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Nonlocal integrable PDEs from hierarchies of symmetry laws: The example of Pohlmeyer–Lund–Regge equation and its reflectionless potential solutions

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ABSTRACT

By following the ideas presented by Fukumoto and Miyajima in Fukumoto and Miyajima (1996) we derive a generalized method for constructing integrable nonlocal equations starting from any bi-Hamiltonian hierarchy supplied with a recursion operator. This construction provides the right framework for the application of the full machinery of the inverse scattering transform. We pay attention to the Pohlmeyer–Lund–Regge equation coming from the nonlinear Schrödinger hierarchy and construct the formula for the reflectionless potential solutions which are generalizations of multi-solitons. Some explicit examples are discussed.

1. Introduction

The hierarchies of bi-Hamiltonian PDEs are sets of commuting evolution equations which can be constructed recursively [1]. Their commutation implies that such flows can be summed preserving the integrability property. Historically, the relevance of commuting flow summations in hydrodynamics goes back to the paper by Fukumoto and Miyazaki written in 1991 [2], where the authors connect the vortex motion in a three dimensional Euler fluid to the Hirota equation, which is the sum of the nonlinear Schrödinger (NLS) and complex modified Korteweg–de Vries Hamiltonian flows. Along the same line of research, in 1996 Fukumoto and Miyajima [3] found an interesting connection between the NLS hierarchy and the Pohlmeyer-Lund-Regge (PLR) equation: the PLR equation can be obtained as a suitable infinite sum of commuting flows in the NLS hierarchy. This property, *de facto*, is a Hamiltonian proof of the integrability of the PLR equation. In this paper we generalize such construction to any bi-Hamiltonian hierarchy for which can be defined a recursion operator formally inverting one of the Poisson bi-vectors (see e.g. ωN-manifolds in [4]). The result of these infinite summation methods is generically a nonlocal PDE. This construction could appear as an academic exercise but it is an explicit way to construct nonlocal integrable systems whose interest is growing. Moreover, in the inverse scattering transform (IST) framework, it can be used to explicitly find reflectionless solutions. We concretely illustrate the method in the case of soliton-like solutions of the PLR equation.

The PLR equation in a uniform static external field has been proposed in 1976 by Lund and Regge [5] as a possible model describing both motion of extended relativistic strings and (in a particular limit) nonrelativistic vortices in superfluids. It is

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explicitly given by

$$\begin{aligned} &\mathbf{X}_{tt} - \mathbf{X}_{xx} = -2\mathbf{X}_t \times \mathbf{X}_x\,, \\ &{\mathbf{X}_t}^2 + {\mathbf{X}_x}^2 = 1\,, \\ &\mathbf{X}_t \cdot \mathbf{X}_x = 0, \end{aligned} \tag{1.1}$$

where $\mathbf{X} \in \mathbb{R}^3$ is the vector of coordinates of the string. In the same year Pohlmeyer [6] proposed the same equation in the framework of Hamiltonian systems as an integrable generalization of the sine-Gordon equation. The ubiquity of the PLR equations as a model of very different phenomena involves also plasma physics: in a relatively recent paper [7], Schief proved a relation between a particular constrained version of the PLR equation and magnetohydrodynamics.

An interesting mathematical property of the PLR equation (see e.g. [3]) is that, through the Hasimoto map $q = K \exp(\int^x \tau \, dx)$, K and τ being the curvature and torsion of the curve $\mathbf{X}$, respectively, Eq. (1.1) becomes

$$iq_t - \varepsilon q_{xt} + 2\varepsilon kq \int^x |q|_t^2 dx = -q_{xx} + 2k|q|^2 q\,. \tag{1.2}$$

This form of the PLR equation, used throughout the paper, can be viewed as a nonlocal generalization of the NLS equation. Another equation which shares with PLR a similar property, i.e. to be a nonlocal and integrable generalization of NLS, is the Landau–Lifshitz equation [8].

In our paper we derive an explicit multisoliton solution formula for the PLR equation (1.2). In fact, to the best of our knowledge only few examples of soliton (see e.g. [9]) or shape invariant [3] solutions have been obtained for Eq. (1.2) and typically in implicit form.

To get such results we generalize the procedure used in [10] to solve the Hirota equation. This procedure combines the so-called matrix triplet method (which is partially based on the IST; see Section 3 for more details) with the observation that if V_ε denotes the time evolution matrix for the summed flows of the NLS hierarchy, we have

$$V_\varepsilon = \sum_{n=1}^{\infty} \varepsilon^{n-1} V_n, \tag{1.3}$$

where V_n is the evolution matrix of the nth flow of the hierarchy and ε is a small positive parameter. For the class of reflectionless solutions we prove that asymptotically the series (1.3) is absolutely convergent. Moreover, even though the problem of establishing the absolute convergence of the series (1.3) has so far remained unanswered for the class of non reflectionless solutions, it is interesting to observe that the summations truncated after the first N flows give an interesting indication of the qualitative features of the flow summations on a hierarchy and an approximation of the PLR solutions. In fact, as suggested in a purely physical context by the study of the axial velocity in vortex filaments [10], the main effect is the variation of the typical speed of the solution for fixed amplitudes.

To obtain an explicit manageable formula we use the so-called matrix triplet method: this method is based on the observation that the integral kernel of the Marchenko integral equation has separated variables if the reflection coefficient vanishes identically. In that case there exists a triplet of matrices (A, B, C), of sizes $p \times p$, $p \times 1$, and $1 \times p$, such that the Marchenko kernel is given by

$$\Omega_l(x + y, t) = Ce^{tH} e^{-(x+y)A} B,$$

where the $p \times p$ matrices A and H commute and A has only eigenvalues with positive real part. Usually H is a simple function of A. Solving the Marchenko equation by elementary means, we arrive at the solution of the initial-value problem in terms of the matrix triplet (A, B, C) and the matrix H describing the time dependence. The expression obtained can then be written in terms of elementary functions using computer algebra. The matrix triplet method has been applied successfully to the KdV equation [11], the focusing NLS equation [12–16], the sine-Gordon equation [17,18], the modified Korteweg–de Vries (mKdV) equation [19], the Hirota equation [10], and the Heisenberg ferromagnetic equation [20]. In this article we show how to get a solution of the Marchenko equation associated to the PLR equation (1.2).

2. Bi-Hamiltonian structures and nonlocal integrable equations

In this section we present a generalization of the method used in [3] for the construction of the PLR equation. Using the classical bi-Hamiltonian recursion relations, it is possible to construct a nonlocal integrable equation associated to the hierarchy by assuming the existence of a recursion operator. In this framework this requirement is always fulfilled when one of the two Poisson structures is the inverse of a symplectic structure. In infinite dimensional spaces the notion of invertibility of a tensor boils down to a suitable choice of the function space in which the theory is formulated. Actually, the Poisson tensors are differential operators acting on the variation of Hamiltonian functionals. The study of this problem, also in well-known cases such as constant structures, goes beyond the scope of this paper: we refer the reader to the excellent classical paper by Maltsev and Novikov [21], where the NLS case is one of the many cases considered.

We recall that a bi-Hamiltonian differential equation is an evolution equation which is Hamiltonian with respect to two different Poisson structures P_i and Hamiltonian functionals H_i such that

$$q_t = P_0 \delta H_1 = P_1 \delta H_0 \,, \tag{2.1}$$

where $q = q(x, t)$ is a function possibly vector-valued. By δH we mean the variation of the functional H.[1] The apparently strange choice of the indices will be clarified in a moment. Moreover, the Poisson structures have to satisfy another really strict property, called compatibility: the sum of the two structures must be a Poisson structure itself [1]. There is a standard way, called Lennard–Magri recursion, to generate an infinite number of commuting flows [1].

Let us consider a bi-Hamiltonian hierarchy of Hamiltonian flows

$$q_{t_n} = W_n \,, \qquad n \geq 1, \tag{2.2}$$

such that

$$W_n = P_1 \delta H_{n-1} = P_0 \delta H_n. \tag{2.3}$$

As one can see from this construction, the apparently strange scaling in the indices of the Hamiltonians in (2.1) is due to the natural gradation present in the full hierarchy. If P_0 is formally invertible, the recursion operator acting on the flows as

$$W_n = \mathcal{R}\, W_{n-1} \tag{2.4}$$

is given by $\mathcal{R} = P_1 P_0^{-1}$. It is well-known that for the Gelfand–Dickii equation and the NLS equation (see e.g. [21]) the recursion operator is not local, even though every equation of the hierarchy is. In general, it is possible to summarize the full hierarchy in the following way (see for the Langer–Perline hierarchy [23]):

$$\begin{aligned} W^\tau &\equiv \sum_{n=0}^{+\infty} \varepsilon^n W_n = W_0 + \sum_{n=1}^{+\infty} \varepsilon^n \mathcal{R} W_{n-1} = W_0 + \varepsilon \mathcal{R} \sum_{n=1}^{+\infty} \varepsilon^{n-1} W_{n-1} \\ &= W_0 + \varepsilon \mathcal{R} \sum_{n=0}^{+\infty} \varepsilon^n W_n = W_0 + \varepsilon \mathcal{R} W^\tau. \end{aligned} \tag{2.5}$$

Therefore, the flow W^τ can be defined by

$$(1 - \varepsilon P_1 P_0^{-1}) W^\tau = W_0. \tag{2.6}$$

The summed hierarchy can be seen as described by the nonlocal tensor given by

$$I = 1 - \varepsilon P_1 P_0^{-1}, \tag{2.7}$$

acting on the first flow of the hierarchy. In general, it is possible, for every hierarchy having a recursion operator, to construct a "dual" nonlocal hierarchy whose flows are given by

$$W_N^\tau \equiv \sum_{n=N}^{+\infty} \varepsilon^n W_n = W_N + \varepsilon \mathcal{R} W_N^\tau \tag{2.8}$$

or

$$(1 - \varepsilon \mathcal{R}) W_N^\tau = W_N. \tag{2.9}$$

We will call *seed* the flow W_N used for the construction of the dual non-local flow W_N^τ.

Eq. (2.6) is, by construction, non-evolutionary and, in general, non-local. However, a formal Hamiltonian structure can be recovered using the summation rule (2.5). Actually, it is obvious that any equation obtained by means of the previous construction is still bi-Hamiltonian with respect the same structures P_0 and P_1 and Hamiltonians given by the formal infinite sum of all the Hamiltonians of the hierarchy. For the first structure the related Hamiltonian H_1^τ is obtained from (2.5) as

$$W^\tau \equiv \sum_{n=0}^{+\infty} \varepsilon^n W_n = \sum_{n=0}^{+\infty} \varepsilon^n P_0 \delta H_n = P_0 \delta \left(\sum_{n=0}^{+\infty} \varepsilon^n H_n \right) \equiv P_0 \delta\, H_1^\tau \,. \tag{2.10}$$

[1] As standard in the infinite dimensional Hamiltonian systems, the variation symbol δ means

$$\delta \int h(q, q_x, q_{xx}, \ldots)\, \mathrm{d}x \equiv \sum_n (-1)^n \partial^n \frac{\partial h}{\partial (\partial^n q)}, \qquad \partial = \frac{\partial}{\partial x} \,.$$

For the explanation of this notation we refer the interested reader to Dubrovin and Zhang's work [22].

Analogously, for the second structure we have

$$W^\tau \equiv \sum_{n=0}^{+\infty} \varepsilon^n W_n = \sum_{n=1}^{+\infty} \varepsilon^n P_1 \delta H_{n-1} = P_1 \delta \left(\sum_{n=1}^{+\infty} \varepsilon^n H_{n-1} \right) \equiv P_1 \delta\, H_0^\tau \,. \tag{2.11}$$

However, there is a second way to construct a bi-Hamiltonian structure of Eqs. (2.5) by using an infinite sum of Poisson tensors acting on the same variations of Hamiltonians (H_1 and H_0) of the seed flows. Obviously, this procedure is only possible due to the compatibility of the P_0 and P_1 Poisson structures. Actually, starting from

$$W^\tau \equiv \sum_{n=0}^{+\infty} \varepsilon^n W_n = \sum_{n=0}^{+\infty} \varepsilon^n \mathcal{R}^n W_0 = \left(\sum_{n=0}^{+\infty} \varepsilon^n \mathcal{R}^n \right) P_0 \delta H_1 = (1 - \varepsilon \mathcal{R})^{-1} P_0 \delta H_1 \,, \tag{2.12}$$

by the compatibility condition the operator

$$P_0^\tau = (1 - \varepsilon \mathcal{R})^{-1} P_0 \tag{2.13}$$

is a formal nonlocal Poisson operator such that

$$W^\tau = P_0^\tau \, \delta H_1 \,. \tag{2.14}$$

Analogous computations can be done for the Hamiltonian H_0. In general, the infinite sum of compatible Poisson tensors like (2.13) yields a really complicated nonlocal operator whose structure is unclear. However, in some cases the nonlocality does not blow up and the tensor summation yields a nonlocality which is of the same degree as the recursion operator. Some explicit cases such as the KdV and Camassa–Holm equations have been studied in [24,25] and the related P_0^τ tensor is only weakly non-local in the sense of [21]. It is an open question if such a construction is possible also for NLS.

At the end of this general discussion we stress that the weak-nonlocality notion is relevant when one studies the Hamiltonian properties of the system but it is not involved in the analytic method to find solutions. Consequently, the explicit solutions will be found without the introduction of the nonlocality notion because, as we will see in the next section, the proof is based on the explicit summation of the infinite time evolutions of the compatibility problems associated to the flows.

2.1. A simple example: KdV case

The KdV equation

$$q_t = q_{xxx} + 3qq_x \tag{2.15}$$

is bi-Hamiltonian with respect to the Poisson structures

$$P_0 = \partial, \qquad P_1 = \partial^3 + 2q\partial + q_x, \tag{2.16}$$

and the respective Hamiltonians

$$H_1 = \int \left(\frac{q_x^2}{2} + \frac{q^3}{2} \right) \mathrm{d}x, \qquad H_0 = \int \frac{q^2}{2} \mathrm{d}x, \tag{2.17}$$

such that $q_t = P_0 \delta H_1 = P_1 \delta H_0$. Let us now consider the associated nonlocal system. Using the KdV equation as the seed equation one obtains

$$I q_\tau = q_{xxx} + 3qq_x \,, \tag{2.18}$$

where I is the nonlocal tensor

$$I = 1 - \varepsilon \partial^2 - \varepsilon q - \varepsilon \partial q \partial^{-1} \,, \tag{2.19}$$

or more explicitly

$$q_\tau - \varepsilon \left(q_{xx\tau} + 2qq_\tau + q_x \int q_\tau \, dx \right) = q_{xxx} + 3qq_x. \tag{2.20}$$

As described in the general discussion, it is possible to construct an integrable nonlocal evolutionary equation starting from every hierarchy evolution $q_{t_n} = W_n$ whose dual non-local counterpart is

$$I q_{\tau_n} = W_n. \tag{2.21}$$

The first nontrivial commuting flow in the KdV hierarchy is $W_0 = q_x$, which gives

$$q_{\tau_0} - \varepsilon \left(q_{xx\tau_0} + 2qq_\tau + q_x \int q_{\tau_0} \, dx \right) = q_x \,. \tag{2.22}$$

The KdV equation has the peculiar property that in the potential variable $v_x = q$ the equations of the hierarchy become local, e.g. for the first two flows

$$
\begin{aligned}
&v_{x\tau_0} - \varepsilon(v_{xxx\tau_0} + 2v_x v_{x\tau_0} + v_{xx}v_{\tau_0}) = v_{xx}\,,\\
&v_{x\tau} - \varepsilon(v_{xxx\tau} + 2v_x v_{x\tau} + v_{xx}v_\tau) = v_{xxxx} + 3v_{xx}v_x.
\end{aligned}
\tag{2.23}
$$

As we will see in the NLS case, the locality of the dual hierarchy is not general, and we think that it is a quite rare feature.

2.2. PLR equation and NLS hierarchy

In the remainder of this paper we confine our considerations to the PLR equation which is related to the hierarchy of integrable evolution equations, where the NLS equation is the first and the modified Korteweg–de Vries equation is the second equation. Fukumoto and Miyajima showed in [3] that the equation obtained by summing the Langer–Perline hierarchy equations [23] is equivalent to the PLR equation. Here we obtain the same result by acting directly on the NLS hierarchy which is related to the Langer–Perline hierarchy by the Hasimoto map. By using the notations adopted in the introduction, we have [21]

$$
P_0 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \qquad P_1 = \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix} - 2k\begin{pmatrix} -q\partial^{-1}q & q\partial^{-1}q^* \\ q^*\partial^{-1}q & q^*\partial^{-1}q^* \end{pmatrix}.
$$

Using the respective Hamiltonians

$$
H_1 = \int\left(-\frac{|q_x|^2}{2} - k|q|^4\right)\mathrm{d}x\,, \qquad H_0 = \int \frac{i}{2}\left(q_x q^* - q q_x^*\right)\mathrm{d}x\,,
\tag{2.24}
$$

we obtain the standard NLS equation

$$
q_t = iq_{xx} - 2ik|q|^2 q\,, \qquad q_t^* = -iq_{xx}^* + 2ik|q|^2 q^*\,.
\tag{2.25}
$$

Then the recursion operator is given by

$$
\mathcal{R} = P_1 P_0^{-1} = i\begin{pmatrix} -\partial & 0 \\ 0 & \partial \end{pmatrix} + 2ik\begin{pmatrix} q\partial^{-1}q^* & q\partial^{-1}q \\ -q^*\partial^{-1}q^* & -q^*\partial^{-1}q \end{pmatrix}
$$

and the corresponding Eq. (2.6) is

$$
iq_\tau + q_{xx} - 2k|q|^2 q - \varepsilon\left(q_{x\tau} - 2kq\int^x (|q|^2)_\tau\, dx\right) = 0\,,
\tag{2.26}
$$

using as the seed equation the NLS equation itself. Eq. (2.26) is exactly the PLR equation [3]. Using a flow commuting with NLS as the seed equation, one can obtain a flow commuting with PLR by construction. For example, using the momentum conservation as the seed equation one obtains the flow

$$
iq_\sigma + q_x - \varepsilon\left(q_{x\sigma} + 2kq\int^x (|q|^2)_\sigma\, dx\right) = 0
\tag{2.27}
$$

which commutes with PLR.

3. Inverse scattering transform for PLR equation

As already stressed in the introduction, to the best of our knowledge there are only a few explicit solutions of the PLR equation. To construct the general multisoliton solution formula for the PLR equation (2.26), we need to recall some preliminaries on the IST, because the procedure used to establish our main results is essentially based on it. We have already mentioned in the introduction that the IST consists of three parts: the direct scattering problem, the evolution of the scattering data, and the inverse scattering problem. In this section we are going to give more details on each of these parts.

The bi-Hamiltonian structure associated to the NLS equation is closely related to the structure of the compatibility problem useful for the inverse scattering problem.

It is suitable to consider the AKNS pair (see [26]) corresponding to each equation of the hierarchy. In this way we construct a hierarchy of compatibility problems of the following form:

$$
\psi_x = U\psi\,, \qquad \psi_t = V_n\psi\,,
\tag{3.1}
$$

where $U = -i\lambda\sigma_3 + Q$, q is the so-called potential which is assumed to belong to $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and to satisfy $q_x \in L^1(\mathbb{R})$, λ is the spectral parameter, $Q = \begin{pmatrix} 0 & q \\ -q^* & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is the third Pauli matrix, and V_n is a matrix function depending on x, t, and λ (it is important to stress that the matrix V_n depends explicitly on the potential q and its derivative with respect to x which are functions of x and t). Then V_n gives the time evolution matrix for the nth flow of the hierarchy.

The compatibility condition $\psi_{xt} = \psi_{tx}$ leads to the zero-curvature representation

$$U_t - (V_n)_x + UV_n - V_nU = 0_{2\times 2}$$

of the nth equation of the hierarchy. We remark that U remains the same for all hierarchy equations, whereas V_n is the time evolution for the nth (fixed) flow of the hierarchy. For each fixed n, from the NLS hierarchy we get an *integrable equation* in the sense that we know what the AKNS pair [26] associated to the nth flux is. In fact, in [26] it is explained how to construct the time evolution matrix V_n when the matrix U is given. Then for each fixed n the AKNS pair generating the nth flow of the hierarchy is available. It is well-known [26–28] that when an AKNS pair is associated to a given nonlinear PDE, the initial value problem of this equation can be solved by applying the Inverse Scattering Transform (IST) and many authors called the PDEs solvable by applying the IST integrable [26,28–32].

We observe that V_n has the following form:

$$V_n = -2^n i\lambda^{n+1}\sigma_3 + (\ldots), \tag{3.2}$$

where the expression in $(\ldots)$ is a finite sum of terms each of which contains the potential and/or its derivatives. In particular, as $x \to \pm\infty$ (for each fixed t), we have $V_n \to -2^n i\lambda^{n+1}\sigma_3$. It will be proved in Appendix that the time evolution of the scattering data is only determined by the asymptotic behavior of the evolution operator V_n. We note that for $n = 1$ and $n = 2$ the asymptotic behavior of V_n is given by $-2i\lambda^2\sigma_3$ and $-4i\lambda^3\sigma_3$, respectively, which coincides with the asymptotic behavior of the NLS and mKdV time evolution operators [10].

In [10] we have constructed the reflectionless solutions of the Hirota equation by observing that

(1) the Hirota equation can be viewed as a linear combination of the NLS and mKdV equations;
(2) the kernel of the Marchenko integral equation associated to the Hirota equation can be obtained by a (suitable) linear combination of the kernels of the Marchenko equations associated with the NLS and mKdV equations.

Since Eq. (2.26) is obtained by summing the fluxes of the NLS hierarchy, we can obtain the reflectionless solutions of this equation by applying the ideas used in [10]. Here the difference lies in the fact that we have to sum an infinite number of equations/kernels which corresponds to considering a (formal) series. In particular, the time evolution operator associated to the PLR operator is given by

$$V_\varepsilon = \sum_{n=1}^{\infty} \varepsilon^{n-1} V_n,$$

where V_n satisfy (3.2). The natural thing to do is to write the matrix V in the AKNS pair of the PLR equation as

$$V_\varepsilon = \sum_{n=1}^{\infty} \varepsilon^{n-1}(-2^n i\lambda^{n+1}\sigma_3) + (\ldots) = \frac{-2i\lambda^2}{1-\varepsilon\lambda}\sigma_3 + (\ldots),$$

where $(\ldots)$ indicates the part vanishing as $x \to \pm\infty$. This requires interchanging the summation and the $x \to \pm\infty$ limits in V, as well as justifying the convergence of the above geometric series by assuming that $\varepsilon|\lambda| < 1$. So far we have not been able to justify either fact. However, when restricting ourselves to the reflectionless case and seeking soliton solutions, the use of matrix triplets (A, B, C) will allow us to justify these two facts. In fact, it will be sufficient to take the positive ε to be smaller than the reciprocal of the spectral radius of the matrix A [cf. (4.3)].

Let us explain our idea in a more detailed manner. Since the first of Eqs. (3.1) holds for each equation of the hierarchy, i.e., depends only on the hierarchy, the *PLR equation is associated to the Zakharov–Shabat (ZS) system* [27]. It is well-known that the ZS system is given by (3.1). If we want to apply the IST to the PLR equation, we have to associate it to the ZS system (corresponding to the first of Eq. (3.1)) by developing the direct and inverse scattering theory for this system. There is a vast literature on the ZS system, so we can skip the proofs of the statements reported below by referring the reader to [27,30,32] for details. We present the direct and inverse scattering problems of the ZS system and, for the sake of simplicity, we omit the subscripts n and ε (we recall that each flux of the AKNS hierarchy is associated to the ZS system).

Direct Scattering Problem. The direct scattering problem consists of constructing the so-called scattering matrix $S(\lambda)$ (or $\breve{S}(\lambda)$). As it will be explained better later in this section, the matrix $S(\lambda)$ contains part of the scattering data. The knowledge of the scattering data at the time t allows one to find the potential $q(x, t)$ satisfying the PLR equation. It is important to remember that in the direct scattering theory the initial potential $q(x, 0)$ appears as a coefficient in the ZS system (3.1) and, as a consequence, we will first find the scattering data corresponding to $t = 0$. One has to take into account the second equation in the AKNS pair $\psi_t = V_n\psi$ if one wants to know the time evolution of the scattering data (we will deal with this problem after we have completed the study of the direct scattering problem). Essentially, the direct scattering problem can be regarded as the study of the spectral properties of the ZS system and, in this sense, the construction of $S(\lambda)$ represents an important result.

In order to construct the matrix $S(\lambda)$, let us introduce the 2×1 columns known as *Jost functions from the right* $\overline{\psi}(\lambda, x)$ and $\psi(\lambda, x)$, the 2-component vectors known as *Jost functions from the left* $\phi(\lambda, x)$ and $\overline{\phi}(\lambda, x)$, and the 2×2 matrices called

Jost matrices $\Psi(\lambda,x)$ and $\Phi(\lambda,x)$ *from the right and the left* as those solutions of the ZS system satisfying the asymptotic conditions

$$\Psi(\lambda,x)=\begin{pmatrix}\overline{\psi}(\lambda,x) & \psi(\lambda,x)\end{pmatrix}=e^{-i\lambda\sigma_3 x}[I_2+o(1)],\qquad x\to+\infty, \tag{3.3a}$$
$$\Phi(\lambda,x)=\begin{pmatrix}\phi(\lambda,x) & \overline{\phi}(\lambda,x)\end{pmatrix}=e^{-i\lambda\sigma_3 x}[I_2+o(1)],\qquad x\to-\infty, \tag{3.3b}$$

where I_2 is the identity matrix of order 2 (from now on, I_p denotes the identity matrix of order p). The Jost functions are the key instruments to prove the analytic properties of the scattering data associated to the ZS system. Using (3.3a) and (3.3b), we get the Volterra integral equations

$$\Psi(\lambda,x)=e^{-i\lambda\sigma_3 x}+i\sigma_3\int_x^\infty dy\, e^{i\lambda\sigma_3(y-x)}Q(y)\Psi(\lambda,y), \tag{3.4a}$$
$$\Phi(\lambda,x)=e^{-i\lambda\sigma_3 x}-i\sigma_3\int_{-\infty}^x dy\, e^{-i\lambda\sigma_3(x-y)}Q(y)\Phi(\lambda,y). \tag{3.4b}$$

Since the ZS system is first order, there exist matrices $a_l(\lambda)$ and $a_r(\lambda)$ not depending on $x\in\mathbb{R}$ called *transition matrices* from the left and the right, respectively, such that

$$\Phi(\lambda,x)=\Psi(\lambda,x)a_r(\lambda),\qquad \Psi(\lambda,x)=\Phi(\lambda,x)a_l(\lambda), \tag{3.5}$$

where

$$a_r(\lambda)=I_2+i\sigma_3\int_{-\infty}^\infty dy\, e^{i\lambda\sigma_3 y}Q(y)\Psi(\lambda,y),$$
$$a_l(\lambda)=I_2-i\sigma_3\int_{-\infty}^\infty dy\, e^{i\lambda\sigma_3 y}Q(y)\Phi(\lambda,y).$$

It is immediate to verify that $a_l(\lambda)$ and $a_r(\lambda)$ are each others inverses and, from Eqs. (3.3) and (3.4), we easily obtain

$$\Psi(\lambda,x)=e^{-i\lambda\sigma_3 x}\left[a_l(\lambda)+o(1)\right],\quad x\to-\infty, \tag{3.6}$$
$$\Phi(\lambda,x)=e^{-i\lambda\sigma_3 x}\left[a_r(\lambda)+o(1)\right],\quad x\to+\infty. \tag{3.7}$$

It is convenient to use the matrix representations

$$a_l(\lambda)=\begin{pmatrix}a_{l1}(\lambda) & a_{l2}(\lambda)\\ a_{l3}(\lambda) & a_{l4}(\lambda)\end{pmatrix},\qquad a_r(\lambda)=\begin{pmatrix}a_{r1}(\lambda) & a_{r2}(\lambda)\\ a_{r3}(\lambda) & a_{r4}(\lambda)\end{pmatrix},$$

where (cf. [28,30,32]) $a_{l1}(\lambda)$ and $a_{r4}(\lambda)$ are continuous in $\lambda\in\overline{\mathbb{C}^+}$, are analytic in $\lambda\in\mathbb{C}^+$, and tend to 1 as $|\lambda|\to+\infty$ from within $\overline{\mathbb{C}^+}$. Here $\mathbb{C}^\pm$ is the open upper/lower complex plane and $\overline{\mathbb{C}^\pm}=\mathbb{C}^\pm\cup\mathbb{R}$. In the same way we see that $a_{r1}(\lambda)$ and $a_{l4}(\lambda)$ are continuous in $\lambda\in\overline{\mathbb{C}^-}$, are analytic in $\lambda\in\mathbb{C}^-$, and tend to 1 as $|\lambda|\to+\infty$ from within $\overline{\mathbb{C}^-}$. The remaining elements $a_{l2}(\lambda)$, $a_{l3}(\lambda)$, $a_{r2}(\lambda)$, and $a_{r3}(\lambda)$ are continuous in $\lambda\in\mathbb{R}$ and vanish as $\lambda\to\pm\infty$.

The zeros $\lambda\in\mathbb{C}^+$ of $a_{l1}(\lambda)$ and $a_{r4}(\lambda)$ are exactly the discrete eigenvalues of the ZS system in $\mathbb{C}^+$. On the other hand, the zeros $\lambda\in\mathbb{C}^-$ of $a_{r1}(\lambda)$ and $a_{l4}(\lambda)$ are exactly the discrete eigenvalues of the ZS system in $\mathbb{C}^-$ which are the complex conjugates of those in $\mathbb{C}^+$. We call $\lambda\in\mathbb{R}$ a *spectral singularity* if it is a zero of, at least one of the diagonal elements $a_{l1}(\lambda)$, $a_{l4}(\lambda)$, $a_{r1}(\lambda)$, and $a_{r4}(\lambda)$. From now on, *we assume that there are no spectral singularities*. In that case, elementary complex analysis implies that the number of discrete eigenvalues of the ZS system is finite [26]. Moreover, we observe that the discrete eigenvalues are the poles of the transmission coefficient introduced below in (3.10) and defined in terms of the elements of the transition matrices in the Appendix.

To introduce the scattering matrices $S(\lambda)$ and $\breve{S}(\lambda)$, let us introduce the *modified Jost matrices* as follows:

$$F_+(\lambda,x)=\begin{pmatrix}\phi(\lambda,x) & \psi(\lambda,x)\end{pmatrix},\qquad F_-(\lambda,x)=\begin{pmatrix}\overline{\psi}(\lambda,x) & \overline{\phi}(\lambda,x)\end{pmatrix}. \tag{3.8}$$

Then $F_\pm(\lambda,x)e^{-i\lambda x\sigma_3}$ are continuous in $\lambda\in\overline{\mathbb{C}^\pm}$, are analytic in $\mathbb{C}^\pm$, converge to I_2 as $|\lambda|\to+\infty$ from within $\overline{\mathbb{C}^\pm}$, and are related as follows:

$$F_-(\lambda,x)=F_+(\lambda,x)\sigma_3 S(\lambda)\sigma_3,\qquad F_+(\lambda,x)=F_-(\lambda,x)\sigma_3\breve{S}(\lambda)\sigma_3, \tag{3.9}$$

where the scattering matrices $S(\lambda)$ and $\breve{S}(\lambda)$ are each other's inverses. Writing

$$S(\lambda)=\begin{pmatrix}T_r(\lambda) & L(\lambda)\\ R(\lambda) & T_r(\lambda)\end{pmatrix},\qquad \breve{S}(\lambda)=\begin{pmatrix}\breve{T}_l(\lambda) & \breve{R}(\lambda)\\ \breve{L}(\lambda) & \breve{T}_l(\lambda)\end{pmatrix}, \tag{3.10}$$

we get the reflection coefficients $R(\lambda)$ and $\breve{R}(\lambda)$ from the right, the reflection coefficients $L(\lambda)$ and $\breve{L}(\lambda)$ from the left, the transmission coefficient $\breve{T}(\lambda)$ (which is meromorphic in $\lambda\in\mathbb{C}^-$), and the transmission coefficient $T(\lambda)$ (which is meromorphic in $\lambda \in \mathbb{C}^+$). Moreover, it is easily verified that

$$\breve{S}(\lambda) = S(\lambda)^{-1} = \sigma_3 S(\lambda)^{\dagger}\sigma_3, \quad \text{for } \lambda \in \mathbb{R},$$

where the dagger denotes the matrix complex conjugate transpose. Under the assumption that there are no spectral singularities, we also have

$$R(\lambda) = \int_{-\infty}^{\infty} dy\, e^{-i\lambda y}\rho(y), \qquad L(\lambda) = \int_{-\infty}^{\infty} dy\, e^{i\lambda y}\ell(y), \tag{3.11}$$

where ρ, ℓ belong to $L^1(\mathbb{R})$. Furthermore, $\breve{R}(\lambda)$ and $\breve{L}(\lambda)$ have analogous representations, where $\breve{\rho}(y) = -\rho(y)^*$ and $\breve{\ell}(y) = -\ell(y)^*$ replace ρ and ℓ. The scattering data associated with the ZS system consist of:

(a) one reflection coefficient;
(b) the discrete eigenvalues of the ZS system;
(c) a suitable set of nonzero constants associated to the discrete eigenvalues called the *norming constants*.

The construction of the norming constants has been treated in detail in [30], where the case where all the eigenvalues have algebraic multiplicity one is considered. It is important to note that the multiplicity of the norming constants is not necessarily one (see [12] on this aspect). By the way, we briefly discuss how to introduce the norming constants and how to determine their time evolution in the Appendix.

Having determined the scattering data for the ZS system a natural problem which arises is the research of their time evolution. Before dealing with this problem, we present the Inverse Scattering Problem for the ZS system.

Inverse Scattering Problem. The inverse scattering problem consists of the construction of the potential q corresponding to a given set of scattering data. As already said in the introduction, we will formulate this problem in terms of the Marchenko method. In this method the scattering data are used to construct the kernel of suitable integral equations (the so-called Marchenko integral equations) whose solution is connected by an easy algebraic relation with the potential $q(x)$. In general, it is difficult to write down explicitly the solution of the Marchenko equations but we will see in the next section how this result can be achieved in the reflectionless case (see also [32] for the conditions under which the Marchenko equations are uniquely solvable). We discuss the inverse problem neglecting the time variable, i.e., $t = 0$ but we remark that everything can be repeated also by considering the variable t as a parameter. We underline that the introduction of the time variable requires the knowledge of the evolution of the scattering data. We prefer to postpone this topic because to find the reflectionless solutions of the PLR equation, we will employ the time evolution of the "entire" kernel of the Marchenko equation (see (3.23) and (3.24)). For this pedagogical reason, we prefer to introduce the Marchenko equation before discussing the evolution of the scattering data. The time variable will be reintroduced in the next section.

In order to find the Marchenko integral equations we observe that [10,28,30,31] it is possible to write the Jost solutions by using their Fourier triangular representations

$$\Psi(\lambda, x) = \begin{pmatrix}\overline{\psi}(\lambda, x) & \psi(\lambda, x)\end{pmatrix} = e^{-i\lambda\sigma_3 x} + \int_x^{\infty} dy\, \alpha_l(x, y)e^{-i\lambda\sigma_3 y}, \tag{3.12a}$$

$$\Phi(\lambda, x) = \begin{pmatrix}\overline{\phi}(\lambda, x) & \phi(\lambda, x)\end{pmatrix} = e^{-i\lambda\sigma_3 x} + \int_{-\infty}^{x} dy\, \alpha_r(x, y)e^{-i\lambda\sigma_3 y}, \tag{3.12b}$$

where the following notation is adopted:

$$\alpha_l(x, y) = \begin{pmatrix}\overline{K}(x, y) & K(x, y)\end{pmatrix}, \qquad \alpha_r(x, y) = \begin{pmatrix}M(x, y) & \overline{M}(x, y)\end{pmatrix}. \tag{3.13}$$

Here $\overline{K}(x, y)$, $K(x, y)$, $M(x, y)$, $\overline{M}(x, y)$ are column vectors of length two (up and down will denote the first and second components of such column vectors). Furthermore, $\alpha_l(x, y)$ and $\alpha_r(x, y)$ have to satisfy the following Marchenko integral equations [30,32]:

$$\alpha_l(x, y) + \omega_l(x + y) + \int_x^{\infty} dz\, \alpha_l(x, z)\omega_l(z + y) = 0_{2\times 2}, \tag{3.14a}$$

$$\alpha_r(x, y) + \omega_r(x + y) + \int_{-\infty}^{x} dz\, \alpha_r(x, z)\omega_r(z + y) = 0_{2\times 2}. \tag{3.14b}$$

The kernels $\omega_l(x + y)$, $\omega_r(x + y)$ appearing in the Marchenko equations are called the left and right *Marchenko kernels*, respectively. It is well-known that these kernels can be expressed in terms of the scattering data as follows:

$$\omega_l(x) = \begin{pmatrix} 0 & -\rho(x)^* - \sum_{j=1}^{m}\sum_{s=0}^{n_j} \frac{x^s}{s!} e^{-i\lambda_j^* x}[C_l]_{js}^* \\ \rho(x) + \sum_{j=1}^{m}\sum_{s=0}^{n_j} \frac{x^s}{s!} e^{i\lambda_j x}[C_l]_{js} & 0 \end{pmatrix}, \tag{3.15}$$

$$\omega_r(x) = \begin{pmatrix} 0 & \ell(x) + \sum_{j=1}^{m}\sum_{s=0}^{n_j} \frac{x^s}{s!} e^{-i\lambda_j x}[C_r]_{js} \\ -\ell(x)^* - \sum_{j=1}^{m}\sum_{s=0}^{n_j} \frac{x^s}{s!} e^{i\lambda_j^* x}[C_r]_{js}^* & 0 \end{pmatrix}, \tag{3.16}$$

where λ_j are the distinct discrete eigenvalues in $\mathbb{C}^+$, n_j is the algebraic multiplicity of the eigenvalue λ_j and $[C_{l,r}]_{js}$ are the associated norming constants.

In general, for $q \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ the potential $q(x)$ is related to the Marchenko solutions $\alpha_l(x, y)$ and $\alpha_r(x, y)$ as indicated below [cf. (A.2) and (A.4) in [15]]:

$$\alpha_l(x, x) = -\frac{1}{2} \begin{pmatrix} \int_x^\infty dz\, |q(z)|^2 & q(x) \\ -q(x)^* & \int_x^\infty dz\, |q(z)|^2 \end{pmatrix}, \tag{3.17a}$$

$$\alpha_r(x, x) = -\frac{1}{2} \begin{pmatrix} \int_{-\infty}^x dz\, |q(z)|^2 & -q(x) \\ q(x)^* & \int_{-\infty}^x dz\, |q(z)|^2 \end{pmatrix}. \tag{3.17b}$$

As a result, to recover the potential $q(x)$ we can follow the three steps indicated below:

a. Suppose that the reflection coefficient $R(\lambda)$, the discrete eigenvalues $\{\lambda_j\}_{j=1}^m$, and the norming constants $\left\{\{C_{js}\}_{s=0}^{n_j-1}\right\}_{j=1}^m$ are given, where m denotes the number of discrete eigenvalues in $\mathbb{C}^+$ while n_j is the multiplicity of λ_j as a pole of $T(\lambda)$. By using the scattering data we construct the kernel of the Marchenko equation:

$$\Omega_l(y) \stackrel{\text{def}}{=} -\rho(y) + \sum_{j=1}^{m}\sum_{s=0}^{n_j-1} c_{js} \frac{y^s}{s!} e^{i\lambda_j y}, \tag{3.18}$$

where $\rho(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R(\lambda) e^{i\lambda y} d\lambda$ is the Fourier transform of $R(\lambda)$.

b. Solve the Marchenko equation having as its kernel the function (3.18), i.e., the following integral equation

$$K^{up}(x, y) - \Omega_l^*(x+y) + \int_x^\infty dz \int_x^\infty ds\, K^{up}(x, z)\, \Omega_l(z+s)\, \Omega_l^*(s+y) = 0. \tag{3.19}$$

c. Finally, we get the potential $q(x)$ by using the following formula:

$$q(x) = -2K^{up}(x, x). \tag{3.20}$$

An analogous procedure can be followed by using the right Marchenko kernel.

Time Evolution of the Scattering Data. The considerations made until now involved only the first of Eqs. (3.1). In this paragraph we take into account the second equation in (3.1) to determine how the kernel of the Marchenko equation evolves in time. Following the procedure explained in Appendix, we arrive at the following equation describing the evolution of the reflection coefficient of the nth flux

$$R_{(n)}(\lambda, t) = e^{2^{n+1} i \lambda^{n+1} t} R_{(n)}(\lambda, 0)\,. \tag{3.21}$$

Computing the derivative with respect to the time variable, we obtain

$$\partial_t R_{(n)}(\lambda, t) = 2^{n+1} i \lambda^{n+1} R_{(n)}(\lambda, t)$$

and taking the Fourier transform of the preceding equation we get

$$\partial_t \rho_{(n)}(\alpha, t) = 2^{n+1} (i)^{-n} \partial_\alpha^{(n+1)} \rho_{(n)}(\alpha, t), \tag{3.22}$$

where $\rho_{(n)}(\alpha, t)$ is the Fourier transform of the reflection coefficient of the nth flux.

Furthermore, by applying the procedure shown in Appendix, we can see that the kernel $\Omega_{(n)l}$ of the Marchenko equation associated to the nth flux is given by

$$\partial_t \left(\Omega_{(n)l}(\alpha, t) - \rho_n(\alpha, t)\right) = 2^{n+1} (i)^{-n} \partial_\alpha^{(n+1)} \left(\Omega_{(n)l}(\alpha, t) - \rho_n(\alpha, t)\right). \tag{3.23}$$

From [26,32] we know that the construction of the kernel is linear in the transmission and reflection coefficients and thus, by using Eq. (1.3), we get that the time evolution of the kernel of the Marchenko equation associated to the PLR equation is as follows:

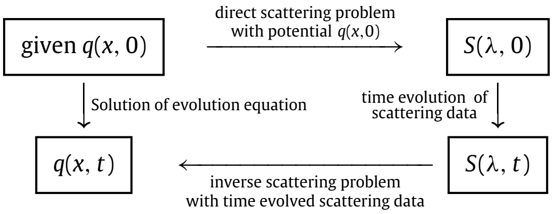
$$\partial_t\left(\Omega_{\varepsilon l}(\alpha,t)-\rho(\alpha,t)\right)-\sum_{n=1}^{\infty}2^{n+1}(i)^{-n}\partial_{\alpha}^{(n+1)}\left(\Omega_{\varepsilon l}(\alpha,t)-\rho(\alpha,t)\right)=0, \tag{3.24}$$

where $\Omega_{\varepsilon l}(\alpha, t)$ and $\rho(\alpha, t)$ represent, respectively, the kernel of the Marchenko equation defined by (4.2) and the Fourier transform of the reflection coefficient associated to the PLR equation.

Inverse Scattering Transform for the PLR equation. By using $q(x, 0)$ as the potential in the ZS system, we develop the direct scattering theory as explained above and construct the scattering data (for $t = 0$). Next, let us evolve the scattering data in time in such a way that Eq. (3.24) is satisfied. Finally, the solution of the PLR equation is obtained by finding the solution $K^{(up)}(x, y; t)$ of the Marchenko equation (3.19) (where $\Omega(\alpha)$ is replaced by $\Omega(\alpha, t)$ with $\Omega(\alpha, t)$ satisfying (3.24)) and by using the formula (3.20):

$$q(x;t)=-2K^{(up)}(x,x;t).$$

The following classical scheme illustrates and summarizes how the IST works:



4. Reflectionless solutions of the PLR equation

In this subsection we construct an explicit soliton solution formula of the PLR equation. To get this result we use the matrix triplet method which has been successfully applied to solve important integrable equations like the NLS [12,16], mKdV [19], sine-Gordon [18], Hirota [10], and Heisenberg ferromagnetic equations [20]. We refer the reader to the papers cited above for details and for the calculations which allow one to solve explicitly the Marchenko equation. Here it is sufficient to mention that the main idea of the matrix triplet method is based on the particular form assumed by the kernel $\Omega_l(\alpha, t)$ of the Marchenko equation when the reflection coefficient $R(\lambda, t)$ vanishes. In this case the kernel of the Marchenko equation can be written in separated form, implying that the Marchenko equation can be explicitly solved by separation of variables. More precisely, to find the soliton solutions of the PLR equation we put $R(\lambda, t) = 0$ in the expression of $\Omega_l(\lambda, t)$ and obtain

$$\Omega_{\varepsilon l}(y;t)=\sum_{j=1}^{m}\sum_{s=0}^{n_j-1}c_{js}(t)\frac{y^s}{s!}e^{i\lambda_j y}=C(t)e^{-yA}B, \tag{4.1}$$

where $\lambda_1, \dots, \lambda_N$ are the discrete eigenvalues, n_j are the orders of the poles of the transmission coefficient at the discrete eigenvalues $i\lambda_j$, and c_{js} are the so-called norming constants. The complex conjugate $\Omega^*_{\varepsilon l}(y; t)$ is obviously obtained as $\Omega^*_{\varepsilon l}(y;t) = (C(t)e^{-yA}B)^\dagger$. It is well-known [32–34] that if a function assumes the form $\sum_{j=1}^{m}\sum_{s=0}^{n_j-1}c_{js}(t)\frac{y^s}{s!}e^{i\lambda_j y}$ then there exists a triplet of matrices (A, B, C) of order $p \times p$, $p \times 1$, $1 \times p$, respectively, where p is a positive integer number and C depends on t such that the second equality in (4.1) holds. It will be clear later that some restrictions on the choice of the matrix triplet have to be imposed [12,33,34].

We assume the following:

a. The eigenvalues of the matrix A have positive real parts;
b. The triplet (A, B, C) provides a minimal representation for the kernel $\Omega_l(y; t)$ in the sense that

$$\bigcap_{r=1}^{+\infty}\left[\ker CA^{r-1}\right]=\bigcap_{r=1}^{+\infty}\left[\ker B^\dagger (A^\dagger)^{r-1}\right]=\{0\},$$

(we refer to [32–34] for more details on minimal representations). Here $\ker S$ denotes the null space of a matrix S.

It is easy to verify that choosing $\Omega_{\varepsilon l}(y, t)$ as follows

$$\Omega_{\varepsilon l}(y,t)=Ce^{-i\phi_\varepsilon(iA)t}e^{-yA}B, \tag{4.2}$$

where

$$\phi_\varepsilon(z)=-\sum_{n=1}^{\infty}\varepsilon^{n-1}2^{n+1}z^{n+1}=\frac{-4z^2}{1-2\varepsilon z}. \tag{4.3}$$

Eq. (3.24) is satisfied. Here we observe that $\det(I - 2i\varepsilon A) \neq 0$, since A has only eigenvalues with positive real parts. The particular form of (4.3) deserves a comment: as is proven in the Appendix, ϕ_ε function is the result of the sum of the constant dominant terms of the T_n matrices. Then Eqs. (4.2) and (4.3) give us the time evolution of the kernel $\Omega_l(y, t)$ of the Marchenko equations associated to the inverse problem for the PLR equation.

To derive the soliton solution formula for the PLR equation we mimic the procedure explained in [10]. First of all, we have to solve the Marchenko equation (3.19), where the kernel $\Omega_l(y)$ is to be replaced by $\Omega_{\varepsilon l}(y, t) = Ce^{-i\phi_\varepsilon(iA)}e^{-yA}B$ and $\phi(z)$ is given by Eq. (4.3). Repeating the calculations in [10], we get the following solutions:

$$K_\varepsilon^{up}(x, y; t) = B^\dagger e^{-A^\dagger x}\Gamma_\varepsilon(x, t)^{-1}e^{-A^\dagger y + i\phi_\varepsilon(-iA^\dagger)t}C^\dagger\,, \tag{4.4}$$

where

$$\Gamma_\varepsilon(x, t) = I_p + e^{-A^\dagger x + i\phi_\varepsilon(-iA^\dagger)t}Qe^{-2Ax - i\phi_\varepsilon(iA)t}Ne^{-A^\dagger x}\,, \tag{4.5}$$

$$Q = \int_0^\infty ds\, e^{-A^\dagger s}C^\dagger Ce^{-As}, \qquad N = \int_0^\infty dr\, e^{-Ar}BB^\dagger e^{-A^\dagger r}\,. \tag{4.6}$$

Recalling the relationship between the solution of the Marchenko equation and the solution of the PLR equation, i.e., Eq. (3.20), we arrive at

$$q_\varepsilon(x, t) = -2B^\dagger e^{-A^\dagger x}\Gamma_\varepsilon^{-1}(x, t)e^{-A^\dagger x + i\phi_\varepsilon(-iA^\dagger)t}C^\dagger\,. \tag{4.7}$$

The solution expressed by (4.7) depends only on the matrix triplet chosen as input. In fact, for a given triplet of matrices (A, B, C) satisfying the conditions a. and b. above, we can calculate Q, N, and $\Gamma_\varepsilon(x, t)$ and, consequently, the solution $q_\varepsilon(x, t)$. It should be noted that the expressions found for Q and N are the expressions corresponding to the solutions to the Lyapunov equations

$$A^\dagger Q + QA = C^\dagger C, \qquad AN + NA^\dagger = BB^\dagger. \tag{4.8}$$

The Lyapunov equations are studied in detail in [33,34] where the proof that the matrices Q and N introduced through (4.5) satisfy Eqs. (4.8), respectively, can be found. Moreover, under the above hypotheses a. and b., the Lyapunov equations are uniquely solvable. Of course, the solutions expressed by (4.7) hold only if the integrals defining Q and N are convergent and the matrix $\Gamma_\varepsilon(x, t)$ is invertible. It should be proven that the convergence of these integrals and the invertibility of matrix $\Gamma_\varepsilon(x, t)$ are equivalent to requiring the condition a., while the hypothesis of minimality is convenient to prove that $\Gamma_\varepsilon^{-1}(x, t) \to 0$ as $x \to \pm\infty$ (for each fixed t.)[2] We skip the proof of these statements, because the proofs furnished in [12] can be repeated verbatim. We observe that the solution (4.7) can be obtained starting from different triplets of matrices and this justifies that two triplets of matrices are called *equivalent triplets* if they lead to the same potential $q_\varepsilon(x, t)$.

Furthermore, in [14] a class of matrix triplets (larger than those characterized by properties a. and b.) such that the integrals appearing in (4.5) are convergent and the matrix $\Gamma_\varepsilon(x, t)$ is invertible has been introduced and called the *admissible class*. We refer the interested reader to the paper [12] for the definition of admissible class. The result useful for this paper is given by the proposition below which suggests the "canonical way" of taking the triplet of matrices generating the reflectionless solutions given by (4.7):

Proposition 4.1. *Starting from $(\tilde{A}, \tilde{B}, \tilde{C})$ in the admissible class, it is possible to associate to this triplet an equivalent triplet (A, B, C), where A has the Jordan canonical form with each Jordan block containing a distinct eigenvalue having a positive real part, the column B consists of zeros and ones, and C has real entries. More specifically, for some appropriate positive integer m, we have*

$$A = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_m \end{pmatrix}, \qquad B = \begin{pmatrix} B_1 \\ B_2 \\ \vdots \\ B_m \end{pmatrix}, \qquad C = \begin{pmatrix} C_1 & C_2 & \cdots & C_m \end{pmatrix}, \tag{4.9}$$

where in the case of a real (positive) eigenvalue ω_j of A_j the corresponding blocks are given by

$$A_j := \begin{pmatrix} \omega_j & -1 & 0 & \cdots & 0 & 0 \\ 0 & \omega_j & -1 & \cdots & 0 & 0 \\ 0 & 0 & \omega_j & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \omega_j & -1 \\ 0 & 0 & 0 & \cdots & 0 & \omega_j \end{pmatrix}, \qquad B_j := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix},$$
$$C_j := \begin{pmatrix} c_{jn_j} & \cdots & c_{j2} & c_{j1} \end{pmatrix}, \tag{4.10}$$

[2] It is important to recall that $q_\varepsilon(x, t) \to 0$ as $x \to \pm\infty$ for each fixed t.

A_j having size $n_j \times n_j$, B_j size $n_j \times 1$, C_j size $1 \times n_j$, and the constant c_{jn_j} is nonzero. In the case of complex eigenvalues, which must appear in pairs as $\alpha_j \pm i\beta_j$ with $\alpha_j > 0$, the corresponding blocks are given by

$$A_j := \begin{pmatrix} \Lambda_j & -I_2 & 0 & \dots & 0 & 0 \\ 0 & \Lambda_j & -I_2 & \dots & 0 & 0 \\ 0 & 0 & \Lambda_j & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \Lambda_j & -I_2 \\ 0 & 0 & 0 & \dots & 0 & \Lambda_j \end{pmatrix}, \quad B_j := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \tag{4.11}$$

$$C_j := \begin{pmatrix} \gamma_{jn_j} & \epsilon_{jn_j} & \dots & \gamma_{j1} & \epsilon_{j1} \end{pmatrix},$$

where γ_{js} and ϵ_{js} for $s = 1, \dots, n_j$ are real constants with $(\gamma_{jn_j}^2 + \epsilon_{jn_j}^2) > 0$, each column vector B_j has $2n_j$ components, each A_j has size $2n_j \times 2n_j$, and the 2 × 2 matrix Λ_j is defined as

$$\Lambda_j := \begin{pmatrix} \alpha_j & \beta_j \\ \\ -\beta_j & \alpha_j \end{pmatrix}. \tag{4.12}$$

Proof. The triplet (A, B, C) can be chosen as in Section 3 of [12]. ■

Even though the choice of the triplet shown in Proposition 4.1 helps significantly in the classification of the reflectionless solutions, however, in the next section we do not always consider the triplet in this form in order to get clearer plots and to reduce computing time.

4.1. Examples of reflectionless solutions of the PLR equation

In this subsection, we give examples of solutions obtained starting from (4.7) and choosing the matrix triplet (A, B, C) as indicated by Proposition 4.1. The plots in this section (obtained with the help of the software Mathematica 9) display the curvature and torsion of the PLR solution as defined in the introduction

$$K = |q|, \qquad \tau = \frac{1}{2i}\left(\frac{q_x}{q} - \frac{q_x^*}{q^*}\right), \tag{4.13}$$

where q is explicitly given by

$$\begin{aligned} q_\varepsilon(x,t) = -2B^\dagger e^{-A^\dagger x} \Bigg(I_p + e^{-A^\dagger x + i\phi_\varepsilon(-iA^\dagger)t} \int_0^\infty ds\, e^{-A^\dagger s} C^\dagger C e^{-As} e^{-2Ax - i\phi_\varepsilon(iA)t} \times \\ \int_0^\infty dr\, e^{-Ar} BB^\dagger e^{-A^\dagger r} e^{-A^\dagger x} \Bigg)^{-1} e^{-A^\dagger x + i\phi_\varepsilon(-iA^\dagger)t} C^\dagger . \end{aligned} \tag{4.14}$$

Example 1 (*One Soliton Solution*)**.** Let us consider the triplet

$$A = (1 + 3i), \quad B = (1), \quad C = (1 + 2i). \tag{4.15}$$

Then it is easily verified that

$$Q = \begin{pmatrix} 1/2 \end{pmatrix}, \qquad N = \begin{pmatrix} 1/2 \end{pmatrix},$$

are the unique solutions to the Lyapunov equations

$$A^\dagger Q + QA = C^\dagger C, \qquad AN + NA^\dagger = BB^\dagger.$$

In the plot below we compare a one soliton solution obtained by using the Hasimoto map of the NLS solution with the PLR solution with $\varepsilon = 0.1$. The perturbation affects both the amplitude of the soliton curvature and the velocity of the bump. In this simple case the torsion remains constant in both cases (see Fig. 4.1). In the following two examples the effects will be more evident.

Example 2 (*Two Soliton Solution*)**.** Let us take the triplet

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \quad C = \begin{pmatrix} 3 & -2 \end{pmatrix}. \tag{4.16}$$

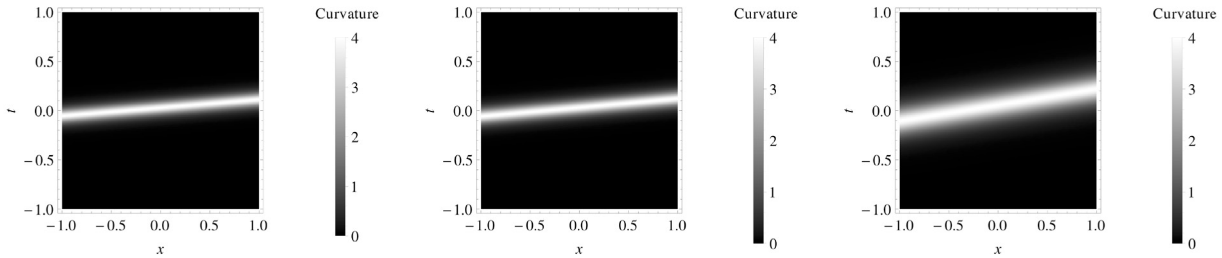


Fig. 4.1. Curvature of a one soliton solution corresponding to: on the left to NLS, in the center to the PLR with $\varepsilon = 0.01$ and on the right to the PLR with $\varepsilon = 0.1$. The triplet is given in (4.15).

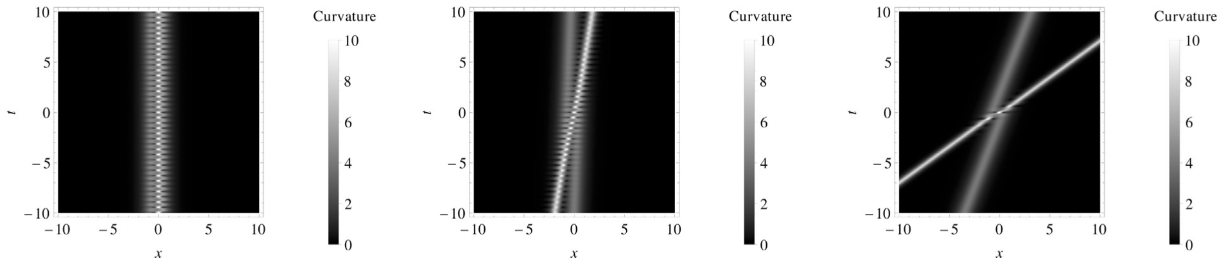


Fig. 4.2. Curvature of a two soliton solution corresponding to: on the left to NLS, in the center to the PLR with $\varepsilon = 0.01$ and on the right to the PLR with $\varepsilon = 0.1$. The triplet used to generate these solutions is (4.16).

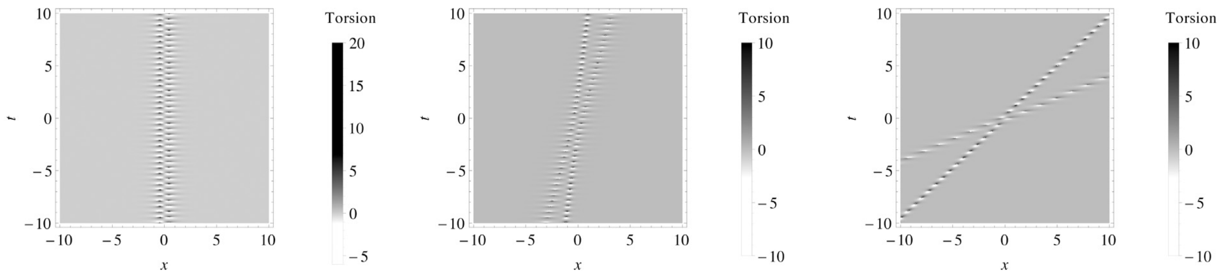


Fig. 4.3. Torsion of a two soliton solution corresponding to: on the left to NLS, in the center to the PLR with $\varepsilon = 0.01$ and on the right to the PLR with $\varepsilon = 0.1$. The triplet used to generate these solutions is (4.16).

The unique solutions of the Lyapunov equations

$$A^\dagger Q + QA = C^\dagger C, \qquad AN + NA^\dagger = BB^\dagger,$$

are given as follows

$$Q = \begin{pmatrix} \frac{9}{4} & -2 \\ -2 & 2 \end{pmatrix}, \qquad N = \begin{pmatrix} \frac{9}{4} & 2 \\ 2 & 2 \end{pmatrix}.$$

Below we compare the plots of curvature (see Fig. 4.2) and torsion (see Fig. 4.3) of the NLS equation with those of the PLR equation for different values of ε.

In this case the qualitative effects of the perturbation are evident. In fact, we choose a particular two soliton solution of NLS related to a coupled state with fixed mean position: the main qualitative effect of the PLR perturbation is the decoupling of the solution that now appears to behave as an asymptotically free two soliton solution.

Example 3 (*Double Pole Soliton Solution*)**.** Let us consider the following triplet

$$A = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 \end{pmatrix}, \tag{4.17}$$

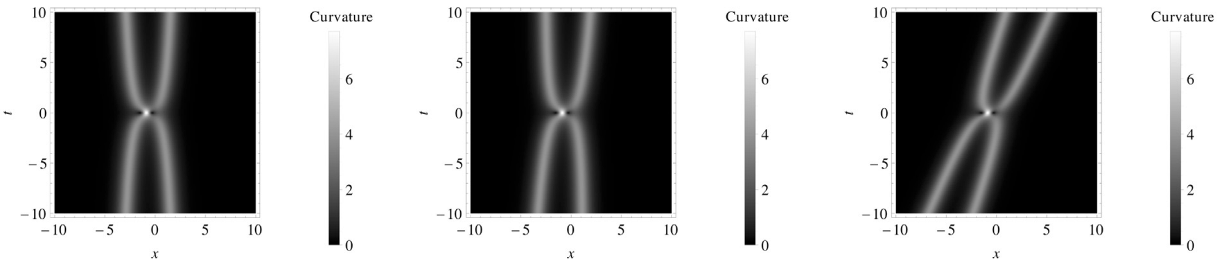


Fig. 4.4. Curvature of a double pole solution corresponding to: on the left to NLS, in the center to the PLR with $\varepsilon = 0.01$ and on the right to the PLR with $\varepsilon = 0.1$. The triplet used to generate these solutions is (4.17).

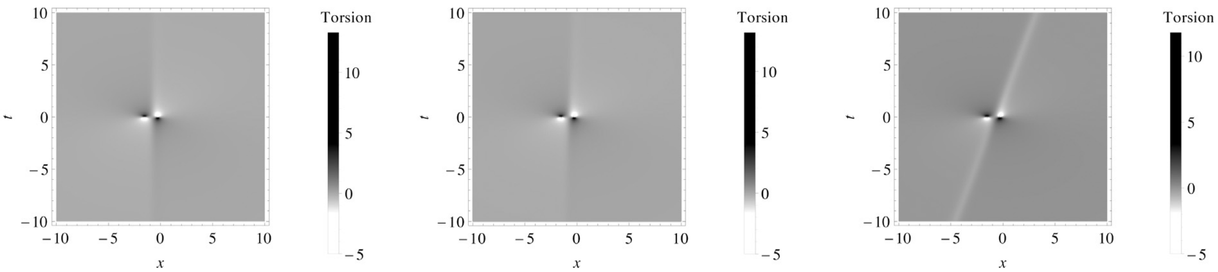


Fig. 4.5. Torsion of a double pole solution corresponding on the left to NLS, on the center to the PLR with $\varepsilon = 0.01$ and on the right to the PLR with $\varepsilon = 0.1$. The triplet used to generate these solutions is (4.17).

where the matrix A is not diagonalizable, namely it is a Jordan block of dimension two. It is easy to verify that Q and N are given by

$$Q = \begin{pmatrix} \frac{1}{2} & \frac{3}{4} \\ \frac{3}{4} & \frac{5}{4} \end{pmatrix}, \qquad N = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix}.$$

Below we compare the plots of curvature (see Fig. 4.4) and torsion (see Fig. 4.5) of the NLS equation with those of the PLR equation for different values of ε.

In this case the correction does not change the qualitative pole-like behavior of the solution. In the celebrated paper [27] the authors remark that the distance between the two bumps in the modulus of the a two-pole solution grows logarithmically in time. In the paper [10] the authors showed that a similar behavior is present also for the Hirota equation: the main difference is that in this case the "center of the mass" of the solution[3] is not at rest as in the NLS case. The PLR equation displays a qualitative behavior similar to the Hirota equation: the curvature of the solution the reciprocal distance of the two bumps grows sub-linearly in time and the velocity of the center of mass in not zero.

5. Conclusions and further developments

In this paper we combine two rather different approaches to integrability: Bi-Hamiltonian structures and inverse scattering transform. Such a combination allows us to explicitly find soliton-like solutions of the Pohlmeyer–Lund–Regge model. The interest for this equation (born in string-theory) has been renewed by the fact that it is the prototype of a nonlocal equation obtained as a sum of infinite commuting flows (a whole hierarchy in fact). The paper [2] has been one of the first works on this subject. In the fluid-dynamics context, the authors of [2] studied the effect to the NLS evolution given by the sum of the complex modified Korteweg–de Vries equation which is its first symmetry. The key property of this system is that such sum preserves the integrability of the evolution. A natural question, addressed in [3] for the focusing NLS, is what happens if one sums a whole hierarchy. In this work we contribute to this research line studying the solutions of the PLR equation and generalizing the construction to any bi-Hamiltonian hierarchy.

[3] As usual the center of mass X_f of a function f is defined as

$$X_f = \left(\int_{\mathbb{R}} x f(x) \mathrm{d}x\right) / \left(\int_{\mathbb{R}} f(x) \mathrm{d}x\right).$$

.

Even though in this work we focus our attention on PLR due to its relevance in the literature, our long term interest is the analysis of nonlocality properties of integrable systems. We do not see any obstruction to applying our approach to all of the nonlocal integrable systems obtained using the method of Section 2 starting, for example, from the system (2.20). We hope that such constructions could give some hint to the study of the Landau–Lifshitz equation, which is another nonlocal generalization of the nonlinear Schrödinger equation. The main step towards this goal is to find a "decomposition" of the nonlocality in an infinite number of local flows where the inverse scattering transform applies.

Another interesting research direction involves the study of the nonlocal operators (2.7) naturally arising in the framework of [2]. They seem to share some properties of the so-called "inertia operators" naturally arising in the study of some classes of non-evolutionary integrable systems such as the Camassa–Holm equation and, more generally, the tri-Hamiltonian dual construction discovered by Olver and Rosenau [35]. The study of these operators following the lines suggested in [35] could lead to the existence of solutions of PLR which are not deformations of solutions of the NLS equation.

Finally, a perhaps more technical open question could be the study of nonlocal Poisson structures (2.13) naturally arising from the flow summation procedure and, in particular, their relation to the analogue generator of Poisson structures related to the algebra sl_2 presented in [24,25].

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Appendix. Time evolution

In this subsection, we establish the time evolution of the scattering data associated to the PLR equation.

Assume that q and its successive (weak) derivatives belong to $L^1(\mathbb{R})$, so the potentials q and their successive derivatives vanish as $x \to \pm\infty$. Let us define the Jost matrices as those solutions of the ZS system $Z_x = (-i\lambda\sigma_3 + Q)Z$ which satisfy the asymptotic conditions (3.3), (3.6) and (3.7). Then the Jost matrices are given by $\Psi = ZC_\Psi^{-1}$ and $\Phi = ZC_\Phi^{-1}$, where $Z = Z(\lambda, x; t)$ is an invertible matrix function satisfying the AKNS pair $Z_x = UZ$ and $Z_t = VZ$ and C_Ψ and C_Φ do not depend on x. Then

$$\Psi_t = V\Psi - \Psi[C_\Psi]_t C_\Psi^{-1}, \tag{A.1a}$$

$$\Phi_t = V\Phi - \Phi[C_\Phi]_t C_\Phi^{-1}, \tag{A.1b}$$

implying

$$[C_\Psi]_t C_\Psi^{-1} = \Psi^{-1}V\Psi - \Psi^{-1}\Psi_t, \tag{A.2a}$$

$$[C_\Phi]_t C_\Phi^{-1} = \Phi^{-1}V\Phi - \Phi^{-1}\Phi_t. \tag{A.2b}$$

Now note that the left-hand sides of (A.2) do not depend on $x \in \mathbb{R}$, whereas the asymptotic forms of Ψ and V as $x \to +\infty$ and of Φ and V as $x \to -\infty$ are diagonal matrices. To see this, note that $V \sim \beta(\lambda)\sigma_3$ for a convenient scalar function $\beta(\lambda)$, because the potential q and its successive derivatives vanish as $x \to \pm\infty$. Consequently,

$$[C_\Psi]_t C_\Psi^{-1} = [C_\Phi]_t C_\Phi^{-1} = \sum_{j=0}^{\infty} \begin{pmatrix} (-i\lambda)^j \alpha_j & 0 \\ 0 & (+i\lambda)^j \delta_j \end{pmatrix} = \begin{pmatrix} A(\lambda) & 0 \\ 0 & D(\lambda) \end{pmatrix}, \tag{A.3}$$

where

$$A(\lambda) = \sum_{n=0}^{\infty} A_{(n)}(\lambda) = \sum_{j=0}^{\infty} (-i\lambda)^j \alpha_j, \qquad D(\lambda) = \sum_{n=0}^{\infty} D_{(n)}(\lambda) = \sum_{j=0}^{\infty} (+i\lambda)^j \delta_j.$$

$A_{(n)}(\lambda)$ and $D_{(n)}(\lambda)$ describe the asymptotic behavior as $x \to \infty$ of the evolution operator associated to the nth flux of the NLS hierarchy. For example, it is easy to verify that the expression of $A_1(\lambda)$ (NLS flow) and $A_2(\lambda)$ (mKdV flow) are given by, respectively $A_1(\lambda) = -D_1(\lambda) = 2i\lambda^2,\ A_2(\lambda) = -D_2(\lambda) = 4i\lambda^3$.

Therefore,

$$\begin{aligned}
[a_l]_t &= [\Phi^{-1}\Psi]_t = \Phi^{-1}\Psi_t - \Phi^{-1}\Phi_t\Phi^{-1}\Psi \\
&= \Phi^{-1}\left\{V\Psi - \Psi[C_\Psi]_t C_\Psi^{-1}\right\} - \Phi^{-1}\left\{V\Phi - \Phi[C_\Phi]_t C_\Phi^{-1}\right\} a_l \\
&= -a_l[C_\Psi]_t C_\Psi^{-1} + [C_\Phi]_t C_\Phi^{-1} a_l, \\
[a_r]_t &= [\Psi^{-1}\Phi]_t = \Psi^{-1}\Phi_t - \Psi^{-1}\Psi_t\Psi^{-1}\Phi \\
&= \Psi^{-1}\left\{V\Phi - \Phi[C_\Phi]_t C_\Phi^{-1}\right\} - \Psi^{-1}\left\{V\Psi - \Psi[C_\Psi]_t C_\Psi^{-1}\right\} a_r \\
&= -a_r[C_\Phi]_t C_\Phi^{-1} + [C_\Psi]_t C_\Psi^{-1} a_r.
\end{aligned}$$

Then (A.3) implies that the diagonal elements of the transmission matrices $a_l(\lambda;t) = \begin{pmatrix} a_{l1}(\lambda;t) & a_{l2}(\lambda;t) \\ a_{l3}(\lambda;t) & a_{l4}(\lambda;t) \end{pmatrix}$ and $a_r(\lambda;t) = \begin{pmatrix} a_{r1}(\lambda;t) & a_{r2}(\lambda;t) \\ a_{r3}(\lambda;t) & a_{r4}(\lambda;t) \end{pmatrix}$ are time independent, while

$$a_{l2}(\lambda;t) = e^{[A(\lambda)-D(\lambda)]t} a_{l2}(\lambda;0), \qquad a_{l3}(\lambda;t) = e^{[D(\lambda)-A(\lambda)]t} a_{l3}(\lambda;0),$$
$$a_{r2}(\lambda;t) = e^{[A(\lambda)-D(\lambda)]t} a_{r2}(\lambda;0), \qquad a_{r3}(\lambda;t) = e^{[D(\lambda)-A(\lambda)]t} a_{r3}(\lambda;0).$$

Defining the transmission coefficient $T_r(\lambda;t)$, the reflection coefficient from the right $R(\lambda;t)$, and the reflection coefficient from the left $L(\lambda;t)$ by $T_r = [1/a_{r1}] = [1/a_{l4}]$, $R = [a_{l2}/a_{l4}] = -[a_{r2}/a_{r1}]$, and $L = [a_{r3}/a_{r1}] = -[a_{l3}/a_{l4}]$, we see that

$$T_r(\lambda;t) = T_r(\lambda;0), \tag{A.4a}$$
$$R(\lambda;t) = e^{[A(\lambda)-D(\lambda)]t} R(\lambda;0), \qquad L(\lambda;t) = e^{[D(\lambda)-A(\lambda)]t} L(\lambda;0). \tag{A.4b}$$

Recalling the Fourier representations (3.11)

$$R(\lambda;t) = \int_{-\infty}^{\infty} dy\, e^{-i\lambda y} \rho(y;t), \qquad L(\lambda;t) = \int_{-\infty}^{\infty} dy\, e^{i\lambda y} \ell(y;t),$$

we obtain with the help of (A.4b)

$$\begin{aligned}
\rho_t(y;t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda\, e^{i\lambda y} e^{[A(\lambda)-D(\lambda)]t} \left(\sum_{j=0}^{\infty} [(-1)^j \alpha_j - \delta_j](i\lambda)^j \right) R(\lambda;0) \\
&= \sum_{j=0}^{\infty} [(-1)^j \alpha_j - \delta_j] \left(\frac{d}{dy}\right)^j \rho(y;t) = \sum_{j=0}^{\infty} [(-1)^j \alpha_j - \delta_j] \rho^{[j]}(y;t), \\
\ell_t(y;t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda\, e^{-i\lambda y} e^{[D(\lambda)-A(\lambda)]t} \left(\sum_{j=0}^{\infty} [(-1)^j \delta_j - \alpha_j](-i\lambda)^j \right) L(\lambda;0) \\
&= \sum_{j=0}^{\infty} [(-1)^j \delta_j - \alpha_j] \left(\frac{d}{dy}\right)^j \ell(y;t) = \sum_{j=0}^{\infty} [(-1)^j \delta_j - \alpha_j] \ell^{[j]}(y;t).
\end{aligned} \tag{A.5}$$

Let us assume that there are finitely many poles $ik_1, \ldots, ik_n$ of the transmission coefficient $T_r(\lambda)$ in the upper plane $\overline{\mathbb{C}}^+$ all of which are assumed to be simple. Following [29] and [31], we let θ_j stand for the residue of $T_r(\lambda)$ at $\lambda_j = ik_j$, i.e.,

$$\begin{aligned}
\theta_j &= \operatorname*{Res}_{\lambda=ik_j} (T_r(\lambda)) = \lim_{\lambda\to ik_j} (\lambda - ik_j) T_r(\lambda) \\
&= \lim_{\lambda\to ik_j} \frac{\lambda - ik_j}{a_{r1}(\lambda) - a_{r1}(ik_j)} = \left(\left.\frac{\mathrm{d}a_{r1}}{\mathrm{d}\lambda}\right|_{\lambda=ik_j} \right)^{-1}.
\end{aligned} \tag{A.6}$$

We then introduce the *norming constants* c_j such that

$$\phi(x,ik_j)\theta_j = \psi(x,ik_j)\left(ic_j\right), \qquad j = 1,2,\ldots,n. \tag{A.7}$$

Differentiating (A.7) with respect to t, we obtain

$$\phi_t(x,ik_j)\theta_j = \psi_t(x,ik_j)\left(ic_j\right) + \psi(x,ik_j)\, i[c_j]_t\,. \tag{A.8}$$

Using (A.1) and (A.2), we get

$$\begin{aligned}
\left\{ V(ik_j)\,\phi(x,ik_j) - A(ik_j)\,\phi(x,ik_j) \right\} \theta_j &= \left\{ V(ik_j)\,\psi(x,ik_j) - D(ik_j)\,\psi(x,ik_j) \right\} \left(i\,c_j\right) \\
&\quad + \psi(x,ik_j)\, i[c_j]_t.
\end{aligned}$$

Using (A.6) in the preceding equation we obtain

$$[c_j]_t = \left(D(ik_j) - A(ik_j)\right)\, c_j\,. \tag{A.9}$$

From (A.9) we easily obtain the time evolution of the norming constants:

$$c_j(t) = e^{-\left(A(ik_j)-D(ik_j)\right)t}\, c_j(0)\,. \tag{A.10}$$

An analogous relation to (A.10) can be established also for the norming constants corresponding to the poles of the transmission coefficient $\breve{T}_l = [1/a_{r1}] = [1/a_{l4}]$ (for the sake of readability, we omitted the argument λ in the preceding expression).

As a result of (A.5) and (4.1), we obtain

$$[\Omega_l]_t(y;t) = \sum_{j=0}^{\infty} \left[(-1)^j - \delta_j\right] \Omega_l^{(j)}(y;t) \tag{A.11}$$

and analogously for Ω_r. Using continuous approximation of multiple pole models by simple pole models while keeping invariant the reflection coefficient $R(\lambda, t)$, we can prove (A.11) in the multiple pole case as well.

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