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A stationary criticality problem in general L_p-space for energy dependent neutron transport in cylindrical geometry *)

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1. Introduction

The theoretical interest and the great deal of literature which has been published on criticality problems in multiplying media are justified by their significance with respect to nuclear reactor applications. The criticality problem is usually formulated as arising from the stationary linearized Boltzmann equation, either in integro-differential or integral form, and it is studied as an eigenvalue problem in a suitable functional space. In integral equation form one has to face the eigenvalue problem for an integral operator, which has a singular kernel depending on the geometry of the medium.

Much of the literature has been devoted to these topics. In most cases one is searching for the eigensolutions of the criticality problem in a Hilbert space of square integrable functions.

Recently some authors have formulated the problem in a space of continuous functions (see, for example, [4], [6], [21], [22]).

In the criticality analysis for energy dependent transport, the integral operator is not symmetric with respect to the couple x, E (position, energy) and hence an L_2 analysis based on the self-adjointness of the operator is no longer possible. On the other hand, the Perron-Fröbenius-Jentsch theory for positive operators, as developed in the fundamental paper by Krein and Rutman [14], supplies a powerful tool in order to search for eigensolutions in a cone with interior, such as the positive cone of C.

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However, it is our opinion that a space of summable functions is the most appropriate one from a physical point of view, because the L_1 norm of the neutron flux is the total number of particles. Our opinion is shared by several authors (see [16], [18], for instance).

In this paper we study the stationary energy dependent problem in integral form for a multiplying cylindrical medium. This geometry, which leads to a rather complicated singular kernel of the integral operator, suffers from a considerable theoretical gap in comparison with slab and spherical geometry, in spite of its actual importance in reactor analysis.

Our study is carried out in an L_1 space, in which the positivity properties of the operator still supply a suitable tool for establishing criticality, although the positive cone of L_1 has no interior. The concept of u_0-positivity, as formulated by Krasnosel'skii, replaces the one of strong positivity, and supplies the criticality analysis in L_1 space (or, in general, L_p spaces) with a new basis. On the other hand, the strict positivity of the transport operator (namely, its kernel is a.e. positive) enables us to show that the critical solution is positive (a.e.).

Consider a multiplying homogeneous cylinder of infinite height, embedded in vacuum or in a purely absorbing medium; under assumptions of isotropic scattering the stationary transport equation for the total neutron flux $\varphi(x, E)$ reads as follows, [3]:

$$\varphi(x, E) = c \int_0^1 \int_{E_m}^{E_M} T(x, x', E)\, S(E, E')\, \varphi(x', E')\, \mathrm{d}E'\, \mathrm{d}x', \tag{1}$$

$$T(x, x', E) = \frac{1}{4\pi} \int_0^{2\pi} \int_{-\infty}^{+\infty} \frac{x' \exp\left[-\sigma(E)\left(x^2 + x'^2 + z^2 - 2xx'\cos\theta\right)^{1/2}\right]}{x^2 + x'^2 + z^2 - 2xx'\cos\theta}\, \mathrm{d}z\, \mathrm{d}\theta \tag{2}$$

$$S(E, E') = (E'/E)^{1/2}\, \sigma(E')\, f(E, E'); \tag{3}$$

where the cylinder radius is normalized to one and a finite interval of admissible energies is considered: $0 < E_m \le E \le E_M < \infty$. Here $\sigma(E)$ is the total optical thickness, which is equal to the total cross-section $\Sigma(E)$ multiplied by the cylinder radius R; c is the average number of secondaries per collision which is supposed to be independent of energy.

$f(E, E')$ is the energy transfer function and $f(E, E')\, \mathrm{d}E\, \mathrm{d}E'$ is the probability that a neutron (a neutron packet) with energy in $(E', E' + \mathrm{d}E')$ emerges with energy in $(E, E + \mathrm{d}E)$ after collision, and it is chosen such that

$$\int_{E_m}^{E_M} f(E, E')\, \mathrm{d}E = 1, \qquad \forall E' \in [E_m, E_M]. \tag{4}$$

If we recall the definition of c: $c\,\sigma(E) = \sigma_s(E) + \nu\,\sigma_f(E)$, $(\sigma_s(E) = R\,\Sigma_s(E)$, Σ_s being the macroscopic scattering cross-section, $\sigma_f(E) = R\,\Sigma_f(E)$, Σ_f being the macroscopic fission cross-section, and ν the average number of secondaries per fission), (3) can be explicited by means of the energy transfer functions for scattering and fission:

$$S(E,E') = (E'/E)^{1/2}\,(\sigma_s(E')\,f_s(E,E') + \nu\,\sigma_f\,(E')\,f_f\,(E,E')),$$

where both $f_s(E,E')$ and $f_f\,(E,E')$ are normalized with respect to emerging energy.

The existence and uniqueness of the solution of (1) in L_1 space rely on the following assumptions on $\sigma(E)$ and $S(E,E')$, (in order to simplify notations, let $I=[0,1]$, $U=[E_m,E_M]$):

A.1 $\sigma(E)$ is a real-valued, essentially bounded function of E, such that

$$0<\sigma_m\le\sigma(E)\le\sigma_M<\infty,\quad \text{a.e. on } U.$$

A.2 $S(E,E')$ is a non-negative measurable function from $U\times U$ into $\mathbb{R}$.

A.3 A real-valued, a.e. positive, Lebesgue integrable function $r(E)$ exists such that

$$S(E,E')\le r(E),\quad \text{for a.e. } E,E'\in U.$$

A.4 $S(E,E')$ has the properties

$$0<m\le \operatorname{ess\,inf}\left\{\int_U S(E,E'')\,S(E'',E')\,\mathrm{d}E'';\,E,E'\in U\right\},$$

$$\operatorname{ess\,sup}\left\{\int_U S(E,E'')\,S(E'',E')\,\mathrm{d}E'';\,E,E'\in U\right\}\le M<+\infty.$$

Assumptions 1 and 2 are essentially technical; one can remark that A.2 implies $S(E,E')\in L_1(U\times U)$, because of (4). Assumptions 3 and 4 are necessary in order to work in an L_1 setting; A.3 is anyway consistent with (4) and gives a sufficient condition for the weak compactness of the operator S:

$$S: L^1(U)\to L^1(U),$$

$$(S\,g)(E)=\int_U S(E,E')\,g(E')\,\mathrm{d}E';$$

A.4 gives a sufficient condition for the u_0-positivity of S in L_1. As we shall see in the last section, A.4 can be made far less restrictive, but for the sake of simplicity we shall give all proofs with A.4.

We now give the abstract formulation of our problem, introducing the functional space $L_1(V)$, $V=I\times U$, in which we shall look for solutions to Eq. (1).

Eq. (1) in $L_1(V)$ reads as follows:

$$\varphi = c\,K\,\varphi, \tag{5}$$

where

$$K: L_1(V)\to L_1(V),$$

$$(K\,f)(x,E)=\int_I\int_U T(x,x',E)\,S(E,E')\,f(x',E')\,\mathrm{d}E'\,\mathrm{d}x'$$

with $T(x,x',E)$ and $S(E,E')$ defined by means of (2) and (3).

The present work is a continuation of [5] and draws back on some techniques exploited in [17]. The plan of this paper is the following. In Sec. 2 we shall prove that K is a weakly compact operator on $L_1(V)$ and, hence, that K^2 is compact on $L_1(V)$; moreover, we shall prove that K is a u_0-positive operator. Section 3 is devoted to the investigation of the eigenvalue problem; the positivity properties of K enable us to state the existence of an a.e. positive eigenfunction, corresponding to a (first, simple) dominating eigenvalue λ_0.

In Sec. 4, among other results, we obtain the continuous and monotonical dependence of λ_0 on the radius R characterizing the geometry of the system.

As far as the physical problem is concerned, we recall that the average number c_0 of secondaries per collision, required to keep critical a cylinder of radius R, is equal to $1/\lambda_0$. Thus, we give a sufficient condition to warrant that the critical solution exists in a multiplying medium situation, i.e. the critical eigenfunction corresponds to some $c_0 = 1/\lambda_0 > 1$.

In the fifth section, after introducing some definitions concerning Riesz operators, we shall conclude that the spectrum of K on L_p $(1 \le p < \infty)$ does not depend on p. The last section is a discussion about the generality of our assumptions. Here we shall point out how Assumption A.4 can be generalized.

2. Properties of the operators K and K^2

Before studying properties of the integral operator K, let us recall the following inequality for $T(x, x', E)$, [4], [20]:

$$0 \le x' J \le T(x, x', E) \le K_0(\sigma_m |x - x'|) \quad \text{for a.e.} \quad E \in U, \quad x \neq x', \tag{6}$$

where K_0 is the modified Bessel function of zero order, [1], and

$$2J = \int_{-\infty}^{+\infty} \exp(-\sigma_M (4 + z^2)^{1/2})/(4 + z^2)\, dz > 0.$$

Inequality (6) states that $T(x, x', E)$ is a nonnegative kernel with a weak singularity, since $K_0(u) \sim -\ln u$ as $u \to 0$.

First of all, we state the following proposition:

Proposition 1. i) K is weakly compact as an operator acting on $L_1(V)$, hence K^2 is compact on $L_1(V)$; ii) K^2 is bounded as an operator acting from $L_1(V)$ into $L_\infty(V)$.

Proof: i) Since the Lebesgue measure of V is finite, it is sufficient to prove that for each $\varepsilon > 0$, there exists $\delta > 0$ such that:

$$\iint_W |T(x, x', E)\, S(E, E')|\, dE\, dx < \varepsilon,$$

uniformly with respect to $(x', E') \in V$, for each measurable set $W \subset V$, with $\operatorname{mes}(W) < \delta$, [19].

Recalling inequality (6) and applying A.3, one has

$$|T(x, x', E)\, S(E, E')| \le K_0(\sigma_m |x - x'|)\, r(E), \quad \text{for a.e.} \quad E' \in U, \quad x \ne x'.$$

Hence, putting $\sigma_m(x - x') = s$, we have for each measurable set $W \subset V$

$$\iint_W |T(x, x', E)\, S(E, E')|\, \mathrm{d}E\, \mathrm{d}x \le \sigma_m^{-1} \iint_{T(W)} K_0(|s|)\, r(E)\, \mathrm{d}E\, \mathrm{d}s, \tag{7}$$

where $T(W)$ means the transformation of W, because of the change of variable, with $\operatorname{mes}(T(W)) = \sigma_m \operatorname{mes}(W)$, $K_0(|s|)\, r(E) \in L_1(T(V))$, because of the logarithmic singularity of $K_0(|s|)$; therefore, the absolute continuity of the Lebesgue integral ensures that, for each $\varepsilon < 0$, there exists $\delta > 0$ such that (7) is less than ε, for each W with $\operatorname{mes}(W) < \delta$.

So K is a weakly compact operator on $L_1(V)$ and hence, K^2 is compact, [8].

ii) Directly from the definition of K, one has

$$\begin{aligned}
|(K^2 f)(x, E)| = \Big|\int_I \int_U \Big\{\int_I \int_U T(x, x'', E)\, T(x'', x', E'')\, S(E, E'')\, S(E'', E')\, \mathrm{d}E''\, \mathrm{d}x''\Big\} \\
\cdot f(x' E')\, \mathrm{d}E'\, \mathrm{d}x'\Big| \le \int_I \int_U \Big\{\int_I K_0(\sigma_m |x - x''|)\, K_0(\sigma_m |x'' - x'|)\, \mathrm{d}x''\Big\} \\
\cdot \Big\{\int_U S(E, E'')\, S(E'', E')\, \mathrm{d}E''\Big\} \cdot |f(x', E')|\, \mathrm{d}E'\, \mathrm{d}x',
\end{aligned} \tag{8}$$

where we used (6). Consider now the following inequality

$$\int_I K_0(\sigma_m |x - x''|)\, K_0(\sigma_m |x'' - x'|)\, \mathrm{d}x'' \le \int_0^{+\infty} K_0^2(\sigma_m y)\, \mathrm{d}y = \pi^2/4\sigma_m,$$

where, for the computation of the latter integral, see [9], page 693. Substituting this into (8), one obtains

$$|(K^2 f)(x, E)| \le (\pi^2/4\sigma_m) \int_I \int_U \Big\{\int_U S(E, E'')\, S(E'', E')\, \mathrm{d}E''\Big\}\, |f(x', E')|\, \mathrm{d}E'\, \mathrm{d}x'.$$

Hence, from A.4 one finally has

$$|(K^2 f)(x, E)| \le \pi^2 M \|f\|/4\sigma_m, \quad \text{for a.e.} \quad (x, E) \in V,$$

and the proposition is completely proved. □

Let us denote by L^+ the usual cone of a.e. nonnegative functions in $L_1(V)$. We now formulate the following proposition:

Proposition 2. K is a u_0-positive operator, with $u_0 \equiv 1$ and exponent equal to 2, [11], [12]; i.e., for each $f \in \mathrm{L}^+$ (non-zero), there exist two positive numbers α, β (depending on f), the exponent $p = 2$ and the positive vector $u_0 \equiv 1$, such that

$$\alpha(f)\, u_0 \le K^p f \le \beta(f)\, u_0.$$

Proof: Let us start proving that $\alpha(f)$ exists: $\alpha(f)\, u_0 \leq K^2 f, \forall f \in L^+$. Let f be nonnegative; one has:

$$(K^2 f)(x, E) \geq \int_I \int_U \left\{\int_I J^2 x' x'' \,\mathrm{d}x''\right\} \left\{\int_U S(E, E'')\, S(E'', E')\,\mathrm{d}E''\right\} f(x', E')\,\mathrm{d}E'\,\mathrm{d}x'$$

where inequality (6) is used. Then, making use of A.4, we obtain for each $f \in L^+$, non-zero,

$$K^2 f \geq J^2 m\, \alpha'(f)/2 = \alpha(f) > 0,$$

where

$$\alpha'(f) = \int_I \int_U x' f(x', E')\,\mathrm{d}E'\,\mathrm{d}x' > 0.$$

The proof that $K^2 f \leq \beta(f)\, u_0, \forall f \in L^+$, with a suitable β, follows directly from ii) of Proposition 1, putting $\beta(f) = \pi^2 M \,\|f\|/4\,\sigma_m$. □

3. The eigenvalue problem

The properties we proved in the preceding section enable us to solve the eigenvalue problem in L^+ connected with Eq. (5):

$$(\lambda I - K)\,\varphi = 0. \tag{10}$$

In fact, the positivity of K and the other forementioned properties of K confirm the effectiveness of the theory of positive operators in Banach spaces for the criticality analysis of a stationary transport process. As regards to the existence of a dominant eigenvalue with the corresponding positive eigenfunction, we state the following main theorem:

Theorem 1. Let $K: L_1(V) \to L_1(V)$ be a linear integral operator with a.e. positive kernel, and let the following assumptions be satisfied:

i) K is weakly compact;
ii) for each $f \in L^+$ (non-zero), there exist two positive numbers $\alpha = \alpha(f)$, $\beta = \beta(f)$ such that

$$\alpha(f) \leq K^2 f \leq \beta(f);$$

iii) K^2 maps $L_1(V)$ into the subspace of bounded functions:

$$|K^2 f| \leq M_1 \|f\|, \quad \forall f \in L_1(V).$$

Then K has one and only one eigenfunction $f_0 \in L^+$, a.e. positive, and a corresponding dominant eigenvalue $\lambda_0 > 0$ (λ_0 is simple and greater in modulus than any other eigenvalue of K).

Proof: Assumption i) means that K^2 is a compact operator on $L_1(V)$. Assumption ii) means that K^2 is u_0-positive, with $u_0 \equiv 1$ (see Propositions 1 and 2). It is well known that a compact linear u_0-positive operator has a unique nonnegative eigenfunction ([11], Th. 2.2). Let f_0 be such an eigenfunction of the operator K^2, in the cone L^+, with the corresponding eigenvalue $\mu_0 > 0$.

Since K^2 is strictly L^+-positive, namely it has an a.e. positive kernel, f_0 is such that

$$f_0(x, E) = \mu_0^{-1} (K^2 f_0)(x, E) > 0, \quad \text{for a.e.} \quad (x, E) \in V.$$

Let us now consider the problem of uniqueness. From assumption iii), for each $f \in L_1(V)$, non-zero, there exist a natural number $p = 1$ and a real number $v = v(f) > 0$, such that

$$v(K^2)^p f = v(f) K^2 f \leq 1,$$

where 1 is the function with respect to which K^2 is u_0-positive, and where $v(f) = 1/M_1 \|f\|$. This fact guarantees that μ_0, the positive eigenvalue of K^2 corresponding to f_0, is simple and greater in modulus than the remaining eigenvalues; so, in other words, μ_0 is the dominant eigenvalue.

Let us now consider the operator K.

If we apply K to both sides of the following equation:

$$K^2 f_0 = \mu_0 f_0, \qquad \text{one has}$$

$$K^2(K f_0) = \mu_0 K f_0,$$

namely, $K f_0$ also is an eigenfunction of K^2, corresponding to μ_0. From the geometric simplicity of μ_0, $K f_0$ must be a multiple of f_0: $K f_0 = \lambda_0 f_0$.

If we apply K again, we have

$$\mu_0 f_0 = K^2 f_0 = \lambda_0 K f_0 = \lambda_0^2 f_0,$$

and hence

$$\lambda_0 = (\mu_0)^{1/2} > 0.$$

So K has the same positive eigenfunction f_0 as K^2, with the corresponding positive eigenvalue $\lambda_0 = (\mu_0)^{1/2}$. We now apply Theorems 2.10, 2.11 and 2.13 of [12] and draw the following conclusions: i) λ_0 is an eigenvalue of K of geometric multiplicity one (as well as of rank one), ii) any other eigenvalue λ of K has the property $|\lambda| < \lambda_0$, and iii) λ_0 is the only eigenvalue of K for which there is a corresponding nonnegative eigenfunction. Herewith we have completed the proof. □

4. Monotonicity of λ_0 and critical solution

First of all, let us remark that in the eigenvalue problem (10) one has $\lambda = 1/c$, where c is the average number of secondaries per collision. Thus, Theorem 1 also states the existence of $c_0 = 1/\lambda_0$, where the properties of c_0 can be derived directly from those already proved for λ_0.

Let us now investigate the dependence of $\lambda_0 = \lambda_0(R)$ on the cylinder radius R.

To this purpose we transfer our problem to the space $L_1((0, R) \times (E_m, E_M))$, by the change of variable $y = Rx$. This leads us to introduce the following scaling transformation:

$$S_R : L_1((0,1) \times (E_m, E_M)) \to L_1((0, R) \times (E_m, E_M)),$$

defined by

$$(S_R \varphi)(y, E) = \varphi(y/R, E).$$

Its inverse is given by

$$(S_R^{-1} \psi)(x, E) = \psi(Rx, E).$$

The original operator K, (5), enables us to define the following operator, acting in the space $L_1((0, R) \times (E_m, E_M))$:

$$\hat{K} = S_R K S_R^{-1},$$

given by

$$(\hat{K}\psi)(y, E) = \int_0^R \int_{E_m}^{E_M} \hat{T}(y, y', E)\, \hat{S}(E, E')\, \psi(y', E')\, \mathrm{d}E'\, \mathrm{d}y', \tag{11}$$

where

$$\hat{T}(y, y', E) = \frac{1}{4\pi} \int_0^{2\pi} \int_{-\infty}^{+\infty} \frac{y' \exp[-\Sigma(E)(y^2 + y'^2 + w^2 - 2yy'\cos\theta)^{1/2}]}{y^2 + y'^2 + w^2 - 2yy'\cos\theta}\, \mathrm{d}w\, \mathrm{d}\theta;$$

$$\hat{S}(E, E') = (E'/E)^{1/2}\, \Sigma(E')\, f(E, E').$$

It is relevant remarking that the kernel of $\hat{K}$ does not depend on R.

We now pay attention to the eigenvalue problem for the operator $\hat{K}$, in order to find nonnegative $\psi \in L_1((0, R) \times (E_m, E_M))$, such that

$$(\lambda I - \hat{K})\psi = 0. \tag{12}$$

It is easy to transfer to the operators $\hat{K}$ and $\hat{K}^2$ the results of Propositions 1 and 2 obtained for the operators K and K^2. Thus $\hat{K}^2$ is a compact operator on $L_1((0, R) \times (E_m, E_M))$ and it is bounded as an operator acting from $L_1((0, R) \times (E_m, E_M))$ into $L_\infty((0, R) \times (E_m, E_M))$; moreover $\hat{K}$ is v_0-positive with exponent equal to 2 and $v_0 \equiv 1$.

Theorem 1 enables us to draw the analogous conclusions for $\hat{K}$ and to state the following proposition:

Proposition 3. There exists a unique positive eigenvalue λ_0 of $\hat{K}$ (and also K), which is dominant and corresponds to a positive eigenfunction $\psi_0 = S_R f_0$, where f_0 is the positive eigenfunction of K, corresponding to λ_0.

We shall now establish two propositions, in order to compare the values assumed by the dominant eigenvalue in correspondence to the variation of the cylinder radius. With this aim we list here some results for the non-homogeneous equation connected with the eigenvalue problem (10).

Proposition 4. Consider the equation

$$(\lambda I - K)\,\varphi = f \tag{13}$$

on $L_1((0,1)\times(E_m, E_M))$. Then

i) for $\lambda > \lambda_0(R)$ Equation (13) is uniquely solvable; if $f \in L^+$, then also $\varphi \in L^+$;
ii) for $0 < \lambda \leq \lambda_0(R)$ and $f \in L^+$, non-zero, Equation (13) does not have solutions $\varphi \in L^+$.

Proof: i) Note that $\lambda_0(R) = r(K)$, where $r(K)$ is the spectral radius of K. As $\lambda > r(K)$, one has the invertibility of $(\lambda I - K)$ and

$$\varphi = (\lambda I - K)^{-1} f = \sum_{n=0}^{\infty} \lambda^{-(n+1)} K^n f \geq 0 \quad \text{whenever} \quad f \geq 0.$$

ii) Recall that K is u_0-positive with respect to a cone on $L_1((0,1)\times(E_m, E_M))$, which is reproducing; then the result follows immediately from [12], Th. 2.16. □

Proposition 5. Consider the equation

$$(\lambda I - \hat{K})\,\psi = g \tag{14}$$

on $L_1((0,R)\times(E_m, E_M))$. Then

i) for $\lambda > \lambda_0(R)$ Equation (14) is uniquely solvable; if $g \in L^+$, then also $\psi \in L^+$.
ii) for $0 < \lambda \leq \lambda_0(R)$ and $g \in L^+$, non-zero, Equation (14) does not have solutions $\psi \in L^+$.

Proof: It is sufficient to apply the scaling transformation S_R to the previous Proposition. □

Propositions 4 and 5 enable us to state Theorem 2, in which one establishes properties of $\lambda_0 = \lambda_0(R)$.

Note that from now on we shall write K_R, $\hat{K}_R$ instead of K, $\hat{K}$ in order to display explicitly the dependence of such operators on the cylinder radius.

Theorem 2. The function $R \to \lambda_0(R)$ is:

i) continuous (in fact, C^∞-function);
ii) strictly monotonically increasing;
iii) satisfying the following properties:

$$\lim_{R \to 0^+} \lambda_0(R) := \lambda_0(0^+) = 0; \qquad 0 < \lim_{R \to +\infty} \lambda_0(R) := \lambda_0(+\infty) < +\infty.$$

Proof: i) Let us generalize the operator K_R to complex ϱ in the open right half-plane, and let us denote its kernel by $N_\varrho(x, x', E, E')$. Then

$$|N_\varrho(x, x', E, E')| = |\varrho| N_R(x, x', E, E')/R$$

where $R = \operatorname{Re} \varrho$. We can now repeat the proof of Proposition 1 and conclude that in this more general situation K_ϱ^2 is compact too. We notice that K_ϱ is a bounded operator on $L_1(V)$, which depends analytically on the parameter ϱ. For any real $\varrho > 0$ the dominating positive eigenvalue $\lambda_0(\varrho)$ is simple in both geometric and algebraic sense. Using [10], Theorem VII 1.8, we find that for every $R > 0$ the function $\varrho \to \lambda_0(\varrho)$ has an analytic continuation to a neighbourhood of R. Hence, $\lambda_0(R)$ is a C^∞-function on $(0, +\infty)$, which proves the continuity.

ii) Let $0 < R_1 < R_2 < +\infty$ and consider the eigenvalue problem (12) for the operator $\hat{K}_{R_2}$, corresponding to $R = R_2$, with $\lambda = \lambda_0(R_2)$ and ψ the corresponding eigenfunction:

$$(\lambda_0(R_2) I - \hat{K}_{R_2}) \psi = 0. \tag{15}$$

Written explicitly, this also reads:

$$\lambda_0(R_2) \psi(y, E) - \int_0^{R_2} \int_{E_m}^{E_M} \hat{T}(y, y', E) \hat{S}(E, E') \psi(y', E') \, \mathrm{d}E' \, \mathrm{d}y' = 0,$$

$$y \in [0, R_2], E \in [E_m, E_M].$$

It is easy to rearrange (15) as follows:

$$(\lambda_0(R_2) I - \hat{K}_{R_1}) \psi = g, \tag{16}$$

where $g(y, E) = \int_{R_1}^{R_2} \int_{E_m}^{E_M} \hat{T}(y, y', E) \hat{S}(E, E') \psi(y', E') \, \mathrm{d}E' \, \mathrm{d}y'$.

If the right-hand side of (16) would be zero a.e., then

$$\hat{T}(y, y', E) \hat{S}(E, E') \psi(y', E') \equiv 0 \quad \text{for a.e.} \quad y' \in [R_1, R_2]$$
$$y \in [0, R_2], E, E' \in [E_m, E_M].$$

This fact, together with the positivity assumption on the energy transfer kernel, would imply that ψ is a.e. zero on a set of positive measure contained in $[0, R_2] \times [E_m, E_M]$, which is a contradiction. Thus the right-hand side is non-negative and different from zero. According to Proposition 5, one has

$$\lambda_0(R_2) > \lambda_0(R_1),$$

which proves the strictly increasing monotonicity.

iii) Since $\lambda_0(R) = r(\hat{K}_R)$ and $\lim r(\hat{K}_R) = 0$ as $R \to 0^+$, it is immediate that $\lambda_0(0^+) = 0$. To study $\lambda_0(+\infty)$, one considers $\hat{K}_\infty$, defined on $L_1((0, +\infty) \times (E_m, E_M))$ by

$$(\hat{K}_\infty \psi)(y, E) = \int_0^\infty \int_{E_m}^{E_M} \hat{T}(y, y', E)\, \hat{S}(E, E')\, \psi(y', E')\, \mathrm{d}E'\, \mathrm{d}y'.$$

Now $\hat{K}_\infty$ is a bounded operator and $0 < r(\hat{K}_\infty) < \infty$.
So $0 < \lambda_0(R) = r(\hat{K}_R) \leq r(\hat{K}_\infty) < \infty$, which implies $0 < \lambda_0(+\infty) < \infty$. □

We are now able to discuss the solution of our original physical equation (1), in its abstract formulation (5). We remark that the previously established eigensolution ψ_0 corresponds to $c_0(R) = 1/\lambda_0(R) > 1$, for each value of R. At this point note that assumptions A.1 and A.3 imply the following inequalities:

$$\operatorname*{ess\,sup}_{E' \in U} \int_U \frac{1}{\Sigma(E)} S(E, E')\, \mathrm{d}E \leq \int_U \frac{1}{\Sigma(E)} r(E)\, \mathrm{d}E \leq \frac{1}{\Sigma_m} \int_U r(E)\, \mathrm{d}E < +\infty,$$

because $r(E) \in L_1(U)$.

Since $\lambda_0(R) < \lambda_0(+\infty) \leq \|\hat{K}_\infty\|$, we can now give a sufficient assumption for the existence of a critical solution ψ_0, with corresponding $c_0(R)$, for $R \in (0, +\infty)$, namely the non-multiplying medium condition

A.5 $\int_U S(E, E')/\Sigma(E)\, \mathrm{d}E \leq 1$ for a.e. $E' \leq [E_m, E_M]$.

Note that A.5 merely is a sufficient condition in order that the spectral radius $r(S/\Sigma(E)) \leq 1$, where S is the operator defined in the Introduction. Condition A.5 is satisfied, in particular, if

$$\int_U r(E)/\Sigma(E)\, \mathrm{d}E \leq 1.$$

Let us now consider the following inequalities, which can be directly established by norm calculations:

$$\|\hat{K}_\infty\| \leq \operatorname*{ess\,sup}_{E'} \int_U \frac{1}{\Sigma(E)} S(E, E')\, \mathrm{d}E \leq 1,$$

where the last inequality is permitted by A.5. Finally, the results can be summarized in the following

Theorem 3. Under the assumptions A.1–A.5, concerning the physical problem (1), there exist a unique critical value $c_0(R) = 1/\lambda_0(R) > 1$, for each $R \in (0, +\infty)$ and a unique critical neutron flux $\psi_0 = \psi_0(y, E)$ in the cylinder such that

$$\psi_0(y, E) > 0 \quad \text{a.e., in} \quad L_1(0, R) \times (E_m, E_M), \quad \text{and} \quad \|\psi_0\| = 1.$$

The function $c_0 = c_0(R)$ is continuous and strictly monotonically decreasing in R.

5. The eigenvalue problem in L_p spaces, $1 \leq p < \infty$.

The aim of this section is to extend the whole formulation of the problem from L_1 to every L_p, $1 \leq p < \infty$.

First of all, we recall some definitions and we list without proof some results about Riesz operators. For an account of Riesz operators theory, the reader is referred to Part 2 of [7].

Let $L(X)$ be the Banach algebra of bounded linear operators on the Banach space X, and let $T \in L(X)$. T is called a Riesz operator if it has the following properties:

i) for every $0 \neq \lambda \in \mathbb{C}$ (the set of complex numbers) and each positive integer n, the solutions of $(\lambda I - T)^n x = 0$ form a finite-dimensional subspace of X, which is independent of n provided that n is sufficiently large;
ii) for every $\lambda \neq 0$ and each positive integer n, Im $(\lambda I - \mathrm{T})^n$ is a closed subspace of X which is independent of n provided that n is sufficiently large;
iii) the eigenvalues of T have at most one cluster point 0.

Let $K(X)$ the closed two-sided ideal in $L(X)$ formed by the set of compact operators on X. T is called asymptotically quasi-compact if

$$[\inf\{\|T^n - C\| : C \in K(X)\}]^{1/n} \to 0, \quad \text{as} \quad n \to \infty.$$

It is known that the class of Riesz operators coincides with the class of asymptotically quasi-compact operators ([7], Th. 3.12, page 73).

Let $T \in L(X)$. T is said to be a Fredholm operator if Ker T is finite dimensional and Im T is closed and finite codimensional. Let $n(T)$ and $d(T)$ denote, respectively, dim Ker T and codim Im T; then we define the index of the Fredholm operator T to be ind $T = n(T) - d(T)$. $n(T)$ is called the nullity of T, $d(T)$ is called the deficiency of T, [15].

We return to our problem again.

Because the operator K is not well-defined on L_∞, we introduce the following assumption:

A.6 The nonnegative measurable function $S(E, E')$ is such that

$$\operatorname*{ess\,sup}_{E} \int_U S(E, E')\, \mathrm{d}E' = N < +\infty.$$

Then K is bounded as an operator acting from L_∞ to L_∞; moreover, for a.e. $(x, E) \in I \times U$

$$|(K^2 f)(x, E)| \leq \|f\|_\infty \cdot \operatorname*{ess\,sup}_{x,E} \int_I \mathrm{d}x' \int_U \mathrm{d}E'$$
$$\cdot \int_I \mathrm{d}x'' \int_U K_0(\sigma_m |x - x''|)\, K_0(\sigma_m |x'' - x'|)\, S(E, E'')\, S(E'', E')\, \mathrm{d}E''$$
$$\leq M(E_m - E_M)\, \pi^2 \|f\|_\infty / 4\, \sigma_m,$$

where $\|f\|_\infty$ denotes the norm of f in $L_\infty(I\times U)$, and hence K^2 is also bounded on L_∞. (Note: this does not follow from A.6).

As far as K is concerned as an operator from L_1 to L_1, we recall that K is bounded and K^2 is compact.

This permits us to interpolate the continuity property of the operator K and the compactness property of the operator K^2 and to prove that K is bounded and K^2 is compact as operators acting from L_p to L_p, with $1 \le p < \infty$, ([13], Th. 3.10, page 57).

Now we state the following

Proposition 6. i) K as an operator from L_p to L_p, with $1 \le p < \infty$, is a Riesz operator; ii) for every λ, $I - \lambda K$ is a Fredholm operator of index 0.

Proof: Recalling the definition of asymptotically quasi-compact operators and that the square of K is compact, point i) readily follows.

It is well-known that if T is a Riesz operator then for every λ the operator $I - \lambda T$ is a Fredholm operator, [2]. Moreover, if $T \in L(X)$ is a Riesz operator, then

$$\dim \operatorname{Ker} (I - \lambda T) = \operatorname{codim} \operatorname{Im} (I - \lambda T),$$

([7], Th. 3.26, page 83). Thus point ii) can be established immediately. □

We apply the preceding results to prove the independence of the spectral properties of K on the specific L_p setting.

Proposition 7. The spectrum of the operator K acting on L_p, with $1 \le p < \infty$, does not depend on p.

Proof: Observe that $L_{p_1} \supseteq L_{p_2}$ for $1 \le p_1 \le p_2 < \infty$. Denote by T_i, an operator T as one acting on L_{p_i}, $i = 1, 2$). Then, for every $\lambda \ne 0$,

$$\operatorname{Ker} (I - \lambda K)_1 \supseteq \operatorname{Ker} (I - \lambda K)_2$$
$$\operatorname{Im} (I - \lambda K)_1 \supseteq \operatorname{Im} (I - \lambda K)_2 .$$

Thus recalling the definition of n and d,

$$n (I - \lambda K)_1 \ge n (I - \lambda K)_2$$
$$d (I - \lambda K)_1 \le d (I - \lambda K)_2 .$$

Since on L_{p_1} and L_{p_2} the operator $I - \lambda K$ is Fredholm of index 0, we have also

$$n (I - \lambda K)_1 = d (I - \lambda K)_1$$
$$n (I - \lambda K)_2 = d (I - \lambda K)_2$$

Hence, $n (I - \lambda K)$ is the same on L_{p_1} and L_{p_2} and $d (I - \lambda K)$ is the same on L_{p_1} and L_{p_2}. Moreover, $\lambda^{-1} \notin \sigma(K)$ on L_{p_1} if and only if $n (I - \lambda K)_1 = d (I - \lambda K)_1 = 0$, and also $\lambda^{-1} \notin \sigma(K)$ on L_{p_2} if and only if $n(I - \lambda K)_2 = d(I - \lambda K)_2 = 0$.Thus from the equivalence of $n(I - \lambda K) = d(I - \lambda K) = 0$ in both L_{p_1} and L_{p_2}, it follows that $\sigma(K)$ is the same on L_{p_1} and L_{p_2}. This completes the proof. $\square$

We have actually proved that

$$n(I - \lambda K) = d(I - \lambda K)$$

is independent of $p \in [1, \infty)$. Hence, the dominant eigenvalue (i.e. simple and greater in modulus than the remaining eigenvalues) of K on L_1 space also is the dominant eigenvalue of K on L_p, $1 \leq p < \infty$. The solution of our cylinder criticality problem therefore does not depend on the specific L_p space setting. If we only consider the independence from the L_p setting for $1 \leq p \leq 2$, we may drop the assumption A.6, because we would not have to interpolate between L_2 and L_∞.

6. Discussion

It is relevant to remark that the research of the existence of a dominant eigenvalue, and a corresponding a.e. strictly positive eigenfunction is done under a set of assumptions (A.1–A.3) which are suitable (not too strong) in an L_1-setting and which guarantee that K^2 is a compact operator on $L_1(V)$ and K is a Riesz operator on $L_p(V)$ $(1 \leq p \leq 2)$. A.4 may appear as a rather restrictive assumption, since it physically means that the energy interval is completely filled up after only two successive collisions. The choice of A.4, however, was made for the sake of simplicity.

A more physically appropriate assumption would be, for some n,

$$\text{A.4 bis} \quad 0 < m \leq \operatorname{ess\,inf} \{\ker S^n(E, E'); E, E' \in U\}$$
$$\leq \operatorname{ess\,sup} \{\ker S^n(E, E'); E, E' \in U\} \leq M < \infty,$$

since in this way the energy interval is completely filled up after n successive collisions. Here S was defined in the introduction.

On the other hand, this more general assumption would not change anything in our arguments since K^n is compact for any $n \geq 2$, and, moreover, (under A.4 bis) K is u_0-positive. Thus we could proceed to obtain our results in perfect analogy with what we have proved above. In a different geometry an assumption of the type A.4 bis was used before by Victory [21].

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Abstract

The energy-dependent neutron transport integral equation in a homogeneous cylinder of radius R and infinite height with isotropic scattering is studied as an abstract equation $f = Kf$ in the space $L_1((0, 1)\times(E_m, E_M))$. By means of techniques based on the theory of positive operators in Banach spaces, we prove that the eigenvalue problem for the integral operator K admits as a solution a unique a.e. positive eigenfunction to which the leading eigenvalue λ_0 corresponds.

After establishing continuity and strictly increasing monotonicity of λ_0 in R we discuss and solve the criticality problem under the assumption of subcriticality for a non-multiplying medium.

The formulation of the eigenvalue problem for K is finally extended to any L_p space, $1 \leq p < \infty$. Recalling that K is a Riesz operator in L_p, we prove, as a general result, that the spectrum of K, acting on L_p, is independent of p.

Résumé

On étudie un faisceau de neutrons d'une énergie E (comprise entre deux bornes E_m et E_M), dans un cylindre de rayon $0 \leqslant r \leqslant R$ de hauteur infinie; on considère la diffraction comme isotrope. L'équation intégrale du transport de neutrons est formuleé abstraitement par $f = Kf$, où K est un opérateur dans l'espace $L_1(r, E)$ des fonctions intégrables. La théorie des opérateurs positifs dans les espaces de Banach nous permet de démontrer que l'opérateur intégral K possède une fonction propre unique, positive presque partout, correspondant à la valeur propre dominante λ_0.

Après avoir démontré que λ_0 est continue et strictement croissante par rapport à R, on discute et résout le problème critique sous une hypothèse bien motivée physiquement.

La formulation du problème aux valeurs propres est généralisée dans un espace L_p, p quelconque, K étant un opérateur de Riesz; on obtient comme résultat que le spectre de K dans L_p est indépendant de p.

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