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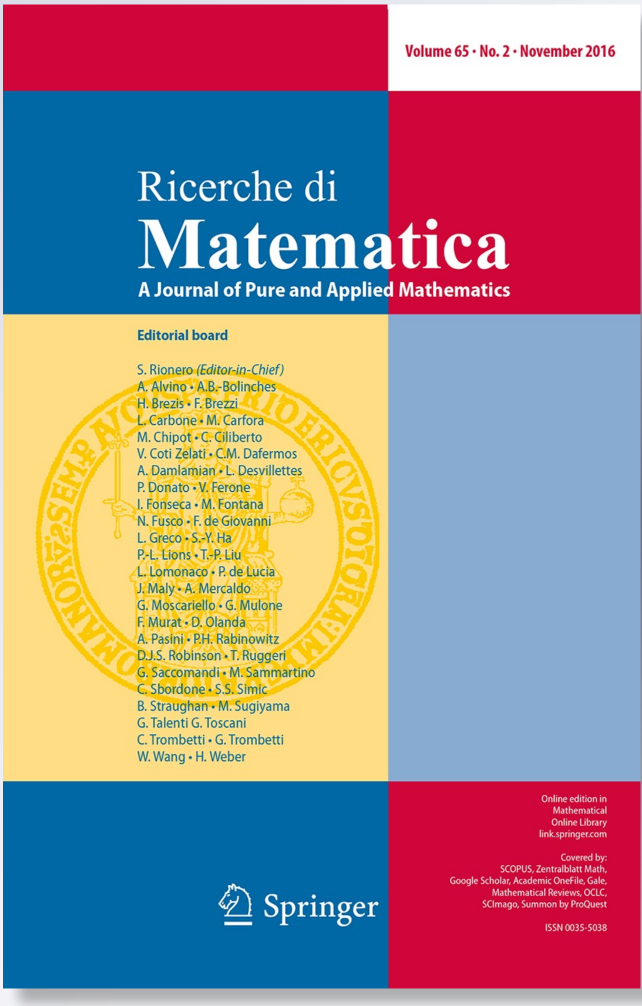

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Nonsmooth spin densities for continuous Heisenberg spin chains

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Abstract In this article we derive the triangular integral representations of the Jost matrix solutions of the continuous Heisenberg spin chain equation, as proposed by Zakharov and Takhtajan, without making smoothness assumptions on the spin density.

Keywords Volterra integral equations · Jost solutions · Continuous Heisenberg spin chain equation

Mathematics Subject Classification 45D05 · 35Q60

1 Introduction

The experimental observation of solitary waves in ferromagnetic materials at the nanoscale length [16] has sparked an increasing interest in such phenomena. These solitary waves are called magnetic droplets [10] and represent the analogues of the solitons propagating in nonlinear media. Magnetic droplets can be controlled by using both electric currents and magnetic fields [11] and have been proposed as a convenient way to achieve significant progress in nanomagnetism [14] and spintronics [3]. In particular, the magnetization dynamics of a thin, one-dimensional, unbounded,

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damped, and anisotropic ferromagnet are described by the Landau-Lifshitz (LL) equation [4,12,13]

$$\boldsymbol{m}_t = -\boldsymbol{m} \wedge \boldsymbol{h}_{eff} - \alpha \boldsymbol{m} \wedge (\boldsymbol{m} \wedge \boldsymbol{h}_{eff}), \tag{1}$$

where the spin density $\boldsymbol{m}(z,t)$ is a unit vector in $\mathbb{R}^3$ depending on position z and time t, $\boldsymbol{h}_{eff} = -(\boldsymbol{m}_{zz} + \boldsymbol{h}_0 + J\boldsymbol{m})$ is the effective magnetic field, and $\alpha \geq 0$ is the so-called "damping" parameter. Here $\boldsymbol{m}_{zz}$ represents the exchange term, $\boldsymbol{h}_0$ is the external magnetic field, and $J\boldsymbol{m}$ is the anisotropy term, where $J = \mathrm{diag}(J_1, J_2, J_3)$ accounts for the relative anisotropies.

Assuming isotropy (i.e., $J_1 = J_2 = J_3$), the absence of damping (i.e., $\alpha = 0$), and the absence of an external magnetic field (i.e., $\boldsymbol{h}_0 = 0$), Eq. (1) reads

$$\boldsymbol{m}_t = \boldsymbol{m} \wedge \boldsymbol{m}_{zz}, \tag{2}$$

where it is assumed that $\boldsymbol{m}(z,t) \to \boldsymbol{e}_3$ as $z \to \pm\infty$, $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ being the canonical basis of $\mathbb{R}^3$. Equation (2) is the well-known continuous Heisenberg ferromagnetic chain equation [2,17] first introduced by Weiss [21] to explain the spontaneous magnetization observed in ferromagnetic materials. In fact, Weiss suggested that the magnetic moments of atoms in ferromagnetic materials become oriented, even if there is no external magnetic field. Twenty years later it was proved by Heisenberg [9] that this spontaneous magnetization is mainly caused by a quantum mechanical effect called exchange interaction. About 50 years after Heisenberg's paper, Eq. (2) became the object of intensive study after Tjon and Wright [19] found its travelling wave solutions and Takhtadzhyan [18,22] showed it to be an integrable PDE in the sense that it can be solved by the so-called inverse scattering transform (IST) method. Let us recall [1,8,20] that the IST method consists of associating to the nonlinear integrable PDE a linear eigenvalue equation having the nonlinear PDE solution as a coefficient (or "potential") in such a way that the evolution of the nonlinear PDE solution is converted into the linear and often elementary evolution of its asymptotic (or "scattering") properties. To apply the IST method to Eq. (2), it is necessary (a) to develop the direct and inverse scattering theory of the associated linear eigenvalue problem

$$\boldsymbol{m}_z = i\lambda(\boldsymbol{m} \cdot \sigma), \tag{3}$$

where $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ contains the Pauli matrices $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, and $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\lambda \in \mathbb{R}$ is a spectral parameter, and (b) to specify the time evolution of the scattering data.

Introducing, for $\lambda \in \mathbb{R}$, the *Jost matrices* from the right $\Psi_H(z,\lambda)$ and from the left $\Phi_H(z,\lambda)$ as those 2×2 matrix solutions to Eq. (3) which behave as $e^{i\lambda z\sigma_3}$ as $z \to +\infty$ and $z \to -\infty$, respectively, we can derive the triangular representations [7]

$$\Psi_H(z,\lambda) = \boldsymbol{H}(z) \left\{ e^{i\lambda z\sigma_3} + \int_z^{\infty} d\hat{z}\, \boldsymbol{K}(z,\hat{z}) e^{i\lambda\hat{z}\sigma_3} \right\}, \tag{4a}$$

$$\Phi_H(z,\lambda) = \boldsymbol{J}(z) \left\{ e^{i\lambda z\sigma_3} + \int_{-\infty}^{z} d\hat{z}\, \boldsymbol{N}(z,\hat{z}) e^{i\lambda\hat{z}\sigma_3} \right\}, \tag{4b}$$

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where

$$\sup_{z\in\mathbb{R}}\left\{\int_z^{\infty} d\hat{z}\,\|\boldsymbol{K}(z,\hat{z})\| + \int_{-\infty}^{z} d\hat{z}\,\|\boldsymbol{N}(z,\hat{z})\|\right\} < +\infty$$

and $\boldsymbol{H}(z)$ and $\boldsymbol{J}(z)$ are continuous matrix functions of $z \in \mathbb{R}$ which converge to the identity matrix I_2 as $z \to +\infty$ and $z \to -\infty$, respectively. The expressions in (4) between braces are known to satisfy the focusing Zakharov-Shabat system [18,22]

$$\boldsymbol{V}_z = (i\lambda\sigma_3 + \boldsymbol{Q})\boldsymbol{V}, \tag{5}$$

where the potential $\boldsymbol{Q}(z,t)$ is skew-hermitian and anticommutes with σ_3, and $\int_{-\infty}^{\infty} dz\,\|\boldsymbol{Q}(z,t)\|$ converges for each $t \in \mathbb{R}$. Denoting the Jost matrices of the Zakharov-Shabat system (5) by $\Psi_{\text{zs}}(z,\lambda)$ and $\Phi_{\text{zs}}(z,\lambda)$ (see, e.g., [1,8,20]), we obtain the gauge transformations [18,22]

$$\boldsymbol{m}(z)\cdot\boldsymbol{\sigma} = \Psi_{\text{zs}}(z,0)^{-1}\sigma_3\Psi_{\text{zs}}(z,0) = \Phi_{\text{zs}}(z,0)^{-1}\sigma_3\Phi_{\text{zs}}(z,0). \tag{6}$$

Using that $\Psi_H(z,0) = \Phi_H(z,0) = I_2$, we easily see that $\Psi_{\text{zs}}(z,0) = \boldsymbol{H}(z)^{-1}$ and $\Phi_{\text{zs}}(z,0) = \boldsymbol{J}(z)^{-1}$. Under the hypothesis that $\boldsymbol{m}(z) - \boldsymbol{e}_3$ and its distributional derivative $\boldsymbol{m}_z$ have their entries in $L^1(\mathbb{R})$, a consistent IST method for Eq. (2) was formulated in [7] which relies on the so-called triangular representations (4) and lead to explicit multisoliton solutions.

The seemingly natural triangular representations (4) have been dubbed "alternative," because they differ from those previously introduced by Zakharov and Takhtajan [18,22]. In fact, instead of (4) these authors have proposed the triangular representations

$$\Psi_H(z,\lambda) = e^{i\lambda z\sigma_3} + \lambda\int_z^{\infty} d\hat{z}\,\breve{\boldsymbol{K}}(z,\hat{z})e^{i\lambda\hat{z}\sigma_3}, \tag{7a}$$

$$\Phi_H(z,\lambda) = e^{i\lambda z\sigma_3} + \lambda\int_{-\infty}^{z} d\hat{z}\,\breve{\boldsymbol{N}}(z,\hat{z})e^{i\lambda\hat{z}\sigma_3}. \tag{7b}$$

Either equation (7) contains a factor λ in front of the integral sign which makes it inconvenient to study the $\lambda \to \pm\infty$ behavior of the Jost matrices. These representations were shown to be inconsistent with some of their other equations relevant to applying the IST method [7].

In this article we shall derive the triangular representations (7) under the sole assumption that $\boldsymbol{m}(z) - \boldsymbol{e}_3$ has its entries in $L^2(\mathbb{R})$. The main result will be stated and proved in Sect. 2. Although these representations do not allow one to develop an obvious direct and inverse scattering theory of the linear eigenvalue problem (3), they imply $\lambda \to \pm\infty$ asymptotics of the Jost solutions and scattering coefficients, where the leading terms are almost periodic functions of $\lambda \in \mathbb{R}$ [5]. Although we shall not pursue such an asymptotic analysis in this paper, it could in principle be used to analyze Heisenberg spin chains, where the spin density has jump discontinuities. We observe that recently the study of the long time behavior of the Volterra equations in

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a different, but significative, context has been performed in [6,15]. At the end of the paper we prove the representations (7) under the assumptions made in [7] instead of under the more general assumptions of Sect. 2.

2 Main theorem

In this section we prove the following main theorem.

Theorem 1 *Suppose $\boldsymbol{m}(z)$ is a unit vector function of $z \in \mathbb{R}$ such that*

$$\int_{-\infty}^{\infty} dz\, \|\boldsymbol{m}(z) - \boldsymbol{e}_3\|^2 < +\infty. \tag{8}$$

Then the Jost matrices $\Psi_H(z,\lambda)$ and $\Phi_H(z,\lambda)$ have the triangular representations (7), where

$$\int_{z}^{\infty} d\hat{z}\, \|\breve{\boldsymbol{K}}(z,\hat{z})\|^2 \le \int_{z}^{\infty} d\hat{z}\, \|\boldsymbol{m}(\hat{z}) - \boldsymbol{e}_3\|^2, \tag{9a}$$

$$\int_{-\infty}^{z} d\hat{z}\, \|\breve{\boldsymbol{N}}(z,\hat{z})\|^2 \le \int_{-\infty}^{z} d\hat{z}\, \|\boldsymbol{m}(\hat{z}) - \boldsymbol{e}_3\|^2. \tag{9b}$$

We first remark that, for $(x,\lambda) \in \mathbb{R}^2$, $\Psi(x,\lambda)$ and $\Phi(x,\lambda)$ are unitary matrices of determinant 1 and hence belong to the group $SU(2)$. Thus if (7) is true, then

$$\breve{\boldsymbol{K}}(z,\hat{z}) = \begin{pmatrix} \breve{K}_1(z,\hat{z}) & -\breve{K}_2(z,\hat{z})^* \\ \breve{K}_2(z,\hat{z}) & \breve{K}_1(z,\hat{z})^* \end{pmatrix}, \quad \breve{\boldsymbol{N}}(z,\hat{z}) = \begin{pmatrix} \breve{N}_1(z,\hat{z})^* & \breve{N}_2(z,\hat{z}) \\ -\breve{N}_2(z,\hat{z})^* & \breve{N}_1(z,\hat{z}) \end{pmatrix}, \tag{10}$$

where the asterisk denotes the complex conjugate. We therefore work with the first column of $\breve{\boldsymbol{K}}(z,\hat{z})$ and the second column of $\breve{\boldsymbol{N}}(z,\hat{z})$ instead of with the 2×2 matrices $\breve{\boldsymbol{K}}(z,\hat{z})$ and $\breve{\boldsymbol{N}}(z,\hat{z})$ themselves. Due to the symmetry (10), the euclidean norms of the columns of $\breve{\boldsymbol{K}}(z,\hat{z})$ and $\breve{\boldsymbol{N}}(z,\hat{z})$ coincide with the spectral norms of the corresponding 2×2 matrices.

We shall first prove Theorem 1 for piecewise constant spin densities coinciding with $\boldsymbol{e}_3$ outside a set of compact support. We shall consider values defined on intervals of lengths that are integer multiples of a given positive d. The density of such vector functions $\boldsymbol{m}(z) - \boldsymbol{e}_3$ in $L^2(\mathbb{R})$ will then allow us to derive the most general result by continuous extension.

Proof **Part 1**. Let us first analyze how $\breve{K}_1(z,\hat{z})$ and $\breve{K}_2(z,\hat{z})$ change if $\boldsymbol{m}(z) - \boldsymbol{e}_3$ is replaced by a constant vector on a finite interval without modifying it on the half-line to its right. More precisely, let $\boldsymbol{m}(z) = \tilde{\boldsymbol{m}}(z)$ be known for $z > M$ and equal the constant vector $\boldsymbol{\mu} = \begin{pmatrix}\mu_1 & \mu_2 & \mu_3\end{pmatrix}^T$ for $L < z < M$ and the canonical basis vector $\boldsymbol{e}_3$ for $z < L$. Then, using the continuity of $\Psi_H(z,\lambda)$ in $(z,\lambda) \in \mathbb{R}^2$, we get

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$$\Psi_H(z,\lambda)=\begin{cases}\tilde{\Psi}_H(z,\lambda), & z\ge M,\\ e^{-i\lambda(M-z)[\boldsymbol{\mu}\cdot\boldsymbol{\sigma}]}\tilde{\Psi}_H(M,\lambda), & L\le z\le M,\\ e^{-i\lambda(L-z)\sigma_3}e^{-i\lambda(M-L)[\boldsymbol{\mu}\cdot\boldsymbol{\sigma}]}\tilde{\Psi}_H(M,\lambda), & z\le L,\end{cases}$$

where

$$\tilde{\Psi}_H(z,\lambda)=e^{i\lambda z\sigma_3}+\lambda\int_z^\infty d\hat{z}\,\tilde{\boldsymbol{K}}(z,\hat{z})e^{i\lambda\hat{z}\sigma_3}.$$

Then $\breve{\boldsymbol{K}}(z,\hat{z})=\tilde{\boldsymbol{K}}(z,\hat{z})$ for $\hat{z}\ge z\ge M$. For $z\le L$ we have

$$\begin{aligned}\Psi_H(z,\lambda)=e^{-i\lambda(L-z)\sigma_3}\Psi_H(L,\lambda)&=e^{i\lambda z\sigma_3}+\lambda\int_L^\infty dy\,e^{-i\lambda(L-z)\sigma_3}\breve{\boldsymbol{K}}(L,y)e^{i\lambda y\sigma_3}\\ &=e^{i\lambda z\sigma_3}+\lambda\int_z^\infty d\hat{z}\,\breve{\boldsymbol{K}}(z,\hat{z})e^{i\lambda\hat{z}\sigma_3},\end{aligned}$$

so that

$$\breve{K}_1(z,\hat{z})=\breve{K}_1(L,L+\hat{z}-z),\qquad \breve{K}_2(z,\hat{z})=\breve{K}_2(L,z+\hat{z}-L).$$

Next, for $L\le z\le M$ we have

$$\begin{aligned}\frac{\Psi_H(z,\lambda)-e^{i\lambda z\sigma_3}}{\lambda}&=e^{-i\lambda(M-z)[\boldsymbol{\mu}\cdot\boldsymbol{\sigma}]}\frac{\tilde{\Psi}_H(M,\lambda)-e^{i\lambda M\sigma_3}}{\lambda}\\ &\quad+\frac{e^{-i\lambda(M-z)[\boldsymbol{\mu}\cdot\boldsymbol{\sigma}]}-e^{-i\lambda(M-z)\sigma_3}}{\lambda}e^{i\lambda M\sigma_3}\\ &=e^{-i\lambda(M-z)[\boldsymbol{\mu}\cdot\boldsymbol{\sigma}]}\int_M^\infty dy\,\tilde{\boldsymbol{K}}(M,y)e^{i\lambda y\sigma_3}\\ &\quad-\tfrac{i}{2}[(\boldsymbol{\mu}\cdot\boldsymbol{\sigma})-\sigma_3]\int_z^{2M-z}d\hat{z}\,e^{i\lambda\hat{z}\sigma_3}.\end{aligned}$$

By using that $e^{-i\lambda(M-z)[\boldsymbol{\mu}\cdot\boldsymbol{\sigma}]}=\cos[\lambda(M-z)]I_2-i\sin[\lambda(M-z)](\boldsymbol{\mu}\cdot\boldsymbol{\sigma})$, for $L\le z\le M$ we have

$$\begin{aligned}\breve{K}_1(z,\hat{z})&=\tfrac12\Big[(1+\mu_3)\tilde{K}_1(M,M+\hat{z}-z)+(\mu_1-i\mu_2)\tilde{K}_2(M,M+\hat{z}-z)\Big]\\ &\quad+\tfrac12\Big[(1-\mu_3)\tilde{K}_1(M,z+\hat{z}-M)-(\mu_1-i\mu_2)\tilde{K}_2(M,z+\hat{z}-M)\Big]\\ &\quad+\tfrac{i}{2}(1-\mu_3)\chi_{(z,2M-z)}(\hat{z}),\\ \breve{K}_2(z,\hat{z})&=\tfrac12\Big[(1-\mu_3)\tilde{K}_2(M,M+\hat{z}-z)+(\mu_1+i\mu_2)\tilde{K}_1(M,M+\hat{z}-z)\Big]\\ &\quad+\tfrac12\Big[(1+\mu_3)\tilde{K}_2(M,z+\hat{z}-M)-(\mu_1+i\mu_2)\tilde{K}_1(M,z+\hat{z}-M)\Big]\\ &\quad-\tfrac{i}{2}(\mu_1+i\mu_2)\chi_{(z,2M-z)}(\hat{z}),\end{aligned}$$

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where $\chi_E(z) = 1$ for $z \in E$ and $\chi_E(z) = 0$ for $z \notin E$. In other words,

$$\begin{pmatrix} \breve{K}_1(z,\hat{z}) \\ \breve{K}_2(z,\hat{z}) \end{pmatrix} = P_{\boldsymbol{\mu}}^{+} \begin{pmatrix} \tilde{K}_1(M, M+\hat{z}-z) \\ \tilde{K}_2(M, M+\hat{z}-z) \end{pmatrix} + \tfrac{i}{2} \begin{pmatrix} 1-\mu_3 \\ -(\mu_1+i\mu_2) \end{pmatrix}, \tag{11a}$$

if $L \le z < \hat{z} \le 2M - z$, and

$$\begin{pmatrix} \breve{K}_1(z,\hat{z}) \\ \breve{K}_2(z,\hat{z}) \end{pmatrix} = P_{\boldsymbol{\mu}}^{+} \begin{pmatrix} \tilde{K}_1(M, M+\hat{z}-z) \\ \tilde{K}_2(M, M+\hat{z}-z) \end{pmatrix} + P_{\boldsymbol{\mu}}^{-} \begin{pmatrix} \tilde{K}_1(M, \hat{z}+z-M) \\ \tilde{K}_2(M, \hat{z}+z-M) \end{pmatrix}, \tag{11b}$$

if $L \le z \le 2M - z < \hat{z}$. Here

$$P_{\boldsymbol{\mu}}^{+} = \tfrac{1}{2} \begin{pmatrix} 1+\mu_3 & \mu_1 - i\mu_2 \\ \mu_1 + i\mu_2 & 1-\mu_3 \end{pmatrix}, \qquad P_{\boldsymbol{\mu}}^{-} = \tfrac{1}{2} \begin{pmatrix} 1-\mu_3 & -\mu_1 + i\mu_2 \\ -\mu_1 - i\mu_2 & 1+\mu_3 \end{pmatrix},$$

are complementary orthogonal projections and hence have unit norm.

Part 2 . Let us compute $\breve{K}_1(z,\hat{z})$ and $\breve{K}_2(z,\hat{z})$ for piecewise constant $\boldsymbol{m}(z) - \boldsymbol{e}_3$. More precisely, assume that $I_0 = (-\infty, J)$ and $I_j = (M_j, M_{j+1})$, where $M_j = J + (j-1)d$ $(j = 1, \ldots, n)$, and $I_{n+1} = (J + nd, +\infty)$. Put

$$\boldsymbol{m}(z) = \begin{cases} \boldsymbol{m}_j, & z \in I_j, \;\; j = 1, \ldots, n, \\ \boldsymbol{e}_3, & z \in I_0 \cup I_{n+1}. \end{cases}$$

Writing $\boldsymbol{E}_\mu$ for the inhomogeneous column vector term in (11a), we immediately see that, for $z \in I_1 \cup \cdots \cup I_n$, $\breve{\boldsymbol{K}}(z,\hat{z})$ is the zero matrix for $\hat{z} < z$ and for $\hat{z}+z > 2[J+nd]$ and is constant on each of the n^2 congruent triangles produced by drawing the lines $z = M_j$, $\hat{z} = z + 2jd$, and $\hat{z} + z = 2M_{j+1}$ $(j = 0, 1, \ldots, n-1)$; the values on two triangles having a vertical segment as a common boundary are identical. The rectangular triangle with vertices (J, J), $(J+nd, J+nd)$, and $(J, J+2nd)$ has thus been subdivided into $\frac{1}{2}n(n-1)$ squares having vertical and horizontal diagonals and n triangles having one side along $z = M_0$ on which $\breve{K}_1(z,\hat{z})$ and $\breve{K}_2(z,\hat{z})$ take constant values (an example is shown in Fig. 1). Writing $\boldsymbol{Z}(s,r)$ $(r = 1, \ldots, n-s+1)$ for the constant value of $\breve{\boldsymbol{K}}(z,\hat{z})$ on the square/triangle having $\{(M_{s-1}, y) : M_{s+r-2} < y < M_{s+r}\}$ as its vertical diagonal/side, the $\frac{1}{2}n(n+1)$ values $\boldsymbol{Z}(s,r)$ can be expressed in $E_{\boldsymbol{m}_s}$ $(s = 1, \ldots, n)$ as follows:

$$\boldsymbol{Z}(s,1) = P_{\boldsymbol{m}_s}^{+} \boldsymbol{Z}(s+1,1) + \boldsymbol{E}_{\boldsymbol{m}_s}, \tag{12a}$$

$$\boldsymbol{Z}(s,r) = P_{\boldsymbol{m}_s}^{+} \boldsymbol{Z}(s+1,r) + P_{\boldsymbol{m}_s}^{-} \boldsymbol{Z}(s+1,r-1), \quad r = 2, \ldots, n-s, \tag{12b}$$

$$\boldsymbol{Z}(s,n-s+1) = P_{\boldsymbol{m}_s}^{-} \boldsymbol{Z}(s+1,n-s), \tag{12c}$$

where $s = 1, \ldots, n-1$ and $\boldsymbol{Z}(n,1) = \boldsymbol{E}_{\boldsymbol{m}_n}$. Equation (12a) follows by applying (11a) to the squares/triangles bordering $\hat{z} = z$, Eq. (12c) follows by applying (11b) to

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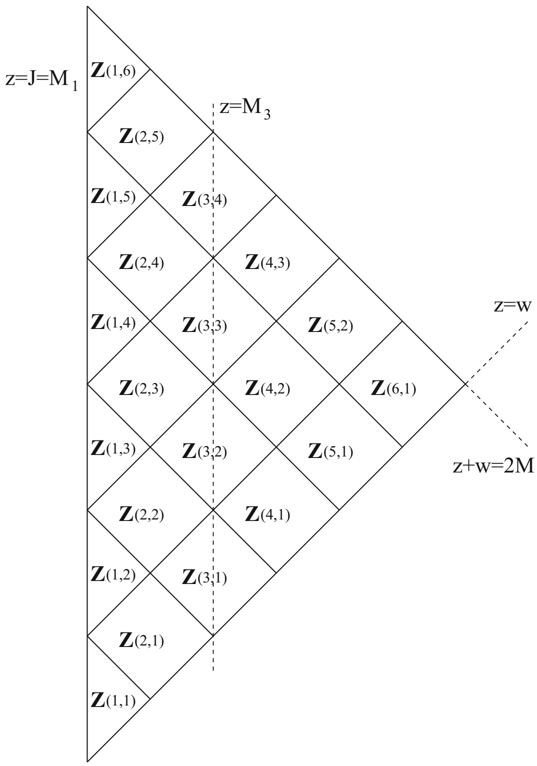
Fig. 1 For $n = 6$ the *triangle* with vertices (J, J), (M, M), and $(J, 2M - J)$ has been subdivided into *squares* and *triangles* in which $\breve{\boldsymbol{K}}(z, w)$ has the constant value $\mathbf{Z}(s, r)$



the squares/triangles bordering $z + \hat{z} = 2M$, and Eq. (12b) follows by applying (11b) to the remaining squares/triangles.

For $M_{s-1} \leq z \leq M_s$ we thus have

$$\int_z^\infty d\hat{z}\, \|\breve{\boldsymbol{K}}(z, \hat{z})\|^2 = 2(M_s - z)\Sigma_{s-1} + 2(z - M_{s-1})\Sigma_s, \tag{13}$$

where $\Sigma_s = \sum_{r=1}^{n-s+1} \|\boldsymbol{Z}(s, r)\|^2$ and $\Sigma_{n+1} = 0$. On the other hand, using that $\|P_{\boldsymbol{m}_s}^+ \boldsymbol{Z}\|^2 + \|P_{\boldsymbol{m}_s}^- \boldsymbol{Z}\|^2 = \|\boldsymbol{Z}\|^2$, we get by adding the squared norms in (12)

$$\begin{aligned} \Sigma_s &= \Sigma_{s+1} + \|P_{\boldsymbol{m}_s}^+ \boldsymbol{Z}(s+1, 1) + \boldsymbol{E}_{\boldsymbol{m}_s}\|^2 - \|P_{\boldsymbol{m}_s}^+ \boldsymbol{Z}(s+1, 1)\|^2 \\ &= \Sigma_{s+1} + \|\boldsymbol{E}_{\boldsymbol{m}_s}\|^2 + 2\,\mathrm{Re}\langle P_{\boldsymbol{m}_s}^+ \boldsymbol{Z}(s+1, 1), \boldsymbol{E}_{\boldsymbol{m}_s}\rangle, \end{aligned} \tag{14}$$

where

$$\boldsymbol{Z}(s, 1) = \boldsymbol{E}_{\boldsymbol{m}_s} + \sum_{j=s+1}^{n} P_{\boldsymbol{m}_s}^+ \dots P_{\boldsymbol{m}_{j-1}}^+ \boldsymbol{E}_{\boldsymbol{m}_j}.$$

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Applying (14) exactly $(n-s)$ times, we now observe that

$$\Sigma_s = \Sigma_n + \|\boldsymbol{E}_{m_s}\|^2 + \cdots + \|\boldsymbol{E}_{m_{n-1}}\|^2 + F_s + \cdots + F_{n-1},$$

where $F_s = 2\,\mathrm{Re}\langle P_{m_s}^{+}\boldsymbol{Z}(s+1,1), \boldsymbol{E}_{m_s}\rangle$. Hence, using Cauchy's inequality, we obtain

$$\Sigma_s \leq \Sigma_n + 2\left(\|\boldsymbol{E}_{m_s}\|^2 + \cdots + \|\boldsymbol{E}_{m_{n-1}}\|^2\right) + \sum_{r=s+1}^{n} \|\boldsymbol{Z}(r,1)\|^2.$$

Consequently, using (13) we get

$$\int_{M_s+d}^{\infty} d\hat{z}\, \|\breve{\boldsymbol{K}}(M_s+d,\hat{z})\|^2 \leq \int_{M_s}^{\infty} d\hat{z}\, \|\boldsymbol{e}_3 - \boldsymbol{m}(\hat{z})\|^2, \tag{15}$$

where we note that $\|\boldsymbol{E}_{\boldsymbol{\mu}}\|^2 = \frac{1}{2}(1-\mu_3)$.

Part 3. Let us now prove the main theorem for (special) piecewise constant $\boldsymbol{m}(z) - \boldsymbol{e}_3$. In fact, in (15) we replace d by $d/2^k$ for $k = 0, 1, 2, \ldots$. Then

$$\int_{J+\frac{jd}{2^k}}^{\infty} d\hat{z}\, \|\breve{\boldsymbol{K}}(J+\tfrac{jd}{2^k},\hat{z})\|^2 \leq \int_{J+\frac{(j-1)d}{2^k}}^{\infty} d\hat{z}\, \|\boldsymbol{e}_3 - \boldsymbol{m}(\hat{z})\|^2,$$

where $j = 1, 2, \ldots$ and d is an arbitrary positive number. Therefore, we get

$$\int_{z+\delta}^{\infty} d\hat{z}\, \|\breve{\boldsymbol{K}}(z+\delta,\hat{z})\|^2 \leq \int_{z}^{\infty} d\hat{z}\, \|\boldsymbol{m}(\hat{z}) - \boldsymbol{e}_3\|^2,$$

where $z = J + (jd/2^k)$ and $\delta = (d/2^k)$. Letting $k \to +\infty$ and hence $\delta \to 0^+$, we obtain

$$\int_{z}^{\infty} d\hat{z}\, \|\breve{\boldsymbol{K}}(z,\hat{z})\|^2 \leq \int_{z}^{\infty} d\hat{z}\, \|\boldsymbol{m}(\hat{z}) - \boldsymbol{e}_3\|^2. \tag{16}$$

Part 4. Let us now extend the main result from piecewise constant $\boldsymbol{m}(z) - \boldsymbol{e}_3$ to arbitrary $\boldsymbol{m}(z) - \boldsymbol{e}_3$ with entries in $L^2(\mathbb{R})$. In fact, Eq. (16) is extended to real unit vectors $\boldsymbol{m}(z)$ such that

$$\int_{-\infty}^{\infty} dz\, \|\boldsymbol{m}(z) - \boldsymbol{e}_3\|^2$$

converges by approximating $\boldsymbol{m}(z) - \boldsymbol{e}_3$ by piecewise constant vector functions of compact support where the inverse images of the values are intervals having lengths that are integer multiples of the same positive number. □

Using Plancherel's theorem, we obtain

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Corollary 1 *Suppose* $\boldsymbol{m}(z)$ *is a unit vector function of* $z \in \mathbb{R}$ *satisfying* (8). *Then the Jost matrices* $\Psi_H(z,\lambda)$ *and* $\Phi_H(z,\lambda)$ *satisfy*

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda \left\| \frac{\Psi_H(z,\lambda) - e^{i\lambda z \sigma_3}}{\lambda} \right\|^2 \leq \int_z^{\infty} d\hat{z}\, \|\boldsymbol{m}(\hat{z}) - \boldsymbol{e}_3\|^2,$$
$$\frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda \left\| \frac{\Phi_H(z,\lambda) - e^{i\lambda z \sigma_3}}{\lambda} \right\|^2 \leq \int_{-\infty}^{z} d\hat{z}\, \|\boldsymbol{m}(\hat{z}) - \boldsymbol{e}_3\|^2.$$

If we assume that $\boldsymbol{m}(z) - \boldsymbol{e}_3$ and its distributional derivative $\boldsymbol{m}_z$ have their entries in $L^1(\mathbb{R})$ (which is the assumption made in [7]), it is possible to prove that for every $z \in \mathbb{R}$ the distributional partial derivatives of $\boldsymbol{K}(z,\hat{z})$ and $\boldsymbol{N}(z,\hat{z})$ with respect to $\hat{z}$ have their entries in $L^1(z,+\infty)$ and $L^1(-\infty,z)$, respectively [7]. Putting

$$\breve{\boldsymbol{K}}(z,\hat{z}) = +i\boldsymbol{H}(z)\int_{\hat{z}}^{\infty} dw\, \boldsymbol{K}(z,w)\sigma_3,$$
$$\breve{\boldsymbol{N}}(z,\hat{z}) = -i\boldsymbol{J}(z)\int_{-\infty}^{\hat{z}} dw\, \boldsymbol{N}(z,w)\sigma_3,$$

we easily write the representations (4) in the form (7), where

$$\boldsymbol{H}(z) = I_2 + i\breve{\boldsymbol{K}}(z,z)\sigma_3, \qquad \boldsymbol{J}(z) = I_2 - i\breve{\boldsymbol{N}}(z,z)\sigma_3. \tag{17}$$

Here $\breve{\boldsymbol{K}}(z,z)$ and $\breve{\boldsymbol{N}}(z,z)$ are the limits of $\breve{\boldsymbol{K}}(z,\hat{z})$ and $\breve{\boldsymbol{N}}(z,\hat{z})$ as $\hat{z} \to z^{\pm}$, respectively. Equations (17) and (6) are easily seen to imply the gauge transformations

$$\boldsymbol{m}(z)\cdot\boldsymbol{\sigma} = \left[I_2 + i\breve{\boldsymbol{K}}(z,z)\sigma_3\right]^{-1}\sigma_3\left[I_2 + i\breve{\boldsymbol{K}}(z,z)\sigma_3\right]$$
$$= \left[I_2 - i\breve{\boldsymbol{N}}(z,z)\sigma_3\right]^{-1}\sigma_3\left[I_2 - i\breve{\boldsymbol{N}}(z,z)\sigma_3\right]$$

formulated by Zakharov and Takhtajan [18,22].

We observe that the above hypotheses on $\boldsymbol{m}(z)$ imply that $\boldsymbol{m}(z) - \boldsymbol{e}_3$ is a continuous function of $z \in \mathbb{R}$ having its entries in $L^2(\mathbb{R})$. We have thus proven Theorem 1 again, though under the more restrictive assumptions of [7].

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