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An abstract approach to evaporation models in rarefied gas dynamics

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I. Introduction

Linear kinetic equations describing evaporation of a liquid into a half space vacuum may be formulated as abstract differential equations of the form

$$(Tf)'(x) = -Af(x), \qquad 0 < x < \infty, \tag{1}$$

where T and A are self-adjoint operators on a complex Hilbert space H, zero is not an eigenvalue of T, and A is positive with closed range and finite dimensional kernel. In typical strong evaporation problems (cf. [1, 2, 3]) the Hilbert space H is an L_2-space of square integrable functions weighted by a Maxwellian distribution, T is an operator of multiplication by an independent variable plus drift velocity, $I - A$ is a compact operator and the null space $\operatorname{Ker} A$ of A is non-trivial. The solution $f(x)$ represents the deviation of the velocity distribution from the drift Maxwellian. On Eq. (1) one usually imposes the boundary conditions

$$Q_+ f(0) = f_+ \tag{2 a}$$

$$\lim_{x\to\infty} \|f(x)\| = 0, \tag{2 b}$$

where Q_+ is the orthogonal projection onto the maximal positive T-invariant subspace of H, and f_+ is a given vector in the range $\operatorname{Ran} Q_+$ of Q_+. Additional constraints are imposed corresponding to the relevant conservation laws.

Cercignani [1] first conjectured that the breakdown of existence of stationary solutions of strong evaporation models would show up in the linear theory as a non-completeness result. Such concrete evaporation models have been analyzed recently by Arthur and Cercignani [1], and by Siewert and Thomas [2, 3]. These three papers add up to over 40 pages of mostly computational analysis. Still a major purpose is to verify the existence and uniqueness of solutions of

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the problem (under additional constraints) for drift velocities below the speed of sound of the vapor, and to back up analytically the numerically observed breakdown of the existence of solutions for drift velocities above this threshold. Our objective will be a concise derivation of these results in a general setting which allows for the routine analysis of a large class of similar problems, i.e., one dimensional stationary problems with (linear) self-adjoint collision operators. In fact we obtain a complete existence and uniqueness theory for the abstract boundary value problem (1)–(2), with a simple algorithm for determining the location of transitions from existence to non-existence of solutions.

The mathematical foundations on which the present study is based were laid mainly by Beals, Greenberg, van der Mee and Zweifel [4, 5, 6, 7, 8]. These authors studied abstract boundary value problems of the type (1)–(2), but with condition (2 b) replaced by the condition $\|f(x)\| = 0(1)$ as $x \to \infty$. (In [8] both asymptotic conditions are studied.) Nevertheless, the analysis of [4, 5, 6] can be adapted to meet the present situation. As pointed out in [5, 7], the existence and uniqueness problem turns out to be a scrutiny of the structure of $\operatorname{Ker} A$. Our main task will, indeed, consist of unraveling the structure of this finite dimensional space.

In Section II we deal with the boundary value problem (1)–(2) in an abstract way, utilizing a reduction procedure appearing in [5, 6, 7] to treat the problem of non-trivial $\operatorname{Ker} A$. For clarity of exposition we will assume A to be bounded, although it is seen at once from [5, 6] that the results carry over in toto to the case A unbounded (but see [6] for restrictions on T when both T, A unbounded). We shall actually establish under what conditions the boundary value problem is uniquely solvable. The main result of this article, the measure of non-completeness theorem presented in Section II, provides a concise computational prescription for determining existence of solutions. Section III is devoted to the applications treated in [1, 2, 3], and we conclude with some remarks related to the general abstract problem.

II. Solutions in abstract Hilbert spaces

Throughout this section T and A will be self-adjoint operators on a complex Hilbert space H with inner product $(.,.)$, $\operatorname{Ker} T = \{0\}$, and A bounded, positive and Fredholm. Thus A has closed range and a kernel of finite dimension. Let $K = T^{-1} A$ be the operator on the (dense) domain $\{x \in H \mid Ax \in \operatorname{Ran} T\}$ satisfying $TK = A$. The closedness of T and the boundedness of A imply that K is a closed operator. Thus the adjoint $K^* \supseteq AT^{-1}$ is closed and densely defined. In addition we assume that the zero root linear manifold

$$Z_0(K) = \{x \in D(K) \mid \exists n : K^n x = 0\} \supseteq D(T)$$

and is non-degenerate:

$$\{h \in Z_0(K) \mid (Th, k) = 0 \quad \text{for all} \quad k \in Z_0(K)\} = \{0\}.$$

Under these hypotheses we obtain the following result (cf. [5, 6]):

Lemma. We have the decompositions

$$Z_0(K) \oplus Z_0(K^*)^\perp = H, \qquad Z_0(K^*) \oplus Z_0(K)^\perp = H,$$

where

$$A[Z_0(K)] \subseteq Z_0(K^*) = T[Z_0(K)],$$

$$A[Z_0(K^*)^\perp] = Z_0(K)^\perp = \overline{T[Z_0(K^*)^\perp \cap D(T)]}.$$

Furthermore, if B is an invertible operator on $Z_0(K)$ satisfying $(TB^{-1}x, x) \geqq 0$ for $x \in Z_0(K)$ and P denotes the projection of H onto $Z_0(K^*)^\perp$ along $Z_0(K)$, then

$$A_B = TB^{-1}(I - P) + AP$$

is a strictly positive (un)bounded operator satisfying

$$A_B^{-1} T = B \oplus (T^{-1} A|_{Z_0(K^*)^\perp})^{-1}.$$

The solution of the abstract boundary value problem is not found in general in H. We refer to [5, 6, 8] for a discussion of "weak" solutions. To describe solutions we introduce the inner product

$$(x, y)_T = (|T| x, y), \qquad x, y \in D(T),$$

and denote the completion of $D(T)$ with respect to it by H_T. The orthogonal projection Q_+ onto the maximal T-invariant T-positive subspace of H leaves invariant $D(T)$, and its restriction to $D(T)$ extends to an orthogonal projection on H_T, which we also denote by Q_+. Then for every $g_+ \in Q_+[H_T]$ there exists a unique solution g of the modified half space problem

$$(Tg)'(x) = -A_B g(x), \qquad 0 < x < \infty,$$

$$Q_+ g(0) = g_+, \qquad \lim_{x\to\infty} \|g(x)\|_T = 0; \tag{3}$$

this solution can be described by an invertible operator $E_B: H_T \to H_T$, the so-called albedo operator, and is written as (cf. [4])

$$g(x) = e^{-xT^{-1}A_B} E_B g_+, \qquad 0 < x < \infty. \tag{4}$$

Once one has specified B according to the statement of the lemma, every solution of Eqs. (1)–(2) has the form (cf. [5, 6])

$$f(x) = e^{-xT^{-1}A} P E_B g_+, \qquad 0 < x < \infty, \tag{5}$$

where g_+ is some vector in $Q_+[H_T]$ such that

$$Q_+ P E_B g_+ = f_+. \tag{6}$$

Theorem. Given $f_+ \in Q_+[H_T]$, the problem (1)–(2) has at most one solution. There is a unique solution for every $f_+ \in Q_+[H_T]$ if and only if $\operatorname{Ker} A$ has the property that $(Tx, x) < 0$ for all $0 \neq x \in \operatorname{Ker} A$. In general, the measure of non-completeness for the solution of the problem coincides with the maximal number m of linearly independent $x_1, \ldots, x_m \in \operatorname{Ker} A$ such that

(i) $(Tx_j, x_k) = 0$ for $j \neq k$,

(ii) $(Tx_i, x_i) \geqq 0$ for $i = 1, 2, \ldots, m$.

Proof. To establish uniqueness we consider Eqs. (1)–(2) for $f_+ = 0$ and prove that the solution is $f = 0$. However, every solution f of Eq. (1) with boundary conditions (2 a) and

$$\|f(x)\|_T = 0(1) \; (x \to \infty) \tag{7}$$

is uniquely specified by f_+ and its value at infinity ([6], Sec. IV), which implies the uniqueness.

Let us investigate the existence. Suppose that for given $f_+ \in Q_+[H_T]$ there exists a solution of Eq. (1)–(2). Then this solution has the form (5), where g_+ satisfies (6). Now let Q_- denote the orthogonal projection onto the maximal negative T-invariant subspace of H. As $\operatorname{Ker} T = \{0\}$, one has $Q_- = I - Q_+$. As with Q_+ we may define Q_- on H_T. Further we observe that $A_B^{-1} T$ is self-adjoint with respect to the inner product [9]

$$(x, y)_{A_B} = (A_B x, y),$$

which, because of the invertibility of A_B, is complete on H. Let P_+ denote the $(.,.)_{A_B}$-orthogonal projection onto the maximal positive $A_B^{-1} T$-invariant subspace of H. As shown in [4], P_+ leaves invariant $D(A_B^{-1} T) = D(T)$, and its restriction to $D(T)$ can be extended to a bounded projection on H_T, also denoted by P_+. We recall (cf. [4, 5, 6]) that

$$P_+ E_B Q_+ x = E_B Q_+ x, \qquad x \in H_T.$$

Returning to Eqs. (1)–(2), we directly infer from Eq. (6) that

$$f_+ = P E_B g_+ - Q_- P E_B g_+ \in \operatorname{Ran} P P_+ \oplus \operatorname{Ran} Q_- \subseteq H_T.$$

Thus the measure of non-completeness δ_0^+ for the solution of the problem equals

$$\delta_0^+ = \dim \frac{Q_+[H_T]}{(\operatorname{Ran} P P_+ \oplus \operatorname{Ran} Q_-) \cap Q_+[H_T]} = \dim \frac{H_T}{\operatorname{Ran} P P_+ \oplus \operatorname{Ran} Q_-}. \tag{8}$$

Because Eq. (1) with boundary conditions (2 a) and (7) always has at least one solution [6, 7] and E_B is an invertible operator on H_T [4], one also has

$$\operatorname{Ran} P P_+ \oplus \operatorname{Ran} Q_- + \operatorname{Ker} A = H_T. \tag{9}$$

Exploiting Eqs. (8) and (9) one obtains

$$\delta_0^+ = \dim \frac{\operatorname{Ran} P P_+ \oplus \operatorname{Ran} Q_- + \operatorname{Ker} A}{\operatorname{Ran} P P_+ \oplus \operatorname{Ran} Q_-} = \dim \frac{\operatorname{Ker} A}{[\operatorname{Ran} P P_+ \oplus \operatorname{Ran} Q_-] \cap \operatorname{Ker} A}.$$

It should be observed that the subspace

$$N_- = [\operatorname{Ran} PP_+ \oplus \operatorname{Ran} Q_-] \cap \operatorname{Ker} A$$

has the property that $(Tx, x) < 0$ for all $0 \neq x \in N_-$. Furthermore, any subspace L_- with the same property and satisfying $N_- \subseteq L_- \subseteq \operatorname{Ker} A$ necessarily coincides with N_- [5, 6, 7]. Using standard Pontrjagin space theory ([10], Chapter IX) we conclude that δ_0^+ coincides with the (uniquely defined) dimension of a subspace N_+ of $\operatorname{Ker} A$, which satisfies $(Tx, x) \geqq 0$ for all $x \in N_+$ and is maximal in this respect. This characterization of δ_0^+ completes the proof of the theorem.

The set of all $f_+ \in Q_+[H_T]$ for which Eqs. (1)–(2) have a solution has been shown to coincide with the finite-codimensional subspace $\operatorname{Ran} PP_+ \oplus \operatorname{Ran} Q_-$ of H_T. Let us find a more expedient way to specify those f_+ for which a solution exists. Let us put

$$M_\pm = [\operatorname{Ran} PP_\mp \oplus \operatorname{Ran} Q_\pm] \cap Z_0(K), \tag{10}$$

where P_- is the $(.,.)_{A_B}$-orthogonal projection onto the maximal negative $A_B^{-1}T$-invariant subspace of H and where the restrictions of $P_\pm$, P and $Q_\pm$ to $D(T)$ have been continuously extended to H_T with all notation preserved. It is known [5, 6, 7] that $(Tx, x) > 0$ for all $0 \neq x \in M_+$ and M_+ is maximal in this respect, while

$$(T[M_+])^\perp = M_- \oplus Z_0(K^*)^\perp$$

(where the orthogonal complement is understood in H with respect to $(.,.)$). Thus, one finds

$$(T[\operatorname{Ran} PP_+ \oplus \operatorname{Ran} Q_-])^\perp = Q_+[M_+] \oplus \operatorname{Ran} Q_-.$$

Hence, given $f_+ \in Q_+[D(T)]$ the boundary value problem (1)–(2) has a solution if and only if

$$(Tf_+, Q_+ g) = 0, \qquad g \in M_+. \tag{11}$$

If one tries to solve Eqs. (1)–(2) with the additional piece of information that $f_+ \in Q_+[\operatorname{Ker} A]$, then, of course, $f_+ = Q_+ f$ where

$$f \in [\operatorname{Ran} PP_+ \oplus \operatorname{Ran} Q_-] \cap \operatorname{Ker} A = M_- \cap \operatorname{Ker} A. \tag{12}$$

Conversely, if $f_+ \in Q_+ f$ with f taken as above, then $f_+ \in \operatorname{Ran} PP_+ \oplus \operatorname{Ran} Q_-$ and Eqs. (1)–(2) have a solution. We now establish

Corollary. Let $\operatorname{Ker} A \cap \operatorname{Ran} Q_- = \{0\}$. Then the codimension in $\operatorname{Ker} A$ of the linear set of those $f \in \operatorname{Ker} A$ for which $f_+ = Q_+ f$ has a solution coincides with the measure of non-completeness δ_0^+.

Proof. It can be shown (see [5], Sec. V) that $M_\pm \cap \operatorname{Ker} A$ has the property $(Tx, x) \gtrless 0$ for $0 \neq x \in M_\pm \cap \operatorname{Ker} A$, and is maximal in this respect. Again from standard Pontrjagin space theory [10] one finds that the codimension of $M_- \cap \operatorname{Ker} A$ in $\operatorname{Ker} A$ equals the maximal number m of linearly independent $x_1, \ldots, x_m \in \operatorname{Ker} A$ satisfying properties (i) and (ii) in the statement of the theorem, which establishes the corollary.

III. Applications to evaporation problems

This section is devoted to the evaporation models studied in [1, 2, 3]. We shall specify H, T and A, and unravel the structure of $Z_0(K)$.

1. The one-dimensional BGK model equation

$$(c+d)\frac{\partial f}{\partial x}(x,c) + f(x,c)$$
$$= \pi^{-1/2} \int_{-\infty}^{\infty} \left\{1 + 2cc' + 2\left(c^2 - \frac{1}{2}\right)\left(c'^2 - \frac{1}{2}\right)\right\} e^{-c'^2} f(x,c')\,dc'$$

where $d > 0$ is a fixed positive drift velocity. We study this equation on the Hilbert space $H = L_2(\mathbb{R})_\delta$ of measurable functions $h, k\colon \mathbb{R} \to \mathbb{C}$ with inner product

$$(h,k) = \pi^{-1/2} \int_{-\infty}^{\infty} h(c)\overline{k(c)}\,e^{-c^2}\,dc,$$

and define T and A by

$$(Th)(c) = (c+d)\,h(c)$$

$$(Ah)(c) = h(c) - \pi^{-1/2} \int_{-\infty}^{\infty} \{1 + 2cc' + 2(c^2 - \tfrac{1}{2})(c'^2 - \tfrac{1}{2})\}\,e^{-c'^2} h(c')\,dc'.$$

Then T is unbounded self-adjoint with zero null space, A bounded positive and (incidentally) $I - A$ compact. The kernel of A is the linear span of the vectors 1, c and $c^2 - \frac{1}{2}$. So $\operatorname{Ker} A \cap \operatorname{Ran} Q_- = \{0\}$. Now we introduce the sesquilinear form

$$[h,k] = (Th,k) = \pi^{-1/2} \int_{-\infty}^{\infty} (c+d)\,h(c)\overline{k(c)}\,e^{-c^2} h(c)\,dc.$$

One easily finds that $\{1, c^2 - \frac{1}{2}, dc - c^2\}$ forms a basis of $\operatorname{Ker} A$ of mutually $[.,.]$-orthogonal vectors, while

$$[1,1] = d, \quad [c^2 - \tfrac{1}{2}, c^2 - \tfrac{1}{2}] = \tfrac{5}{4}d, \quad [dc - c^2, dc - c^2] = \tfrac{1}{2}d(d^2 - \tfrac{3}{2}).$$

The only value of $d > 0$ for which $[dc - c^2, dc - c^2] = 0$ is $d = \sqrt{3/2}$, and for this sole value of d the zero root linear manifold $Z_0(K)$ will strictly contain $\operatorname{Ker} A$; in this case one has $T^{-1}A(c^3 - \frac{3}{2}c) = dc - c^2$.

Let us apply the main theorem. Then for all cases Eqs. (1)–(2) have at most one solution, but for $0 < d < \sqrt{3/2}$ the measure of non-completeness equals 2 and for $d \geqq \sqrt{3/2}$ it equals 3. The existence of solutions for all $f_+ \in Q_+[H_T]$ depends only upon the behavior of $f_+ \in Q_+[H_T] \cap Q_+[\operatorname{Ker} A] = Q_+[\operatorname{Ker} A]$, and so we may study (for $c > -d$, of course)

$$f_+(c) = \Delta\varrho + 2c(d_0 - d) + (c^2 - \tfrac{1}{2})\Delta T, \tag{13}$$

where $\Delta\varrho$, d_0 and ΔT are dimensionless physical quantities with $\Delta\varrho$ and ΔT to be fixed by the conservation laws. Indeed, if, for $d < \sqrt{3/2}$, one chooses d_0 (and thus $d_0 - d$) arbitrarily, there will be precisely one $\Delta\varrho$ and one ΔT such that Eqs. (1)–(2) with f_+ given by Eq. (13) have a solution. To see this we observe that for $d < \sqrt{3/2}$ one has $[dc - c^2, dc - c^2] < 0$, $[1,1] > 0$ and $[c^2 - \frac{1}{2}, c^2 - \frac{1}{2}] > 0$, whence given d_0 there exist unique $\Delta\varrho$ and ΔT such that $f_+ = Q_+ f$ with $f \in M_- \cap \operatorname{Ker} A$. For $d \geqq \sqrt{3/2}$ the subspace $\operatorname{Ker} A$ satisfies $(Tx, x) \geqq 0$ for $x \in \operatorname{Ker} A$, whereas $(Ty, y) < 0$ for all $0 \neq y \in M_-$. Thus $M_- \cap \operatorname{Ker} A = \{0\}$ for $d \geqq \sqrt{3/2}$. Hence for $d \geqq \sqrt{3/2}$ Eqs. (1)–(2) with boundary data f_+ as in Eq. (13) are solvable only for $f_+ = 0$. We therefore recover the results of Siewert and Thomas [2].

2. The three-dimensional BGK model equation

$$(c_x + d)\frac{\partial f}{\partial x}(x, \vec{c}) + f(x, \vec{c})$$
$$= \pi^{-3/2} \int_{-\infty}^{\infty} f(\vec{c}\,')\left[1 + 2\vec{c}\cdot\vec{c}\,' + \frac{2}{3}\left(c^2 - \frac{3}{2}\right)\left(c'^2 - \frac{3}{2}\right)\right] e^{-c'^2}\, d^3c'$$

where $d > 0$ is a drift velocity in the x-direction. This equation is studied on the Hilbert space $H = L_2(\mathbb{R}^3)_\delta$ of measurable functions h, k: $\mathbb{R} \to \mathbb{C}$ with inner product

$$(h, k) = \pi^{-3/2} \int_{-\infty}^{\infty} h(\vec{c})\overline{k(\vec{c})}\, e^{-c^2}\, d^3c,$$

while T and A are defined by

$$(Th)(\vec{c}) = (c_x + d)\, h(\vec{c})$$

$$(Ah)(\vec{c}) = h(\vec{c}) - \pi^{-3/2} \int_{-\infty}^{\infty} h(\vec{c}\,')[1 + 2\vec{c}\cdot\vec{c}\,' + \tfrac{2}{3}(c^2 - \tfrac{3}{2})(c'^2 - \tfrac{3}{2})]\, e^{-c'^2}\, d^3c'.$$

Then T is unbounded self-adjoint with zero null space, A bounded positive and (again incidentally) $I - A$ compact. The kernel of A is spanned by the vectors 1, c_x, c_y, c_z and $c^2 - \frac{3}{2}$. With respect to the sesquilinear form

$$[h, k] = (Th, k) = \pi^{-3/2} \int_{-\infty}^{\infty} (c_x + d)\, h(\vec{c})\overline{k(\vec{c})}\, e^{-c^2}\, d^3c$$

the set $\{1, c^2 - 3\,d\,c_x, c_y, c_z, c^2 - \frac{3}{2}\}$ is a basis of $\operatorname{Ker} A$ of mutually orthogonal vectors, while

$$[1,1] = d, \quad [c^2 - 3\,d\,c_x, c^2 - 3\,d\,c_x] = \tfrac{9}{2} d\,(d^2 - \tfrac{5}{6}),$$
$$[c_y, c_y] = [c_z, c_z] = \tfrac{1}{2} d, \quad [c^2 - \tfrac{3}{2}, c^2 - \tfrac{3}{2}] = \tfrac{3}{2} d.$$

The only value of $d > 0$ for which $\frac{9}{2} d\,(d^2 - \frac{5}{6}) = 0$ is $d = \sqrt{5/6}$, and for this sole value the space $Z_0(K)$ strictly contains $\operatorname{Ker} A$, as we may compute:

$$T^{-1} A\,\{d\,(c^2 - 3\,c_x^2) + c_x (c^2 - \tfrac{5}{2})\} = c^2 - 3\,d\,c_x.$$

For this model Eqs. (1)–(2) have at most one solution, but for $0 < d < \sqrt{5/6}$ the measure of non-completeness equals 4 and for $d \geqq \sqrt{5/6}$ it equals 5. Let us consider initial values $f_+ \in Q_+ [\operatorname{Ker} A]$ written in the form

$$f_+(\vec{c}) = \Delta\varrho + 2\,(d_0 - d)\,c_x + (c^2 - \tfrac{3}{2})\,\Delta T + 2\,d_y c_y + 2\,d_z c_z,$$

where we anticipate that $\Delta\varrho$, ΔT, d_y and d_z must be determined by conservation of mass, energy, and the transverse momenta. Then f_+ is an arbitrary vector in $Q_+ [\operatorname{Ker} A]$, while $\operatorname{Ker} A \cap \operatorname{Ran} Q_- = \{0\}$. The corollary does, in fact, imply that for $d < \sqrt{5/6}$ and given d_0 Eqs. (1)–(2) have a solution for unique values of the parameters $\Delta\varrho$, ΔT, d_y and d_z, whereas for $d \geqq \sqrt{5/6}$ Eqs. (1)–(2) are not (except for the trivial case) solvable.

3. The three-dimensional BGK model in moment form

$$(c + d)\frac{\partial}{\partial x}\begin{bmatrix} f_+(x,c) \\ f_-(x,c) \end{bmatrix} + \begin{bmatrix} f_+(x,c) \\ f_-(x,c) \end{bmatrix} = \pi^{-1/2} \int_{-\infty}^{\infty} D(c,\hat{c}) \begin{bmatrix} f_+(x,\hat{c}) \\ f_-(x,\hat{c}) \end{bmatrix} e^{-\hat{c}^2}\, d\hat{c}$$

with

$$D(c,\hat{c}) = \begin{bmatrix} 1 + 2\,c\,\hat{c} + \frac{2}{3}(c^2 - \frac{1}{2})(\hat{c}^2 - \frac{1}{2}) & \frac{2}{3}(c^2 - \frac{1}{2}) \\ \frac{2}{3}(\hat{c}^2 - \frac{1}{2}) & \frac{2}{3} \end{bmatrix}$$

where $d > 0$ again is a drift velocity. We analyze this equation on the Hilbert space $H = L_2(\mathbb{R})_\delta \oplus L_2(\mathbb{R})_\delta$, where $L_2(\mathbb{R})_\delta$ is the space defined in Sec. III.1. We define T and A on H by

$$(T\boldsymbol{h})(c) = (c + d)\,\boldsymbol{h}(c)$$

$$(A\boldsymbol{h})(c) = \boldsymbol{h}(c) - \pi^{-1/2} \int_{-\infty}^{\infty} D(c,\hat{c})\,\boldsymbol{h}(\hat{c})\,e^{-\hat{c}^2}\, d\hat{c},$$

where $\boldsymbol{h}(c)$ is a column vector. Then T is unbounded self-adjoint with zero null space, A bounded positive and (again incidentally) with zero null space, A bounded positive and (again incidentally) $I - A$ compact. The kernel of A has the form

$$\operatorname{Ker} A = \operatorname{Span}\left\{\alpha^{(1)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \alpha^{(2)} = \begin{bmatrix} 1 - 3\,d\,c + c^2 \\ 1 \end{bmatrix}, \alpha^{(3)} = \begin{bmatrix} c^2 - \frac{1}{2} \\ 1 \end{bmatrix}\right\},$$

where the vectors on the right-hand side are mutually orthogonal with respect to the inner product $[\boldsymbol{h}, \boldsymbol{k}] = (T\boldsymbol{h}, \boldsymbol{k})$. One computes that

$$[\alpha^{(1)}, \alpha^{(1)}] = d, \qquad [\alpha^{(2)}, \alpha^{(2)}] = \tfrac{9}{2} d\,(d^2 - \tfrac{5}{6}), \qquad [\alpha^{(3)}, \alpha^{(3)}] = \tfrac{3}{2} d.$$

The only $d > 0$ for which one of these inner products vanishes is $d = \sqrt{5/6}$, and for this value the space $Z_0(K)$ is strictly larger than Ker A; indeed,

$$T^{-1} A \begin{bmatrix} c^2(c - 3d) \\ c \end{bmatrix} = \alpha^{(2)}, \qquad d = \sqrt{5/6}.$$

Eqs. (1)–(2) have at most one solution, but for $0 < d < \sqrt{5/6}$ the measure of non-completeness equals 2 and for $d \geqq \sqrt{5/6}$ it equals 3. Let us consider

$$f_0 = \Delta\varrho\alpha^{(1)} + 2(d_0 - d)\begin{bmatrix} c \\ 0 \end{bmatrix} + \Delta T\alpha^{(3)}.$$

We observe $\operatorname{Ker} A \cap \operatorname{Ran} Q_- = \{0\}$ and apply the corollary. For $0 < d < \sqrt{5/6}$ and arbitrary d_0 there exist unique $\Delta\varrho$ and ΔT for which Eqs. (1)–(2) have a solution; for $d \geqq \sqrt{5/6}$ Eqs. (1)–(2) do not have non-trivial solutions. These are the results of Siewert and Thomas [3]. Further, the Wiener-Hopf factorization indices of the dispersion matrix of the problem are $\varkappa_1 = \varkappa_2 = 1$ for $d < \sqrt{5/6}$, and $\varkappa_1 = 1$ and $\varkappa_2 = 2$ for $d \geqq \sqrt{5/6}$ (see [3], Appendix C). But for $d \neq \sqrt{5/6}$ the sum $\varkappa_1 + \varkappa_2$ is known to be the measure of non-completeness (cf. [11]; for $d = \sqrt{5/6}$ the dispersion matrix has zeros on the extended imaginary axis and the situation is more complicated) and this corresponds to our result.

IV. Discussion

Considerable effort over the past ten years has been devoted to the rigorous treatment of various specific linear transport models. We now have available an existence and uniqueness theory for "half range" boundary value problems related to the abstract equation (1), for T and A self-adjoint (both possibly unbounded) and A positive Fredholm [5, 6]. The abstract equation models numerous transport phenomena in addition to rarefied gas dynamics, including electron and phonon transport, radiative transfer, neutron transport, etc. The linear problems in gas dynamics illustrated in this article represent, as is typical, perturbations of density from the equilibrium distribution. For that reason the asymptotic condition (2 b) has been studied. Equally it is possible to obtain an analysis of the boundary value problem (1)–(2 a)–(7). However, in this case we have:

(i) existence is guaranteed, i.e., measure of non-completeness is zero.
(ii) non-uniqueness: measure of non-uniqueness may be described in terms of the form $x \to (Tx, x)$ on a basis of $\operatorname{Ker} A$.

The abstract problem with T bounded and A (possibly unbounded) with finite dimensional negative part has also been studied [8]. Such problems are typical of one-speed (or symmetric multigroup) neutron transport in supercritical media, T representing an angular variable and supercriticality reflected in the non-positivity of A. Again, results of the sort above are available. In general, for this case both existence and uniqueness are lost, and the analysis of measure of non-completeness and measure of non-uniqueness is substantially more complicated. We note in particular that complex zeros of the dispersion function, on and off the imaginary axis, also non-simple, are all possible.

Finally, we note that the compactness of $I - A$ in the examples discussed, while irrelevant to the application of the theorem and corollary derived in Sec. II, does in fact have the consequence that Eq. (1) can be solved in H rather than H_T ("strong" solutions, cf. [4, 7, 12]).

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Abstract

An efficient algorithm is presented for determining the unique solvability of certain one-dimensional stationary transport problems. The non-existence of stationary evaporation states with supersonic drift velocities for one and three dimensional BGK model is recovered.

Sommario

È presentato un algoritmo efficiente per determinare la solubilità univoca di alcuni problemi unidimensionali di trasporto stazionario. È ripresa la non esistenza di stati stazionari di evaporazione con velocità di deriva supersoniche per modelli BGK a una e tre dimensioni.

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