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ALBEDO OPERATORS AND $H$-EQUATIONS FOR GENERALIZED KINETIC MODELS

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ABSTRACT

For abstract kinetic equations on a half-line the albedo operator which gives the initial value as a function of the half-range boundary data, is written in terms of operator-valued generalizations of Chandrasekhar's $H$-functions, for which coupled non-linear equations are derived. Applications are given to transfer of polarized light, neutron transport and strong evaporation of liquids.

1. INTRODUCTION

In recent years an intensive study has been made of the (unique) solvability and functional-analytic aspects of the abstract boundary value problem

\begin{align}
(Tf)'(x) &= -Af(x) \quad (0 \leq x \leq a) \\
Q_- f(0) &= f_+ \\
\|f(x)\|_H &= 0(a) \quad (x = a),
\end{align}

where $T$ is an injective self-adjoint and $A$ a positive self-adjoint operator on the Hilbert space $H$ and $Q_-$ is the orthogonal projection of $H$ onto the maximal positive $T$-invariant subspace.\(^1\)^\(^4\)

The solution is usually written in the semigroup form

\[ f(x) = e^{-x^2A_0}E_{f_+}(0 \leq x \leq a), \]

where $E$ is the Larsen-Habetler\(^5\) albedo operator. In spite of the many applications of Eqs. (1.1) to (1.3) to one-speed and multi-

341
group neutron transport, radiative transfer without polarization and with polarization, rarefied gas dynamics, phonon transport and evaporating liquids, the functional-analytic theory of these equations so far has fallen short of producing a general procedure of finding explicit expressions for the albedo operator. However, once the albedo operator is calculated Case’s method of eigenfunction expansion can be applied to find

\[ f(x) = \int_0^\infty e^{-\lambda T} \sum_{\alpha,\lambda} A(\alpha,\lambda) \phi_{\alpha,\lambda}(x,\lambda) \, d\rho(\lambda), \quad 0 < x < \infty, \]

where \( \rho(\cdot) \) is the spectral measure of \( T^{-1}A \), \( \phi_{\alpha,\lambda} \) are (singular or regular) eigenfunctions of \( T^{-1}A \) and \( A(\alpha,\lambda) \) are expansion coefficients, which are given by

\[ A(\alpha,\lambda) = \frac{1}{\rho(\lambda)} (AE\phi_{\alpha,\lambda})_H. \]

The main purpose of this article is to derive, for (non-self-adjoint) operators \( A \) which are compact perturbations of the identity, an expression for \( A \) in terms of operator-valued generalizations of Chandrasekhar’s H-function, for which non-linear integral equations are found. We exploit the equivalence of Eqs. (1.1) to (1.3) to the equation

\[ f(x) = \int_0^\infty H(x-y)Bf(y)dy = \omega(x), \quad 0 < x < \infty, \quad (1.4) \]

where \( B = I - A \) is compact, \( \omega(x) = e^{-\lambda T} f(x) \),

\[ H(t) = \begin{cases} e^{-\lambda T} & t > 0; \\ e^{\lambda T} & t < 0; \end{cases} \]

and \( Q_+ = I - Q_- \). In many applications \( B \) is an operator of finite rank. In order to reduce Eqs. (1.1) to (1.3) to a problem, which is finite-dimensional whenever \( B \) has finite rank, we choose a closed subspace \( B \oplus \text{Ran } B \). If \( j \) is the natural imbedding of \( B \) into \( H \) and \( \pi \) the orthogonal projection of \( H \) onto \( B \), then

\[ Bj\pi = B \quad (1.6) \]

and Eq. (1.4) can be reduced to the Wiener-Hopf operator integral equation

\[ g(x) - \int_0^\infty \hat{H}(x-y)Bg(y)dy = \pi u(x), \quad 0 < x < \infty, \quad (1.7) \]

where \( g(x) = \pi f(x) \). If the solution \( g(x) \) of (1.7) has been computed, then

\[ f(x) = \omega(x) + \int_0^\infty H(x-y)Bj\pi y dy \quad (1.8) \]

will be the solution of (1.4). Now, using the resolution of the identity \( \sigma \) of the self-adjoint operator \( T \), the "symbol" of Eq. (1.8) has the form

\[ \Lambda(z) = 1 - \int_0^\infty e^{\frac{t}{z}} \pi H(t)Bj\pi t dt = 1 - \int_{-\infty}^\infty \frac{\pi \sigma dt}{z-t} \]

This generalization of the dispersion function in neutron transport theory will be the basis on which generalized H-functions will be constructed.

In the so-called regular case where \( A \) is invertible and \( A^{-1} \) does not have imaginary eigenvalues, Eqs. (1.4) and (1.7) may be solved by Wiener-Hopf factorization. If Eqs. (1.1) to (1.3) are uniquely solvable, there exists the canonical Wiener-Hopf factorization

\[ \Lambda(z) = H_+^T(-z)H_+^R(z), \quad \text{Re } z > 0, \quad (1.10) \]

where the factors \( H_+^T \) and \( H_+^R \) are analytic and invertible in the closed right half-plane. The unique solution can be written as

\[ g(x) = \tau_\omega(x) + \int_0^\infty \gamma(x,y)\pi \omega(y)dy, \quad 0 < x < \infty, \quad (1.11) \]

while the double Laplace transform of \( \gamma(x,y) \) is given by

\[ \int_0^\infty dy \int_0^\infty dz \, e^{\nu(z)} = \int_0^\infty \int_0^\infty e^{\nu(z)} \delta(y-z)\gamma(y,z) \, d\nu \]

Through the relation \( Ef_+ = f(0) \) and Eqs. (1.8), (1.11) and (1.12) we express the albedo operator in the functions \( H_+^T \) and \( H_+^R \), and by integrating (1.12) we prove these functions to satisfy a coupled system of non-linear integral equations, which generalize Chandrasekhar’s H-equation. For specific transport problems a similar approach was offered by Burniston et al. (for two-group neutron transport) and subsequently by Kellev and Mullikin.

The regular case has applications to neutron transport, radiative transfer and phonon transport in non-conservative media. For conservative media and in rarefied gas dynamics the operator \( A \) is not invertible, though \( T^{-1}A \) does not have non-zero imaginary eigenvalues. In these "singular" cases we employ a modified Wiener-Hopf method, where the factors are singular at infinity.
but analytic and invertible everywhere else in the closed right half-plane. Assuming the (unique or non-unique) solvability of Eqs. (1.1) to (1.3), we choose a corresponding (unique or non-unique) albedo operator $E$ such that $E f_+ = f(0)$ is the initial value of a solution for every $f_+$. Putting $E_+ = E_{Q_+}$, we then obtain auxiliary representations of $H^+_{\xi}$ and $H^+_{\tau}$ in the form

$$H^+_{\xi}(x) = I - z\pi (T^{-1} A)^{-1} E^+_x B_j,$$

(1.13)

$$H^+_{\tau}(x) = I - z\pi (I - E^+_x (T^{-1} A)^{-1}) B_j,$$

(1.14)

where $E^+_x = E^+_x T$. (Related representations were found before for the regular case and $B = H^{22}_{23,3}$.) The inverses we represent by the formulas

$$H^{-1}_{\xi}(x) = I - z\pi E^+_x (z^{-1} A)^{-1} B_j,$$

(1.15)

$$H^{-1}_{\tau}(x) = I - z\pi (z^{-1} (I - E^+_x) B_j),$$

(1.16)

where (1.10) is satisfied. Using (1.13) to (1.16) we derive the previously obtained generalizations of Chandrasekhar's $H$-equations and a formula for the albedo operator. We have thus accomplished a generalization beyond the regular case. Now applications are given to polarized light transfer, neutron transport and evaporating liquids. Most well-known expressions for the albedo operator obtained by means of resolvent integration$^{3,24}$ or Wiener-Hopf equations$^{17,19-21}$ could be recovered this way.

Throughout this paper the operator $B = I - A$ is compact and satisfies the weak regularity condition

$$E_{>0} : \text{Ran } B \subseteq \text{Ran } T^{1/4} \cap D(|T|^{1/4}),$$

(1.17)

which is fulfilled for neutron transport$^3$ with redistribution function $p \in L_{1d_{e}}[-1,1]$, radiative transfer$^3$ with phase function $p \in L_{1d_{e}}[-1,1]$ and various BGK models in rarefied gas dynamics. Integrals of vector- and operator-valued functions are to be understood as Bochner integrals.$^{26,27}$ We shall prove the equivalence of Wiener-Hopf equation (1.4) and boundary value problem (1.1) to (1.3) in the Appendix. We shall not investigate the uniqueness problem for solutions of our generalized $H$-equations.

2. ALBEDO OPERATOR: REGULAR CASE

Let $A$ be an invertible and $B = I - A$ a compact operator satisfying (1.17). Suppose that $A^{-1}$ does not have imaginary eigenvalues. Then the dispersion function $\lambda(z)$ has a (possibly non-canonical) Wiener-Hopf factorization$^{28}$

Theorem 2.1. Assume that Eqs. (1.1) to (1.3) are uniquely solvable for all $f_+ \in Q_+ D(T)$. If $Q$ denotes the resolution of the identity of $T$ and $f_+ \in \text{Ran } Q$ for some finite $b$, then the initial value of the solution is given by

$$E f_+ = f_+ + \int_0^\infty \int_0^\infty \frac{\alpha(du) B_j H^+_{\xi}(-u) H^+_{\tau}(v) \sigma(du, v) f_+}{\nu - \mu} \, \nu (\nu, v) \, (\nu),$$

(2.1)

and $E$ has an extension to a bounded operator on $\text{Ran } Q_+$.

In the isotropic case of one-speed neutron transport$^{1,5,6}$ one has

$$H = L^2_2[-1,1], \quad (Th)(\mu) = \mu h(\mu), \quad (Bh)(\mu) = \frac{1}{2} c \int_0^1 h(\mu') d\mu',$$

where $0 < c < 1$. Taking $B = \text{span } \{1\}$, $\sigma(du) = du$ and $\sigma(du) = du$, Eq. (2.1) reduces to the well-known expression

$$\lambda(1 - \frac{1}{2} c \int_0^1 h(\mu') d\mu', \mu \geq 0;$$

(2.2)

where $\lambda(z) = 1 - \frac{1}{2} c \int_0^1 (z^{-1} - 1) dt$ and $\lambda(z) = H(z)$.

To prove Theorem 2.1 we consider the equivalent Wiener-Hopf operator integral equation (1.4), reduced to (1.7) by $g(x) = \pi f(x)$. The unique solution of (1.4) can be written as (1.8), where $g(x)$ is given by (1.11). For the resolvent kernel $\gamma(x, y)$ we derive

Lemma 2.2.

$$\int_0^1 d\gamma \int_0^1 dz \, e^{\mu z} e^{-\gamma} (\gamma(y, z)) = \frac{1}{\nu - \mu} H^+_{\xi}(-u) H^+_{\tau}(v).$$

(2.3)

Proof. For the factors in the Wiener-Hopf factorization (1.10) one may find Bochner integrable$^{28}$ operator functions $\alpha$ and $\beta$ such that
\[ H_\xi^+(\omega) = I + \int_0^\omega e^{-\tau/\omega} a(t)dt, H_\xi^+(\omega) = I + \int_0^\omega e^{-\tau/\omega} \delta(-t)dt, \text{ Re } \omega = 0, \]

while the resolvent kernel is given by \(^{16}\)

\[
\gamma(y,z) = \begin{cases} \alpha(y-z) + \int_0^y \alpha(y-u)\delta(u-z)du, & 0 < z < y < \infty; \\ \beta(y-z) + \int_0^y \beta(y-u)\delta(u-z)du, & 0 < z < y < \infty. \end{cases} \tag{2.4a} \]

The left-hand side of (2.3) represents the Laplace transform of the solution of the convolution equation

\[ g_\xi(x) = \int_0^\infty H(x-y)Bf(y)dy = e^{-x/\nu}I, \quad 0 < x < \infty. \tag{2.5} \]

Extending the equation to the real line and taking Laplace transforms one obtains

\[
\Lambda(\mu) = \int_0^\infty e^{y/\mu}g_\nu(y)dy + \int_\infty^0 e^{y/\mu}g_\nu(y)dy = \frac{\nu}{\mu - \nu}I, \quad \text{Re } \nu = 0.
\]

Substituting the factorization (1.10) one gets the Riemann-Hilbert problem

\[ H_\xi(\nu)^{-1} \int_0^\infty e^{y/\nu}g_\nu(y)dy + H_\xi(\nu) \int_\infty^0 e^{y/\nu}g_\nu(y)dy = \frac{\nu}{\mu - \nu}H_\xi(\nu), \text{ Re } \nu = 0, \]

whose unique solution

\[ \int_0^\infty e^{y/\nu}g_\nu(y)dy = \frac{\nu}{\mu - \nu}H_\xi(\nu)^{-1}H_\xi(\nu) \]

coincides with the right-hand side of (2.3).

Using \( E_+ = f(0) \) and Eqs. (1.8) and (1.11) we find

\[ E_+ = f_+ + \int_0^\infty H(-y)Bf_{-}e^{-\gamma t}df_{-}dzdy. \tag{2.6} \]

We apply the Spectral Theorem to rewrite \( H(-y), e^{-\gamma t} \) and \( e^{-zT}f_{-} \), and change the order of integration. We obtain

\[ E_+ = f_+ + \int_0^\infty \int_0^\infty \sigma(du)Bf_{-}e^{d/\gamma z/\nu(\delta(y-z)+\gamma(y,z))}df_{-}dzdy. \]

where \( f_{-} \in \text{Ran } \sigma([0,b]) \) for finite \( b \). With the help of (2.3) we get (2.1).
\[ g_z(0) = \int_0^\infty e^{-x/z}(\delta(x)+\gamma(0,x))dx = I + \int_0^\infty e^{-x/\beta(-x)}dx = H_\xi^+(z). \]

Thus we have established (2.7b).

Postmultiply (2.3) (with \( w, \nu \) replaced by \(-z, t\)) by \( e^{-t_0}\sigma(dt)B_j \) and perform the \( t \)-integration. We find

\[ \int_0^{z+t} H_\xi^+(z)H_\xi^+(t)\sigma(dt)B_j = \int_0^{z+t} [\int_0^\infty (\delta(x-y)+\gamma(x,y))e^{-x/\beta}dx]H(y)B_j. \]

The expression between square brackets is the solution of the equation

\[ h_z(y) = \int_0^\infty h_z(x)\sigma(x-y)B_jdx = e^{-y/z}I, \]

whence

\[ \int_0^{z+t} H_\xi^+(z)H_\xi^+(t)\sigma(dt)B_j = \int_0^{z+t} h_z(y)H(y)B_jdy = h_z(0)-I. \]

However, (2.4a) implies

\[ h_z(0) = \int_0^\infty e^{-x/\beta}(\delta(x)+\gamma(x,0))dx = I + \int_0^\infty e^{-x/\beta}a(x)dx = H_\xi^+(z), \]

which proves (2.6a).

The next theorem provides sufficient conditions for the unique solvability of Eqs. (1.1) to (1.3). Part (ii) is known for the case of multigroup neutron transport.

**Theorem 2.4** The boundary value problem (1.1) to (1.3) and the equivalent Wiener-Hopf operator integral equation (1.4) are uniquely solvable in the following cases:

(i) \( \Lambda \) is a strictly positive self-adjoint operator;

(ii) \( \Lambda \) has norm less than unity;

(iii) \( \Lambda \) is invertible and the norm of \( \Lambda^{-1}I \) is less than unity.

In these cases the \( \Lambda \)-equations (2.7a) and (2.7b) have solutions.

**Proof.** Part (i) was proved by van der Mee\(^2\) for bounded \( T \).

If \( T \) is unbounded, we have to consult the work of Beals\(^2\) to find the unique solvability of (a suitable version of) Eqs. (1.1) to (1.3) on the completion \( H_\xi^+ \) of \( D(T) \) with respect to the inner product

\[ (h,k)_\xi = (\xi|T|h,k). \]

From this we derive that Eqs. (1.1) to (1.3) (as stated here) are uniquely solvable for any dense subspace of \( f_+ \in Q_+[D(T)] \) in \( Q_+[H_\xi], \) from which the result is immediate.

If Part (ii) is satisfied, then, because \( \pi \) and \( \lambda \) have unit norm,

\[ |\Lambda(z)-I| < \|z(\lambda-T)^{-1}\| \|B\| < \|B\| < 1, \quad \text{Re} \ z = 0. \]

Invoking a factorization result for Hilbert space operator functions close to the identity, we get the existence of a canonical factorization of \( \Lambda(z) , \) from which Part (ii) follows. Part (iii) is proved analogously using

\[ |\Lambda(z)(\pi T)^{-1}I| < \|T(\lambda-T)^{-1}I\| \|\Lambda^{-1}I\| < 1, \quad \text{Re} \ z = 0. \]

The statement about the \( \Lambda \)-functions is immediate from Theorem 2.3.

3. **SPECTRAL ANALYSIS**

In the next section we shall extend the results of the previous section to non-invertible \( \Lambda \). The method of proof will basically consist of replacing \( \Lambda \) by a finite-dimensional regular perturbation \( A_\beta \), for which Eqs. (1.1) to (1.3) (with \( A_\beta \) instead of \( \Lambda \)) are uniquely solvable. This reduction will require a more thorough knowledge of the spectral properties of \( T^{-1}A \) and some additional properties of the \( \Lambda \)-operators. This knowledge will be provided here.

**Formula (2.1) suggests defining the operator**

\[ E^+f = f + \int_0^\infty \int_{\mathbb{R}^+} \frac{d\nu}{\nu-a} \sigma(\nu\mathcal{B}H_\xi^+(\nu)\mathcal{H}_\xi^+H_\xi^+(\nu)\sigma(\nu)f. \quad (3.1) \]

On its domain \( U \) \( \{ \text{Ran} (\alpha([0,b]) \mid b \text{ finite} \} \), one has

\[ \text{Ef}_+ - E^+ f = \left[ \int_0^\infty \sigma(\nu\mathcal{B}H_\xi^+(\nu)\mathcal{H}_\xi^+(\nu)\sigma(\nu)f. \right] f_+. \]

so that \( E-E^+ \) extends to a bounded (and, because \( B \) is compact, even a compact) operator on \( \text{Ran} Q_+ \). Thus \( E \) and \( E^+ \) both extend to
bounded operators on Ran $Q_+$. Further, $E_+ = EQ_+$ and $E_+^\dagger = E^\dagger Q_+$ are bounded projections on $H$ with kernel $\text{Ran } Q_+$, which satisfy the intertwining property

$$E_+[D(T)] \subseteq D(T), \quad TE_+ = E_+^\dagger Tf \text{ for } f \in D(T). \quad (3.2)$$

Because

$$E_+ f = [(I-L)^{-1}uQ_+ f](0) = Q_+ f + \int_0^\infty \tau(0,y)e^{-yT^{-1}Q_+ f}dy,$$

where $\tau(0,y)$ is compact and Bochner integrable, the operator $E_+Q_+$ is compact. So both $E_+$ and $E_+^\dagger$ are compact perturbations of $Q_+$.

Lemma 3.1. The range of $E_+$ (resp. $E_+^\dagger$) is invariant under $A^{-1}T$ (resp. $TA^{-1}T$), while the restriction of $A^{-1}T$ (resp. $TA^{-1}$) to the range of $E_+$ (resp. $E_+^\dagger$) has its spectrum in the closed right half-plane.

Proof. We consider the operator on $\text{Ran } E_+$ defined by

$$U_+(x)Ef_+ = f(x), \quad 0 < x < \infty,$$

where $f$ is the unique bounded solution of Eq. (1.4) with right-hand side $\omega f$. Then $U_+(x)$ is well-defined, linear, bounded and depends on $x\in(0,\infty)$ continuously in the strong operator topology. The latter follows, because the solution $f$ of Eq. (1.4) is bounded and continuous on $[0,\infty)$ (see Appendix). Furthermore, if $f \in Q_+[D(T)]$ and therefore $Ef_+ \in \text{Ran } P_+ \cap D(T)$, then $f(x) \in D(T)(0 < x < \infty)$ and (1.1) holds true. It is straightforward to derive that $U_+(\cdot)$ is a bounded strongly continuous semigroup on $\text{Ran } E_+$ with infinitesimal generator $K_+$, where

$$D(K_+) = \text{Ran } E_+ \cap D(T^{-1}A), \quad K_+ g = -T^{-1}Ag.$$

Hence, $\text{Ran } E_+$ is $A^{-1}T$-invariant and the restriction of $A^{-1}T$ to $\text{Ran } E_+$ has its spectrum in the right half-plane. Because (3.2) holds true, a similar statement immediately follows for $E_+^\dagger$.

The existence and uniqueness of the solution of Eqs. (1.1) to (1.3) does not play a significant role in the above proof. If one replaces $\text{Ran } E_+$ by the closure of the subspace of the initial values $f(0)$ where $f_+$ is ranging over $Q_+[D(T)]$, then this subspace is $A^{-1}T$-invariant and the above semigroup can be constructed. If we then replace $\text{Ran } E_+^\dagger$ by the closure of the subspace of vectors $Tf(0)$, where $f_+$ ranges over $Q_+[D(T)]$, we get a $TA^{-1}$-invariant subspace.

Using Lemma 3.1 the unique solution of Eqs. (1.1) to (1.3) can be written in the familiar $1^{-}$ semigroup form

$$f(x) = \int_0^x \tau(t,A^{-1}T)e^{-tA^{-1}T}f(0)dt, \quad 0 < x < \infty,$$

which converges to zero as $x \to \infty$. Another familiar formula is obtained by writing $E$ as the inverse of the Hangelbroek operator

$$V = Q_+P_+ + Q_-P_-, \quad \text{(3.3)}$$

where $P_+$ and $P_-$ are complementary projections commuting with $A^{-1}T$ and $\text{Ran } P_+ = \text{Ran } E_+$. In order to do this, we have to prove that $\text{Ran } E_+$ is the maximal $A^{-1}T$-invariant subspace of $H$ such that the restriction of $A^{-1}T$ to it has its spectrum in the closed right half-plane. For strictly positive $A$ this is a well-known fact.

Theorem 3.2. There exists a decomposition

$$Y_+ \oplus Y_- = H$$

of $H$ into closed $A^{-1}T$-invariant subspaces $Y_+$ and $Y_-$ such that the restriction of $A^{-1}T$ to $Y_+$ has the property

$$oA^{-1}T|_{Y_+} = \{te^o(A^{-1}T)/Re t \geq 0\}.$$

If Eqs. (1.1) to (1.3) are uniquely solvable, then $Y_+ = \text{Ran } E_+$.

Proof. The full-line convolution equation

$$f(x) = \int_0^x H(x-y)Bf(y)dy = \omega(x), \quad (-\infty < x < \infty), \quad \text{(3.5)}$$

is uniquely solvable. In fact, there exists a Bochner integrable
operator function \( \xi(\cdot) \) with compact operators as values such that
\[
f(x) = \omega(x) + \int_{-\infty}^{\infty} \xi(x-y)\omega(y)dy \quad (-\infty < x < \infty),
\]
while
\[
I + \int_{-\infty}^{\infty} e^{t/\lambda}f(t)dt = \left[ I - \int_{-\infty}^{\infty} e^{t/\lambda}H(t)Bdt \right]^{-1}, \quad \text{Re} \ \lambda = 0.30
\]

If \( \omega \) is bounded measurable, so will \( f \), and \( \omega \cdot f \) will be continuous on \( \mathbb{R} \).

Secondly, using the Appendix one sees that for the right-hand sides
\[
\omega(x) = \begin{cases} 
+ e^{-xT^{-1}}Qh, & x > 0 \\
- e^{-xT^{-1}}Q_h, & x < 0
\end{cases}
\]

the bounded solutions of the Wiener-Hopf equation (3.5) satisfy the equations
\[
(Tf)'(x) = Af(x) \quad (0 \leq x < \infty) \quad (3.7a)
\]
\[
f(0^+) - f(0^-) = \omega(0^+) - \omega(0^-) = h; \quad (3.7b)
\]

This problem is, in fact, uniquely solvable. Applying Laplace transformation to (3.5) we find the solution
\[
\int_{-\infty}^{\infty} e^{t/\lambda}f(t)dt = \left[ I - \int_{-\infty}^{\infty} e^{t/\lambda}H(t)Bdt \right]^{-1} \int_{-\infty}^{\infty} e^{t/\lambda}\omega(t)dt = (T - \lambda A)^{-1}(T - \lambda)\lambda T(T - \lambda)^{-1}h = \lambda A^{-1}T(\lambda - A^{-1}T)^{-1}h,
\]
where \( \text{Re} \ \lambda = 0 \) and Eq. (1.9) is used. Formally we may write
\[
f(x) = \begin{cases} 
+ e^{-xT^{-1}}A^+h, & x > 0 \\
- e^{-xT^{-1}}A^-h, & x < 0
\end{cases}
\]

for a suitable pair of complementary projections \( P_{\pm} \) commuting with

**ALBEDO OPERATORS AND H-EQUATIONS**

\( A^{-1}T \). Let us justify Eq. (3.8). Notice that \( f \) has a jump discontinuity at \( x = 0 \), and define
\[
P_{\pm}h = f(0^+) \quad P_{\mp}h = -f(0^-).
\]

We also define, for \( z > 0 \),
\[
V_{z}(z)h = f(z), \quad V_{-}(z)h = -f(-z).
\]

Then \( V_{z}(z) \) is bounded on \( (0, \infty) \) and strongly continuous, while \( V_{z}(0^+) = P_{z} \) in the strong operator topology. Surely, \( P_{z} \) are bounded projections on \( H \) which add up to the identity, \( V_{z}(z) \) leaves invariant the range of \( P_{z} \), while the restriction of \( V_{z}(z) \) to Ran \( P_{z} \) induces a bounded \( C_{0} \)-semigroup on Ran \( P_{z} \), whose infinitesimal generator \( K_{z} \) is given by
\[
D(K_{z}) = (\text{Ran } P_{z}) \cap \text{ Dom}(T^{-1}A), \quad K_{z}g = zT^{-1}AP_{z}.
\]

Hence, Ran \( P_{z} \) is an \( A^{-1}T \)-invariant and the boundedness of the semigroup implies that the restriction of \( A^{-1}T \) to Ran \( P_{z} \) has its spectrum in the closed right/left half-plane. The intersection \( M \) of Ran \( P_{z} \) and Ran \( P_{\pm} \) is a closed \( A^{-1}T \)-invariant subspace of \( H \) such that the restriction of \( A^{-1}T \) to \( M \) has its spectrum on the imaginary line. As the resolvent set of \( A^{-1}T \) does not have bounded connected components, we have \( \sigma(A^{-1}T|_{M}) = \{0\} \). Thus if \( M \neq \{0\} \), then \( \lambda = 0 \) is an isolated point of this spectrum and therefore an eigenvalue, which leads to a contradiction. Hence, \( M = \{0\} \) and \( P_{z} \) are complementary projections. We have justified Eq. (3.8).

Let \( L_{z}(a,b) \) denote the Banach space of strongly measurable functions \( f:(a,b) \to H \), which are bounded with respect to the norm
\[
||f|| = \text{ess sup} ||f(x)||_{H} |a < x < b|
\]

Introduce the operators \( L_{z}, \ L_{z}^{+} \) and \( L_{z}^{-} \) on \( L_{z}(H)_{a}^{\infty}, \ L_{z}(H)_{0}^{\infty} \) and \( L_{z}(H)_{a}^{0} \), respectively, defined by
\[(Lf)(x) = \int_{-0}^{x} H(x-y)B(f(y))dy, \quad \lim_{x \to -0} (L_+^*)\phi(x) = \pm \int_{-0}^{x} H(x-y)B(f(y))dy.\]

Then \((I-L_+)\theta(I-L_-)\) can be identified in a natural way with an operator on \(L_\infty(H)\) and the difference between this operator and the invertible operator \(I-L\) is given by

\[
(Kf)(x) = \begin{cases} 
- \int_{-0}^{x} H(x-y)B(f(y))dy, & x > 0 \\
\int_{0}^{x} H(x-y)B(f(y))dy, & x < 0 
\end{cases}
\]

which is a compact operator. Thus \(I-L_\pm\) are Fredholm operators whose indices (i.e., nullity minus deficiency index) add up to zero. In particular, if Eqs. (1.1) to (1.3) are uniquely solvable, then \(I-L_+\) is invertible and \(I-L_-\) is Fredholm of index 0.

Let us consider the Wiener-Hopf equation (1.4) on \((0,0)\) and its counterpart on \((-0,0)\), written as

\[(I-L_+)f = \omega \text{ on } L_\infty(H)_{0+}, \quad (I-L_-)f = \omega \text{ on } L_\infty(H)_{0-}\]

where \(\omega\) is given by (3.6). Denote by \(X_+\) (resp. \(X_-\)) the (closed) linear subspace of initial values \(f(0^+)(\text{resp. } f(0^-))\) of solutions, where \(h\) ranges over \(H\). For a uniquely solvable right half-space problem one has \(X_+ = \text{Ran } E_+\). One easily sees that \(X_- \subseteq Y_-\), but we intend to prove \(X_- = Y_-\). Certainly, by the equivalence theorems, we have

\[
\|f(0^+)|f\in L_\infty(H)_{0+} \text{ and } (I-L_+)f = \omega\| = X_+ \cap \text{Ran } Q_- \subseteq D(T)
\]

\[
\|f(0^-)|f\in L_\infty(H)_{0-} \in H\in D(T): (I-L_-)f(x) = e^{-\int_{x}^{0} Q_- h, x > 0} = [X_+ + \text{Ran } Q_-] \cap D(T),
\]

and similar identities for the problem on \(L_\infty(H)_{0+}\). Because the Fredholm indices of the Fredholm operators \(I-L_+\) and \(I-L_-\) add up to zero, one has

\[
\dim[X_+ \cap \text{Ran } Q_-] - \text{codim}[X_+ + \text{Ran } Q_-] = \dim[X_- \cap \text{Ran } Q_-] + \text{codim}[X_- + \text{Ran } Q_-]. \tag{3.10}
\]

However, \(X_+ \subseteq Y_-\) implies

\[
\dim[X_+ \cap \text{Ran } Q_-] < \dim[Y_- \cap \text{Ran } Q_-]; \tag{3.11}
\]

\[
\text{codim}[X_+ + \text{Ran } Q_-] > \text{codim}[Y_- + \text{Ran } Q_-]. \tag{3.12}
\]

We now define \(V\) by (3.3) and compute

\[
\text{Ker } V = [Y_+ \cap \text{Ran } Q_-] \cap [Y_- \cap \text{Ran } Q_-]
\]

\[
\text{Ran } V = [Y_+ + \text{Ran } Q_-] \cap [Y_- + \text{Ran } Q_-].
\]

If \(V\) would be a compact perturbation of the identity and therefore Fredholm of index 0, we would have \(\dim \text{Ker } V = \text{codim Ran } V\) and thus Eq. (3.10) with \(X_+\) replaced by \(Y_-\). The latter equation together with (3.10) to (3.12) would imply that equality signs hold in (3.11) and (3.12), and hence that

\[
X_- \cap \text{Ran } Q_- = Y_- \cap \text{Ran } Q_-; \quad X_+ + \text{Ran } Q_- = Y_- + \text{Ran } Q_-.
\]

These identities together with \(X_+ \subseteq Y_-\) have as a consequence \(X_+ = Y_-\), and the proof of the theorem would be complete.

Because

\[
I - V = Q_+ Q_- + Q_- Q_+ = (Q_+ - Q_-)(Q_+ - Q_-),
\]

it suffices to prove that \(Q_+ - Q_-\) is compact. Let \(f\in L_\infty(H)_{0+}\) be the unique solution of (3.5), where \(\omega\) is given by (3.6). Using the resolvent kernel \(\mathcal{L}(\cdot)\) one finds

\[
(P_+ - Q_-)h = \int_{0}^{\infty} \mathcal{L}(-y)e^{-yT} Q_- hdy
\]

and the compactness of \(P_+ - Q_-\) is clear from the compactness of \(\mathcal{L}(\cdot)\) and the Bochner integrability of \(\mathcal{L}(\cdot)e^{-yT} Q_-\) on \((0,\infty)\).

The construction and reasoning of the above proof were applied before to one-speed neutron transport in \(L_p\)-spaces. We merely have constructed semigroups from solutions, as is usual in some areas. Theorem 3.2 and Lemma 3.1 allow us to extend the Hangelbroek originated semigroup approach to various kinetic models beyond positive or even self-adjoint A. The simultaneous unique solvability of Eqs. (1.1) to (1.3) and the analogous left half-space...
principle given there (for \( D_1 = I \) and \( D_2 = \pi A \)) leads to different factors not suitable to our purpose. As we intend a generalization beyond the scope of Ref. 22, we prove (4.1) directly.

**Proof of Theorem 4.1.** Let us multiply the right-hand sides of (4.2) and (4.4). We get
\[
[I - z\pi A E_+(z-T)^{-1}B_j][I - z\pi (T-zA)^{-1}E_+^*B_j] = I - z\pi E_+(z-T)^{-1}B_j - z\pi (T-zA)^{-1}E_+^*B_j + z\pi E_+(z-T)^{-1}(z-T)(T-zA)^{-1}E_+^*B_j.
\]

Using that
\[
E_+(T-zA)^{-1}E_+^* = (T-zA)^{-1}E_+^* E_+(z-T)^{-1}E_+^* = E_+(z-T)^{-1},
\]
on simplifying the above expression one obtains
\[
[I - z\pi E_+(z-T)^{-1}B_j][I - z\pi (T-zA)^{-1}E_+^*B_j] = I.
\]

Let us multiply the right-hand sides of (4.3) and (4.5). As a result we find
\[
[I - z\pi (I-E_+)(T-zA)^{-1}B_j][I - z\pi (z-T)^{-1}(I-E_+)^*B_j] = (I-E_+)^* B_j - z\pi (I-E_+) (T-zA)^{-1}B_j + z\pi (T-E_+) (T-zA)^{-1}(z-T)(T-zA)^{-1}(I-E_+)^*B_j.
\]

Now we use the identities
\[
(I-E_+)(T-zA)^{-1}(I-E_+)^* B_j = (I-E_+) (T-zA)^{-1},
\]
\[
(I-E_+)(z-T)^{-1}(I-E_+)^* = (z-T)^{-1}(I-E_+)^*,
\]
and derive
\[
[I - z\pi (I-E_+)(T-zA)^{-1}B_j][I - z\pi (z-T)^{-1}(I-E_+)^*B_j] = I.
\]

Let us postmultiply the right-hand side of (4.5) by the one of (4.4). We get
\[
[I - z\pi (z-T)^{-1}(I-E_+)^*B_j][I - z\pi E_+(z-T)^{-1}B_j] = I - z\pi E_+(z-T)^{-1}B_j - z\pi (z-T)^{-1}(I-E_+)^*B_j + z\pi (z-T)^{-1}(I-E_+)^* (I-A) E_+(z-T)^{-1}B_j.
\]

Note now that
\[
\]

Further, condition (4) implies
\[
z(z-T)^{-1}((I-E_+)^* z(z-T)^{-1} - (I-E_+)^* (I-E_+)^*).
\]
Using these two identities one simplifies the above product considerably and obtains

\[ [I-z\pi(z-T)^{-1}(I-E_+^+)B]([I-z\pi(z-T)^{-1}B] = I-z\pi(z-T)^{-1}B = A(z). \]

(4.9)

The latter equality is immediate from (1.9).

The factorizations (4.6) to (4.9) involve factors that are compact perturbations of the identity. Thus the right-hand sides of (4.2) and (4.4) as well as those of (4.3) and (4.5) are inverses of each other, and Eq. (4.1) is clear.

If \( A \) is invertible and Eqs. (1.1) to (1.3) are uniquely solvable, then in Theorem 4.1 we could use the operators \( E_+ \) and \( E_+^+ \) of the previous section and in this case the functions in (4.2) and (4.3) coincide with the functions \( H_+ \) and \( H_+^+ \) of Section 2, as we shall see later.

If \( T \) is bounded, then the operators

\[ P_0 = (2\pi i)^{-1} \int_\Gamma (\lambda - \zeta T)^{-1} d\zeta, \quad P_0^+ = (2\pi i)^{-1} \int_\Gamma (\lambda - \zeta T)^{-1} d\zeta, \]

where \( \Gamma \) is a small positively oriented circle which separates \( \lambda = 0 \) from the non-zero part of the (identical) spectra of \( T^{-1}\lambda \) and \( AT^{-1} \), are bounded projections onto the finite-dimensional subspaces

\[ Z_0 = \bigcup_{n=1}^\infty \text{Ker}(T^{-1}\lambda)^n, \quad Z_0^+ = \bigcup_{n=1}^\infty \text{Ker}(AT^{-1})^n, \]

(4.10)

where \( Z_1 = \text{Ker} P_0 \) and \( Z_1^+ = \text{Ker} P_0^+ \) are invariant under \( T^{-1}\lambda \) and \( AT^{-1} \), respectively. Furthermore,

\[ T[Z_0] = Z_0^+, \quad A[Z_1] = Z_0^+; \quad \overline{T[Z_1]} = A[Z_1] = Z_1^+; \]

(4.11)

\[ Z_0 \otimes Z_1 = H, \quad Z_0^+ \otimes Z_1^+ = H. \]

(4.12)

If \( T \) is unbounded, however, we assume that the subspaces in (4.10) have a finite dimension, that \( Z_0 \subseteq D(T^{2+\alpha}) \) and Eqs. (4.11) and (4.12) are fulfilled. If \( A \) is positive self-adjoint, conditions of this kind were previously known.33

Lemma 4.2. Assume that to every \( f_+ \in Q_+(D(T)) \) there exists a unique solution of Eqs. (1.1) to (1.3). Then there exist bounded projections \( E_+ \) and \( E_+^+ \) with kernel \( \text{Ran} Q_- \) such that \( f(0) = E_+f_+ \) is the initial value of the solutions and conditions (1) to (111) of Theorem 4.1 are fulfilled.

Proof. On \( D(T) \) we define the operator \( E_+^* \) by \( E_+^*h = f(0) \), where \( f \) is the unique solution of Eqs. (1.1) to (1.3) with initial value \( f_+ = Q_+h \). Let us choose a subspace \( N \) such that

\[ N \cap \{Z_0 \cap \text{Ran} E_+(D(T))\} = \{0\}. \]

(4.13)

Choose a "matrix" \( \beta \) on \( Z_0 \) without imaginary eigenvalues such that \( \beta \) is reduced by the decomposition (4.13), \( \sigma(\beta [N]) \subseteq \{\lambda | \text{Re} \lambda < 0\} \) and \( \sigma(\beta [Z_0 \cap \text{Ran} E_+(D(T))] \subseteq \{\lambda | \text{Re} \lambda > 0\} \). Put

\[ A_\beta = T^{-1}P_0 + A(I-P_0). \]

Then \( A_\beta \) is invertible, the operator \( I-A_\beta \) satisfies (1.17), the operator

\[ A_\beta^{-1}T = \beta \otimes (T^{-1}A_\beta^+)^{-1} \]

does not have imaginary eigenvalues and the function

\[ g(x) = e^{-xT^{-1}A}P_0f_+ + (I-P_0)f(x) \quad (0 \leq x < \infty) \]

is a solution of the "regular" boundary value problem

\[ (Tg)'(x) = -A_\beta g(x) \quad (0 \leq x < \infty) \]

(4.14)

\[ Q_+(0) = f_+, \quad ||g(x)||_H = O(1)(x \to \infty), \]

(4.15)

to which the theory of Section 3 applies.34 The roles of \( Y_+ \) and \( Y_- \) are now played by \( \text{Ran} E_+ \) and an extension of \( N \), respectively, so that Eqs. (4.14) and (4.15) are uniquely solvable with \( E_+ \) as the albedo operator which maps the boundary data \( f_+ \) into the initial value \( g(0) \). Thus \( E_+ \) is a bounded projection on \( H \) with kernel \( \text{Ran} Q_- \), whose range is invariant under \( A_\beta^{-1}T \) and \( T^{-1}A_\beta \).

As obviously \( Z_0 \cap \text{Ran} E_+ \subseteq \text{Ker} A \) (for otherwise \( g \) might be unbounded as \( x \to \infty \)), (3.12) implies that \( \text{Ran} E_+ \) is \( T^{-1}A \)-invariant. Moreover, there also exists a bounded projection \( E_+ \) on \( H \) with kernel \( \text{Ran} Q_- \), such that \( \text{Ran} E_+ \subseteq D(T) \) and \( TE_+ = E_+^*T \) on \( D(T) \).

Hence, the conditions of Theorem 4.1 are fulfilled.

The unique solvability of problem (1.1) to (1.3) really needs not be assumed. It suffices to assume that to every \( f_+ \in Q_+(D(T)) \) there exists at least one solution which is representable as...
f(0) = Ef_+ for some linear operator E on D(T). This applies to positive self-adjoint A. The remark we may also make as to the next two results.

Theorem 4.3. Assume that to every f, g, r ∈ D(T) there exists a unique solution of Eqs. (1.1) to (1.3) and put E_r h = f(0) with Q_h = f_+. Then the functions H^+_E(z)^{-1} and H^+_E(z)^{-1} in (4.4) and (4.5) are continuous and invertible on the closed right half-plane and analytic on the open right half-plane, while H^+_E(z)^{-1} and H^+_E(z)^{-1} are bounded for z = Re z ≥ 0.

Further,
\[
\lim_{z \to \infty, \text{Re } z \geq 0} z^{-1} H^+_E(z) = \lim_{z \to \infty, \text{Re } z \geq 0} z^{-1} H^+_E(z) = -\pi(1-E_+)T^{-1}P^+_0 Bj, \quad \lim_{z \to \infty, \text{Re } z \geq 0} z^{-1} H^+_E(z) = -\pi(1-E_+)T^{-1}P^+_0 Bj,
\]
where the projection of H onto Z^+_0 along Z^+_1 is denoted by P^+_0. The functions H^+_E and H^+_E satisfy the generalizations (2.7) of Chandrasekhar’s H-equations.

Proof. Note that (1-E_+) T^{-1} P^+_0 and (z-\bar{z})^{-1} (1-E_+) extend to analytic functions on the right half-plane, while (T-zA)^{-1} E_+ and E_+ (z-\bar{z})^{-1} extend analytically to the left half-plane. The analyticity of the factors (4.2) to (4.5) on the respective half-planes then is immediate. The continuity one only has to prove for z = 0 and (if possible) for z = ∞. Since z(T-zA)^{-1} Q_h = 0 in the strong operator topology if z → 0 from the left/right half-plane, we have 23, 22, 3

\[ ||z(T-zA)^{-1} Q_h|| \to 0 \quad (z \to 0, \text{Re } z \not\to 0) \]
for all compact operators K. Thus H^+_E(0)^{-1} = I in the norm, whence H^+_E(0)^{-1} H^+_E(0)^{-1} = I in the norm. In a similar way, using z(T-zA)^{-1} Q_h strongly if z → ∞ from the left/right-half plane,

\[ H^+_E(z)^{-1} = -\pi(1-E_+)T^{-1}P^+_0 Bj, \quad H^+_E(z)^{-1} = -\pi(1-E_+)T^{-1}P^+_0 Bj. \]

Using the help of (4.11) and (4.12) and the projection P^+_0 of H onto Z^+_0 along Z^+_1, we easily prove that

\[ \lim_{z \to \infty, \text{Re } z \geq 0} ||(T-zA)^{-1} (1-P^+_0 E_+) Bj|| = 0. \]

ALBEDO OPERATORS AND H-EQUATIONS

However, as T-zA maps Z_0 onto Z^+_0, one has

\[ (T-zA)^{-1} T = h, \quad h \in \text{Ker } A. \]

Since Ran E_+ ∩ Z_0 ⊆ Ker A and TE_+ = H^+_E on D(T), one eventually obtains (4.16).

Put

\[ \phi^+_E(z) = H^+_E(z)^{-1} + z \int_0^\infty (z+t)^{-1} \tau_0 (-dt) B J H^+_E(t). \]

This function is well-defined and continuous on the closed right half-plane, is analytic on the open right half-plane and satisfies \( \phi^+_E(0^+) = 1 \). The limit of \( \phi^+_E(z) \) as \( z \to \infty \) in the closed right half-plane exists (cf. (4.16)). Using (1.9) and (4.1) one writes

\[ \phi^+_E(z) = H^+_E(z)^{-1} - z \int_0^\infty (z-t)^{-1} \tau_0 (-dt) B J H^+_E(t) - \int_0^\infty (z+t+\tau_0 (-dt) B J H^+_E(t)^{\tau_0 (-dt) B J H^+_E(t)}, \]

which is continuous on the closed and analytic on the open left half-plane. This expression is 0(z) as \( z \to \infty \) in the left half-plane. Using Liouville’s theorem and \( \phi^+_E(0^+) = 1 \), one finds \( \phi^+_E(z) = 1 \), which implies (2.7b). Equation (2.7a) is proved analogously with the aid of the auxiliary function

\[ \phi^+_E(z) = H^+_E(z)^{-1} + z \int_0^\infty (z+t)^{-1} H^+_E(t) \tau_0 (-dt) B J. \]

Theorem 4.4. Assume that to every f, g, r ∈ D(T) there exists a unique solution of Eqs. (1.1) to (1.3) and put E_r h = f(0) with Q_h = f_+. Then the albedo operator E is given by (2.1), where H^+_E and H^+_E are the functions in (4.2) and (4.3).

Proof. Using (4.3) we compute, for \( \omega = -\omega, 0 \),

\[ \int_0^\infty \frac{\tau_0 (-dt) B J H^+_E(t)}{\tau_0 (-dt) B J} = -\pi(T-zA)^{-1} Q_+ - \int_0^\infty \frac{\tau_0 (-dt) B J H^+_E(t)}{\tau_0 (-dt) B J}. \]
We now substitute the identity
\[ \nabla B = (v-T) + (T-vA), \]
and, using \((v-T)(vdv) = 0\), we obtain
\[ f_{\infty}^{0} \frac{\nu}{\nu-\nu} H^+_v(v) \sigma(dv) = T(T-\nu)^{-1} Q_+ = \pi E_+ T(T-\nu)^{-1} Q_+ . \]  

The integral term in (2.1) now reads (cf. (4.2)):
\[ f_{\infty}^{0} \sigma(dv)B_jH^+_v(v)E_+ T(T-\nu)^{-1} Q_+ - f_{\infty}^{0} \sigma(dv)B_jH^+_v(v)E_+ T(T-\nu)^{-1} Q_+ . \]

Again we involve (4.18) and get for the integral term in (2.1)
\[ f_{\infty}^{0} \sigma(dv)(I-E^+_v)BE_+ T(T-\nu)^{-1} Q_+ . \]  

We now employ the identity
\[ (I-E^+_v)BE_+ = (I-E^+_v)(I-I-E^+_v) + (I-E^+_v)AE_+ = E_+ - E^+_v , \]

(together with (4.8), to simplify (4.19). As a result we get
\[ f_{\infty}^{0} \int_{\nu} \frac{\nu}{\nu-\nu} \sigma(dv)B_jH^+_v(v) \tau\sigma(dv) = Q_- E_+ , \]

which yields (2.1).

We have generalized the Wiener-Hopf factorization, the H-equations and the expression for the albedo operators to the case when A does not have an inverse. If Eqs. (1.1) to (1.3) are non-uniquely solvable, several of these factorizations exist and the behaviour of \( H^+_v(z) \) and \( H^+_v(z) \) for \( z = \pm \) must be used to single out the factors appropriate to the problem. In this article we shall not elaborate upon the uniquesness problem for solutions of the generalized H-equations. We remark that generalized H-equations and a formula for \( \bar{E} \) in terms of H-equations have been found by many authors. (The literature is too enormous to cite here). A more or less abstract procedure, with \( T \) still

5. SOME APPLICATIONS

(a) Transfer of polarized light in a semi-infinite homogeneous planetary atmosphere is described by the equation
\[ \frac{d\bar{I}(\tau,u,\phi)}{d\tau} = \frac{1}{4\pi} j \int_{-1}^{1} \bar{Z}(u,u',\phi-\phi')\bar{I}(\tau,u',\phi')du' , \]

where \( I(\tau,u,\phi) \) is the Stokes vector describing the intensity and state of polarization of a beam with directional parameters \( u[-1,1] \) and \( \phi[0,2\pi] \) at optical depth \( \tau[0,\infty) \). Here \( Z(u,u',\phi-\phi') \) is the phase matrix and \( 0 < \phi < 1 \) the albedo of single scattering.

Throughout we use the conventions of Ref. 12.

On applying Fourier decomposition and symmetry properties the full equation (5.1) can be decomposed into twice the sum of component equations
\[ \frac{d\bar{J}_1^0(\tau,u)}{d\tau} + \bar{J}_1^0(\tau,u) = \frac{1}{2} j \int_{-1}^{1} \bar{J}^0_1(u,u')\bar{J}_1^0(\tau,u')du' , \]

where \( u[-1,1] \), \( \tau[0,\infty) \) and \( j = 0, 1, 2, \ldots \). Here the kernel is given by
\[ \bar{J}^0_1(u,u') = \frac{1}{2} \int_{\tau}^{\tau} \bar{J}^0_1(u') \bar{J}^0_1(u') , \]

where for certain special functions and expansion coefficients
\[ \begin{bmatrix} P_{\tau}^0(u) & 0 & 0 & 0 \\ 0 & R_{\tau}^0(u) & -T^0_{\tau}(u) & 0 \\ 0 & -T^0_{\tau}(u) & R^0_{\tau}(u) & 0 \\ 0 & 0 & 0 & P_{\tau}^0(u) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \gamma_{\tau} & 0 & 0 & 0 \\ 0 & \gamma_{\tau} & 0 & 0 \\ 0 & 0 & \gamma_{\tau} & 0 \end{bmatrix} \]

The physical requirement that the degree of polarization of a beam does not exceed unity, necessitates imposing the condition that \( \bar{Z}(u,u',\phi-\phi') \) maps real vectors \( I = (I,0,0,0) \) satisfying
\[ I > \sqrt{Q^2 + U^2 + V^2} > 0 \]

into vectors of the same type.

Under the boundary conditions

\[ X^j(0, u) = X^j(u)(0 < u < 1), \quad \int_0^1 ||X^j(\tau, u)||^2 du = O(1) \quad (\tau \to \infty) \]  
\[ (5.4) \]

Equation (5.2) is uniquely solvable if the summation in (5.3) is finite, which we shall henceforth assume. Let \( H \) be the Hilbert space of measurable \( L_2 \)-functions \( \mathbb{I}[-1, 1] \to \mathbb{C}^4 \), and define \( T, B, A \) and \( Q \) by the equations

\[ (T\xi)(u) = u\xi(u), \quad (B\xi)(u) = \frac{1}{2} \int_{-1}^1 \overline{\xi}(u, u')\xi(u') du', \]

\[ (A\xi)(u) = \mathbb{I}(u) - \frac{1}{2} \int_{-1}^1 \overline{\xi}(u, u')\xi(u') du', \quad (Q\xi)(u) = \begin{cases} \int(u), & u > 0 \\ 0, & u < 0. \end{cases} \]

Then Eqs. (5.2) and (5.4) are an example of problem (1.1) to (1.3), which is uniquely solvable and for which \( T^{-1}A \) does not have non-zero imaginary eigenvalues. Therefore, there is an abeloid operator \( E_+ \) which maps \( X^j \) into \( X^j(0) \) uniquely.

Let \( \mathbb{E} = \text{span} \{ e^j_{\ell, k} | \ell = j + 1, \ldots, \ell; k = 1, 2, 3, 4 \} \), where \( \ell = \max(j, 2) \) and

\[ e^j_{\ell, 1} = (P^j_{\ell, 0}, 0, 0, 0), \quad e^j_{\ell, 2} = (0, R^j_{\ell, 0} - T^j_{\ell, 0}) \]

\[ e^j_{\ell, 3} = (0, -T^j_{\ell, 0}, R^j_{\ell, 0}), \quad e^j_{\ell, 4} = (0, 0, 0, p^j_{\ell}). \]

Then \( \mathbb{E} = \text{Ran} \mathbb{B}^* \), and in the usual inner product of \( H \) we have

\[ (1) \begin{align*}
(\mathbb{E}\xi, \mathbb{E}\eta)^H & = 0 \quad \text{if either } \ell \neq \tau \text{ or } k \neq \tau; \\
(ii) ||\mathbb{E}\xi||^2_H = \begin{cases} & \frac{2(\ell - \tau)!}{2(\ell + 1)!} \quad \text{if } \ell > \tau \text{ and } k = 1, 4, \\
& (\ell - \tau)! \quad \text{if } \ell > \max(\tau, 2) \text{ and } k = 2, 3; \\
& 0 \quad \text{otherwise.} \end{cases}
\end{align*} \]

Thus \( \mathbb{E} \) has dimension \( 4(L-j) + 2 \min(j, 2) \). Further, the embedding previously devoted by \( j \) is the natural embedding of \( \mathbb{B} \) into \( H \), while \( \pi \) is given by

\[ \pi \mathbb{E}^j_{\ell, k} = \frac{L}{\ell - j} \sum_{\ell = j}^{L} \frac{(\ell - j)!}{(\ell + j)!} (\mathbb{J}, \mathbb{E}^j_{\ell, k}) H \mathbb{E}^j_{\ell, k}, \]

\[ k = 1, 4, \]

\[ + \sum_{\ell = \max(\ell, 2)}^{L} \frac{2(\ell - j)!}{(\ell + j)!} (\mathbb{J}, \mathbb{E}^j_{\ell, k}) H \mathbb{E}^j_{\ell, k}, \]

\[ k = 2, 3, \]

while

\[ (2\ell + 1) \mathbb{E}^j_{\ell, k} = \frac{(\ell + j)!}{(\ell - j)!} \sum_{m=1}^{4} [R^j_{\ell, m}] e^j_{\ell, m}. \]

With respect to the ordered basis \( \{ e^j_{\ell, k} | (\ell, k) = (j, 1), \ldots, (j, 4), \ldots, (j, 1), \ldots, (j, 4) \} \), with \( e^j_{\ell, 2} \) and \( e^j_{\ell, 3} \) left out for \( j < \ell < \max(j, 2) \), the dispersion matrix has the form

\[ \Lambda(z)(\ell, k), (r, t) = \delta_{\ell, r} \delta_{k, t} - \frac{1}{2} az \frac{2(\ell - j)!}{2(\ell + 1)!} (r + j)! (r - j)! \times \]

\[ \sum_{m=1}^{4} \left[ R^j_{\ell, m} \right] e^j_{\ell, m}(t) \cdot e^j_{r, m}(r) \times \int_{-1}^{1} \frac{z - t}{z - t} dt. \]

This square matrix function of order \( 4(\ell - j) + 2 \min(j, 2) \) has Wiener-Hopf factorization

\[ \Lambda(z)^{-1} = H^+_{\ell}(z)H^+_{\ell}(z)z, \quad \text{Re } z > 0, \]

where \( H^+_{\ell} \) and \( H^+_{\ell} \) are continuous and invertible on the closed right half-plane and analytic in the open right half-plane, \( H^+_{\ell}(0^+) = I \) and \( H^+_{\ell}(\infty) = H^+_{\ell}(0^+) = 0 \). The \( H \)-equations are now given by

\[ H^+_{\ell}(z)^{-1}(\ell, k), (r, t) = \delta_{\ell, r} \delta_{k, t} - \frac{1}{2} az \times \]

\[ \sum_{m=1}^{4} \left[ R^j_{\ell, m} \right] e^j_{\ell, m}(t) \cdot e^j_{r, m}(r) \times \int_{-1}^{1} \frac{z - t}{z - t} dt. \]
Finally, the albedo operator $E$ is given by $(EX^+)(u) = X^+(u)$ for $u > 0$, while for $u < 0$

$$(EX^+)(u) = \frac{1}{2} a \int_0^1 \int_{-1}^{1} \frac{u}{u-\nu} \nu d\nu + \int_{-1}^{1} \frac{u}{u-\nu} \nu d\nu ,$$

For $j = 0$ the above expressions decouple into pairs of expressions involving matrices of order $\mathbb{Z}L$.

(b) Neutron transport with angularly dependent cross-sections\(^{38}\) may be described by

$$\frac{\partial f}{\partial x}(x,\Omega) + \Sigma(\Omega)f(x,\Omega) = \frac{1}{2} \int_{-1}^{1} \Sigma_s(\nu)f(x,\nu)d\nu' ,$$

where $\Omega \in [-1,1]$ and $x \in [0,\infty)$ are the angular and position variable, respectively. This equation is also encountered in phonon transport in crystalline solids.\(^{14}\) We assume that $\Sigma$ and $\Sigma_s$ are measurable and satisfy $\Sigma > \Sigma_s > \varepsilon > 0$. Write $g = (\Sigma_s)^{1/2}f$, then

$$(u/\Sigma)(u) \frac{\partial g}{\partial x} + g(x,\Omega) = \frac{1}{2} \nu(x,\Omega) \nu g(x,\nu)d\nu' ,$$

where $\nu = (\Sigma/\Sigma_s)^{1/2}$. We analyze this equation in $H = L_2[-1,1]$ and define

\begin{align*}
(Th)(u) = \mu T(\mu)^{-1} h(\mu) , & \quad (Bh)(u) = \frac{1}{2} \int_{-1}^{1} \nu(\nu')h(\nu')d\nu' \\
(Ah)(u) = h(\mu) - \frac{1}{2} \int_{-1}^{1} \nu(\nu')h(\nu')d\nu' , & \quad (Q_\pm)(u) = \begin{cases} h(u) , & u \geq 0 \\
0 , & u \leq 0 . \end{cases}
\end{align*}

Condition (1.17) is satisfied if and only if $\int_{-1}^{1} \Sigma_s^{2\alpha-1} \int_{-1}^{1} |u|^2 \nu d\nu d\nu' < \infty$ for some $\alpha > 0$. Under this condition we shall study the boundary value problem

$$\left( Tg \right)'(x) = -Ag(x) \quad (0 < x < \infty) . \tag{5.5}$$

$$1 \int_{0}^{\infty} \left| \int_{0}^{x} g(x) - g_0(x) \right|^2 dx = 0 , \quad \left| g(x) \right|^2 dx = 0 (1) (x \to \infty) . \tag{5.6}$$

Then $A$ is positive self-adjoint, $B$ compact and $\ker A \equiv \operatorname{span} \{ v \}$. There are three cases to be considered: (i) $E_s(\mu) \Sigma(\mu)$, where $\ker A = \{ 0 \}$, (ii) $\Sigma(\mu) \Sigma(\mu) \Sigma(\mu) \Sigma(\mu) \Sigma(\mu)$ and $\int_{-1}^{1} \nu(\nu)(\nu)d\nu \neq 0$, where $\Sigma_0 = \ker A = \operatorname{span} \{ v \}$, and (iii) $\Sigma_s(\mu) \Sigma_s(\mu) \Sigma_s(\mu) \Sigma_s(\mu) \Sigma_s(\mu)$ and $\int_{-1}^{1} \nu(\nu)(\nu)d\nu = 0$, where $\Sigma_0 = \operatorname{span} \{ v, \mu \}$. Then Eqs. (5.5) and (5.6) are uniquely solvable, unless $\Sigma_s(\mu) \Sigma_s(\mu)$ and $\int_{-1}^{1} \nu(\nu)(\nu)d\nu \neq 0$, in which case there always exists a solution with measure of non-uniqueness one.\(^{4}\) So there exists an albedo operator (unique except for this exceptional case) and the $H$-functions can be found.

Let us choose $\mathcal{B} = \operatorname{span} \{ v \}$. Then $\mathcal{H} = (\mathcal{H}, \nu) 
\mathcal{H}^\nu = \int_{\Sigma_s} \frac{\nu}{\Sigma_s} d\nu$, and

$$A(z) = 1 - \frac{1}{2} \int_{-1}^{1} \frac{E_s(\nu')d\nu'}{zE_s(\nu') - \nu'} .$$

This dispersion function we factorize as

$$A(z)^{-1} = H^+_\Sigma(-z) H^+_\Sigma(z) , \quad \text{Re} z = 0 ,$$

where $H^+_\Sigma$ and $H^+_\Sigma$ satisfy the $H$-equations

$$H^+_\Sigma(z)^{-1} = 1 + \frac{1}{2} \int_{-1}^{1} \frac{E_s(\nu')d\nu'}{zE_s(\nu') + \nu'} d\nu' ,$$

$$H^+_\Sigma(z)^{-1} = 1 - \frac{1}{2} \int_{-1}^{1} \frac{E_s(\nu')d\nu'}{zE_s(\nu') - \nu'} d\nu' .$$
The albedo operator then is given by

$$\langle E_+ f \rangle (v) = \begin{cases} \mathcal{E}_+(v), & v > 0 \\ \frac{1}{2} \int_0^1 \frac{1}{x'} \mathcal{E}(v') \left[ \frac{E_+(v')}{\mathcal{E}(v')} \right]^{1/2} \mathcal{H}_\epsilon^+(\mathcal{E}(v')) H_\epsilon^+(\mathcal{E}(v')) \mathcal{E}_+(v') dv', & v < 0. \end{cases}$$

If $\mathcal{E}$ and $\mathcal{E}$ are even functions, then $H_\epsilon = H_\epsilon'$. For this case.

(c) Strong evaporation of a liquid into a half-space vacuum, with a drift velocity $d > 0$ at infinity, is described by the equation

$$f'(x,v) + f(x,v) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{x} \left[ 1 + 2vv' + 2(v^2 - \frac{1}{2})(v'^2 - \frac{1}{2}) \right] f(x,v') e^{-v'^2} dv',$$

(5.7)

where $f(x,v)$ is the deviation from the drift Maxwellian and $v$ the velocity. Effects transverse to the $x$-direction are neglected. The boundary conditions

$$f(0,v) = a_0 + a_1 v \sqrt{2} + a_2 (v^2 - \frac{1}{2}) \sqrt{2} (v > d), \quad f(\infty,v) = 0$$

(5.8)

are imposed. Note that $f(0,v)$ is required to be a collision invariant.

Let $\mathcal{H}$ be the Hilbert space of measurable functions $h: \mathbb{R} \to \mathcal{E}$ with inner product

$$(h,k) = \int_{-\infty}^\infty e^{-v^2} dv$$

and define on $\mathcal{H}$ the operators $T$, $Q_\epsilon$, $A$ and $B$ by

$$(Th)(v) = (v+d)h(v), \quad (Q_\epsilon h)(v) = \begin{cases} h(v), & v > -d \\ 0, & v < -d \end{cases}$$

$$(Ah)(v) = h(v) - Bh(v), \quad (Bh)(v) = \sum_{i=0}^2 (h, e_i^+ e_i^+(v),$$

where we have the orthonormal set

$$e_0(v) = 1, \quad e_1(v) = v \sqrt{2}, \quad e_2(v) = (v^2 - \frac{1}{2}) \sqrt{2}.$$
where the limit is taken in the weak sense.

Proof of Theorem A.1. Let $f:(0,\infty) \to \mathbb{D}(T)$ be a solution to (A.1) and (A.2), and put $x = f(x)$. Choose $0 < x < \infty$, and take $0 < x_1 < x_2 < x_3 < \infty$. Then

$$
\int_0^{x_2} H(x-y)Bf(y)dy = \int_0^{x_1} H(x-y)(T+y/1'+H(y)\{T(x_{21})\}'(x+y))dy = \left[e^{-(x-y)T}Q_+(x+y)\right]_{x_1}^{x_2}.
$$

The left-hand sides have strong limits for $x_1 \downarrow x$, $x_2 \downarrow x$, and $x_3 \uparrow \infty$ (see Lemma A.2, together with the boundedness of $f$). Thus the right-hand sides have strong limits for $x_1 \downarrow x$, $x_2 \downarrow x$, and $x_3 \uparrow \infty$. As $Q_+(x+y)\to 0$ for $y \to 0$ strongly and Lemma A.4 holds, one obtains (A.3)
as a result, where \( f \in L^\infty_\omega (H)_0 \).

Conversely, let \( f \) be a solution in \( L^\infty(H)_0 \) of \( (A.3) \). Because 
\( f - \omega \) is the convolution product of the Bochner integrable function \( H(\cdot)B \) and a function \( f \in L^\infty_\omega (H)_0 \), it is bounded and continuous on \([0, \infty)\). Using Lemma A.3 we prove that

\[
g(x) = \int_0^\infty H(x-y)Bf(y)dy \in D(T) \quad (\text{see also Lemma A.2}),
\]

while

\[
T \int_0^\infty H(x-y)Bf(y)dy = \int_0^\infty TH(x-y)Bf(y)dy, \quad 0 < x < \infty.
\]

Repeatedly using Lemma A.3 we get for all \( \varepsilon > 0 \):

\[
T[g(x+\varepsilon)-g(x)]/\varepsilon = h_1 + h_2 + h_3 + h_4,
\]

where

\[
\begin{align*}
h_1 &= \varepsilon^{-1} \int_0^\infty \left( e^{-cT} - I \right) H(x-y)Bf(y)dy \\
h_2 &= -\int_0^\infty \left( e^{-cT} - I \right) H(x-y)Bf(y)dy \\
h_3 &= \varepsilon^{-1} \int_x^\infty TH(x+y)Bf(y)dy \\
h_4 &= -\varepsilon^{-1} \int_x^\infty TH(x-y)Bf(y)dy.
\end{align*}
\]

Let us take \( \varepsilon \downarrow 0 \). Using simple semigroup theory, \( h_1 = \int_0^x H(x-y)Bf(y)dy \). By the continuity of \( f \), dominated convergence and the same semigroup property, \( h_2 = \int_0^\infty H(x-y)Bf(y)dy \). By the continuity of the integrands, \( h_3 + Q_Bf(x) \) and \( h_4 + Q_Bf(x) \). Thus \( Tg \) is strongly differentiable on \([0, \infty)\) from the right and

\[
(Tg)'(x) = -g(x)+Bf(x) \quad (A.4)
\]

Similarly, one proves strong differentiability from the left. Hence, \( Tg \) is strongly differentiable on \([0, \infty)\) and \( (A.4) \) holds true. However, \( (A.1) \) implies \( g = f - \omega \), and the function \( f \) therefore satisfies \( (A.1) \).

The first one of Eqs. \( (A.2) \) follows by substitution of \( x = 0 \) into \( (A.3) \).

ALBEDO OPERATORS AND H-EQUATIONS

If \( f_+ \in Q_1(D(T)) \) and \( \omega(x) = e^{-xT}f_+ \), then \( (A.3) \) is equivalent to problem \((1.1)\) to \((1.3)\), as one easily sees.

The next theorem is stated without proof. For \( \omega \) as in \((3.6)\) we find that Eq. \((3.5)\) implies problem \((3.7)\).

Theorem A.5. Let \( \omega : \mathbb{R} \to H \) be bounded and continuous, except for a possible jump discontinuity at \( x = 0 \). Let \( \omega(x) \in D(T) \) and \( T \omega \) be strongly differentiable on \( \mathbb{R}\setminus\{0\} \). Then a solution \( \phi \in L^\infty_\omega (H)_0 \) of the Wiener-Hopf equation

\[
f(x) - \int_0^\infty H(x-y)Bf(y)dy = \omega(x) \quad (x \in \mathbb{R}) \quad (A.5)
\]

is bounded and continuous on \( \mathbb{R} \), except possibly for a jump discontinuity at \( x = 0 \) of size

\[
\psi(0^+) - \psi(0^-) = \omega(0^+) - \omega(0^-),
\]

and satisfies the vector-valued differential equation

\[
(Th)'(x) = -Af(x) + (Tu)'(x) + \omega(x) \quad (0 \neq x \in \mathbb{R}).
\]

This result can easily be deduced from Theorem A.4 by decomposing \((A.5)\) in equations on \((0, \infty)\) and \((-\infty, 0)\).

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ALBEDO OPERATORS AND H-EQUATIONS


25. Condition (1.17) implies \( \int_0^\infty ||H(t)B|| dt < \infty \), see also the Appendix.


28. The convolution kernel \( H(t)B \) is compact and Bochner integrable on \( (-\infty, \infty) \) (see Lemma A.2), while \( \lambda(z) \) is invertible for \( \text{Re} \, z > 1 \). According to Theorem 4.3 (as 4.4) of I. C. Gohberg and J. Leiterer, Math. Nachrichten. 55, 33 (1973), \( \lambda(z) \) has, indeed, a Wiener-Hopf factorization.


30. Ibid., Appendix, where a new proof appears of a result of Allan, Bochner and Phillips. See Ref. 29 for the references to the latter work.

31. Note the following: \( (1-L)f)(x) = e^{-xt} \cdot \overline{Q}(x) \); for some \( h \in D(T) \), if and only if \( f(0) \in \text{Ran} \, \mathcal{X}_h \cap D(T) \) and \( Q_h f(0) = Q_h \cdot f(0) \). So \( Q_h = f(0) = Q_h \cdot f(0) \) belongs to \( [X_h \cap \text{Ran} \, Q_h] \cap D(T) \). Further, if \( (1-L)f) = 0 \), then \( f(x) \in D(T) \), and thus both \( f(0) \in \mathcal{X}_h \cap D(T) \) and \( Q_h f(0) = 0 \). So \( f(0) \in \mathcal{X}_h \cap \text{Ran} \, Q_h \subseteq D(T) \).


34. A reduction of this kind appeared before in Refs. 3, 4 and 33.


ABSTRACT

In this paper, we present a new numerical method for solving the transport equation in a slab geometry. The method is based on the characteristic finite difference method and includes a correction term to improve the accuracy of the solution. We compare our results with those obtained using the conventional discrete ordinates method and show that our method provides a significant improvement in accuracy, especially for problems with high-scattering. The method is also shown to be more stable and robust than the traditional discrete ordinates method.

INTRODUCTION

The characteristic method is a powerful tool for solving the transport equation in slab geometry. It is particularly useful for problems with high scattering, where the discrete ordinates method may not be accurate. The characteristic method directly solves the transport equation and is not subject to the limitations of the angular discretization of the discrete ordinates method.

ERROR BOUNDS FOR SOME CHARACTERISTIC METHODS

We analyze the error bounds for the characteristic method and compare them with those for the discrete ordinates method. The results indicate that the characteristic method is more accurate and has lower error bounds than the discrete ordinates method. The characteristic method also has the advantage of being independent of the angular discretization, which makes it more flexible and easier to implement.

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