Inverse Scattering Transform and Solitons for Square Matrix Nonlinear Schrödinger Equations

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The inverse scattering transform (IST) is developed for a class of matrix nonlinear Schrödinger-type systems whose reductions include two equations that model certain hyperfine spin $F = 1$ spinor Bose–Einstein condensates, and two novel equations that were recently shown to be integrable, and that have applications in nonlinear optics and four-component fermionic condensates. In addition, the general behavior of the soliton solutions for all four reductions is analyzed in detail, and some novel solutions are presented.

1. Introduction

The study of multicomponent Bose–Einstein condensates (BECs) has been a very active field of research in the last two decades. These systems can be derived within mean-field theory, and the static and dynamical properties of the nonlinear excitations that they exhibit are well described by a system of coupled Gross–Pitaevskii equations [1–4], which is a variant of the so-called defocusing vector nonlinear Schrödinger (NLS) equation [5, 6], to which it reduces in the absence of a confining potential when the repulsive interactions within and between the atomic species are of equal strength (the integrable, or so-called Manakov limit [7]). In particular,

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optical trapping techniques have made it possible to realize spinor BECs in which atoms can be confined regardless of their spin hyperfine state [8, 9]. In the homogeneous setting (i.e., in the absence of a confining potential), and for suitable choices of the interaction coefficients, $F = 1$ spinor BECs can be described by the matrix NLS equation:

$$iQ_t + Q_{xx} - 2\nu QQ^\dagger Q = 0_{2\times 2}\,, \qquad \nu = \pm 1\,, \tag{1}$$

where $Q(x,t)$ is a 2×2 complex matrix valued function and $Q^\dagger$ denotes the Hermitian conjugate of Q. When $\nu = -1$ (respectively, $\nu = +1$), the system is referred to as being in a self-focusing (respectively, self-defocusing) regime. If the 2×2 matrix potential $Q(x,t)$ is chosen to be a complex symmetric matrix, i.e., $Q(x,t) = \mathrm{diag}(q_1, q_{-1}) + q_0\sigma_1$ (here and in the following σ_j for $j = 1, 2, 3$ denote the Pauli matrices), the system (1) can be used as a model to describe hyperfine spin $F = 1$ spinor BECs; the self-defocusing case ($\nu = 1$) corresponds to repulsive interatomic interactions and antiferromagnetic spin-exchange interactions, while $\nu = 1$ accounts for attractive interatomic interactions and ferromagnetic spin-exchange interactions ($\nu = -1$). In both cases, the functions q_1, q_0, q_{-1} are related to the vacuum expectation values of the three components of the quantum field operator in the spin configurations $1, 0, -1$ [10, 11].

This work deals with a generalization of the above matrix NLS equation, namely:

$$iQ_t + Q_{xx} - 2Q\,\Sigma\,Q^\dagger\,\Omega\,Q = 0_{2\times 2}, \tag{2}$$

where $Q = Q(x,t)$ is a 2×2 matrix, which was shown to be integrable for any choice of 2×2 Hermitian matrices Σ, Ω [12]. The latter can be chosen without loss of generality to be in canonical form, i.e., diagonal and with diagonal entries equal to 0 or ± 1. Since we are interested in a fully coupled system, rather than a triangular one, we will assume that Σ and Ω are 2×2 diagonal matrices with entries ± 1. If we denote $\Sigma = \mathrm{diag}(\sigma_{11}, \sigma_{22})$ and $\Omega = \mathrm{diag}(\omega_{11}, \omega_{22})$, with $\sigma_{11}^2 = \sigma_{22}^2 = \omega_{11}^2 = \omega_{22}^2 = 1$, and assume that $Q(x,t)$ is a symmetric matrix:

$$Q(x,t) = \begin{pmatrix} q_1(x,t) & q_o(x,t) \\ q_o(x,t) & q_{-1}(x,t) \end{pmatrix},$$

the compatibility of the off-diagonal terms in Eq. (2) requires $\sigma_{11}\omega_{22} = \sigma_{22}\omega_{11}$. It is then clear that one can have:

Case 1:

$$\Sigma = \Omega = I_2$$

(or, equivalently, $\Sigma = \Omega = -I_2$), which yields:

$$i\partial_t q_1 + \partial_x^2 q_1 - 2q_1\left[|q_1|^2 + 2|q_o|^2\right] - 2q_o^2 q_{-1}^* = 0\,, \tag{3a}$$

$$i\partial_t q_{-1} + \partial_x^2 q_{-1} - 2q_{-1}\left[|q_{-1}|^2 + 2|q_o|^2\right] - 2q_o^2 q_1^* = 0\,, \qquad (3b)$$

$$i\partial_t q_o + \partial_x^2 q_o - 2q_o\left[|q_1|^2 + |q_o|^2 + |q_{-1}|^2\right] - 2q_1 q_o^* q_{-1} = 0\,, \qquad (3b)$$

corresponding to Eq. (1) with $\nu = 1$, so a matrix NLS system of defocusing type.

Case 2:

$$\Sigma = -\Omega = I_2$$

(or, equivalently, $\Omega = -\Sigma = I_2$), which yields:

$$i\partial_t q_1 + \partial_x^2 q_1 + 2q_1\left[|q_1|^2 + 2|q_o|^2\right] + 2q_o^2 q_{-1}^* = 0\,, \qquad (4a)$$

$$i\partial_t q_{-1} + \partial_x^2 q_{-1} + 2q_{-1}\left[|q_{-1}|^2 + 2|q_o|^2\right] + 2q_o^2 q_1^* = 0\,, \qquad (4b)$$

$$i\partial_t q_o + \partial_x^2 q_o + 2q_o\left[|q_1|^2 + |q_o|^2 + |q_{-1}|^2\right] + 2q_1 q_o^* q_{-1} = 0\,, \qquad (4c)$$

corresponding to Eq. (1) with $\nu = -1$, so a matrix NLS system of focusing type.

As mentioned before, the above equations are well known and well studied in the literature, and soliton solutions (both bright and dark, i.e., both with zero and nonzero boundary conditions) have been derived in the context of spinor BECs [10, 11, 13–18]. Two other choices are possible, though.

Case 3:

$$\Sigma = \Omega = \sigma_3$$

(or, equivalently, $\Sigma = \Omega = -\sigma_3$), which yields:

$$i\partial_t q_1 + \partial_x^2 q_1 - 2q_1\left[|q_1|^2 - 2|q_o|^2\right] - 2q_o^2 q_{-1}^* = 0\,, \qquad (5a)$$

$$i\partial_t q_{-1} + \partial_x^2 q_{-1} - 2q_{-1}\left[|q_{-1}|^2 - 2|q_o|^2\right] - 2q_o^2 q_1^* = 0\,, \qquad (5b)$$

$$i\partial_t q_o + \partial_x^2 q_o - 2q_o\left[|q_1|^2 - |q_o|^2 + |q_{-1}|^2\right] + 2q_1 q_o^* q_{-1} = 0\,. \qquad (5c)$$

Case 4:

$$\Sigma = -\Omega = \sigma_3$$

(or, equivalently, $\Sigma = -\Omega = -\sigma_3$), which corresponds to:

$$i\partial_t q_1 + \partial_x^2 q_1 + 2q_1\left[|q_1|^2 - 2|q_o|^2\right] + 2q_o^2 q_{-1}^* = 0\,, \qquad (6a)$$

$$i\partial_t q_{-1} + \partial_x^2 q_{-1} + 2q_{-1}\left[|q_{-1}|^2 - 2|q_o|^2\right] + 2q_o^2 q_1^* = 0\,, \qquad \text{(6b)}$$

$$i\partial_t q_o + \partial_x^2 q_o + 2q_o\left[|q_1|^2 - |q_o|^2 + |q_{-1}|^2\right] - 2q_1 q_o^* q_{-1} = 0\,. \quad \text{(6c)}$$

It is worth mentioning that one could choose $Q(x,t)$ to be an antisymmetric matrix instead, but the corresponding equations can be obtained from the above cases 1–4 by simply changing $q_j(x,t)$ into $-q_j(x,t)$ for either $j=1$ or $j=-1$. Therefore, there is no need to consider these equations separately.

Both equations for cases 3 and 4 above correspond to what for coupled NLS is referred to as the "mixed sign" case, when one has a nonlinearity in the norm that is of Minkowski-type, instead of Euclidean-type. Soliton solutions and their interactions for the mixed sign vector NLS equation have been derived both with zero and with nonzero boundary conditions [19–23]. In the two-component case, the mixed sign NLS equation models the propagation of a light beam with arbitrary polarization when the wave–wave interaction exhibits a large-phase mismatch [19]. It can also be obtained as a model to describe the dynamics of vector solitons in waveguide arrays. Another relevant application of the mixed sign two-component coupled NLS is a chain of drops of a binary BEC trapped in an optical lattice, where an external magnetic field can be used to change the values and signs of the nonlinear coefficients, a feature known as the Feschbach resonance [19].

The situation is different in the matrix case, however, because for the spinor model, the signs of the coupling constants, which are related to the s-wave scattering lengths accounting for inter- and intraspecies atomic interactions, cannot be chosen as in cases 3 and 4. Although there are no foreseeable physical realizations of three-component (spin-1) bosonic condensate for the matrix equation (2) with the choices of signs in cases 3 and 4, the corresponding equations can model two other classes of physical problems: nonlinear optics and four-component fermionic condensates. In the context of nonlinear optics, three-component copropagating electromagnetic waves and their mutual intensity transfers have been studied in [24] and [25]. The nonlinear dynamics of the energy transfer process between the fundamental and second harmonic fields in the presence of the phase matched direct current field has been investigated in one-dimensional geometries. The emerging spatio-temporal phenomena include localized soliton-like excitations and can be modeled by the equations corresponding to our cases 3 and 4 by appropriate choices of signs in the nonlinear susceptibility tensor. These phenomena of switching and downfrequency conversion are finding exciting applications to integrated optoelectronic devices. The mixed signs of cases 3 and 4 can also model multicolor optical spatio-temporal solitary waves created by interaction of light at a central frequency with two sideband waves both through cross-phase modulation and parametric four-wave mixing of opposite signs [26]. In this work, different families of multicolor bright spatial optical solitons were found by numerical integration of the corresponding stationary equations only. Cases 3 and 4 also emerge in the context of fermionic condensates of ultracold atoms [27–30], although there are some difficulties in both the justification of mean-field condensate wave function for fermions as well as possible pairing instabilities of superconducting types. Nevertheless, with sufficient care, our cases 3 and 4 can model four-component spin-3/2 cold atomic systems under special circumstances. Specifically, spin-3/2 systems have four components but only even total spin channels are open by $SO(5)$ symmetry, and, unlike the bosonic case, the signs of scattering amplitudes can be controlled independently. When in the so-called quartetting phase [30], spin-3/2 systems exhibit a three-component condensate similar to our cases 3 and 4. Moreover, when fermionic condensates are placed in optical lattices (rather than optical traps), interacting Heisenberg-like models can arise with rich structures of nonlinear soliton-like excitations [26].

In light of its potential applicative relevance, in this work, we develop the inverse scattering transform (IST) for the system (2) as a tool to solve the initial-value problem, as well as to obtain explicit soliton solutions. While the IST for "unreduced" matrix NLS systems, and for the "canonical" reductions corresponding to cases 1 and 2 (focusing and defocusing matrix NLS), is well established, both with zero and nonzero boundary conditions (see, for instance, [5, 17, 31–34] and references therein), the IST and the soliton solutions corresponding to the reductions in cases 3 and 4 described above are novel, and present some interesting aspects and additional challenges with respect to the other two cases in that one needs to impose suitable constraints on the norming constants to guarantee that the soliton solutions are smooth for all $x, t \in \mathbb{R}$. As a matter of fact, this work also provides several advances as far as the IST for general matrix NLS systems is concerned, including the well-known focusing and defocusing matrix NLS systems corresponding to cases 1 and 2. Specific focus of the work is to: (i) provide a rigorous definition of the norming constants that does not require any unjustified analytic extension of the scattering relations, and clarify the role of the rank of the norming constants in the spectral characterization of the corresponding solutions; (ii) properly account for all the symmetries in the potential matrix, and derive the corresponding symmetries in the scattering data (reflection coefficients and norming constants); (iii) formulate the inverse problem as a Riemann–Hilbert problem (RHP), instead of in terms of Marchenko equations; (iv) obtain novel soliton solutions for the reductions of Eq. (2) corresponding to cases 3 and 4, and specify the necessary and sufficient conditions for which the solutions are regular for all $x, t \in \mathbb{R}$; (v) discuss the reductions of one-soliton solutions, and identify conditions on the norming constants for which the solutions are unitarily equivalent to diagonal ones.

The paper is organized as follows. Section 2 is devoted to the direct scattering problem for the general matrix NLS equation (2). In Section 3, we formulate the inverse scattering problem for the eigenfunctions as an RHP with poles, provide the formal solution of the latter in the case of simple poles, and the reconstruction formula of the potential in terms of eigenfunctions and scattering data. In Section 4, we discuss soliton solutions, and derive the necessary and sufficient conditions on the norming constants that guarantee that the solutions in cases 3 and 4 are smooth for all $x, t \in \mathbb{R}$. In Section 5, we discuss the reductions of the one-soliton solutions to unitarily equivalent diagonal solutions. Section 6 is devoted to some concluding remarks, while Appendices A and B provide discussions of the resolvent operator for the scattering problem, and multiple poles in the RHP and corresponding solutions.

2. Direct scattering

2.1. Lax pair and Jost solutions

It is well known that the matrix NLS equation is equivalent to the compatibility condition of a Lax pair for a potential matrix $Q(x, t)$, and the first equation in the Lax pair is the so-called Zakharov-Shabat (ZS) or Ablowitz-Kaup-Newell-Segur (ANKS) equation. Specifically, Eq. (2) admits the Lax pair

$$\varphi_x = U\,\varphi\,, \qquad \varphi_t = V\,\varphi\,, \tag{7}$$

with

$$U(x,t,k) = -ik\boldsymbol{\sigma}_3 + \boldsymbol{Q}(x,t)\,, \qquad V(x,t,k) = -2ik^2\boldsymbol{\sigma}_3 + 2k\,\boldsymbol{Q} + i\boldsymbol{\sigma}_3\,\boldsymbol{Q}_x - i\boldsymbol{\sigma}_3\,\boldsymbol{Q}^2, \tag{8a}$$

$$\boldsymbol{\sigma}_3 = \begin{pmatrix} I_2 & 0_{2\times2} \\ 0_{2\times2} & -I_2 \end{pmatrix}, \qquad \boldsymbol{Q} = \begin{pmatrix} 0_{2\times2} & Q \\ R & 0_{2\times2} \end{pmatrix}, \qquad R = \Sigma\, Q^{\dagger}\Omega. \tag{8b}$$

(Here and in the following, I_n and $0_{n\times n}$ denote the $n \times n$ identity and zero matrices, respectively, and boldface fonts are used to distinguish the 2×2 matrices from the corresponding 4×4 extensions. When otherwise clear from the context, we will sometimes omit the subscripts to specify the sizes of the matrices involved.)

The first equation in the Lax pair (7) is usually referred to as the scattering problem, and the Jost solutions are defined as usual in terms of the asymptotic eigenvectors of the scattering problem. Here, we assume

$\boldsymbol{Q} \to 0$ as $x \to \pm\infty$ sufficiently rapidly, and in this case, the Lax pair asymptotically reduces to: $\varphi_x \sim -ik\boldsymbol{\sigma}_3\varphi$ and $\varphi_t \sim -2ik^2\boldsymbol{\sigma}_3\varphi$.

Then, for all $k \in \mathbb{R}$, the Jost eigenfunctions $\Phi(x,t,k)$ and $\Psi(x,t,k)$ are defined as the *simultaneous* solutions of both parts of the Lax pair such that

$$\Phi(x,t,k) \equiv (\phi(x,t,k)\ \bar{\phi}(x,t,k)) = I_4\,\mathrm{e}^{-i\theta(x,t,k)\sigma_3} + o(1) \qquad \text{as } x \to -\infty\,, \tag{9a}$$

$$\Psi(x,t,k) \equiv (\bar{\psi}(x,t,k)\ \psi(x,t,k)) = I_4\,\mathrm{e}^{-i\theta(x,t,k)\sigma_3} + o(1) \qquad \text{as } x \to +\infty\,, \tag{9b}$$

where

$$\theta(x,t,k) = k(x+2kt)\,, \tag{10}$$

and $\phi(x,t,k)$ and $\bar{\phi}(x,t,k)$ (respectively, $\bar{\psi}(x,t,k)$ and $\psi(x,t,k)$) are 4×2 matrices collecting the first two and last two column vectors of the 4×4 matrix solutions $\Phi(x,t,k)$ (respectively, $\Psi(x,t,k)$). One can then introduce modified eigenfunctions defined as follows:

$$\big(M(x,t,k)\ \bar{M}(x,t,k)\big) = \Phi(x,t,k)e^{i\theta(x,t,k)\boldsymbol{\sigma}_3}, \tag{11a}$$

$$\big(\bar{N}(x,t,k)\ N(x,t,k)\big) = \Psi(x,t,k)e^{i\theta(x,t,k)\boldsymbol{\sigma}_3}, \tag{11b}$$

such that

$$(M(x,t,k)\ \bar{M}(x,t,k)) \sim I_4 \quad \text{as } x \to -\infty, \tag{12a}$$

$$(\bar{N}(x,t,k)\ N(x,t,k)) \sim I_4 \quad \text{as } x \to +\infty\,. \tag{12b}$$

The modified eigenfunctions satisfy the following integral equations:

$$(M(x,t,k)\ \bar{M}(x,t,k)) = I_4 + \int_{-\infty}^{x} e^{ik(y-x)\boldsymbol{\sigma}_3}\,\boldsymbol{Q}(y,t) \times (M(y,t,k)\ \bar{M}(y,t,k))e^{ik(x-y)\boldsymbol{\sigma}_3}dy\,, \tag{13a}$$

$$(\bar{N}(x,t,k)\ N(x,t,k)) = I_4 - \int_{x}^{\infty} e^{ik(y-x)\boldsymbol{\sigma}_3}\,\boldsymbol{Q}(y,t) \times (\bar{N}(y,t,k)\ N(y,t,k))e^{ik(x-y)\boldsymbol{\sigma}_3}dy\,, \tag{13b}$$

and standard techniques (see, for instance, [5]) allow one to prove that if the entries of $Q(\cdot,t)$ belong to $L^1(\mathbb{R})$ for all $t \geq 0$, $M(x,t,k)$ and $N(x,t,k)$ can be analytically extended in the upper half-plane (UHP) of k, and $\bar{M}(x,t,k)$ and $\bar{N}(x,t,k)$ can be analytically extended in the lower half-plane (LHP) of k, and they all are continuous up to $k \in \mathbb{R}$. The analyticity properties of the columns of $\Phi(x,t,k)$ and $\Psi(x,t,k)$ are an obvious consequence of the above. Finally, it easily follows with the help of Gronwall's inequality that the relations (12a) are valid for each $t \in \mathbb{R}$ uniformly in $k \in \mathbb{R}$.

2.2. Scattering coefficients

Since U and V in (7) are traceless, Jacobi's formula implies that any matrix solution $\varphi(x,t,k)$ of (7) satisfies $\partial_x(\det\varphi) = \partial_t(\det\varphi) = 0$. Since for all $k \in \mathbb{R}$, one has $\lim_{x\to-\infty} \Phi(x,t,k)\,\mathrm{e}^{i\theta\sigma_3} = \lim_{x\to+\infty} \Psi(x,t,k)\,\mathrm{e}^{i\theta\sigma_3} = I_4$, it then follows that

$$\det\Phi(x,t,k) = \det\Psi(x,t,k) = 1\,, \qquad x,t,k \in \mathbb{R}\,, \tag{14}$$

which implies that both Φ and Ψ are fundamental matrix solutions of the scattering problem. Hence, there exists a 4×4 scattering matrix $S(k)$ independent of x and t such that

$$\Phi(x,t,k) = \Psi(x,t,k)\,S(k)\,, \qquad S(k) = \begin{pmatrix} a(k) & \bar b(k) \\ b(k) & \bar a(k) \end{pmatrix}, \qquad x,t,k \in \mathbb{R}\,, \tag{15}$$

where $a(k), b(k), \bar a(k), \bar b(k)$ are the 2×2 blocks of the scattering matrix in (15). Using the analytic groups of columns introduced in (9), one can then write:

$$\phi = \psi b + \bar\psi a\,, \qquad \bar\phi = \psi\bar a + \bar\psi\bar b\,. \tag{16}$$

Note that the entries of $S(k)$ are independent of time (this is a consequence of the fact that Φ and Ψ are chosen to be simultaneous solutions of both parts of the Lax pair), and the same holds for the norming constants (see Section 2.4). Moreover, (14) and (15) imply that $\det S(k) = 1$ for all $k \in \mathbb{R}$.

Using (15) one can easily verify that:

$$\det a(k) = \mathrm{Wr}(\phi,\psi)/\mathrm{Wr}(\bar\psi,\psi) \equiv \det(\phi\ \psi)\,, \tag{17a}$$

$$\det \bar a(k) = \mathrm{Wr}(\bar\psi,\bar\phi)/\mathrm{Wr}(\bar\psi,\psi) \equiv \det(\bar\psi\ \bar\phi)\,. \tag{17b}$$

Finally, for $k \in \mathbb{R}$, we can express

$$M(x,t,k)a^{-1}(k) = \bar N(x,t,k) + e^{2i\theta(x,t,k)}N(x,t,k)\rho(k)\,, \tag{18a}$$

$$\bar M(x,t,k)\bar a^{-1}(k) = N(x,t,k) + e^{-2i\theta(x,t,k)}\bar N(x,t,k)\bar\rho(k)\,, \tag{18b}$$

where $M(x,t,k)a^{-1}(k)$ and $\bar M(x,t,k)\bar a^{-1}(k)$ are meromorphic in the UHP and LHP of k, respectively, and we introduce (matrix) reflection coefficients

$$\rho(k) = b(k)a^{-1}(k)\,, \qquad \bar\rho(k) = \bar b(k)\bar a^{-1}(k) \qquad k \in \mathbb{R}. \tag{19}$$

2.3. Symmetries

As it is well known in the IST framework, each symmetry in the potential of the Lax pair directly induces symmetries in the eigenfunctions, and the latter, in turn, induce corresponding symmetries in the scattering data. In the case at hand, one has to account for a generalization of the usual conjugation symmetry $R = \pm Q^\dagger$, which now becomes $R = \Sigma\, Q^\dagger \Omega$, and which for $\boldsymbol{Q}$ reads:

$$\boldsymbol{Q}^\dagger = -\Xi^{-1}\,\boldsymbol{Q}\,\Xi\,, \qquad \Xi = \begin{pmatrix} \Omega^{-1} & 0_{2\times 2} \\ 0_{2\times 2} & -\Sigma \end{pmatrix}, \tag{20}$$

as well as an additional symmetry that takes into account that Q is assumed to be a symmetric matrix, namely, $Q^T = Q$, which in terms of the 4×4 matrix potential $\boldsymbol{Q}$ can be written as

$$\boldsymbol{Q} = \begin{pmatrix} 0_{2\times 2} & \Sigma^{-1}\Omega \\ -I_2 & 0_{2\times 2} \end{pmatrix} \boldsymbol{Q}^T \begin{pmatrix} 0_{2\times 2} & \Sigma\,\Omega^{-1} \\ -I_2 & 0_{2\times 2} \end{pmatrix}. \tag{21}$$

Note that Σ and Ω are assumed to be diagonal matrices; hence, they commute. Note also that in all the cases discussed in Section 1, $\Sigma\,\Omega^{-1}$ and $\Sigma^{-1}\Omega$ are either both equal to I_2 (cases 1 and 3) or both equal to $-I_2$ (cases 2 and 4).

First symmetry—conjugation: $\boldsymbol{Q}^\dagger = -\Xi^{-1}\,\boldsymbol{Q}\,\Xi$, corresponding to $k \mapsto k^*$ (UHP/LHP). To determine how the eigenfunctions and the scattering data are related when the above symmetry in the potential is imposed, we will follow the same approach as in [5]. To this aim, we introduce for $k \in \mathbb{R}$

$$f(x,t,k) = \Phi^\dagger(x,t,k^*)\,\Xi^{-1}\Phi(x,t,k),$$
$$g(x,t,k) = \Psi^\dagger(x,t,k^*)\,\Xi^{-1}\Psi(x,t,k)\,.$$

One can easily verify that f, g are x-independent, and their limits as $x \to \pm\infty$ from (9) yield

$$\Phi^\dagger(x,t,k^*)\,\Xi^{-1}\Phi(x,t,k) = \Psi^\dagger(x,t,k^*)\,\Xi\,\Psi(x,t,k) = \Xi^{-1}. \tag{22}$$

On the one hand, the above relationships can be written as

$$\Phi^{-1}(x,t,k) = \Xi\,\Phi^\dagger(x,t,k^*)\Xi^{-1}, \tag{23a}$$

$$\Psi^{-1}(x,t,k) = \Xi\,\Psi^\dagger(x,t,k^*)\Xi^{-1}, \tag{23b}$$

and the latter provide the following representations for the scattering matrix:

$$S(k) = \Psi^{-1}(x,t,k)\Phi(x,t,k) \equiv \Xi\,\Psi^\dagger(x,t,k^*)\Xi^{-1}\Phi(x,t,k)\,. \tag{24}$$

Let us now introduce the following notation for the upper and lower blocks of the eigenfunctions:

$$\Phi(x,t,k) = \begin{pmatrix} \phi^{\rm up} & \bar\phi^{\rm up} \\ \phi^{\rm dn} & \bar\phi^{\rm dn} \end{pmatrix}, \qquad \Psi(x,t,k) = \begin{pmatrix} \bar\psi^{\rm up} & \psi^{\rm up} \\ \bar\psi^{\rm dn} & \psi^{\rm dn} \end{pmatrix},$$

where each of $A^{\rm up}$, $A^{\rm dn}$ is a 2×2 matrix, and on the right-hand side, we have omitted the x,t dependence for shortness. Computing the diagonal 2×2 blocks of $S(k)$ in (15) using (24) then gives:

$$a(k) = \Omega^{-1}\left(\bar\psi^{\rm up}(x,t,k^*)\right)^\dagger \Omega\, \phi^{\rm up}(x,t,k) - \Omega^{-1}\left(\bar\psi^{\rm dn}(x,t,k^*)\right)^\dagger \Sigma^{-1}\phi^{\rm dn}(x,t,k) \tag{25a}$$

$$\equiv \Omega^{-1}\left(\bar N^{\rm up}(x,t,k^*)\right)^\dagger \Omega\, M^{\rm up}(x,t,k) - \Omega^{-1}\left(\bar N^{\rm dn}(x,t,k^*)\right)^\dagger \Sigma^{-1} M^{\rm dn}(x,t,k),$$

$$\bar a(k) = \Sigma\left(\psi^{\rm dn}(x,t,k^*)\right)^\dagger \Sigma^{-1}\bar\phi^{\rm dn}(x,t,k) - \Sigma\left(\psi^{\rm up}(x,t,k^*)\right)^\dagger \Omega\,\bar\phi^{\rm up}(x,t,k) \tag{25b}$$

$$\equiv \Sigma\left(N^{\rm dn}(x,t,k^*)\right)^\dagger \Sigma^{-1}\bar M^{\rm dn}(x,t,k) - \Sigma\left(N^{\rm up}(x,t,k^*)\right)^\dagger \Omega\,\bar M^{\rm up}(x,t,k).$$

Based on the analyticity properties established for the eigenfunctions, the above expressions show that $a(k)$ can be analytically continued in the UHP of k, and $\bar a(k)$ can be analytically continued in the LHP. The off-diagonal blocks $b(k)$ and $\bar b(k)$ of the scattering matrix, on the other hand, are only defined on the continuous spectrum (i.e., for $k\in\mathbb{R}$), and, in general, are nowhere analytic. Also, note that the above relationships provide yet another representation for the analytic scattering coefficients $a(k)$ and $\bar a(k)$, namely,

$$a(k) = \lim_{x\to+\infty} M^{\rm up}(x,t,k) = \lim_{x\to-\infty} \Omega^{-1}\left(\bar N^{\rm up}(x,t,k^*)\right)^\dagger \Omega\,, \quad k\in\mathbb{C}^+, \tag{26a}$$

$$\bar a(k) = \lim_{x\to+\infty} \bar M^{\rm dn}(x,t,k) = \lim_{x\to-\infty} \Sigma\left(N^{\rm dn}(x,t,k^*)\right)^\dagger \Sigma^{-1}\,, \quad k\in\mathbb{C}^-. \tag{26b}$$

Note, for future reference, that explicitly computing the upper and lower blocks of (22) yields:

$$\left(\phi^{\rm up}(x,t,k^*)\right)^\dagger \Omega\,\bar\phi^{\rm up}(x,t,k) = \left(\phi^{\rm dn}(x,t,k^*)\right)^\dagger \Sigma^{-1}\bar\phi^{\rm dn}(x,t,k)\,, \tag{27a}$$

$$\left(\psi^{\rm up}(x,t,k^*)\right)^\dagger \Omega\,\bar\psi^{\rm up}(x,t,k) = \left(\psi^{\rm dn}(x,t,k^*)\right)^\dagger \Sigma^{-1}\bar\psi^{\rm dn}(x,t,k)\,. \tag{27b}$$

Moreover, (22) implies

$$S^\dagger(k^*)\,\Xi^{-1}S(k) = \Xi^{-1} \qquad k\in\mathbb{R}\,, \tag{28}$$

which in terms of the individual 2×2 blocks reads:

$$a^\dagger(k^*)\Omega\, a(k) - b^\dagger(k^*)\Sigma^{-1}b(k) = \Omega\,, \tag{29a}$$

$$a^{\dagger}(k^*)\Omega\,\bar{b}(k) - b^{\dagger}(k^*)\Sigma^{-1}\bar{a}(k) = 0_{2\times 2}\,, \tag{29b}$$

$$\bar{b}^{\dagger}(k^*)\Omega\,a(k) - \bar{a}^{\dagger}(k^*)\Sigma^{-1}b(k) = 0_{2\times 2}\,, \tag{29c}$$

$$\bar{b}^{\dagger}(k^*)\Omega\,\bar{b}(k) - \bar{a}^{\dagger}(k^*)\Sigma^{-1}\bar{a}(k) = -\Sigma^{-1}\,. \tag{29d}$$

As a consequence, the reflection coefficients introduced in (19) satisfy the following symmetry:

$$\bar{\rho}(k) = \Omega^{-1}\rho^{\dagger}(k^*)\Sigma^{-1}\,, \qquad k \in \mathbb{R}\,. \tag{30}$$

Note that one can rewrite the above equations for the diagonal blocks $a(k)$ and $\bar{a}(k)$ in terms of the reflection coefficients introduced in (19) as follows:

$$a(k)\Omega^{-1}a^{\dagger}(k^*)\Omega = \left[I_2 - \Omega^{-1}\rho^{\dagger}(k^*)\Sigma^{-1}\rho(k)\right]^{-1}\,, \tag{31a}$$

$$\bar{a}(k)\,\Sigma\,\bar{a}^{\dagger}(k^*)\Sigma^{-1} = \left[I_2 - \Sigma\,\bar{\rho}^{\dagger}(k^*)\Omega\,\bar{\rho}(k)\right]^{-1}\,. \tag{31b}$$

Furthermore, if $\Omega = \Sigma = \pm I_2$ (i.e., in case 1), from (29a) and (29d), it follows that for any $\xi \in \mathbb{C}^2$ and $k \in \mathbb{R}$:

$$||a(k)\xi||^2 = ||\xi||^2 + ||b(k)\xi||^2\,, \qquad ||\bar{a}(k)\xi||^2 = ||\xi||^2 + ||\bar{b}(k)\xi||^2\,, \tag{32}$$

so that $a(k)\xi = 0$ or $\bar{a}(k)\xi = 0$ necessarily implies $\xi = 0$. Consequently, one can conclude that in case 1, $\det a(k) \neq 0$ and $\det \bar{a}(k) \neq 0$ for all $k \in \mathbb{R}$. (This is exactly the same as in the scalar case.) Note that Eq. (32) implies that in case 1 for any $k \in \mathbb{R}$, the norms of the reflection coefficients $\rho(k)$ and $\bar{\rho}(k)$ are strictly less than 1.

Note that (28) implies

$$S^{-1}(k) = \Xi\,S^{\dagger}(k^*)\,\Xi^{-1}\,, \qquad S^{-1}(k) = \begin{pmatrix} \bar{c}(k) & d(k) \\ \bar{d}(k) & c(k) \end{pmatrix}, \tag{33}$$

and therefore the blocks of $S(k)$ and $S^{-1}(k)$ for $k \in \mathbb{R}$ are related as follows:

$$\bar{c}(k) = \Omega^{-1}\,a^{\dagger}(k^*)\,\Omega\,, \qquad c(k) = \Sigma\,\bar{a}^{\dagger}(k^*)\,\Sigma^{-1}\,, \tag{34a}$$

$$d(k) = -\Omega^{-1}b^{\dagger}(k^*)\,\Sigma^{-1}\,, \qquad \bar{d}(k) = -\Sigma\,\bar{b}^{\dagger}(k^*)\,\Omega\,. \tag{34b}$$

As usual, Eqs. (34) can be extended to $\mathbb{C}^{\pm}$ by Schwarz reflection principle, but (34b), in general, only hold for $k \in \mathbb{R}$.

In turn, the analog of (17) for $\Psi(x,t,k) = \Phi(x,t,k)S^{-1}(k)$, namely,

$$\det c(k) = \mathrm{Wr}(\phi, \psi)/\mathrm{Wr}(\phi, \bar{\phi}) \equiv \det(\phi\ \psi)\,, \tag{35a}$$

$$\det \bar{c}(k) = \mathrm{Wr}(\bar{\psi}, \bar{\phi})/\mathrm{Wr}(\phi, \bar{\phi}) \equiv \det(\bar{\psi}\ \bar{\phi})\,, \tag{35b}$$

allows one to conclude that

$$\det c(k) = \det a(k) \quad \text{for } k \in \mathbb{C}^+ \cup \mathbb{R}\,, \qquad \det \bar{c}(k) = \det \bar{a}(k) \quad \text{for } k \in \mathbb{C}^- \cup \mathbb{R}\,. \tag{36}$$

Finally, from (34), it follows that

$$\det \bar{a}(k) = \det a^\dagger(k^*) \equiv (\det a(k^*))^* \qquad \text{for } k \in \mathbb{C}^- \cap \mathbb{R}\,. \tag{37}$$

Second symmetry: $Q^T = Q$. To account for the second symmetry in the potential, we introduce for $k \in \mathbb{R}$:

$$\tilde{f}(x,t,k) = \Phi^T(x,t,k)\, F\, \Phi(x,t,k)\,, \qquad \tilde{g}(x,t,k) = \Psi^T(x,t,k)\, F\, \Psi(x,t,k),$$

where

$$F = \begin{pmatrix} 0_{2\times 2} & -\Omega\,\Sigma^{-1} \\ \Sigma^{-1}\Omega & 0_{2\times 2} \end{pmatrix}. \tag{38}$$

As mentioned above, in all the cases discussed in Section 1, $\Sigma\,\Omega^{-1}$ and $\Sigma^{-1}\Omega$ are either both equal to I_2 (cases 1 and 3) or both equal to $-I_2$ (cases 2 and 4).

Again, it is easy to verify that with this choice of F, $\tilde{f}$ and $\tilde{g}$ are independent of x, and taking into account Eq. (9), the limits of $\tilde{f}$ and $\tilde{g}$ as $x \to \pm\infty$ yield

$$\Phi^T(x,t,k)\, F\, \Phi(x,t,k) = \Psi^T(x,t,k)\, F\, \Psi(x,t,k) = F\,, \tag{39}$$

which, in turn, implies

$$S^T(k)\, F\, S(k) = F \qquad k \in \mathbb{R}\,. \tag{40}$$

The above equation can be written down explicitly in terms of the blocks of the scattering matrix (cf. (15)) as follows:

$$b^T(k)\,\Sigma^{-1}\Omega\, a(k) = a^T(k)\,\Omega\,\Sigma^{-1} b(k)\,,$$
$$\bar{b}^T(k)\,\Omega\,\Sigma^{-1}\bar{a}(k) = \bar{a}^T(k)\,\Sigma^{-1}\Omega\,\bar{b}(k)\,,$$
$$a^T(k)\,\Omega\,\Sigma^{-1}\bar{a}(k) - b^T(k)\,\Sigma^{-1}\Omega\,\bar{b}(k) = \Omega\,\Sigma^{-1}\,,$$
$$\bar{a}^T(k)\,\Sigma^{-1}\Omega\, a(k) - \bar{b}^T(k)\Omega\Sigma^{-1} b(k) = \Sigma^{-1}\Omega\,,$$

which then, in particular, imply:

$$\rho^T(k) = \Omega\,\Sigma^{-1}\rho(k)\,\Omega^{-1}\Sigma\,, \qquad \bar{\rho}^T(k) = \Sigma^{-1}\Omega\,\bar{\rho}(k)\,\Sigma\,\Omega^{-1}\,, \tag{41}$$

as well as

$$a(k)\,\Omega^{-1}\Sigma\,\bar{a}^T(k)\,\Sigma^{-1}\Omega = \left[I_2 - \Omega^{-1}\Sigma\bar{\rho}^T(k)\,\Omega\,\Sigma^{-1}\rho(k)\right]^{-1} \qquad k \in \mathbb{R}. \quad (42)$$

Because $\Omega^{-1}\Sigma$ and $\Sigma^{-1}\Omega$ are either both equal to I_2 or both equal to $-I_2$, the above relationships can be simplified to:

$$\rho^T(k) = \rho(k)\,, \qquad \bar{\rho}^T(k) = \bar{\rho}(k) \qquad k \in \mathbb{R}\,, \quad (43)$$

and

$$a(k)\,\bar{a}^T(k) = \left[I_2 - \bar{\rho}^T(k)\,\rho(k)\right]^{-1} \qquad k \in \mathbb{R}\,. \quad (44)$$

Finally, (40) also implies $S^{-1}(k) = A\,S^T(k)\,A^{-1}$ for $k \in \mathbb{R}$, i.e.,

$$\bar{c}^T(k) = \Omega\,\Sigma^{-1}\bar{a}(k)\,\Omega^{-1}\Sigma\,, \qquad c^T(k) = \Sigma^{-1}\Omega\,a(k)\,\Omega^{-1}\Sigma\,,$$
$$d^T(k) = -\Sigma^{-1}\Omega\,\bar{b}(k)\,\Omega^{-1}\Sigma\,, \qquad \bar{d}^T(k) = -\Omega\,\Sigma^{-1}b(k)\,\Omega^{-1}\Sigma\,,$$

and, again, the latter can be simplified to:

$$\bar{c}^T(k) = \bar{a}(k)\,, \qquad c^T(k) = a(k)\,, \qquad d^T(k) = -\bar{b}(k)\,, \qquad \bar{d}^T(k) = -b(k)\,, \quad (45)$$

which, combined with (34), yield

$$\bar{a}^T(k) = \Sigma^{-1}a^\dagger(k^*)\,\Sigma \qquad k \in \mathbb{C}^-\,, \quad (46a)$$

$$\bar{b}^T(k) = \Sigma^{-1}b^\dagger(k^*)\,\Omega^{-1} \qquad k \in \mathbb{R}\,. \quad (46b)$$

2.4. Discrete spectrum, norming constants, and residue conditions

The discrete spectrum is the set of all values $k \in \mathbb{C} \setminus \mathbb{R}$ for which the scattering problem admits eigenfunctions in $L^2(\mathbb{R})$. We show below that these values coincide with the zeros of $\det a(k)$ in $\mathbb{C}^+$ and with those of $\det \bar{a}(k)$ in $\mathbb{C}^-$ (see also Appendix A for a discussion on the resolvent operator of the ZS/AKNZ scattering problem). In general, except for case 1 (see Section 2.3 with regard to the first symmetry), one cannot exclude the possible presence of such zeros on the real axis (so-called spectral singularities): as shown in [35], there exist Schwartz class potentials for which discrete eigenvalues accumulate to spectral singularities, which, in turn, accumulate on the continuous spectrum. For the 2×2 ZS/AKNS system, real spectral singularities have been extensively studied, and, in particular, sufficient conditions on the potential $Q(x,t)$ have been identified to guarantee their absence (e.g., single lobe potentials when $2||Q||_1/\pi$ is not an odd integer, and also certain double and multiple lobe potentials [36–39]). For the general $2m \times 2m$ ZS/AKNS system, in [40], it is shown that if $\|Q\|_1 < \pi/2$, the scattering problem has no discrete eigenvalues or spectral singularities (a result known as the "area theorem"), while if $\|Q\|_1 = \pi/2$, one has no discrete eigenvalues, but cannot exclude the presence of a spectral singularity.

As to the location of proper discrete eigenvalues in the complex plane (i.e., off the real axis), in case 1, the scattering problem is self-adjoint, like for the scalar defocusing NLS, and therefore when the potential is assumed to rapidly vanish as $x \to \pm\infty$ eigenvalues corresponding to square integrable eigenfunctions need to be real. Since we have already excluded zeros of $\det a(k)$ and $\det \bar{a}(k)$ for $k \in \mathbb{R}$, it follows that no discrete eigenvalues exist for case 1. In the remaining three cases, there is no constraint *a priori* on the location of discrete eigenvalues (except for the fact that they appear in complex conjugate pairs).

To properly define the discrete eigenvalues, it is convenient to introduce the 4×4 matrix solutions of (7):

$$P(x,t,k) = (\phi(x,t,k)\ \psi(x,t,k)), \qquad \bar{P}(x,t,k) = \left(\bar{\psi}(x,t,k)\ \bar{\phi}(x,t,k)\right). \tag{47}$$

Clearly, $P(x,t,k)$ is analytic for $k \in \mathbb{C}^+$, and $\bar{P}(x,t,k)$ is analytic for $k \in \mathbb{C}^-$; consequently, the bilinear combinations $A(k) = \bar{P}^\dagger(x,t,k^*)\,\Xi^{-1} P(x,t,k)$ and $\bar{A}(k) = P^\dagger(x,t,k^*)\,\Xi^{-1}\bar{P}(x,t,k)$ with Ξ defined in (20) are analytic in $\mathbb{C}^+$ and $\mathbb{C}^-$, respectively, and they are independent of x and t (as a result of the same argument used in Section 2.3 when discussing the first symmetry). Taking into account (25) and (27), one can explicitly compute the 2×2 blocks of these bilinear combinations of $P(x,t,k)$ and $\bar{P}(x,t,k)$ in terms of the eigenfunctions, and find:

$$A(k) = \bar{P}^\dagger(x,t,k^*)\Xi^{-1}P(x,t,k) \equiv \begin{pmatrix} \Omega\, a(k) & 0_{2\times2} \\ 0_{2\times2} & -\bar{a}^\dagger(k^*)\Sigma^{-1} \end{pmatrix} \quad k \in \mathbb{C}^+, \tag{48a}$$

$$\bar{A}(k) = A^\dagger(k^*) = P^\dagger(x,t,k^*)\Xi^{-1}\bar{P}(x,t,k) \equiv \begin{pmatrix} a^\dagger(k^*)\,\Omega & 0_{2\times2} \\ 0_{2\times2} & -\Sigma^{-1}\bar{a}(k) \end{pmatrix} \quad k \in \mathbb{C}^-. \tag{48b}$$

(Note that because of the symmetries (34), the lower diagonal block of $A(k)$ could equivalently be expressed as $-\Sigma^{-1}c(k)$, and the upper diagonal block of $\bar{A}(k)$ as $\Omega\,\bar{c}(k)$.) From (17) and (47), it follows that

$$\det P(x,t,k) = \det a(k), \qquad \det \bar{P}(x,t,k) = \det \bar{a}(k), \tag{49}$$

showing that the zeros of $\det a(k)$ in $\mathbb{C}^+$ are precisely the points where $\phi(x,t,k)$ and $\psi(x,t,k)$ become linearly dependent. The same, of course, holds for the zeros of $\det \bar{a}(k)$ in $\mathbb{C}^-$, where $\bar{\phi}(x,t,k)$ and $\bar{\psi}(x,t,k)$ become linearly dependent.

Let us start by assuming that $\det a(k)$ has a finite number $\mathcal{N}$ of zeros $k_1, \ldots, k_{\mathcal{N}}$ in $\mathbb{C}^+$. The first symmetry (37) implies $\det \bar{a}(k_n^*) = 0$, and we will denote the set of such zeros in either half-plane as $Z = \{k_n, k_n^*\}_{n=1}^{\mathcal{N}}$. For any $k_n \in \mathbb{C}^+ \cap Z$, one has

$$\operatorname{rank} P(x,t,k_n) = 2, 3\,, \qquad \operatorname{rank} A(k_n) = 0, 2\,, \tag{50}$$

because: (i) $\det P(x,t,k_n) = 0$ (cf. (49)), but the first two and last two columns of $P(x,t,k_n)$ are linearly independent; (ii) $A(k)$ is a block-diagonal matrix, and both diagonal blocks are singular at $k = k_n$ (cf. (48)). In particular, $\operatorname{rank} A(k) = \operatorname{rank} a(k) + \operatorname{rank} \bar{a}^\dagger(k^*)$, and since $\operatorname{rank} a(k) = \operatorname{rank} \bar{a}^\dagger(k^*)$ for all $k \in \mathbb{C}^+$ (cf., for instance, (46a)), it then follows that $\operatorname{rank} A(k_n)$ is either 0 (if and only if $a(k_n) = 0$) or 2 (if and only if $\operatorname{rank} a(k_n) = 1$).

Moreover, from (49), if follows that $k_n \in \mathbb{C}^+ \cap Z$ is a zero of order $m > 0$ of $\det a(k)$ if and only if it is a zero of order m of $\det P(x,t,k)$, and if $\det P(x,t,k_n) = \det a(k_n) = 0$, then 0 is an eigenvalue of both $P(x,t,k_n)$ and $a(k_n)$. Finally, note that the algebraic multiplicity of 0 as an eigenvalue of $a(k_n)$ can be either 1 or 2, and in the latter case, this implies $a(k_n) = 0$.

In the following proposition, we show that $\operatorname{rank} P(x,t,k)$ and $\operatorname{rank} a(k)$ are in one-to-one correspondence for any $k \in \mathbb{C}^+$, and $\operatorname{rank} \bar{P}(x,t,k)$ and $\operatorname{rank} \bar{a}(k)$ are in one-to-one correspondence for any $k \in \mathbb{C}^-$.

Proposition 1. *At a discrete eigenvalue $k_n \in Z \cap \mathbb{C}^+$, one has $\operatorname{rank} P(x,t,k_n) = 2$ if and only if $a(k_n) = 0_{2\times 2}$ and, consequently, $\operatorname{rank} P(x,t,k_n) = 3$ if and only if $\operatorname{rank} a(k_n) = 1$. The same holds for $k_n^* \in Z \cap \mathbb{C}^-$, relatively to the ranks of $\bar{P}(x,t,k_n^*)$ and $\bar{a}(k_n^*)$.*

Proof. Let us first prove that $\operatorname{rank} P(x,t,k_n) = 2 \Rightarrow a(k_n) = 0_{2\times 2}$. If $\operatorname{rank} P(x,t,k_n) = 2$, then there exist two linearly independent vectors $e_j = (\eta_j, -\xi_j)^T \in \mathbb{C}^4$ such that $P(x,t,k_n)e_j = 0_{4\times 1}$ for $j = 1, 2$. From the definition (48), it then follows that $A(k_n)e_j = 0$, and taking into account the explicit block diagonal expression of $A(k_n)$ in (48), we have

$$a(k_n)\eta_j = 0_{2\times 1}\,, \qquad \bar{a}^\dagger(k_n^*)\Sigma^{-1}\xi_j = 0_{2\times 1} \qquad \text{for } j = 1, 2\,. \tag{51}$$

If η_1 and η_2 are linearly independent vectors, this proves that the kernel of $a(k_n)$ has dimension 2, hence $a(k_n) = 0$. Let us then consider $\eta_2 = \alpha\eta_1$ for some $\alpha \in \mathbb{C}$. Using the symmetry (46a) in the second equation in (51), we obtain

$$\xi_j^\dagger \Sigma^{-1} \bar{a}(k_n^*) = \xi_j^\dagger \Sigma^{-1} \Sigma\, a^*(k_n^*)\, \Sigma^{-1} = 0_{1\times 2} \qquad \text{for } j = 1, 2\,,$$

which then is equivalent to

$$a^T(k_n)\xi_j = 0_{2\times 1} \qquad \text{for } j = 1, 2\,.$$

As before, if ξ_1 and ξ_2 are linearly independent, this implies $a^T(k_n) = 0_{2\times 2}$, which is equivalent to $a(k_n) = 0_{2\times 2}$. So, we assume $\xi_2 = \beta\xi_1$. However, since $e_j = (\eta_j, -\xi_j)^T \in \ker P(x,t,k_n)$ for $j = 1, 2$, we have

$$\phi(x,t,k_n)\eta_1 = \psi(x,t,k_n)\xi_1\,, \qquad \alpha\phi(x,t,k_n)\eta_1 = \beta\psi(x,t,k_n)\xi_1$$

yielding $\psi(x,t,k_n)\xi_1(1-\beta/\alpha) = 0_{4\times 1}$ (or $\phi(x,t,k_n)\eta_1(1-\alpha/\beta) = 0_{4\times 1}$). Since the column vectors in both $\psi(x,t,k_n)$ and $\phi(x,t,k_n)$ are linearly independent for all x,t, it follows that $\alpha = \beta$. However, this is a contradiction because it implies that e_1 and e_2 are proportional to each other.

Let us now prove that $a(k_n) = 0_{2\times 2} \Rightarrow \operatorname{rank} P(x,t,k_n) = 2$. If $a(k_n) = 0$, then the symmetry relation (46a) implies $\bar{a}(k_n^*) = 0$, and therefore $A(k_n) = 0$. Recall that if A and B are an $m \times n$ and an $n \times k$ matrices, respectively, with entries in some field, then $\operatorname{rank}(AB) \geq \operatorname{rank} A + \operatorname{rank} B - n$. Using this result, from (48), we have

$$0 = \operatorname{rank} A(k_n) \geq \operatorname{rank} P(x,t,k_n) + \operatorname{rank} \bar{P}(x,t,k_n^*) - 4\,,$$

and because both matrices $P(x,t,k_n)$ and $\bar{P}(x,t,k_n^*)$ have rank that is either 2 or 3, the above inequality requires $\operatorname{rank} P(x,t,k_n) = \operatorname{rank} \bar{P}(x,t,k_n^*) = 2$. (Note that in the proof of Proposition 1, we have used both the first and the second symmetry of the scattering data. The result can actually be proved using the first symmetry alone, and therefore also holds for the case in which $Q(x,t)$ is not necessarily a symmetric matrix, although at the expense of a much harder proof.) ■

If $k_n \in \mathbb{C}^+ \cap Z$, then $\det P(x,t,k_n) = 0$; on the other hand, the two columns in $\phi(x,t,k_n)$ are linearly independent, and so are the two columns in $\psi(x,t,k_n)$, and therefore we conclude that there exist $\xi_n, \eta_n \in \mathbb{C}^2 \setminus \{0\}$ such that

$$\psi(x,t,k_n)\xi_n = \phi(x,t,k_n)\eta_n\,. \tag{52a}$$

Note that such vectors are not uniquely defined, because one can divide both sides of the above equation by any of the nonzero components of either ξ_n or η_n. For any $k_n \in \mathbb{C}^+$ such that $\det a(k_n) = 0$, one also has $\det \bar{a}(k_n^*) = 0$ for $k_n^* \in \mathbb{C}^-$ (cf. (37)), and hence

$$\bar{\psi}\left(x,t,k_n^*\right)\bar{\xi}_n = \bar{\phi}\left(x,t,k_n^*\right)\bar{\eta}_n\,, \tag{52b}$$

for some $\bar{\xi}_n, \bar{\eta}_n \in \mathbb{C}^2 \setminus \{0\}$.

For any $k_n \in \mathbb{C}^+ \cap Z$ from (9) and (52a), it then follows

$$\psi(x,t,k_n)\xi_n \sim \begin{pmatrix} 0_{2\times 1} \\ \xi_n \end{pmatrix} e^{ik_n(x+2k_nt)} \qquad \text{as } x \to +\infty\,,$$

$$\psi(x,t,k_n)\xi_n = \phi(x,t,k_n)\eta_n \sim \begin{pmatrix} \eta_n \\ 0_{2\times 1} \end{pmatrix} e^{-ik_n(x+2k_nt)} \qquad \text{as } x \to -\infty\,,$$

and since $\operatorname{Im} k_n > 0$, the linear combination of eigenfunctions in $\psi(x,t,k_n)$ is exponentially decaying as $x \to \pm\infty$, in what is often referred to as a bound state (continuity of the eigenfunctions for all $x \in \mathbb{R}$ is established by means of the integral equations (13) using standard techniques).

It is important to stress that, while common in the literature, assuming

$$\phi(x,t,k_n) = \psi(x,t,k_n)\, b_n\,, \qquad \bar{\phi}\left(x,t,k_n^*\right) = \bar{\psi}\left(x,t,k_n^*\right)\bar{b}_n\,, \tag{53}$$

where $b_n, \bar{b}_n$ are 2×2 constant, nonzero matrices, is in general not equivalent to (52). Importantly, b_n and $\bar{b}_n$ must be nonsingular (for instance, if $\det b_n = b_n^{1,1}b_n^{2,2} - b_n^{1,2}b_n^{2,1} = 0$, then one can show that (53) implies $\phi_1(x,t,k_n)\, b_n^{2,2} = \phi_2(x,t,k_n)\, b_n^{2,1}$ for the columns $\phi_1(x,t,k_n)$ and $\phi_2(x,t,k_n)$ of $\phi(x,t,k_n)$, which is a contradiction because the two columns are linearly independent).

As a matter of fact, (53) is a stronger condition than (52), because it implies that each column of ϕ (respectively, $\bar{\phi}$) is a linear combination of the two columns of ψ (respectively, $\bar{\psi}$), and it corresponds to assuming that at $k = k_n$, (52a) holds both with $\eta_n = (1\ 0)^T$ and ξ_n given by the first column of b_n, and with $\eta_n = (0\ 1)^T$ and ξ_n given by the second column of b_n (and similarly for k_n^*). Conversely, suppose two linearly independent conditions such as (52a) hold at the same k_n, say

$$\psi(x,t,k_n)\xi_n = \phi(x,t,k_n)\eta_n\,, \qquad \psi(x,t,k_n)\tilde{\xi}_n = \phi(x,t,k_n)\tilde{\eta}_n\,,$$

with $\beta_n = \mathrm{Wr}(\eta_n, \tilde{\eta}_n) \neq 0$; then one can solve the above equations with respect to $\phi(x,t,k_n)$ and obtain a relation like (53) with $b_n = (\tilde{\eta}_n^{(2)}\xi_n - \eta_n^{(2)}\tilde{\xi}_n\,, -\tilde{\eta}_n^{(1)}\xi_n + \eta_n^{(1)}\tilde{\xi}_n)/\beta_n$, where superscripts $^{(j)}$ denote the jth component of the vectors η_n and $\tilde{\eta}_n$.

Equations (52) correspond to a situation in which $\operatorname{rank} P(x,t,k_n) = \operatorname{rank} \bar{P}(x,t,k_n^*) = 3$, and as a consequence of Proposition 1, this implies $a(k_n) \neq 0_{2\times 2}$. On the other hand, under the assumption in (53), because b_n and $\bar{b}_n$ are invertible, then $\operatorname{rank} P(x,t,k_n) = \operatorname{rank} \bar{P}(x,t,k_n^*) = 2$, and as a consequence of Proposition 1 in this case $a(k_n) = 0_{2\times 2}$. Below, we define the norming constants and determine the residue conditions in each of the two cases for $\operatorname{rank} P(x,t,k_n) = \operatorname{rank} \bar{P}(x,t,k_n^*)$ at a discrete eigenvalue pair k_n, k_n^*.

2.4.1. Norming constants and residue conditions when $\operatorname{rank} P(x,t,k_n) = 3$. Let us start by considering the case in which $k_n \in \mathbb{C}^+$ is a simple zero of $\det a(k)$ (in which case $(\det a)'(k_n) \neq 0$ where the prime denotes differentiation with respect to k), and $\operatorname{rank} P(x,t,k_n) = 3$. Then the first symmetry (cf. Section 2.3) implies $\det \bar{a}(k_n^*) = 0$, with $(\det \bar{a})'(k_n^*) \neq 0$. Let $\chi_n \in \mathbb{C}^4 \setminus \{0\}$ be a right null vector of $P(x,t,k_n)$, i.e., $\chi_n \in \ker P(x,t,k_n)$, and let

$$\chi_n = \begin{pmatrix} \chi_n^{\mathrm{up}} \\ \chi_n^{\mathrm{dn}} \end{pmatrix} \qquad \chi_n^{\mathrm{up}}, \chi_n^{\mathrm{dn}} \in \mathbb{C}^2\,,$$

then from (47), it follows that

$$\phi(x,t,k_n)\chi_n^{\rm up} + \psi(x,t,k_n)\chi_n^{\rm dn} = 0_{4\times 2}\,,$$

and therefore any right null vector of $P(x,t,k_n)$ implies (52a), with $\eta_n = \chi_n^{\rm up}$ and $\xi_n = -\chi_n^{\rm dn}$. Note that $\eta_n = \chi_n^{\rm up} \neq 0$ and $\xi_n = -\chi_n^{\rm dn} \neq 0$, because the first two columns as well as the last two columns of $P(x,t,k_n)$ are linearly independent. Vice versa, given ξ_n and η_n as in (52a), the 4×1 vector $\chi_n = (\eta_n\,, -\xi_n)^T$ belongs to $\ker P(x,t,k_n)$. Similar statements can be proved for $k_n^* \in \mathbb{C}^- \cap Z$ and $\bar{P}(x,t,k)$.

If $\xi_n, \eta_n \in \mathbb{C}^2 \setminus \{0\}$ satisfy (52a), then $\chi_n = (\eta_n\,, -\xi_n)^T$ is a right null vector of $A(k) = \bar{P}^\dagger(x,t,k_n^*)\Xi^{-1}P(x,t,k_n)$, and from (48), it then follows that

$$a(k_n)\eta_n = 0_{2\times 1}\,, \qquad \bar{a}^\dagger(k_n^*)\,\Sigma^{-1}\xi_n = 0_{2\times 1}\,, \tag{54a}$$

showing that η_n belongs to $\ker a(k_n)$ and $\Sigma^{-1}\xi_n$ belongs to $\ker \bar{a}^\dagger(k_n^*)$. The converse is also true, i.e., vectors in $\ker a(k_n)$ and $\ker \bar{a}^\dagger(k_n^*)$ provide vectors that satisfy (52a). The analog can easily be shown for any nonzero vector $\bar{\chi}_n = (\bar{\xi}_n, -\bar{\eta}_n) \in \text{kern}\,\bar{P}(x,t,k_n^*)$, for which Eq. (52b) holds; moreover,

$$a^\dagger(k_n)\,\Omega\,\bar{\xi}_n = 0_{2\times 1}\,, \qquad \bar{a}(k_n^*)\bar{\eta}_n = 0_{2\times 1}\,, \tag{54b}$$

so that $\Omega\bar{\xi}_n \in \ker a^\dagger(k_n)$ and $\bar{\eta}_n \in \ker \bar{a}(k_n^*)$.

For any $m\times m$ matrix A, one has $\det(\text{cof}A) = (\det A)^{m-1}$, where cof A is the adjugate matrix of A. Thus, if $\alpha(k)$ denotes the adjugate matrix of $a(k)$, for which $a(k)\alpha(k) = \alpha(k)a(k) = \det a(k)I_2$, it follows that

$$\det\alpha(k) = \det a(k)\,,$$

and hence $\det\alpha(k)$ and $\det a(k)$ have a zero of the same order for each $k_n \in \mathbb{C}^+ \cap Z$. Moreover, since they are both 2×2 matrices, one obviously has $\text{rank}\,a(k) = \text{rank}\,\alpha(k)$, and therefore, as a consequence of Proposition 1, $\alpha(k_n) \neq 0_{2\times 2}$ because we are assuming $\text{rank}\,P(x,t,k_n) = 3$. Similarly, denoting by $\bar{\alpha}(k)$ the adjugate matrix of $\bar{a}(k)$, it follows that $\det\bar{\alpha}(k)$ has a zero of the same order as $\det\bar{a}(k)$ for each $k_n^* \in \mathbb{C}^- \cap Z$.

Since $a(k_n)\alpha(k_n) = \alpha(k_n)a(k_n) = 0_{2\times 2}$ and $\bar{a}(k_n^*)\bar{\alpha}(k_n^*) = \bar{\alpha}(k_n^*)\bar{a}(k_n^*) = 0_{2\times 2}$, each column of $\alpha(k_n)$ is both a left and a right null vector of $a(k_n)$, and each column of $\bar{\alpha}(k_n^*)$ is both a left and a right null vector of $\bar{a}(k_n^*)$. Of course, the two columns of $\alpha(k_n)$ and $\bar{\alpha}(k_n^*)$ are proportional to each other, since $\det\alpha(k_n) = \det\bar{\alpha}(k_n^*) = 0$. Therefore, one can choose two vectors in $\ker P(x,t,k_n)$ with the first two components of each vector given by the first and the second columns of $\alpha(k_n)$, and the remaining two components, columnwise, denoted by $-c_n$:

$$0_{4\times 2} = P(x,t,k_n)\begin{pmatrix}\alpha(k_n)\\ -c_n\end{pmatrix} \Leftrightarrow \phi(x,t,k_n)\alpha(k_n) = \psi(x,t,k_n)c_n\,. \tag{55}$$

Since in this case, we are assuming $\ker P(x,t,k_n)$ is one-dimensional (because $\operatorname{rank} P(x,t,k_n) = 3$), then the two columns of the matrix multiplying $P(x,t,k_n)$ in (55) must be proportional to each other, which then implies $\operatorname{rank} c_n = 1$. Also, considering that $\alpha(k) = a^{-1}(k)/\det a(k)$, if k_n is a simple zero of $\det a(k)$, we have

$$\operatorname{Res}_{k=k_n}\left[\phi(x,t,k)\, a^{-1}(k)\right] = \phi(x,t,k_n)\alpha(k_n)\operatorname{Res}_{k=k_n}\frac{1}{\det a(k)}$$

and from (11a) and (55), it then follows

$$\operatorname{Res}_{k=k_n}\left[M(x,t,k)a^{-1}(k)\right] = \mathrm{e}^{2i\theta(x,t,k_n)}N(x,t,k_n)\,C_n\,, \qquad \det C_n = 0\,, \tag{56a}$$

where $\det C_n = 0$ follows since $C_n = c_n/(\det a)'(k_n)$ and by construction $\det c_n = 0$. Equation (56) defines the norming constant C_n associated with a simple discrete eigenvalue k_n, i.e., a simple zero of $\det a(k)$, in the rank 3 case for $P(x,t,k_n)$, i.e., when $a(k_n) \neq 0$. Similarly, one obtains

$$\operatorname{Res}_{k=k_n^*}\left[\bar{M}(x,t,k)\bar{a}^{-1}(k)\right] = \mathrm{e}^{-2i\theta\left(x,t,k_n^*\right)}\bar{N}\left(x,t,k_n^*\right)\bar{C}_n\,, \qquad \det \bar{C}_n = 0\,. \tag{56b}$$

As mentioned above, $\det \alpha(k)$ and $\det a(k)$ have a zero of the same order at each $k_n \in \mathbb{C}^+ \cap Z$, and similarly $\det \bar{\alpha}(k)$ and $\det \bar{a}(k)$ have a zero of the same order at each $k_n^* \in \mathbb{C}^- \cap Z$. Moreover, for any $k \in \mathbb{C}^+ \setminus Z$, one has $a^{-1}(k) = \alpha(k)/(\det a(k))$, and since $\alpha(k)$ is analytic in $\mathbb{C}^+$, then: (i) $a^{-1}(k)$ is meromorphic in $\mathbb{C}^+$; (ii) its poles coincide with the discrete eigenvalues in the UHP; and (iii) the order of each pole at k_n is less than or equal to the order of k_n as a zero of $\det a(k)$. The same result can obviously be proven for $\bar{a}^{-1}(k)$ in $\mathbb{C}^-$.

Finally, we note that k_n is a simple zero (hence, simple pole of $a^{-1}(k)$), one can easily show that $\ker a(k_n) = \{\tau_n \xi \,:\, \xi \in \mathbb{C}^2\} \equiv \operatorname{range} \tau_n$, where $\tau_n = \operatorname{Res}_{k=k_n} a^{-1}(k) \equiv \alpha(k_n)/((\det a)'(k_n))$.

2.4.2. Norming constants and residue conditions when $\operatorname{rank} P(x,t,k_n) = 2$. We now consider $\operatorname{rank} P(x,t,k_n) = \operatorname{rank} \bar{P}(x,t,k_n^*) = 2$, which, according to Proposition 1, is equivalent to $a(k_n) = \bar{a}(k_n^*) = 0$. Let us start by assuming that k_n is a simple zero of $\det a(k)$, so that $(\det a)'(k_n) \neq 0$. We can write (53) equivalently as $M(x,t,k_n) = \mathrm{e}^{2i\theta(x,t,k_n)}N(x,t,k_n)\,b_n$, and

$$\operatorname{Res}_{k=k_n}\left[M(x,t,k)a^{-1}(k)\right] = \mathrm{e}^{2i\theta(x,t,k_n)}N(x,t,k_n)\,C_n\,,$$
$$C_n = \frac{1}{(\det a)'(k_n)} b_n\,\alpha(k_n)\,,$$

where, as before, $\alpha(k)$ denotes the cofactor matrix of $a(k)$. However, in this case, $a(k_n) = \alpha(k_n) = 0$, and hence, $C_n = 0$. This shows that if $\operatorname{rank} P(x,t,k_n) = 2$, no nontrivial norming constant exists at a discrete eigenvalue that is a simple zero of $\det a(k)$. However, the above arguments can be easily generalized to higher order zeros of $\det a(k)$. If, for instance, k_n is a second-order zero of $\det a(k)$, then $\det \alpha(k)$ also has a second-order zero at k_n. In this case, however, in a neighborhood of k_n, one has

$$a^{-1}(k) = \frac{1}{(k-k_n)^2}\tau_{n,2} + \frac{1}{k-k_n}\tau_{n,1} + f(k),$$

where $f(k)$ is analytic at k_n, and

$$\tau_{n,2} = \lim_{k\to k_n}(k-k_n)^2 a^{-1}(k) \equiv \frac{2}{(\det a)''(k_n)}\alpha(k_n)\,, \tag{57}$$

$$\begin{aligned}\tau_{n,1} &= \lim_{k\to k_n}\frac{d}{dk}\left[(k-k_n)^2 a^{-1}(k)\right]\\ &\equiv \frac{2}{(\det a)''(k_n)}\alpha'(k_n) - \frac{2}{3}\frac{(\det a)'''(k_n)}{((\det a)''(k_n))^2}\alpha(k_n)\,.\end{aligned} \tag{58}$$

If $\operatorname{rank} P(x,t,k_n) = 3$, one has $\alpha(k_n) \neq 0_{2\times 2}$ and hence $\tau_{n,2} \neq 0_{2\times 2}$ and $\det \tau_{n,2} = 0$ (because $\det \alpha(k_n) = 0$); $\tau_{n,1}$, on the other hand, might or might not be zero, and one can have $\det \tau_{n,1} \neq 0$. This generic situation will be discussed in Appendix A. However, in the rank 2 case for $P(x,t,k_n)$, we are considering here, $\alpha(k_n) = \tau_{n,2} = 0$; in this case, even though $\det a(k)$ has a double zero at k_n, $a^{-1}(k)$ still has a first-order pole at k_n, with residue

$$\tau_{n,1} = \frac{2}{(\det a)''(k_n)}\alpha'(k_n)\,.$$

Consequently,

$$\operatorname{Res}_{k=k_n}\left[M(x,t,k)a^{-1}(k)\right] = e^{2i\theta(x,t,k_n)}N(x,t,k_n)\,C_n\,, \tag{59a}$$

$$C_n = \frac{2}{(\det a)''(k_n)}b_n\,\alpha'(k_n)\,.$$

Note that, in general, $\det(\alpha')(k_n)$ need not be zero, so $\tau_{n,1}$ needs not be rank one.

Similarly, at $k_n^* \in \mathbb{C}^-$, simple zero of $\det \bar{a}(k)$, we obtain

$$\operatorname{Res}_{k=k_n^*}\left[\bar{M}(x,t,k)\bar{a}^{-1}(k)\right] = e^{-2i\theta\left(x,t,k_n^*\right)}\bar{N}\left(x,t,k_n^*\right)\bar{C}_n\,, \tag{59b}$$

$$\bar{C}_n = \frac{2}{(\det \bar{a})''(k_n^*)}\bar{b}_n\,\bar{\alpha}'(k_n^*)\,.$$

The above residue conditions generalize (56) to the rank 2 case. As one can see, (56) and (59) have exactly the same form, the only difference being that in the rank 2 case one is allowed to relax the constraint that the norming constant be a rank 1 matrix. Therefore, in the formulation of the inverse problem, we will assume at each discrete eigenvalue pair k_n, a residue condition of the form (59a) holds, where both $\det C_n = 0$ and $\det C_n \neq 0$ are possible choices for the associated norming constant. The same obviously is true for (59a) at each eigenvalues $k_n^* \in \mathbb{C}^-$, and associated norming constant $\bar{C}_n$.

2.5. Asymptotics as $k \to \infty$

To properly define the inverse problem, one needs the large k asymptotic behavior of the eigenfunctions and of the scattering matrix.

Assuming the entries of $Q(x,t)$ and $Q_x(x,t)$ are in $L^1(\mathbb{R})$ for each $t \in \mathbb{R}$, standard integration by parts on the integral equations (13) for the modified eigenfunctions yields:

$$M(x,t,k) = \begin{pmatrix} I_2 + \dfrac{i}{2k}\displaystyle\int_{-\infty}^{x} Q(x',t)\,R(x',t)\,dx' + O(1/k^2) \\ \dfrac{i}{2k}R(x,t) + O(1/k^2) \end{pmatrix} \tag{60a}$$

$$k \to \infty, \quad k \in \mathbb{C}^+ \cup \mathbb{R},$$

$$\bar{M}(x,t,k) = \begin{pmatrix} -\dfrac{i}{2k}Q(x,t) + O(1/k^2) \\ I_2 - \dfrac{i}{2k}\displaystyle\int_{-\infty}^{x} R(x',t)\,Q(x',t)\,dx' + O(1/k^2) \end{pmatrix} \tag{60b}$$

$$k \to \infty, \quad k \in \mathbb{C}^- \cup \mathbb{R},$$

and

$$\bar{N}(x,t,k) = \begin{pmatrix} I_2 - \dfrac{i}{2k}\displaystyle\int_{x}^{\infty} Q(x',t)\,R(x',t)\,dx' + O(1/k^2) \\ \dfrac{i}{2k}R(x,t) + O(1/k^2) \end{pmatrix} \tag{60c}$$

$$k \to \infty, \quad k \in \mathbb{C}^- \cup \mathbb{R},$$

$$N(x,t,k) = \begin{pmatrix} -\dfrac{i}{2k}Q(x,t) + O(1/k^2) \\ I_2 + \dfrac{i}{2k}\displaystyle\int_{x}^{\infty} R(x',t)\,Q(x',t)\,dx' + O(1/k^2) \end{pmatrix} \tag{60d}$$

$$k \to \infty, \quad k \in \mathbb{C}^+ \cup \mathbb{R}.$$

The above equations will allow us to reconstruct the scattering potential $Q(x,t)$ from the solution of the inverse problem for the eigenfunctions.

Finally, inserting the above asymptotic expansions for the modified eigenfunctions into (15), it follows that

$$S(k) = I_2 + O(1/k)\,, \tag{61}$$

as $k \to \infty$ in the appropriate regions of the complex k-plane. Explicitly, the above asymptotic estimate holds in $\mathbb{C}^+$ and $\mathbb{C}^-$ for $a(k)$ and $\bar{a}(k)$, respectively, but only on the real axis for $b(k)$ and $\bar{b}(k)$.

3. Inverse problem (Riemann–Hilbert formulation)

The starting point for the formulation of the inverse problem is (18), regarded as relating the eigenfunctions analytic in $\mathbb{C}^+$ and those analytic in $\mathbb{C}^-$. As usual, one introduces the sectionally meromorphic matrices

$$\mu^+(x,t,k) = (M\,a^{-1}\;\; N)\,, \qquad \mu^-(x,t,k) = (\bar{N}\;\; \bar{M}\,\bar{a}^{-1})\,. \tag{62}$$

(Superscripts $\pm$ distinguish between meromorphicity in $\mathbb{C}^+$ and $\mathbb{C}^-$, respectively.) From (16), we then obtain the jump condition

$$\mu^-(x,t,k) = \mu^+(x,t,k)\,(I_4 - G(x,t,k))\,, \qquad k \in \mathbb{R}\,, \tag{63}$$

where the jump matrix is

$$G(x,t,k) = \begin{pmatrix} 0_{2\times 2} & -\mathrm{e}^{-2i\theta(x,t,k)}\bar{\rho}(k) \\ \mathrm{e}^{2i\theta(x,t,k)}\rho(k) & \rho(k)\bar{\rho}(k) \end{pmatrix}. \tag{64}$$

Recalling the asymptotic behavior of the Jost eigenfunctions and scattering coefficients, it is easy to check that the meromorphic eigenfunctions $\mu^\pm$ satisfy the following normalization condition:

$$\mu^\pm = I_4 + O(1/k)\,, \qquad k \to \infty\,. \tag{65}$$

Equations (62)–(64), supplemented with the normalization condition (65), define a matrix RHP, which needs to be regularized by subtracting out the asymptotic behavior and the pole contributions. Assuming simple poles for the meromorphic eigenfunctions, and taking into account the residue conditions (59), one obtains the following expression for the eigenfunctions:

$$N(x,t,k) = \begin{pmatrix} 0_{2\times 2} \\ I_2 \end{pmatrix} + \sum_{j=1}^{\mathcal{N}} \frac{\mathrm{e}^{-2i\theta\left(x,t,k_j^*\right)}}{k-k_j^*}\,\bar{N}\left(x,t,k_j^*\right)\,\bar{C}_j$$
$$-\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{-2i\theta(x,t,\xi)}\bar{N}(x,t,\xi)\bar{\rho}(\xi)}{\xi-(k+i0)}\,\mathrm{d}\xi\,, \tag{66a}$$

$$\bar{N}(x,t,k) = \begin{pmatrix} I_2 \\ 0_{2\times 2} \end{pmatrix} + \sum_{j=1}^{\mathcal{N}} \frac{\mathrm{e}^{2i\theta(x,t,k_j)}}{k-k_j} N(x,t,k_j)\,C_j + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{2i\theta(x,t,\xi)}N(x,t,\xi)\rho(\xi)}{\xi-(k-i0)}\,\mathrm{d}\xi\,. \tag{66b}$$

The derivation of the above formal solution of the RHP follows from a straightforward application of Cauchy projectors to the jump condition (see Appendix B for further details). To close the system, one needs to evaluate the first equation at each $k = k_n \in \mathbb{C}^+$, for $n = 1, \ldots, \mathcal{N}$ and the second equation at each $k = k_n^* \in \mathbb{C}^-$, for $n = 1, \ldots, \mathcal{N}$.

The last task in the IST is the reconstruction of the potential from the solution of the RHP, which is accomplished by simply evaluating the large k asymptotic behavior of the above equations and comparing it with (60), yielding:

$$Q(x,t) = 2i\sum_{n=1}^{\mathcal{N}} \mathrm{e}^{-2i\theta\left(x,t,k_n^*\right)}\,\bar{N}^{\mathrm{up}}\left(x,t,k_n^*\right)\,\bar{C}_n + \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-2i\theta(x,t,\xi)}\bar{N}^{\mathrm{up}}(x,t,\xi)\,\bar{\rho}(\xi)\,\mathrm{d}\xi\,, \tag{67a}$$

$$R(x,t) = -2i\sum_{n=1}^{\mathcal{N}} \mathrm{e}^{2i\theta(x,t,k_n)}\,N^{\mathrm{dn}}(x,t,k_n)\,C_n + \frac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{e}^{2i\theta(x,t,\xi)}N^{\mathrm{dn}}(x,t,\xi)\rho(\xi)\,\mathrm{d}\xi\,. \tag{67b}$$

Note that in the above expressions, the scattering data (reflection coefficients and norming constants) are time-independent, and the time dependence of the solution is entirely accounted for by the time dependence of the eigenfunctions.

Finally, the consistency of the reconstruction formulas (67) with the symmetries in the potentials $Q(x,t)$ and $R(x,t)$ allows one to identify the symmetries of the norming constants. Recalling that $N^{\mathrm{dn}}(x,t,k) \sim I_2$ as $x \to \infty$ for any $k \in \mathbb{C}^+$, and $\bar{N}^{\mathrm{up}}(x,t,k) \sim I_2$ as $x \to \infty$ for any $k \in \mathbb{C}^-$, the comparison of the above two equations in (67) to $R = \Sigma\, Q^\dagger\, \Omega$ yields

$$\bar{C}_n = \Omega^{-1} C_n^\dagger\, \Sigma^{-1} \qquad n = 1, \ldots, \mathcal{N}\,. \tag{68a}$$

Similarly, the condition that $Q(x,t)$ be a symmetric matrix for all $x, t \in \mathbb{R}$ (i.e., $Q^T(x,t) = Q(x,t)$) requires that the norming constants are themselves symmetric matrices:

$$C_n^T = C_n\,, \qquad \bar{C}_n^T = \bar{C}_n\,, \qquad n = 1, \ldots, \mathcal{N}\,. \tag{68b}$$

In Appendix B, we will discuss how the equations of the inverse problem are generalized to include double poles in the RHP.

4. Soliton solutions

We now consider potentials $Q(x,t)$ corresponding to pure soliton solutions, for which the reflection coefficient is identically zero. In this case, there is no jump in the meromorphic eigenfunctions of the RHP across the continuous spectrum, and the inverse problem is reduced to a linear algebraic system, whose solution yields the soliton solutions of the integrable nonlinear equation. Explicitly, in the reflectionless case, the system (66) for the upper blocks of $N(x,t,k_n)$ and $\bar{N}(x,t,k_n^*)$ can be reduced to:

$$\bar{N}^{\mathrm{up}}\left(x,t,k_n^*\right) = I_2 + \sum_{\ell,j=1}^{\mathcal{N}} \frac{e^{2i(\theta(x,t,k_j)-\theta(x,t,k_\ell^*))}}{(k_n^*-k_j)(k_j-k_\ell^*)} \bar{N}^{\mathrm{up}}(x,t,k_\ell^*)\bar{C}_\ell C_j\,, \tag{69}$$

and the solution of this linear system for $\bar{N}^{\mathrm{up}}(x,t,k_n^*)$ into the reconstruction formula (67a) yields the $\mathcal{N}$ soliton solution:

$$Q(x,t) = 2i\sum_{n=1}^{\mathcal{N}} \mathrm{e}^{-2i\theta\left(x,t,k_n^*\right)}\, \bar{N}^{\mathrm{up}}\left(x,t,k_n^*\right)\, \bar{C}_n\,. \tag{70}$$

For a one-soliton solution, we take $k_1 = \xi + i\eta$ with $\eta > 0$ and obtain:

$$\bar{N}^{\mathrm{up}}(x,t,k_1^*) = \left[I_2 - \frac{e^{-4\eta(x+4\xi t)}}{4\eta^2}\Omega^{-1}C_1^\dagger\Sigma^{-1}C_1\right]^{-1},$$

which yields

$$Q(x,t) = 2ie^{-2i(\xi x+2(\xi^2-\eta^2)t)-2\eta(x+4\xi t)}\left[I_2 - \frac{e^{-4\eta(x+4\xi t)}}{4\eta^2}\Omega^{-1}C_1^\dagger\Sigma^{-1}C_1\right]^{-1}\Omega^{-1}C_1^\dagger\Sigma^{-1}\,, \tag{71}$$

for any choice of the 2×2 norming constant C_1 as a complex symmetric matrix, both rank 1 and rank 2 (see the discussion at the end of Section 2.4).

Assuming the inverse matrix in square brackets in (71) exists (we will discuss later on the necessary and sufficient conditions for this to happen), the solution can be written as:

$$Q(x,t) = 4i\eta e^{i\zeta}\frac{y\left[I_2 - y^2\mathrm{cof}\,(C_1)\,\mathrm{cof}\,(\Sigma^{-1})\,\mathrm{cof}\,\left(C_1^\dagger\right)\,\mathrm{cof}\,(\Omega^{-1})\right]}{1-y^2\mathrm{tr}\left(\Omega^{-1}C_1^\dagger\Sigma^{-1}C_1\right)+y^4|\det C_1|^2}\Omega^{-1}C_1^\dagger\Sigma^{-1}\,,$$

where we introduced the short-hand notations

$$\zeta = -2(\xi x + 2(\xi^2 - \eta^2)t), \qquad y = e^{-2\eta(x+4\xi t)}/(2\eta), \tag{72}$$

and, as before, cof (A) denotes the cofactor matrix of A and tr denotes the matrix trace. The last expression can then be further simplified to

$$Q(x,t) = 4i\eta e^{i\zeta} \frac{y}{1 - y^2 \mathrm{tr}(\Omega^{-1}C_1^\dagger \Sigma^{-1} C_1) + y^4 |\det C_1|^2} \times \left[\Omega^{-1}C_1^\dagger\Sigma^{-1} - y^2 (\det C_1)^* \mathrm{cof}\,(C_1)\right], \tag{73}$$

where we have used that $\det\Sigma\ \det\Omega = 1$ for all four choices for Σ and Ω considered in Section 1.

To discuss the regularity of the solution in all four cases with respect to the norming constant C_1, let

$$C_1 = \begin{pmatrix} c_1 & c_0 \\ c_0 & c_{-1} \end{pmatrix},$$

where c_j, $j = 1, 0, -1$ are arbitrary complex numbers (with $c_0^2 = c_1 c_{-1}$, if $\det C_1 = 0$).

Obviously, in order for the solution to be regular for all $x, t \in \mathbb{R}$ one needs

$$\det\left[I_2 - y^2\Omega^{-1}C_1^\dagger\Sigma^{-1}C_1\right] \equiv 1 - y^2 \mathrm{tr}\left(\Omega^{-1}C_1^\dagger\Sigma^{-1}C_1\right) + y^4 |\det C_1|^2 \neq 0, \tag{74}$$

and hence the necessary and sufficient condition for the solution to be regular for all $x, t \in \mathbb{R}$ is that the above biquadratic polynomial in y does not have any real and positive root (by its definition (72) $y > 0$). Note that for all choices of Σ and Ω considered here, $\mathrm{tr}(\Omega^{-1}C_1^\dagger\Sigma^{-1}C_1) \in \mathbb{R}$, so the above polynomial has real coefficients. It is clear that the regularity condition will depend upon whether $\det C_1 = 0$ (rank 1 case) or $\det C_1 \neq 0$ (rank 2 case). We begin by considering the rank 1 case, i.e., $\det C_1 = 0$, where the necessary and sufficient condition for a regular solution reads

$$\mathrm{tr}\left(\Omega^{-1}C_1^\dagger\Sigma^{-1}C_1\right) \leq 0.$$

Considering the four different choices introduced in Section 1 for Σ and Ω, one finds the following:

Case 1: $\mathrm{tr}(\Omega^{-1}C_1^\dagger\Sigma^{-1}C_1) = \mathrm{tr}(C_1^\dagger C_1) \equiv (|c_1|^2 + |c_{-1}|^2 + 2|c_0|^2) \equiv (|c_1| + |c_{-1}|)^2 > 0$, so no regular solution exists with $C_1 \neq 0$ and $\det C_1 = 0$. This is consistent with the fact that in this case, the scattering problem is self-adjoint and no soliton solutions exist.

Case 2: $\mathrm{tr}(\Omega^{-1}C_1^\dagger\Sigma^{-1}C_1) = -\mathrm{tr}(C_1^\dagger C_1) \equiv -(|c_1|^2 + |c_{-1}|^2 + 2|c_0|^2) \equiv -(|c_1| + |c_{-1}|)^2 < 0$, so the solution is regular for any choice of the norming constant $C_1 \neq 0$ with $\det C_1 = 0$.
Case 3: $\mathrm{tr}(\Omega^{-1}C_1^\dagger\Sigma^{-1}C_1) = \mathrm{tr}(\sigma_3 C_1^\dagger \sigma_3 C_1) \equiv (|c_1|^2 + |c_{-1}|^2 - 2|c_0|^2) \equiv (|c_1| - |c_{-1}|)^2 \geq 0$, so the only regular solutions with $C_1 \neq 0$ in this case correspond to having both $\det C_1 = \mathrm{tr}(\Omega^{-1}C_1^\dagger\Sigma^{-1}C_1) = 0$, i.e., $|c_1| = |c_{-1}|$ and $c_0^2 = c_1 c_{-1}$.
Case 4: $\mathrm{tr}(\Omega^{-1}C_1^\dagger\Sigma^{-1}C_1) = -\mathrm{tr}(\sigma_3 C_1^\dagger \sigma_3 C_1) \equiv -(|c_1|^2 + |c_{-1}|^2 - 2|c_0|^2) \equiv -(|c_1| - |c_{-1}|)^2 \leq 0$, so the solution is regular for any choice of the norming constant C_1 with $\det C_1 = 0$.

Note that (73) allows one to easily investigate if $Q(x,t)$ vanishes exponentially as $x \to \pm\infty$: the decay as $x \to +\infty$ is obvious, since $y \to 0$; when $x \to -\infty$, the solution also decays as $1/y$ both when $\det C_1 \neq 0$ and when $\det C_1 = 0$. The only case in which the solution does not decay as $x \to -\infty$ is if both $\det C_1 = 0$ and $\mathrm{tr}(\Omega^{-1}C_1^\dagger\Sigma^{-1}C_1) = 0$. This can only happen in cases 3 and 4, and if $|c_1| = |c_{-1}|$, which corresponds to norming constants and corresponding solutions of the form:

$$C_1 = |c_1|\begin{pmatrix} e^{i\alpha} & \pm e^{i(\alpha+\beta)/2} \\ \pm e^{i(\alpha+\beta)/2} & e^{i\beta} \end{pmatrix}, \qquad Q(x,t) = 2i\tilde{\nu}e^{-2i(\xi x + 2(\xi^2-\eta^2)t) - 2\eta(x+4\xi t)}\sigma_3 C_1^\dagger \sigma_3\,, \tag{75}$$

where $\tilde{\nu} = 1$ in case 3 and $\tilde{\nu} = -1$ in case 4. Although smooth, the solutions are not solitons, so for cases 3 and 4, we only consider rank 1 norming constants such that $|c_1| \neq |c_{-1}|$, or equivalently, that

$$\mathrm{tr}(\Omega^{-1}C_1^\dagger\Sigma^{-1}C_1) < 0\,. \tag{76}$$

(As mentioned above, the condition is satisfied by any rank 1 norming constant in case 2).

The expression (73) is particularly simple in the rank 1 case, i.e., if $\det C_1 = 0$:

$$Q(x,t) = 2i\eta \mathrm{e}^{-2i(\xi x + 2(\xi^2-\eta^2)t)}\mathrm{sech}[2\eta(x + 4\xi t - x_o)]\,\frac{\Omega^{-1}C_1^\dagger\,\Sigma^{-1}}{\sqrt{-\mathrm{tr}(\Omega^{-1}C_1^\dagger\Sigma^{-1}C_1)}}\,, \tag{77}$$

where $4\eta^2 e^{2\eta x_o} = -\mathrm{tr}((\Omega^{-1}C_1^\dagger\Sigma^{-1}C_1))$ (and the trace has to be strictly negative, of course; so this only applies to case 2, and to case 4 provided $|c_1| \neq |c_{-1}|$).

Let us now consider the full rank case, $\det C_1 \neq 0$, and use the expression (73) to derive necessary and sufficient conditions for the existence of regular soliton solutions. Recalling that $y = e^{-2\eta(x+4\xi t)}/(2\eta) > 0$, the solution $Q(x,t)$ is regular for all $x, t \in \mathbb{R}$ if and only if the polynomial (74) is nonzero for each $y \in \mathbb{R}$. In other words, when $\det C_1 \neq 0$, we need to find necessary and sufficient conditions on the norming constant C_1 that guarantee that the second-order polynomial (74) does not have any real and positive zeros y. From the sign of the discriminant of the quadratic polynomial, it is then clear that

$$\operatorname{tr}\left(\Omega^{-1} C_1^\dagger \Sigma^{-1} C_1\right) < 2|\det C_1| \tag{78}$$

is the necessary and sufficient condition for the nonexistence of singularities of the potential $Q(x,t)$ given by (73) for $x, t \in \mathbb{R}$. It is easy to check exponential decay as $x \to \pm\infty$ from (73) whenever $\det C_1 \neq 0$, and hence the condition (78) is clearly a necessary and sufficient condition for having a regular soliton solution in all four cases, and both when $\det C_1 = 0$ and when $\det C_1 \neq 0$.

Now (78) is clearly satisfied if the trace is negative, but it also, in principle, allows for regular solutions to exist if the trace is positive. Again, writing the above inequality explicitly for all four different choices for Σ and Ω, one finds the following.

Case 1:

$$0 < |c_1|^2 + |c_{-1}|^2 + 2|c_0|^2 < 2|c_1 c_{-1} - c_0^2| \leq 2|c_1||c_{-1}| + 2|c_0|^2 ,$$

where triangle inequality has been used. The last inequality implies

$$|c_1|^2 + |c_{-1}|^2 - 2|c_1||c_{-1}|^2 \equiv (|c_1| - |c_{-1}|)^2 < 0 ,$$

which is obviously false. So in case 1, there is no regular solution also when $\det C_1 \neq 0$, again consistent with the fact that there are no soliton solutions that are rapidly decaying as $x \to \pm\infty$ for NLS systems of defocusing type.

Case 2: Since $\operatorname{tr}(\Omega^{-1} C_1^\dagger \Sigma^{-1} C_1) < 0$, the solution is regular for any choice of the norming constant C_1 with $\det C_1 \neq 0$ as well.

Case 3: The necessary and sufficient condition for regularity reads

$$|c_1|^2 + |c_{-1}|^2 - 2|c_0|^2 < 2|c_1 c_{-1} - c_0^2| . \tag{79}$$

Importantly, the above condition is incompatible with $c_0 = 0$, so no diagonal norming constant C_1 will lead to a regular soliton solution in case 3. Equation (79) is clearly satisfied whenever the left-hand side is negative, so a sufficient condition for regular solutions in case 3 is given by

$$|c_0|^2 > \frac{1}{2}\left(|c_1|^2 + |c_{-1}|^2\right) . \tag{80}$$

When the left-hand side is nonnegative, squaring both sides the necessary and sufficient condition (79) can be shown to be equivalent to

$$\left||c_1|^2 - |c_{-1}|^2\right| < 2\left|c_0^* c_1 - c_0 c_{-1}^*\right| . \tag{81}$$

From the reverse triangle inequality, it follows that a sufficient condition for (79) to hold is that

$$|c_1|^2 + |c_{-1}|^2 - 2|c_0|^2 < 2||c_1||c_{-1}| - |c_0|^2| .$$

Now it is easy to check that if $|c_1||c_{-1}| > |c_0|^2$, the above inequality leads to a contradiction, and therefore in order for it to be satisfied, a necessary condition is that $|c_1||c_{-1}| \le |c_0|^2$. In this case, the inequality can be rewritten as

$$(|c_1| + |c_{-1}|)^2 < 4|c_0|^2 ,$$

and therefore, a sufficient condition for regularity is to have

$$|c_0| < \frac{1}{2}(|c_1| + |c_{-1}|) . \tag{82}$$

(Note that the above also implies $|c_0|^2 \ge |c_1||c_{-1}|$; also, note that (82) is obviously less stringent than (80)). We conclude that in case 3, one cannot have a regular solution if $\det C_1 = 0$, but there are regular solutions with $\det C_1 \ne 0$ if the entries of C_1 satisfy the above necessary and sufficient constraint (79), or the (simpler) sufficient (82).

Case 4: The necessary and sufficient condition for regularity reads

$$-|c_1|^2 - |c_{-1}|^2 + 2|c_0|^2 < 2\left|c_1 c_{-1} - c_0^2\right| . \tag{83}$$

Clearly, a sufficient condition in this case is

$$|c_0|^2 < \frac{1}{2}\left(|c_1|^2 + |c_{-1}|^2\right) .$$

It remains to be checked if the solution is regular or not when $2|c_0|^2 \ge |c_1|^2 + |c_{-1}|^2$. To find the class of matrices C_1 satisfying the necessary and sufficient condition (83), we observe that

$$\begin{aligned} 2|c_1 c_{-1} - c_o^2| + |c_1|^2 + |c_{-1}|^2 - 2|c_o|^2 &\ge |c_1|^2 + |c_{-1}|^2 - 2||c_0|^2 \\ - |c_1 c_{-1} - c_o^2|| &\ge |c_1|^2 + |c_{-1}|^2 - 2|c_o^2 + c_1 c_{-1} - c_o^2| \\ = \left(|c_1|^2 - |c_{-1}|^2\right) \ge 0 , \end{aligned}$$

which means that the only situation we need to exclude is when none of the above inequalities is strict. In other words, the solution in case 4 is regular for any choice of a nonsingular norming constant C_1 such that at least one of the following three conditions does not hold:

(i) $|c_1| = |c_{-1}|$,
(ii) $|c_0|^2 \geq |c_1 c_{-1} - c_0^2|$,
(iii) $||c_0|^2 - |c_1 c_{-1} - c_o^2|| = |c_1 c_{-1}|$.

$|c_1| \neq |c_{-1}|$ is clearly a sufficient condition for regularity. Note that taking (i) and (ii) into account, (iii) simply becomes (iii$'$) $|c_o|^2 - |c_1|^2 = |c_1 c_{-1} - c_0|^2$. Also, (iii$'$) necessarily implies (ii) as well as $|c_0| \geq |c_1| \equiv |c_{-1}|$. We then need to exclude nonsingular matrices of the form

$$C_1 = \begin{pmatrix} \epsilon\, e^{i\alpha} & \delta\, e^{i\theta} \\ \delta\, e^{i\theta} & \epsilon\, e^{i\beta} \end{pmatrix},$$

such that $\delta^2 - \epsilon^2 = |\delta^2 e^{2i\theta} - \epsilon^2 e^{i(\alpha+\beta)}|$. If we take into account (75), we can conclude that any matrix C_1 except for those of the form

$$C_1 = \begin{pmatrix} \epsilon\, e^{i\alpha} & \pm\delta\, e^{i(\alpha+\beta)/2} \\ \pm\delta\, e^{i(\alpha+\beta)/2} & \epsilon\, e^{i\beta} \end{pmatrix} \qquad \text{with } \delta \geq \epsilon\,, \tag{84}$$

$\delta > \epsilon$ if $\det C_1 \neq 0$] will provide a regular soliton solution for case 4.

5. Reductions of one-soliton solutions

In [41–43], it was shown that for the focusing spinor equation (corresponding here to case 2), soliton solutions can be written as a “superposition of two oppositely polarized displaced solitons” of the focusing scalar NLS equation, up to a rotation of the quantization axes. Following [18], we show below how this result can be obtained using a “spectral” method that amounts to reducing the norming constant to a diagonal form, and which can, in some instances, be generalized to the soliton solutions of the matrix equations corresponding to our cases 3 and 4.

The key property in this reduction and classification result is the invariance of the matrix NLS equation (2) under unitary transformations: if $Q(x,t)$ is a solution of Eq. (2), then $\tilde{Q}(x,t) = U\,Q(x,t)\,V$ is also a solution, for arbitrary constant unitary matrices U and V. Of course, in order for this invariance to also apply to the system where $Q(x,t)$ is assumed to be a symmetric matrix, the unitary matrices U and V must be chosen so that $\tilde{Q}(x,t)$ is also a symmetric matrix. In the spinor BEC model, such unitary transformations are associated with spin rotations, but, of course, the invariance holds for all four reductions considered in this work, as well as for the unreduced case (generic Q, R systems).

The other key ingredient is the observation that because the norming constant C_1 is a (complex, in general) symmetric matrix, Takagi's factorization [44] ensures that there exists a unitary constant matrix U such that

$$UC_1U^T = \Gamma\,, \qquad \Gamma = \mathrm{diag}(\gamma_1, \gamma_{-1})\,, \tag{85}$$

where $\gamma_j \geq 0$ and γ_j^2 are the eigenvalues of $C_1^\dagger C_1$. Obviously, $C_1^\dagger C_1$ is a Hermitian and positive semidefinite matrix, and hence eigenvalues are real and nonnegative; their explicit expressions are:

$$2\gamma_{\pm 1}^2 = |c_1|^2 + |c_{-1}|^2 + 2|c_0|^2 \pm \sqrt{\left(|c_1|^2 + |c_{-1}|^2 + 2|c_0|^2\right)^2 - 4|c_1c_{-1} - c_0^2|^2}\,. \tag{86}$$

Conversely, any norming constant can be factored as

$$C_1 = U^\dagger \Gamma U^*\,. \tag{87}$$

Substituting Eq. (87) into the one soliton solution for case 2, namely, Eq. (71) with $\Sigma = -\Omega = I_2$, one has

$$Q(x,t) = U^T \tilde{Q}(x,t) U\,, \tag{88}$$

where $\tilde{Q}(x,t)$ is a diagonal matrix given by

$$\tilde{Q}(x,t) = \mathrm{diag}\,(\tilde{q}_1(x,t), \tilde{q}_{-1}(x,t)) = -2i\,\mathrm{e}^{-2i\theta_1^*(x,t)}\left(I_2 + \tilde{c}^\dagger\tilde{c}\right)^{-1}\Gamma^\dagger\,, \tag{89}$$

with $\theta_1(x,t) = \theta(x,t,k_1)$ and $\tilde{c} = \Gamma \mathrm{e}^{2i\theta_1(x,t)}/(k_1^* - k_1)$. Note that $\tilde{Q}(x,t)$ is in the form of Eq. (71), and therefore, it is itself a one-soliton solution of the matrix NLS (1) in case 2, with the same discrete eigenvalue k_1 and a diagonal norming constant Γ. At the same time, because $\tilde{Q}(x,t)$ is diagonal, its diagonal components, i.e., $\tilde{q}_{\pm 1}(x,t)$, are decoupled, and each component satisfies the scalar focusing NLS equation:

$$iq_t + q_{xx} + 2|q|^2 q = 0\,. \tag{90}$$

Then, $\tilde{q}_j(x,t)$ with $j = \pm 1$ is a one-soliton solution of Eq. (90) with discrete eigenvalue k_1 and norming constant γ_j. Denoting the discrete eigenvalue as $k_1 = \xi + i\eta$ with $\eta > 0$, each $\tilde{q}_j$ will have the form of a one-soliton solution of the focusing NLS equation:

$$\tilde{q}_{\mathrm{sech},j}(x,t) = -2i\eta\,\mathrm{sech}[2\eta(x + 4\xi t - x_j)]\,\mathrm{e}^{-2i[\xi x + 2(\xi^2 - \eta^2)t]}\,, \tag{91}$$

where $2\eta x_j = \ln[\gamma_j/(2\eta)]$ (note that here by construction $\gamma_j > 0$).

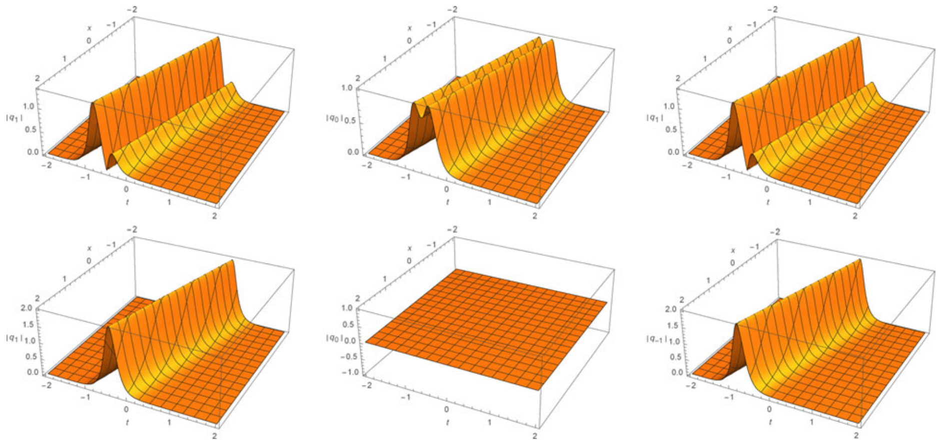

Figure 1. Three components (q_1, q_{-1}, and q_0 from left to right) for case 2: $k_1 = 1 + i$ and in the top panels, the norming constant C_1 has the diagonal entries $c_1 = 1$, $c_{-1} = 3$, and off-diagonal entries $c_0 = 2$; in the bottom panels, the unitarily equivalent diagonal solution with $\Gamma = \mathrm{diag}(2 + \sqrt{5}, 2 - \sqrt{5})$ is plotted.

We conclude that indeed any soliton solution in case 2 can be written as a "superposition of two oppositely polarized displaced solitons" of the focusing scalar NLS equation, up to a rotation of the quantization axes, which is provided by the unitary transformation (88), where U is the unitary matrix that reduces the norming constant to its diagonal form. Furthermore, from Eq. (85), we have

$$|\det C_1| = \gamma_1 \gamma_{-1}\,.$$

As a consequence, if the solution $Q(x,t)$ corresponds to a so-called ferromagnetic state, i.e., with $\det C_1 = 0$, then one of the diagonal entries γ_j must be zero, and one can assume without loos of generality that $\gamma_{-1} = 0$ and $\gamma_1 > 0$ (we can exclude $\gamma_1 = \gamma_{-1} = 0$, because in this case, Eq. (71) implies $Q(x,t) \equiv 0$). On the other hand, if the solution $Q(x,t)$ describes a polar state, i.e., $\det C_1 \neq 0$, then $\gamma_1 > 0$ and $\gamma_{-1} > 0$. Figure 1 shows the amplitudes of the three components of the solution in case 2, for a nondiagonal the norming constant, before (top panels) and after (bottom panels) the reduction to a unitarily equivalent diagonal solution.

For cases 3 and 4, because of the presence of σ_3 in (71) via Σ and Ω, the reduction to a unitarily equivalent diagonal solution is more complicated, but it is still possible in some circumstances. First of all, note that if $\det C_1 = 0$ (which is only applicable to case 4, because we showed in case 3 no regular/smooth soliton solution exists if $\det C_1 = 0$), then the solution reduces to one single sech as given in (77).

When $\det C_1 \neq 0$, to reduce the solution to a unitarily equivalent diagonal one, we have to: either (i) simultaneously unitarily diagonalize both matrices

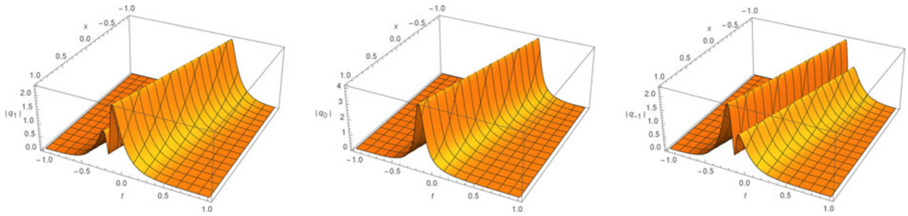

Figure 2. Three components (q_1, q_{-1}, and q_0 from left to right) for case 3: $k_1 = 1 + i$ and the norming constant C_1 has diagonal entries $c_1 = 4 + i$, $c_{-1} = 1 + 3i$ and the off-diagonal entries $c_0 = 1 - 2i$.

C_1 and $\tilde{C}_1 = \pm\sigma_3 C_1 \sigma_3$ in (71); or (ii) reduce both matrices to a diagonal form by means of the same U, U^T that realize the Takagi factorization. Note that (ii) is in principle possible because both matrices are symmetric, but one needs to find conditions under which the same U yields the corresponding Takagi's factorizations.

It is important to point out that whenever C_1 and $\tilde{C}_1$ are simultaneously diagonalizable (either by a unitary similarity transformation, or by a simultaneous Takagi factorization), the necessary and sufficient condition for regularity, namely, Eq. (78), is invariant under the corresponding unitary transformation. As already mentioned in Section 4, in case 3, the necessary and sufficient condition for regularity requires $c_0 \neq 0$ (cf. Eq. (79)). As a consequence, regular soliton solutions in case 3 cannot be reduced to a unitarily equivalent diagonal form. This is consistent with the fact if $Q(x,t)$ is diagonal, in case 3, the system (2) reduces to two decoupled scalar defocusing NLS equations, which do not admit soliton solutions. Figure 2 shows the amplitudes of a one-soliton solution in case 3.

In the remainder of this section, we will therefore investigate the reduction of polar ($\det C_1 \neq 0$) soliton solutions in case 4. It is straightforward to see that C_1 and $\tilde{C}_1 = \sigma_3 C_1 \sigma_3$ have the same eigenvalues. Moreover, they are both normal matrices if and only if their entries satisfy the constraint:

$$c_0^*(c_1 - c_{-1}) = c_o\left(c_1^* - c_{-1}^*\right) .$$

Finally, C_1 and $\tilde{C}_1$ are simultaneously unitarily diagonalizable if and only if they are normal matrices and they commute, and it can be easily verified that a necessary and sufficient condition for this to happen is that either $c_0 = 0$ (which is trivial, because it means that both matrices are already in diagonal form), or $c_1 = c_{-1}$. In this latter case, the eigenvalues of C_1 and $\tilde{C}_1$ are $c_1 \pm c_0$, but the corresponding eigenvectors are switched; in other words, one has

$$C_1 = U^\dagger \Gamma\, U\,, \qquad \tilde{C}_1^\dagger = U^\dagger \tilde{\Gamma}^\dagger U\,, \tag{92}$$

with diagonal matrices $\Gamma = \mathrm{diag}(c_1 + c_0, c_1 - c_0)$ and $\tilde{\Gamma} = \sigma_1 \Gamma \sigma_1 \equiv \mathrm{diag}(c_1 - c_0, c_1 + c_0)$.

If the norming constant has $c_1 = c_{-1}$ and $\det C_1 \neq 0$, then $c_1^2 \neq c_0^2$, i.e., $c_1 \neq \pm c_0$. Moreover, the necessary and sufficient condition for regularity in this case requires $|c_0| < |c_1|$ (cf. (83), or equivalently, (84)).

The corresponding solution in (71) is unitarily equivalent to a diagonal solution like (89), with two shifted sech-like solitons of the form (91) in each of the diagonal components. One important difference with respect to case 2, however, is that x_j in this case is not necessarily real, because the norming constants of the equivalent scalar equation in this case are given by the entries of $\tilde{\Gamma}^\dagger \Gamma$, i.e., $\gamma_{\pm 1}\tilde{\gamma}_{\pm 1}^* = |c_1|^2 - |c_0|^2 \pm 2i\,\mathrm{Im}(c_0 c_1^*)$, and $2\eta x_{\pm 1} = \ln[\sqrt{\gamma_{\pm 1}\tilde{\gamma}_{\pm 1}^*}/(2\eta)]$. Explicitly, one has $Q(x,t) = U^\dagger \tilde{Q}(x,t)\, U$ with $\tilde{Q}(x,t) = \mathrm{diag}(\tilde{q}_{\mathrm{sech},1}(x,t), \tilde{q}_{\mathrm{sech},-1}(x,t))$ and

$$\tilde{q}_{\mathrm{sech},\pm 1}(x,t) = -2i\eta\sqrt{\tilde{\gamma}_{\pm 1}^*/\gamma_{\pm 1}}\,\mathrm{sech}[2\eta(x + 4\xi t - x_{\pm 1})]\,\mathrm{e}^{-2i[\xi x + 2(\xi^2-\eta^2)t]}$$

$$\equiv -2i\eta\,\mathrm{e}^{-2i[\xi x + 2(\xi^2-\eta^2)t]}\frac{\sqrt{\tilde{\gamma}_{\pm 1}^*/\gamma_{\pm 1}}}{\cosh[2\eta(x+4\xi t - x_0)]\cos\chi \mp i\sinh[2\eta(x+4\xi t - x_0)]\sin\chi},$$

with $2\eta x_0 = \ln[\sqrt{|\gamma_{\pm 1}\tilde{\gamma}_{\pm 1}^*|}/(2\eta)]$ and

$$\chi = \begin{cases} \dfrac{1}{2}\tan^{-1}\dfrac{2\,\mathrm{Im}\left(c_0 c_1^*\right)}{|c_1|^2 - |c_0|^2} & \text{if } \mathrm{Im}\left(c_0 c_1^*\right) \geq 0\,, \\ -\dfrac{1}{2}\tan^{-1}\dfrac{2\,\mathrm{Im}\left(c_0 c_1^*\right)}{|c_1|^2 - |c_0|^2} & \text{if } \mathrm{Im}(c_0 c_1^*) \leq 0\,. \end{cases} \tag{93}$$

Here, we have taken into account that the regularity condition requires $|c_1|^2 - |c_0|^2 > 0$.] It is also worth noticing that

$$|\tilde{q}_{\mathrm{sech},\pm 1}(x,t)|^2 = \frac{4\eta^2\left|c_1^2 - c_0^2\right|/|c_1 \pm c_0|^2}{\cosh^2[2\eta(x+4\xi t - x_0)] - \sin^2\chi},$$

and that the solution is always regular ($\sin\chi \neq \pm 1$) whenever the norming constant C_1 satisfies the regularity condition (83), i.e., if $|c_1| > |c_0|$.

Figure 3 shows the amplitudes of the three components of the solution in case 4, when for a nondiagonal norming constant in the normal case, before (top panels) and after (bottom panels) the reduction to a unitarily equivalent diagonal solution.

When C_1 and $\tilde{C}_1$ are not simultaneously unitarily diagonalizable (i.e., if $c_0(c_1 - c_{-1}) \neq 0$), the only option left for reduction to a unitarily equivalent two component solution is for C_1 and $\tilde{C}_1$ to be simultaneously Takagi diagonalizable, i.e., to have:

$$C_1 = U^\dagger \Gamma U^*\,, \qquad \tilde{C}_1 = U^\dagger \tilde{\Gamma}\, U^*$$

with the same unitary matrix U and diagonal matrices $\Gamma = \mathrm{diag}(\gamma_1, \gamma_{-1})$ and $\tilde{\Gamma} = \mathrm{diag}(\tilde{\gamma}_1, \tilde{\gamma}_{-1})$. Note that $\gamma_{\pm 1}$ and $\tilde{\gamma}_{\pm 1}$ are real, as they are by construction the square roots of the real and positive eigenvalues of $C_1^\dagger C_1$

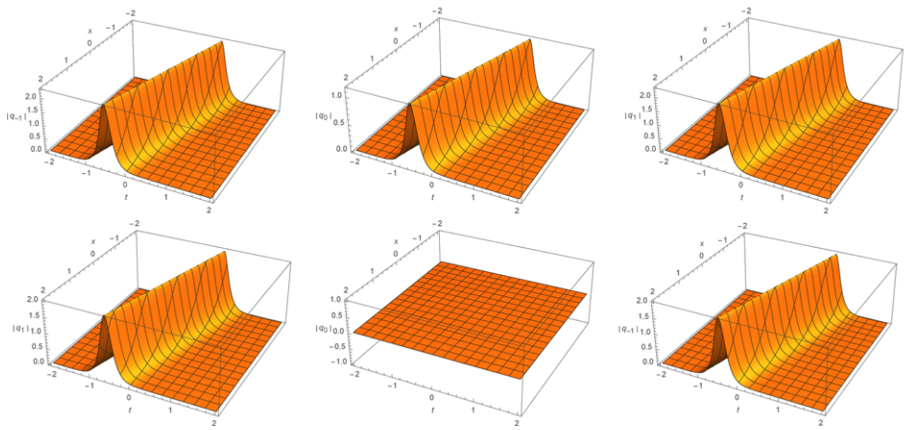

Figure 3. Three components (q_1, q_{-1}, and q_0 from left to right) for case 4: $k_1 = 1 + i$ and in the top panels, the norming constant C_1 has the diagonal entries $c_1 = c_{-1} = 2$, and off-diagonal entries $c_0 = 1$; in the bottom panels, the unitarily equivalent diagonal solution with $\Gamma = \mathrm{diag}(1, 3)$ is plotted.

and $\tilde{C}_1^\dagger \tilde{C}_1$, respectively. In our case, the two matrices $C_1^\dagger C_1$ and $\tilde{C}_1^\dagger \tilde{C}_1$ have the same eigenvalues, which implies that the entries on Γ and $\tilde{\Gamma}$ in the Takagi algorithm can only differ by either the signs or the ordering, or both.

To the best of our knowledge, there is no simple necessary and sufficient condition to guarantee that two symmetric matrices are simultaneously Takagi diagonalizable. Note, however, that if C_1 and $\tilde{C}_1$ are simultaneously Takagi diagonalizable, the solution can be reduced to $Q(x,t) = U^T \tilde{Q}(x,t)\,U$, where $\tilde{Q}(x,t) = \mathrm{diag}(\tilde{q}_1(x,t), \tilde{q}_{-1}(x,t))$ and for $j = \pm 1$

$$\tilde{q}_j(x,t) = -2i\eta\sqrt{\tilde{\gamma}_j/\gamma_j}\,\mathrm{sech}[2\eta(x + 4\xi t - x_j)]\,\mathrm{e}^{-2i[\xi x + 2(\xi^2 - \eta^2)t]}, \quad (94)$$

with $2\eta x_j = \ln[\sqrt{\tilde{\gamma}_j\gamma_j}/(2\eta)]$. As mentioned above, Takagi's algorithm requires $\tilde{\gamma}_j\gamma_j \in \mathbb{R}$ for both $j = \pm 1$, although, in principle, the products can be positive or negative. This implies

$$\ln[\sqrt{\tilde{\gamma}_j\gamma_j}/(2\eta)] = \ln[\sqrt{|\tilde{\gamma}_j\gamma_j|}/(2\eta)] + \frac{i}{2}\mathrm{Arg}\,(\tilde{\gamma}_j\gamma_j),$$

and the argument is either 0 (if $\tilde{\gamma}_j\gamma_j > 0$) or π (if $\tilde{\gamma}_j\gamma_j < 0$). (Note that we are assuming $\det C_1 \neq 0$, so $\gamma_j\tilde{\gamma}_j \neq 0$ for both $j = \pm 1$). The solutions can then be written as

$$\tilde{q}_j(x,t) = -2i\eta\frac{\sqrt{\tilde{\gamma}_j/\gamma_j}}{\cosh[2\eta(x + 4\xi t - \tilde{x}_j)]\cos\chi - i\sinh[2\eta(x + 4\xi t - \tilde{x}_j)]\sin\chi},$$

where $2\eta\tilde{x}_j = \ln[\sqrt{|\tilde{\gamma}_j\gamma_j|}/(2\eta)]$ and $\chi = 0$ if $\tilde{\gamma}_j\gamma_j > 0$, and $\chi = \pi/2$ if $\tilde{\gamma}_j\gamma_j < 0$.

As a consequence, if $\gamma_j\tilde{\gamma}_j < 0$ for either $j = 1$ or $j = -1$, the corresponding solution is singular (because the denominator vanishes for all x, t such that $x + 2\xi t - \tilde{x}_j = 0$), which implies that for regular one-soliton solutions the entries of Γ and $\tilde{\Gamma}$ must have the same sign. In this case, the solutions reduce to

$$\tilde{q}_j(x,t) = -2i\eta \frac{\sqrt{\tilde{\gamma}_j/\gamma_j}}{\cosh[2\eta(x + 4\xi t - \tilde{x}_j)]} .$$

It is easy to check that in order for the above to be a solution of the scalar NLS equations $\tilde{\gamma}_j/\gamma_j = \pm 1$ for both $j = 1$ and $j = -1$, which then requires $\tilde{\gamma}_j = \gamma_j$ for $j = \pm 1$. Therefore, for regular one-soliton solutions in case 4, C_1 and $\tilde{C}_1$ are simultaneously Takagi diagonalizable if and only if $\tilde{\Gamma}_1 = \Gamma_1$, i.e., if and only if $\tilde{C}_1 = C_1$. However, this is trivial, because it corresponds to either $c_0 = 0$ (in which case the solution is already in diagonal form), or to $c_1 = c_{-1} = 0$ (in which case $\tilde{C}_1 = -C_1$ and they are both normal and commuting, so that can be simultaneously unitarily diagonalized). We then conclude that in case 4, the one-soliton solutions corresponding to norming constants for which $c_1 \neq c_{-1}$ cannot in general be reduced to a unitarily equivalent diagonal form.

6. Concluding remarks

In this work, we have developed the IST for a class of matrix NLS equations whose reductions include two equations that have been proposed as a model to describe hyperfine spin $F = 1$ spinor BECs, and two novel equations that were recently shown to be integrable, and that have applications in nonlinear optics and four-component fermionic condensates.

Matrix NLS systems of the form:

$$iQ_t + Q_{xx} + 2Q\,R\,Q = 0\,, \qquad iR_t - R_{xx} + 2R\,Q\,R = 0\,,$$

where $Q(x,t)$ is an $n \times m$ matrix and $R(x,t)$ is an $n \times m$ matrix, have been studied for over 40 years. Although the IST for the "unreduced" matrix NLS systems (Q, R-systems where $Q(x,t)$ and $R(x,t)$ are independent fields), and for the "canonical" reductions corresponding to cases 1 and 2 in this work ($R(x,t) = \pm Q^\dagger(x,t)$, corresponding to focusing and defocusing matrix NLS, respectively) are well established, both with zero and nonzero boundary conditions, this work presents several advances for those systems as well. Specifically: (i) we have provided a rigorous definition of the norming constants that does not require any unjustified analytic extension of the scattering relations, clarified the role that the rank of the norming constants plays and provided a clear spectral characterization of the corresponding solutions; (ii) we have properly accounted for all the symmetries in the potential matrix, and obtained the corresponding symmetries in the scattering data (reflection coefficients and norming constants). The IST and the soliton solutions corresponding to the reductions in cases 3 and 4 ($R(x,t) = \pm\sigma_3 Q(x,t)\sigma_3$) are novel, and present some interesting aspects and additional challenges with respect to the other two cases in that one needs to impose suitable constraints on the norming constants to guarantee that the soliton solutions are smooth for all $(x,t) \in \mathbb{R}$.

We have also studied the one-soliton solutions and showed that: (i) in case 2, all solutions are always unitarily equivalent to a superposition of two oppositely polarized displaced solitons of the focusing scalar NLS equation, up to a rotation of the quantization axes that is provided by the unitary transformation; (ii) in case 3, all regular solutions are irreducible, in the sense that they are not unitarily equivalent to solutions in diagonal form; (iii) in case 4, some solutions (the ones obtained when the associated norming constant is a normal symmetric matrix with the same diagonal entries) are reducible (unitarily equivalent to a diagonal solution), while others are irreducible. We note that even in the reducible case, the solutions are slightly different from the ones in case 2. Further analysis of the soliton solutions, to include double pole solitons and multisoliton interactions, is left for future investigation.

As far as the applications are concerned, it is worth investigating the equations corresponding to cases 3 and 4 in the symmetric case in the framework of multicolor optical spatiotemporal solitary waves created by interaction of light at a central frequency with two sideband waves both through cross-phase modulation and parametric four-wave mixing of opposite signs. On the other hand, the four-component spinor system could have applications to the recently discovered phenomenon of superconductivity in bilayer graphene [45]. The mechanism of superconductivity in this context is yet to be fully understood, and it could be originating from bound states (singlet/triplet) of four-component solitons.

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Appendix A: Resolvent operator and spectrum

In this section, we prove that the discrete eigenvalues $k \in \mathbb{C}^+$ coincide with the poles of $a^{-1}(k)$ in $\mathbb{C}^+$, and those in $\mathbb{C}^-$ with the poles of $\bar{a}^{-1}(k)$ in

$\mathbb{C}^-$. This requires computing the resolvent operator of the AKNS differential operator. As an ancillary result, the domain of the AKNS differential operator will coincide with the range of the resolvent operator.

Given $F \in L^2(\mathbb{R})^{4\times 1}$, let us find $u \in L^2(\mathbb{R})^{4\times 1}$ such that

$$ku - i\boldsymbol{\sigma}_3(I_4\partial_x - \boldsymbol{Q})u = F.$$

When such u cannot be found for each such F in a unique way, then k is said to belong to the *spectrum* of the linear operator $i\boldsymbol{\sigma}_3(I_4\partial_x - \boldsymbol{Q})$. We can write the preceding equation in the form

$$\partial_x u = (-ik\boldsymbol{\sigma}_3 + \boldsymbol{Q})u + i\boldsymbol{\sigma}_3 F\,, \tag{A1}$$

where, for each $t \in \mathbb{R}$, the entries of $\boldsymbol{Q}(\cdot, t)$ belong to $L^1(\mathbb{R})$. We assume that $F(x)$ does not depend on either $t \in \mathbb{R}$ or $k \in \mathbb{C}$. For $k \in \mathbb{C}^+$, we write

$$u(x, t, k) = P(x, t, k)v(x, t, k)\,,$$

where $P(x, t, k)$ is defined in terms of the Jost eigenfunctions analytic in $\mathbb{C}^+$ via (47). Then, $P_x = (-ik\boldsymbol{\sigma}_3 + \boldsymbol{Q}(x))P$ implies that

$$v_x = iP^{-1}\boldsymbol{\sigma}_3 F\,. \tag{A2}$$

Since from (48)

$$P^{-1}(x, t, k) = A^{-1}(k)\bar{P}^\dagger(x, t, k^*)\,\Xi^{-1}$$

for any $k \in \mathbb{C}^+$ such that $\det a(k) \neq 0$, we obtain

$$v^{\mathrm{up}}(x, t, k) = -i\int_x^\infty dy\, a^{-1}(k)\Omega^{-1}\,\bar{\psi}^\dagger(y, t, k^*)\,\Xi^{-1}\boldsymbol{\sigma}_3 F(y)\,,$$

$$v^{\mathrm{dn}}(x, t, k) = -i\int_{-\infty}^x dy\, c^{-1}(k)\Sigma\,\bar{\phi}^\dagger(y, t, k^*)\,\Xi^{-1}\boldsymbol{\sigma}_3 F(y)\,,$$

where we have the integrability of $\bar{\psi}^\dagger(y, t, k^*)$ for $y \geq x$, and of $\bar{\phi}^\dagger(y, t, k^*)$ for $y \leq x$. Consequently, for any $k \in \mathbb{C}^+$ such that if $\det a(k) = \det c(k) \neq 0$, then

$$u(x, t, k) = \int_{-\infty}^\infty dy\, \mathcal{G}(x, y; t, k)F(y), \tag{A3}$$

where

$$\mathcal{G}(x, y; t, k) = \begin{cases} -i\phi(x, t, k)\,a^{-1}(k)\Omega^{-1}\bar{\psi}^\dagger(y, t, k^*)\,\Xi^{-1}\boldsymbol{\sigma}_3, & y > x\,, \\ -i\psi(x, t, k)\,c^{-1}(k)\Sigma\bar{\phi}^\dagger(y, t, k^*)\,\Xi^{-1}\boldsymbol{\sigma}_3, & y < x\,. \end{cases} \tag{A4}$$

In the same way, we prove that for any $k \in \mathbb{C}^-$ such that $\det \bar{a}(k) = \det \bar{c}(k) \neq 0$, then (A3) holds with

$$\mathcal{G}(x, y; t, k) = \begin{cases} i\,\bar{\psi}(x, t, k)\,\bar{c}^{-1}(k)\Omega^{-1}\phi^\dagger(x, t, k^*)\,\Xi^{-1}\boldsymbol{\sigma}_3, & y < x\,, \\ i\,\bar{\phi}(x, t, k)\,\bar{a}^{-1}(k)\,\Sigma\psi^\dagger(x, t, k^*)\,\Xi^{-1}\boldsymbol{\sigma}_3, & y > x\,. \end{cases} \tag{A5}$$

Using (48), (A4), and (A5), we get

$$\mathcal{G}(x,x^+;t,k)-\mathcal{G}(x,x^-;t,k)=\begin{cases} i\boldsymbol{\sigma}_3 & k\in\mathbb{C}^+,\\ -i\boldsymbol{\sigma}_3 & k\in\mathbb{C}^-.\end{cases} \tag{A6}$$

Thus, the Green function $\mathcal{G}(x,y;t,k)$ has a jump discontinuity on the diagonal $y=x$. Equations (A3) and (A4) imply, for $k\in\mathbb{R}$, the existence of $F\in L^2(\mathbb{R})^{4\times 1}$ such that the integral (A3) is not an L^2 vector function of $x\in\mathbb{R}$. In fact, the integral might not even exist. Thus, the spectrum contains the real k axis plus the zeros of $a(k)$ in $\mathbb{C}^+$ and those of $\bar{a}(k)$ in $\mathbb{C}^-$. (The discussion after Eqs. (52) actually shows that all of these zeros belong to the spectrum.)

The spectral projection of the AKNS operator $i\boldsymbol{\sigma}_3(I_4\partial_x-\boldsymbol{Q})$ at the eigenvalue k_n is an integral operator whose integral kernel $\Pi_n(x,y;t)$ is the residue of $\mathcal{G}(x,y;t,k)$ at $k=k_n$. If k_n is a simple pole in $\mathbb{C}^+$, we have

$$\Pi_n(x,y;t)=\begin{cases} -i\phi(x,t,k_n)\tau_n\Omega^{-1}\bar{\psi}^\dagger(y,t,k_n^*)\Xi^{-1}\boldsymbol{\sigma}_3, & y>x,\\ -i\psi(x,t,k_n)\check{\tau}_n\Sigma\,\bar{\phi}^\dagger(y,t,k_n^*)\Xi^{-1}\boldsymbol{\sigma}_3, & y<x,\end{cases}$$

where τ_n and $\check{\tau}_n$ are the residues of $a^{-1}(k)$ and $c^{-1}(k)$ at $k=k_n$. If k_m is a double pole in $\mathbb{C}^+$, we have

$$\Pi_m(x,y;t)=\begin{cases} -i\displaystyle\sum_{j=1}^{2}\sum_{l=0}^{j-1}\phi_{m,l}(x,t)\tau_{m,j}\Omega^{-1}\bar{\psi}^\dagger_{n,j-l-1}(y,t)\Xi^{-1}\boldsymbol{\sigma}_3, & y>x,\\ -i\displaystyle\sum_{j=1}^{2}\sum_{l=0}^{j-1}\psi_{m,l}(x,t)\check{\tau}_{m,j}\Sigma\,\bar{\phi}^\dagger_{n,j-l-1}(y,t)\Xi^{-1}\boldsymbol{\sigma}_3, & y<x,\end{cases}$$

where

$$\phi(x,t,k)=\phi_{m,0}(x,t)+(k-k_m)\phi_{m,1}(x,t)+O((k-k_m)^2),$$
$$\psi(x,t,k)=\psi_{m,0}(x,t)+(k-k_m)\psi_{m,1}(x,t)+O((k-k_m)^2).$$

Here $(k-k_m)^{-1}\tau_{m,1}+(k-k_m)^{-1}\tau_{m,2}$ and $(k-k_m)^{-1}\check{\tau}_{m,1}+(k-k_m)^{-1}\check{\tau}_{m,2}$ are the principal parts of $a^{-1}(k)$ and $c^{-1}(k)$ at $k=k_m$.

Similar expressions hold for higher order poles and for eigenvalues in $\mathbb{C}^-$.

Appendix B: Double pole Riemann–Hilbert problem

In this appendix, we generalize the formulation of the inverse problem as an RHP in Section 4 to the case where the matrix of meromorphic eigenfunctions has double poles.

Let τ_n and $\bar{\tau}_n$ denote the residues of $a^{-1}(k)$ and $\bar{a}^{-1}(k)$ at the simple poles $k_n\in\mathbb{C}^+$ and $k_n^*\in\mathbb{C}^-$. Then,

$$\phi(x,t,k_n)\tau_n=\psi(x,t,k_n)C_n, \tag{B1a}$$

$$\bar{\phi}\left(x,t,k_n^*\right)\bar{\tau}_n = \bar{\psi}\left(x,t,k_n^*\right)\bar{C}_n, \tag{B1b}$$

where C_n and $\bar{C}_n$ are called norming constants.

Equation (B1a) can be phrased in the following way: There exists a unique 2×2 matrix function $C(k)$ meromorphic in a neighborhood of $k=k_n$, and a unique 2×2 matrix function $\bar{C}(k)$ meromorphic in a neighborhood of $k=k_n^*$ such that

$$\phi(x,t,k)a^{-1}(k) = \psi(x,t,k)C(k) + O(1), \quad k\to k_n, \tag{B2a}$$

$$\bar{\phi}(x,t,k)\bar{a}^{-1}(k) = \bar{\psi}(x,t,k)\bar{C}(k) + O(1), \quad k\to k_n^*. \tag{B2b}$$

Then, $C(k)$ has a simple pole at $k=k_n$ and $\bar{C}(k)$ has a simple pole in $k=k_n^*$ and their residues coincide with the norming constants C_n and $\bar{C}_n$.

The above results can be generalized to the case in which $a(k)$ and $\bar{a}^{-1}(k)$ have a double pole:

$$a^{-1}(k) = \frac{\tau_{n,1}}{k-k_n} + \frac{\tau_{n,2}}{(k-k_n)^2} + O(1), \quad k\to k_n,$$
$$\bar{a}^{-1}(k) = \frac{\bar{\tau}_{n,1}}{k-k_n^*} + \frac{\bar{\tau}_{n,2}}{\left(k-k_n^*\right)^2} + O(1), \quad k\to k_n^*,$$

with $\tau_{n,2}, \bar{\tau}_{n,2} \neq 0_{2\times 2}$. We now consider (B2), where

$$C(k) = \frac{C_{n,1}}{k-k_n} + \frac{C_{n,2}}{(k-k_n)^2} + O(1), \quad k\to k_n,$$
$$\bar{C}(k) = \frac{\bar{C}_{n,1}}{k-k_n^*} + \frac{\bar{C}_{n,2}}{\left(k-k_n^*\right)^2} + O(1), \quad k\to k_n^*,$$

with two sets of norming constants, $C_{n,1}, C_{n,2}$ and $\bar{C}_{n,1}, \bar{C}_{n,2}$ for each of the discrete eigenvalues k_n and k_n^*.

Let us now write down the Taylor series expansions

$$\phi(x,t,k) = \sum_{r=0}^{\infty}(k-k_n)^r\phi_{n,r}(x,t), \qquad \psi(x,t,k) = \sum_{r=0}^{\infty}(k-k_n)^r\psi_{n,r}(x,t),$$
$$\bar{\phi}(x,t,k) = \sum_{r=0}^{\infty}\left(k-k_n^*\right)^r\bar{\phi}_{n,r}(x,t), \qquad \bar{\psi}(x,t,k) = \sum_{r=0}^{\infty},\left(k-k_n^*\right)^r\bar{\psi}_{n,r}(x,t),$$

valid if $|k-k_n| < \operatorname{Im} k_n$ and if $|k-k_n^*| < \operatorname{Im} k_n$, respectively. Similarly,

$$M(x,t,k)=\sum_{r=0}^{\infty}(k-k_n)^r M_{n,r}(x,t), \qquad N(x,t,k)=\sum_{r=0}^{\infty}(k-k_n)^r N_{n,r}(x,t),$$

$$\bar{M}(x,t,k)=\sum_{r=0}^{\infty}\left(k-k_n^*\right)^r \bar{M}_{n,r}(x,t), \qquad \bar{N}(x,t,k)=\sum_{r=0}^{\infty}\left(k-k_n^*\right)^r \bar{N}_{n,r}(x,t),$$

valid if $|k-k_n|<\operatorname{Im}k_n$ and if $|k-k_n^*|<\operatorname{Im}k_n$, respectively. We actually only need their coefficients for $r=0,1$.

Now consider the jump conditions (18), written as

$$\bar{N}(x,t,k)=M(x,t,k)a^{-1}(k)-e^{2i\theta(x,t,k)}N(x,t,k)\rho(k)\,, \tag{B3a}$$

$$N(x,t,k)=\bar{M}(x,t,k)\bar{a}^{-1}(k)-e^{-2i\theta(x,t,k)}\bar{N}(x,t,k)\bar{\rho}(k)\,. \tag{B3b}$$

Assuming that, for each $(x,t)\in\mathbb{R}^2$, each term belongs to

$$E^{4\times 1}=E_+^{4\times 1}\oplus E_-\oplus\mathbb{C}^4,$$

where E is a suitable complex Banach space of functions of $k\in\mathbb{R}$ vanishing as $k\to\pm\infty$ and $E_\pm$ are those functions in E that are analytic in $k\in\mathbb{C}^\pm$, we can define the (bounded) projection $\Pi_\pm$ of $E_0=E_+\oplus E_-$ onto $E_\pm$ by the Plemelj formulas

$$(\Pi_\pm f)(k)=\frac{\pm 1}{2\pi i}\int_{-\infty}^{\infty}d\xi\,\frac{f(\xi)}{\xi-(k\pm i0^+)}\,. \tag{B4}$$

As E we can take the constants plus the Fourier transforms of functions in $L^1(\mathbb{R})$, the so-called Wiener algebra. If the potential $Q(\cdot,t)$ has only L^1 entries and there are no spectral singularities, we are always in this situation. Note that the Schwarz reflection principle also implies

$$(\Pi_\pm f^*)(k)=\left[(\Pi_\mp f)(k^*)\right]^*\,. \tag{B5}$$

The coupled singular integral equations for the inverse problem are obtained by applying Π_- to (B3a) and Π_+ to (B3b). If $a^{-1}(k)$ has only simple poles, we arrive at (66), where the definitions of the norming constants are used to replace the residues by norming constants and to get rid of the functions $M(x,t,k)$ and $\bar{M}(x,t,k)$ in favor of the functions $N(x,t,k)$ and $\bar{N}(x,t,k)$. In fact, using that $Ma^{-1}=e^{i\theta}\phi a^{-1}=e^{i\theta}\psi C=e^{2i\theta}NC$ and $\bar{M}\bar{a}^{-1}=e^{-i\theta}\bar{\phi}\bar{a}^{-1}=e^{-i\theta}\bar{\psi}\bar{C}=e^{-2i\theta}\bar{N}\,\bar{C}$, we obtain

$$\begin{aligned}&\Pi_-\left[\bar{N}(x,t,k)+e^{2i\theta(x,t,k)}N(x,t,k)\rho(k)\right]\\&=\Pi_-\left[e^{2i\theta(x,t,k)}N(x,t,k)C(k)+O(1)\right]=\Pi_-\left[e^{2i\theta(x,t,k)}N(x,t,k)C(k)\right],\end{aligned} \tag{B6a}$$

$$\begin{aligned}&\Pi_+\left[N(x,t,k)+e^{-2i\theta(x,t,k)}\bar{N}(x,t,k)\bar{\rho}(k)\right]\\&=\Pi_+\left[e^{-2i\theta(x,t,k)}\bar{N}(x,t,k)\bar{C}(k)+O(1)\right]=\Pi_+\left[e^{-2i\theta(x,t,k)}\bar{N}(x,t,k)\bar{C}(k)\right].\end{aligned} \tag{B6b}$$

For a finite number of simple and double poles, one has

$$C(k)=\sum_{n=1}^{\mathcal{N}}\frac{C_n}{k-k_n}+\sum_{m=1}^{\mathcal{M}}\left[\frac{C_{m,1}}{k-k_m}+\frac{C_{m,2}}{(k-k_m)^2}\right]+\Gamma(k), \tag{B7a}$$

where $\Gamma(k)$ is continuous in $k\in\mathbb{C}^+\cup\mathbb{R}$, analytic in $k\in\mathbb{C}^+$, and has a limit as $k\to\infty$ from within $\mathbb{C}^+\cup\mathbb{R}$, and

$$\bar{C}(k)=\sum_{n=1}^{\mathcal{N}}\frac{\bar{C}_n}{k-k_n^*}+\sum_{m=1}^{\mathcal{M}}\left[\frac{\bar{C}_{m,1}}{k-k_m^*}+\frac{\bar{C}_{m,2}}{\left(k-k_m^*\right)^2}\right]+\bar{\Gamma}(k), \tag{B7b}$$

where $\bar{\Gamma}(k)$ is continuous in $k\in\mathbb{C}^-\cup\mathbb{R}$, analytic in $k\in\mathbb{C}^-$, and has a limit as $k\to\infty$ from within $\mathbb{C}^-\cup\mathbb{R}$.

Using

$$e^{2i\theta(x+2kt)}=e^{2i\theta(x,t,k_n)}\left(1+2i(k-k_n)[x+4k_nt]+O((k-k_n)^2)\right),$$

$$e^{-2i\theta(x+2kt)}=e^{-2i\theta\left(x,t,k_n^*\right)}\left(1-2i\left(k-k_n^*\right)\left[x+4k_n^*t\right]+O\left(\left(k-k_n^*\right)^2\right)\right),$$

and denoting $N_{m,0}(x,t)=N(x,t,k_m)$ and $N_{m,1}(x,t)=(\partial N/\partial k)(x,t,k_m)$, we compute

$$\begin{aligned}
e^{2i\theta(x,t,k)}N(x,t,k)C(k)&=\sum_{n=1}^{\mathcal{N}}\left[\frac{e^{2i\theta(x,t,k)}N(x,t,k)-e^{2i\theta(x,t,k_n)}N_{n,0}(x,t)}{k-k_n}C_n\right.\\
&+\left.\frac{e^{2i\theta(x,t,k_n)}N_{n,0}(x,t)}{k-k_n}C_n\right]+\sum_{m=1}^{\mathcal{M}}\left[\frac{e^{2i\theta(x,t,k)}N(x,t,k)-e^{2i\theta(x,t,k_m)}N_{m,0}(x,t)}{k-k_n}C_{m,1}\right.\\
&+\frac{e^{2i\theta(x,t,k)}N(x,t,k)-e^{2i\theta(x,t,k_m)}N_{m,0}(x,t)-(k-k_m)\frac{\partial}{\partial k}\left[e^{2i\theta(x,t,k)}N(x,t,k)\right]_{k=k_m}}{(k-k_m)^2}\\
&\times C_{m,2}+\frac{e^{2i\theta(x,t,k_m)}N_{m,0}(x,t)C_{m,1}}{k-k_m}+\frac{e^{2i\theta(x,t,k_m)}N_{m,1}(x,t)C_{m,2}}{(k-k_m)^2}\\
&+\left.\frac{2i(x+4k_mt)e^{2i\theta(x,t,k_m)}N_{m,0}(x,t)C_{m,2}}{k-k_m}\right]+e^{2i\theta(x,t,k)}N(x,t,k)\Gamma(k).
\end{aligned}$$

Applying Π_- to the above expression, we get

$$\begin{aligned}
\Pi_-\left[e^{2i\theta(x,t,k)}N(x,t,k)C(k)\right]&=\sum_{n=1}^{\mathcal{N}}\frac{e^{2i\theta(x,t,k_n)}N_{n,0}(x,t)}{k-k_n}C_n\\
&+\sum_{m=1}^{\mathcal{M}}\left[\frac{e^{2i\theta(x,t,k_m)}N_{m,0}(x,t)C_{m,1}}{k-k_m}+\frac{e^{2i\theta(x,t,k_m)}N_{m,1}(x,t)C_{m,2}}{(k-k_m)^2}\right.\\
&+\left.\frac{2i(x+4k_mt)e^{2i\theta(x,t,k_m)}N_{m,0}(x,t)C_{m,2}}{k-k_m}\right].
\end{aligned}$$

In the same way, we get

$$\Pi_+\left[e^{-2i\theta(x,t,k)}\bar{N}(x,t,k)\bar{C}(k)\right] = \sum_{n=1}^{\mathcal{N}} \frac{e^{-2i\theta(x,t,k_n)}\bar{N}_{n,0}(x,t)}{k-k_n^*}\bar{C}_n$$

$$+\sum_{m=1}^{\mathcal{M}}\left[\frac{e^{-2i\theta(x,t,k_m)}\bar{N}_{m,0}(x,t)\bar{C}_{m,1}}{k-k_m^*}\right.$$

$$+\frac{e^{-2i\theta(x,t,k_m)}\bar{N}_{m,1}(x,t)\bar{C}_{m,2}}{(k-k_m^*)^2}$$

$$\left.-\frac{2i(x+4k_m^*t)e^{-2i\theta(x,t,k_m)}\bar{N}_{m,0}(x,t)\bar{C}_{m,2}}{k-k_m^*}\right].$$

Using that

$$\Pi_-\left[\bar{N}(x,t,k)\right] = \bar{N}(x,t,k) - \begin{pmatrix} I_2 \\ 0_{2\times 2}\end{pmatrix}, \quad \Pi_+[N(x,t,k)] = N(x,t,k) - \begin{pmatrix} 0_{2\times 2} \\ I_2\end{pmatrix},$$

as well as (B6a) and (B6b), we obtain the generalizations of the singular integral equations (66) that include double poles:

$$\bar{N}(x,t,k) = \begin{pmatrix} I_2 \\ 0_{2\times 2}\end{pmatrix} + \sum_{n=1}^{\mathcal{N}} \frac{e^{2i\theta(x,t,k_n)}N_{n,0}(x,t)}{k-k_n}C_n$$

$$+\sum_{m=1}^{\mathcal{M}}\left[\frac{e^{2i\theta(x,t,k_m)}N_{m,0}(x,t)C_{m,1}}{k-k_m} + \frac{e^{2i\theta(x,t,k_m)}N_{m,1}(x,t)C_{m,2}}{(k-k_m)^2}\right.$$

$$\left.+\frac{2i(x+4k_mt)e^{2i\theta(x,t,k_m)}N_{m,0}(x,t)C_{m,2}}{k-k_m}\right] + \frac{1}{2\pi i}\int_{-\infty}^{\infty} d\xi\, \frac{e^{2i\theta(x,t,\xi)}N(x,t,\xi)\rho(\xi)}{\xi-(k-i0^+)}, \tag{B8a}$$

$$N(x,t,k) = \begin{pmatrix} 0_{2\times 2} \\ I_2\end{pmatrix} + \sum_{n=1}^{\mathcal{N}} \frac{e^{-2i\theta\left(x,t,k_n^*\right)}\bar{N}_{n,0}(x,t)}{k-k_n^*}\bar{C}_n$$

$$+\sum_{m=1}^{\mathcal{M}}\left[\frac{e^{-2i\theta(x,t,k_m^*)}\bar{N}_{m,0}(x,t)\bar{C}_{m,1}}{k-k_m^*} + \frac{e^{-2i\theta(x,t,k_m^*)}\bar{N}_{m,1}(x,t)\bar{C}_{m,2}}{(k-k_m^*)^2}\right.$$

$$\left.-\frac{2i(x+4k_m^*t)e^{-2i\theta(x,t,k_m^*)}\bar{N}_{m,0}(x,t)\bar{C}_{m2}}{k-k_m^*}\right] - \frac{1}{2\pi i}\int_{-\infty}^{\infty} d\xi\, \frac{e^{-2i\theta(x,t,\xi)}\bar{N}(x,t,\xi)\bar{\rho}(\xi)}{\xi-(k+i0^+)}. \tag{B8b}$$

To "close" the system of equations, one needs to evaluate (B8a) at $k = k_n^*$ and $k = k_m^*$ and the k-derivative of (B8a) at $k = k_m^*$, and to evaluate (B8b) at $k = k_n$ and $k = k_m$ and the k-derivative of (B8b) at $k = k_m$.

The potential is then reconstructed from the solution of the above system by simply evaluating the large k asymptotic behavior of the equations (B8) and comparing it with (60), yielding:

$$\begin{aligned}Q(x,t) &= 2i\sum_{n=1}^{\mathcal{N}} \mathrm{e}^{-2i\theta\left(x,t,k_n^*\right)}\,\bar{N}_{n,0}^{\mathrm{up}}(x,t)\,\bar{C}_n \\ &\quad + 2i\sum_{m=1}^{\mathcal{M}} \mathrm{e}^{-2i\theta\left(x,t,k_m^*\right)}\,\bar{N}_{m,0}^{\mathrm{up}}(x,t)\left[\bar{C}_{m,1} - 2i\left(x+4k_m^* t\right)\bar{C}_{m,2}\right] \\ &\quad + \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-2i\theta(x,t,\xi)}\bar{N}^{\mathrm{up}}(x,t,\xi)\,\bar{\rho}(\xi)\,\mathrm{d}\xi\,, \end{aligned} \tag{B9a}$$

$$\begin{aligned}R(x,t) &= -2i\sum_{n=1}^{\mathcal{N}} \mathrm{e}^{2i\theta(x,t,k_n)}\,N_{n,0}^{\mathrm{dn}}(x,t)\,C_n \\ &\quad - 2i\sum_{m=1}^{\mathcal{M}} \mathrm{e}^{2i\theta(x,t,k_m)}\,N_{m,0}^{\mathrm{dn}}(x,t)\left[C_{m,1} + 2i(x+4k_m t)C_{m,2}\right] \\ &\quad + \frac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{e}^{2i\theta(x,t,\xi)}N^{\mathrm{dn}}(x,t,\xi)\rho(\xi)\,\mathrm{d}\xi\,. \end{aligned} \tag{B9b}$$

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