unbounded) operators on an abstract Hilbert space satisfying: T self-adjoint and injective, A positive, self-adjoint and Fredholm. The "incoming flux" boundary conditions of interest are, depending on the physical context, either

$$\begin{aligned} (Q_+ f)(x=0) &= f_+ \\ \lim_{x\to\infty} \|f(x)\| &= 0 \end{aligned} \tag{2}$$

or

$$\begin{aligned} (Q_+ f)(x=0) &= f_+ \\ \limsup_{x\to\infty} \|f(x)\| &< \infty \end{aligned} \tag{3}$$

for given $f_+$, where $Q_+$ is the maximal T-invariant projection on whose range T is positive. The article of van der Mee in the next issue of T.T.S.P. describes some of the results for subcritical (that is, $A > 0$) and critical ($A \ge 0$) systems, and Hangelbroek's contribution to the *Conference* centers around the subcritical case. Reference [7] presents the complete, and mathematically most complex, theory for supercritical systems ($0 < \dim \operatorname{Ran} \{A-|A|\} < \infty$) and a derivation of the relation between the "complementary" boundary conditions (2)-(3) for subcritical and critical systems as well.

In this note, and on the occasion of the *Eighth International Transport Theory Conference*, we wish to present *eight* unsolved problems connected with the abstract transport equation (1). Perhaps it will be possible to hear of their solution at the *Ninth Conference*; especially we would like to stimulate substantial investigation outside the research groups presently engaged in this area.

We list now the first six of these unsolved problems.

*PROBLEM I. A with finite dimensional negative part, A, $A^{-1}$, T all unbounded.*

Ref. [6] deals with T unbounded, but $A \geq 0$. Ref. [7] allows a finite dimensional negative part of A, but requires T bounded. We note that there is, as yet, no theorem for Pontrjagin spaces classifying the self-adjoint extensions of symmetric operators, which presents an additional difficulty.

*PROBLEM II. A non-Fredholm.*

The Fredholm condition was chosen for convenience. In the notation of the references, it, among other things, guarantees the imbedding $H_A \subseteq H$ and the density of the domain of $T^{-1}A$ in H.

*PROBLEM III. Explicit representation of solutions.*

The connection with the singular eigenfunction expansions of the operator $T^{-1}A$ should be made. In case $I - A$ is a compact operator, the solution f(0) of Eq. (1) at x=0 should be expressed in terms of some generalization of Chandrasekhar's H-functions (In this regard see, for example, Kelley [8].) If one drops such compactness assumptions, it is not clear how to proceed with such an H-function derivation. For instance, if A has compact resolvent (which occurs in many Fokker-Planck type equations), the half space problem does not appear expressible in integral form unless one allows for highly singular kernels (For an example we refer to Ganapol [10]). Indeed, it is not at all clear how to extend notions like dispersion and H-functions to general abstract models and how to exploit such functions to arrive at an expression for f(0).

It is tempting to speculate that the equivalence of the half space problem (1)-(2) to a vector-valued integral equation of convolution type may play a decisive role in establishing the relationship with H-functions. Similar problems exist on the edge of linear systems theory, and parallel to [9] Bart, Gohberg and Kaashoek have developed an abstract half space and slab theory of equivalence (see [11]).

*PROBLEM IV. A non-self-adjoint.*

Refs. [3,4,5,6,7] depend explicitly on the self-adjointness of A. On the other hand, Case's method and the method of resolvent integration [12] work equally well for non-self-adjoint problems. Important applications of this type include non-symmetric multigroup neutron transport, multi-component gas kinetic equations and polarized light transfer.

*PROBLEM V. Banach space theory.*

Again, Case's method and the method of resolvent integration adapt to Banach spece applications. Since f generally represents a particle density, Eq. (1) in a Banach space setting (e.g., $L_1$ setting) is the physically most important problem. In fact, the Hilbert space setting is generally chosen for mathematical convenience. (See the discussion during the *Fourth Transport Theory Conference* [13].)

*PROBLEM VI. Equivalence of theory in $L_p$ spaces.*

Ref. [7] defines measures of non-uniqueness and measures of non-completeness. If the solution of Problem V is realized, it would be useful to know if the aforementioned measures are each equal in all $L_p$ spaces. One could, for instance, prove the equivalence of different $L_p$ settings and transfer the existence and uniqueness theory from the mathematically convenient $L_2$ setting to the physically relevant $L_1$ setting. In [14] this is done for neutron transport, where the half space problem can be written in integral form. If such integral form is not available, it is not clear how to proceed.

The last two open problems we pose are in slab geometry, rather than half space geometry. If we write $Q_- = I - Q_+$, then the slab problem with incoming flux boundary conditions is Eq. (1) with $0 < x < a$ and boundary conditions

$$\begin{aligned} (Q_+f)(x=0) &= f_+ \\ (Q_-f)(x=a) &= f_-. \end{aligned} \tag{4}$$

The solution of this problem in the "enlarged" space $H_K = H_T$ was given by Beals [3] for bounded positive A, and in the original space H by van der Mee [9] for positive A with compactness conditions

$$\begin{aligned} &I - A \ \text{ compact,} \\ &\text{Ran } (I - A) \subseteq \text{Ran } |T|^{\alpha}, \quad 0 < \alpha < 1. \end{aligned} \tag{5}$$

One should mention in this regard also the work of Beals [15], which, although not treating the abstract equation (only positive Sturm-Liouville differential operators are studied), is carried out in a most instructive general setting. Of course, the positivity assumption on A excludes all applications to multiplying media in slab geometry.

The slab problem with various reflective boundary conditions is also of obvious physical importance. Of the results available for special models and special reflective conditions, the recent work of Beals and Protopopescu [16; see also the contribution of Protopopescu to the *Conference*] for certain second order differential operators and absorbing plus specularly reflecting surfaces is again noteworthy for its general setting. The abstract problem with reflection at one surface is described by Eq. (1) with $0 < x < a$ and boundary conditions

$$\begin{aligned} (Q_+ f)(x=0) &= f_+ \\ (Q_- f)(x=a) &= (JRQ_+ f)(x=a) \end{aligned} \qquad (6)$$

where J is an inversion symmetry and R the surface reflection operator. Recently, results have been obtained [17] under the very general assumption

$$0 \le |T|R \le |T| \qquad (7)$$

on the surface reflection operator (including specular, absorbing, diffuse, and most other physical reflection laws), the compactness conditions (5) and A positive. Thus,

*PROBLEM VII. Removal of the boundedness condition on A (respectively, the compactness conditions (5)) for the slab problem with* *incoming flux boundary conditions and with general reflective boundary conditions.*

*PROBLEM VIII. Slab problems for non-positive A.*

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