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Scattering data computation for the Zakharov–Shabat system with non-smooth potentials

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ABSTRACT

In this paper we present a variant of the method for the scattering data computation for the Zakharov–Shabat system, recently proposed by the authors. The algorithm that characterizes this variant allows us to compute the scattering data also in the presence of jump discontinuities of the initial potential.

1. Introduction

Over the last 50 years, due to their relevance in applications, much research has been conducted in the field of nonlinear partial differential equations (NPDEs) of integrable type [1,2]. Among them, a special role is played by the nonlinear Schrödinger (NLS) equation which governs the signal transmission in fiber optics [5,7], as well as in surface waves on deep water [1,2].

The initial value problem (IVP) for the NLS equation can be formulated as follows:

$$\begin{cases} \mathbf{i}u_t + u_{xx} \pm 2|u|^2 u = 0, \quad x \in \mathbb{R}, \quad t > 0 \\ u(x,0) = u_0(x), \quad x \in \mathbb{R} \end{cases} \tag{1}$$

where $\mathbf{i}$ denotes the imaginary unit, $u = u(x,t)$ is the unknown potential, the subscripts x and t designate partial derivatives with respect to position and time, $u_0 \in L^1(\mathbb{R})$ is the initial potential and the $\pm$ sign depends on symmetry properties of u. The plus sign regards the focusing case and the minus sign the defocusing case. As Zakharov and Shabat proved [11], the IVP for the NLS equation can be associated to the first order system of ordinary differential equations

$$\mathbf{i}\mathbf{J}\frac{\partial \mathbf{\Psi}}{\partial x}(\lambda, x) - \mathbf{V}(x)\mathbf{\Psi}(\lambda, x) = \lambda \mathbf{\Psi}(\lambda, x), \quad x \in \mathbb{R} \tag{2}$$

where $\lambda \in \mathbb{C}$ is a spectral parameter and

$$\mathbf{J} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \mathbf{V} = \mathbf{i}\begin{pmatrix} 0 & u_0 \\ v_0 & 0 \end{pmatrix} \tag{3}$$

with $v_0 = u_0^*$ in the focusing case and $v_0 = -u_0^*$ in the defocusing case. Here and in the sequel the asterisk denotes the complex conjugate.

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With the help of this system, known as the ZS system, the solution of (1) can theoretically be obtained from the initial potential u_0 by means of the so-called Inverse Scattering Transform (IST) technique.

An effective method to compute all the scattering data for the ZS system has been recently developed under the hypothesis that $u_0 \in C^0(\mathbb{R})$ [4]. In this paper we propose a variant of this method that allows us to apply the method even in the case $u_0 \notin C^0(\mathbb{R})$.

The paper is organized as follows. In Section 2 we recall the definition of the scattering data and the properties needed to the illustration of our method. In Section 3 we recall the five steps for computing the scattering data [4] under the assumption that $u_0 \in C^0(\mathbb{R})$. The technique that allows us to extend the method to the case $u_0 \notin C^0(\mathbb{R})$ is illustrated in Section 4. In Section 5 we consider an initial potential with a discontinuity jump to illustrate the effectiveness of our method. Finally Section 6 is devoted to the conclusions.

2. Scattering data: definitions and properties

We start by recalling the crucial role played in the computation of the scattering data by the Jost solutions [2], that is by the solutions of the ZS system (2) which satisfy the asymptotic conditions

$$(\bar{\boldsymbol{\Psi}}(\lambda,x),\ \boldsymbol{\Psi}(\lambda,x)) = e^{-\mathbf{i}\lambda\mathbf{J}x}(I+o(1)), \quad x\to+\infty \tag{4}$$

$$(\boldsymbol{\Phi}(\lambda,x),\ \bar{\boldsymbol{\Phi}}(\lambda,x)) = e^{-\mathbf{i}\lambda\mathbf{J}x}(I+o(1)), \quad x\to-\infty \tag{5}$$

where $\lambda\in\mathbb{R}$, I denotes the identity matrix and $\mathbf{J}$ is defined in (3).

Since these solutions satisfy the same linear first order system, there exist transition matrices

$$\mathbf{A}_\ell(\lambda) = \begin{pmatrix} a_{\ell1}(\lambda) & a_{\ell2}(\lambda) \\ a_{\ell3}(\lambda) & a_{\ell4}(\lambda) \end{pmatrix}, \qquad \mathbf{A}_r(\lambda) = \mathbf{A}_\ell(\lambda)^{-1} = \begin{pmatrix} a_{r1}(\lambda) & a_{r2}(\lambda) \\ a_{r3}(\lambda) & a_{r4}(\lambda) \end{pmatrix} \tag{6}$$

such that

$$(\bar{\boldsymbol{\Psi}}(\lambda,x),\ \boldsymbol{\Psi}(\lambda,x)) = (\boldsymbol{\Phi}(\lambda,x),\ \bar{\boldsymbol{\Phi}}(\lambda,x))\mathbf{A}_\ell(\lambda)$$

$$(\boldsymbol{\Phi}(\lambda,x),\ \bar{\boldsymbol{\Phi}}(\lambda,x)) = (\bar{\boldsymbol{\Psi}}(\lambda,x),\ \boldsymbol{\Psi}(\lambda,x))\mathbf{A}_r(\lambda).$$

Denoting by $\mathbb{C}^+$ and $\mathbb{C}^-$ the upper and lower half plane and by $\bar{\mathbb{C}}^+$ and $\bar{\mathbb{C}}^-$ their closures, respectively, the following analytic properties hold true. The Jost functions $\boldsymbol{\Psi}(\lambda,x)$ and $\boldsymbol{\Phi}(\lambda,x)$ are continuous in $\lambda\in\bar{\mathbb{C}}^+$, are analytic in $\lambda\in\mathbb{C}^+$, and behave as $e^{-i\lambda x\mathbf{J}}$ as $\lambda\to\infty$ in $\bar{\mathbb{C}}^+$, whereas $\bar{\boldsymbol{\Psi}}(\lambda,x)$ and $\bar{\boldsymbol{\Phi}}(\lambda,x)$ are continuous in $\lambda\in\bar{\mathbb{C}}^-$, are analytic in $\lambda\in\mathbb{C}^-$, and behave as $e^{-i\lambda x\mathbf{J}}$ as $\lambda\to\infty$ in $\bar{\mathbb{C}}^-$. We can then rewrite (4) and (5) as the Riemann–Hilbert problem

$$(\bar{\boldsymbol{\Psi}}(\lambda,x),\ \bar{\boldsymbol{\Phi}}(\lambda,x)) = (\boldsymbol{\Phi}(\lambda,x),\ \boldsymbol{\Psi}(\lambda,x))\mathbf{J}\mathbf{S}(\lambda)\mathbf{J}$$

where

$$\mathbf{S}(\lambda) = \begin{pmatrix} T(\lambda) & L(\lambda) \\ R(\lambda) & T(\lambda) \end{pmatrix}. \tag{7}$$

In (7) $T(\lambda)$ represents the transmission coefficient, $R(\lambda)$ denotes the reflection coefficients from the right and $L(\lambda)$ stands for the reflection coefficients from the left.

If $T(\lambda)$ has no poles in the complex upper half plane $\mathbb{C}^+$ (as occurs in the defocusing case), the transmission coefficient and the reflection coefficients are the only scattering data to identify. Otherwise, denoting by $\lambda_1,\ \dots,\ \lambda_n$ the so-called bound states, that is the finitely many poles of $T(\lambda)$ in $\mathbb{C}^+$, and by $m_1,\ \dots,\ m_n$ the corresponding multiplicities, we have to identify the parameters $\{n, m_j, \lambda_j\}$ as well as the coefficients $\{(\Gamma_\ell)_{js}, (\Gamma_r)_{js}\}$ of the spectral sums from the left and from the right

$$S_\ell(\alpha) = \sum_{j=1}^{n} e^{i\lambda_j\alpha} \sum_{s=0}^{m_j-1} (\Gamma_\ell)_{js}\frac{\alpha^s}{s!}, \quad \alpha\geq 0,$$

$$S_r(\alpha) = \sum_{j=1}^{n} e^{i\lambda_j^*\alpha} \sum_{s=0}^{m_j-1} (\Gamma_r)_{js}\frac{\alpha^s}{s!}, \quad \alpha\leq 0,$$

where $0!=1$.

The method developed in [4] allows us to compute all the scattering data, i.e. the spectral matrix $\mathbf{S}$ introduced in (7) as well as the discrete scattering data $\{\lambda_j, (\Gamma_\ell)_{js}, (\Gamma_r)_{js}\}$ whenever $S_\ell(\alpha)$ and $S_r(\alpha)$ are known in $N\geq M = m_1+\dots+m_n$ points.

To this end we need to compute the Marchenko kernels from the left $\Omega_\ell(\alpha)$ and from the right $\Omega_r(\alpha)$. These two kernels are, in fact, connected to the above spectral coefficients and spectral sums as follows:

$$\Omega_\ell(\alpha) = \rho(\alpha) + S_\ell(\alpha), \quad \text{for} \quad \alpha \geq 0$$
$$\Omega_r(\alpha) = \ell(\alpha) + S_r(\alpha), \quad \text{for} \quad \alpha \leq 0$$

where

$$\rho(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} R(\lambda) e^{i\lambda\alpha} d\lambda = \mathcal{F}^{-1}\{R(\lambda)\} \tag{8}$$

is the inverse Fourier transform of the reflection coefficient from the right $R(\lambda)$ and

$$\ell(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} L(\lambda) e^{-i\lambda\alpha} d\lambda = \frac{1}{2\pi} \mathcal{F}\{L(\lambda)\}, \tag{9}$$

apart from the factor $1/2\pi$, is the Fourier transform of the reflection coefficient from the left $L(\lambda)$.

We note that $\Omega_\ell(\alpha)$ and $\Omega_r(\alpha)$, respectively, reduce to:

(a) $S_\ell(\alpha)$ and $S_r(\alpha)$ if the reflection coefficients vanish (reflectionless case). This situation occurs for initial potentials leading to N-soliton solutions.
(b) $\rho(\alpha)$ and $\ell(\alpha)$ if there are no bound states. This situation occurs in the defocusing case and whenever $\|u_0\|_1 = \|v_0\|_1 < \frac{\pi}{2}$ [6].

3. The steps for computing the scattering data

1. Auxiliary functions. The first step for computing the Marchenko kernels as well as $\rho(\alpha)$ and $\ell(\alpha)$ consists of the computation of four pairs of auxiliary functions [4]. As explained in [4], for $y \geq x$, we have to compute two pairs of auxiliary functions

$$(\bar{K}^{\mathrm{up}}(x,y), \bar{K}^{\mathrm{dn}}(x,y)) \qquad \text{and} \qquad (K^{\mathrm{up}}(x,y),\ K^{\mathrm{dn}}(x,y))$$

and, for $y \leq x$, two other pairs of auxiliary functions

$$(\bar{M}^{\mathrm{up}}(x,y), \bar{M}^{\mathrm{dn}}(x,y)) \qquad \text{and} \qquad (M^{\mathrm{up}}(x,y),\ M^{\mathrm{dn}}(x,y)).$$

As shown in [4], their computation requires the solution of four systems of structured Volterra systems on unbounded domains.

2. Initial Marchenko kernels. Following [10, 2.50a and 2.50b] we can say that, for $y \geq x \geq 0$, the Marchenko kernel Ω_ℓ is connected to the auxiliary functions K^{dn} and $\bar{K}^{\mathrm{dn}}$ as follows:

$$\Omega_\ell(x+y) + \int_x^\infty K^{\mathrm{dn}}(x,z)\, \Omega_\ell(z+y)\, dz = -\bar{K}^{\mathrm{dn}}(x,y). \tag{10}$$

Similarly, for $y \leq x \leq 0$, the Marchenko kernel Ω_r is connected to the auxiliary functions M^{up} and $\bar{M}^{\mathrm{up}}$ in this way:

$$\Omega_r(x+y) + \int_{-\infty}^x M^{\mathrm{up}}(x,z)\, \Omega_r(z+y)\, dz = -\bar{M}^{\mathrm{up}}(x,y). \tag{11}$$

As a result, assuming known the auxiliary functions, (10) and (11) can be interpreted as structured Volterra integral equations having the initial Marchenko kernels Ω_ℓ and Ω_r as their unknowns.

Let us remark, that although the initial potential u_0 does not appear explicitly in (10) and (11), it is linked to these equations via the couples K and M. Under the assumption that the support of u_0 is bounded, that is

$$u_0(x) = 0, \qquad \text{for} \qquad a < x < b, \tag{12}$$

which could be considered acceptable as $|u_0(x)| \to 0$ for $|x| \to \infty$, provided $|a|$ and $|b|$ are large enough, an effective method to solve numerically (10) and (11) has been developed in [4].

3. Scattering matrix. As proposed in [10], the coefficients of the scattering matrix (7) can be represented as follows:

$$T(\lambda) = \frac{1}{a_{\ell 4}(\lambda)} = \frac{1}{a_{r1}(\lambda)},$$
$$L(\lambda) = \frac{a_{\ell 2}(\lambda)}{a_{\ell 4}(\lambda)} = -\frac{a_{r2}(\lambda)}{a_{r1}(\lambda)}, \qquad R(\lambda) = \frac{a_{r3}(\lambda)}{a_{r1}(\lambda)} = -\frac{a_{\ell 3}(\lambda)}{a_{\ell 4}(\lambda)}, \tag{13}$$

where the $\{a_{\ell j}(\lambda)\}$ and the $\{a_{rj}(\lambda)\}$ denote the entries of the transition matrices (6) from the left and from the right, respectively. The entries of the transition matrix $\{a_{\ell j}(\lambda)\}$ can then be represented in this way:

$$a_{\ell 1}(\lambda) = 1 - \int_{\mathbb{R}^+} e^{-i\lambda z}\,\bar{\Phi}^{\mathrm{dn}}(z)dz, \quad a_{\ell 2}(\lambda) = -\int_{\mathbb{R}} (e^{2i\lambda y}u_0(y) + e^{i\lambda y}\Phi^{\mathrm{dn}}(y))dy, \tag{14}$$

$$a_{\ell 3}(\lambda) = \int_{\mathbb{R}} (e^{-2i\lambda y}v_0(y) + e^{-i\lambda y}\bar{\Phi}^{\mathrm{up}}(y))dy \quad a_{\ell 4}(\lambda) = 1 + \int_{\mathbb{R}^+} e^{i\lambda z}\Phi^{\mathrm{up}}(z)dz, \tag{15}$$

where the four functions $\bar{\Phi}^{\mathrm{up}}$, $\bar{\Phi}^{\mathrm{dn}}$, Φ^{up}, Φ^{dn} are connected with the auxiliary functions as follows:

$$\bar{\Phi}^{\mathrm{dn}}(z) = \int_{\mathbb{R}} u_0(y)\bar{K}^{\mathrm{dn}}(y, y+z)dy, \qquad \Phi^{\mathrm{dn}}(z) = \int_{-\infty}^{\frac{z}{2}} u_0(y)K^{\mathrm{dn}}(y, z-y)dy, \tag{16}$$

$$\Phi^{\mathrm{up}}(z) = \int_{\mathbb{R}} v_0(y)K^{\mathrm{up}}(y, y+z)dy, \qquad \bar{\Phi}^{\mathrm{up}}(z) = \int_{-\infty}^{\frac{z}{2}} v_0(y)\bar{K}^{\mathrm{up}}(y, z-y)dy. \tag{17}$$

Similarly, we can say that

$$a_{r1}(\lambda) = 1 + \int_{\mathbb{R}^+} e^{i\lambda z}\Psi^{\mathrm{dn}}(z)dz, \quad a_{r2}(\lambda) = \int_{\mathbb{R}} (e^{2i\lambda y}u_0(y) + e^{i\lambda y}\bar{\Psi}^{\mathrm{dn}}(y))dy, \tag{18}$$

$$a_{r3}(\lambda) = -\int_{\mathbb{R}} (e^{-2i\lambda y}v_0(y) + e^{-i\lambda y}\Psi^{\mathrm{up}}(y))dy, \quad a_{r4}(\lambda) = 1 - \int_{\mathbb{R}^+} e^{-i\lambda z}\bar{\Psi}^{\mathrm{up}}(z)dz, \tag{19}$$

with

$$\Psi^{\mathrm{dn}}(z) = \int_{\mathbb{R}} u_0(y)M^{\mathrm{dn}}(y, y-z)dy, \quad \bar{\Psi}^{\mathrm{dn}}(z) = \int_{\frac{z}{2}}^{+\infty} u_0(y)\bar{M}^{\mathrm{dn}}(y, z-y)dy, \tag{20}$$

$$\Psi^{\mathrm{up}}(z) = \int_{\frac{z}{2}}^{+\infty} v_0(y)M^{\mathrm{up}}(y, z-y)dy, \quad \bar{\Psi}^{\mathrm{up}}(z) = \int_{\mathbb{R}} v_0(y)\bar{M}^{\mathrm{up}}(y, y-z)dy. \tag{21}$$

While the computation of the scattering matrix (7) is relatively simple by following the steps indicated, the approximation of the functions ρ and ℓ defined in (8) and (9) is more complicated as explained below.

4. Transforms of the reflection coefficients $R(\lambda)$ and $L(\lambda)$

To approximate $\rho(\alpha)$ and $\ell(\alpha)$ we have to compute the scattering coefficients by means of (14)–(21), then the reflection coefficients $R(\lambda)$ and $L(\lambda)$ by using (13) and, finally, $\rho(\alpha)$ and $\ell(\alpha)$ as indicated in (8) and (9). However, this procedure is effective only if the initial potential is smooth enough, that is if $u_0 \in C^0(\mathbb{R})$. If this is not the case, that is if $u_0 \in L^1(\mathbb{R})$ but $u_0 \notin C^0(\mathbb{R})$, as $R(\lambda)$ and $L(\lambda)$ decay too slowly as $\lambda \to \pm\infty$, the numerical computation of the Fourier transforms $\rho(\alpha)$ and $\ell(\alpha)$ requires a too large and expensive integral domain. Then, in order to overcome this problem and accelerate the numerical computation of the scattering data, we propose to approximate the Fourier transforms ρ and ℓ by solving the structured Fredholm integral equations specified in the following theorem [4]:

Theorem 3.1. *The function $\rho(\alpha)$ is the unique solution of the Fredholm integral equation*

$$\rho(\alpha) + \int_0^{\infty} \Phi^{\mathrm{up}}(z)\,\rho(z+\alpha)dz = -\frac{1}{2}v_0\left(\frac{\alpha}{2}\right) - \bar{\Phi}^{\mathrm{up}}(\alpha), \tag{22}$$

and $\ell(\alpha)$ is of the following

$$\ell(\alpha) + \int_0^{\infty} \Phi^{\mathrm{up}}(z)\,\ell(\alpha - z)dz = -\frac{1}{2}u_0\left(\frac{\alpha}{2}\right) - \Phi^{\mathrm{dn}}(\alpha), \tag{23}$$

being Φ^{up}, Φ^{dn} and $\bar{\Phi}^{\mathrm{up}}$ defined in (16)–(17).

5. Bound states and norming constants computation. The bound states with their multiplicities, if they exist, as well as the associated norming constants can be computed by using the matrix pencil method proposed in [3].

4. Our technique for computing $\rho(\alpha)$

Let us now describe the numerical method we propose to approximate $\rho(\alpha)$ by solving (22) under the hypothesis (12). As proved in [4], this assumption allows us to characterize the support of the kernel and of the right-hand side of (22). In fact, we can easily state that

$$\Phi^{\mathrm{up}}(z) \neq 0, \qquad z \in [0,\, 2(b-a)]$$
$$f(\alpha) = -\frac{1}{2} v_0\left(\frac{\alpha}{2}\right) - \bar{\Phi}^{\mathrm{up}}(\alpha) \neq 0, \qquad \alpha \in [2a,\, 2b].$$

Hence equation (22) can be also written as

$$\rho(\alpha) + \int_0^{2(b-a)} \Phi^{\mathrm{up}}(z)\, \rho(z+\alpha) dz = f(\alpha). \tag{24}$$

To solve it, fixing $N \in \mathbb{N}$ and taking $h = \frac{b-a}{N}$, we introduce the collocation points

$$\alpha_i = ih, \quad i = 0, \pm 1, \pm 2, ..., \pm 2N, ...,$$

and write

$$\rho(\alpha_i) + \int_0^{\alpha_{2N}} \Phi^{\mathrm{up}}(z)\rho(z+\alpha_i)dz = f(\alpha_i), \quad i = 0, \pm 1, \pm 2, ..., \pm 2N, ...,$$

where, as noted before, $f(\alpha_i) = 0$ for $\alpha_i < 2a$ and for $\alpha_i > 2b$.

Approximating the integral by the composite Simpson's quadrature formula and setting $\rho_i = \rho(\alpha_i)$ and $f_i = f(\alpha_i)$, we obtain

$$\rho_i + \sum_{j=0}^{2N} \widehat{a}_j \rho_{i+j} = f_i, \quad i \in \mathbb{Z}, \tag{25}$$

where $\widehat{a}_j = w_j \Phi^{\mathrm{up}}(z_j)$ with $w_0 = w_{2N} = \frac{h}{3}$, $w_{2k-1} = \frac{4}{3}h$, $k = 1, ..., N$ and $w_{2k} = \frac{2}{3}h$ for $k = 1, 2, ..., N-1$.

Let us now write (25) as

$$\sum_{j\in\mathbb{Z}} a_j \rho_{i+j} = f_i, \quad \text{with} \quad a_0 = 1 + \widehat{a}_0 \quad \text{and} \quad a_j = \begin{cases} \widehat{a}_j, & j = 1, ..., 2N \\ 0, & \text{otherwise.} \end{cases}$$

The matrix of this system is banded and Toeplitz, which implies that we can solve it by resorting to Fourier analysis. More precisely, multiplying each equation by z^i with $|z| = 1$, we get

$$\sum_{j\in\mathbb{Z}} a_j z^{-j} \rho_{i+j} z^{i+j} = f_i z^i, \quad i \in \mathbb{Z}.$$

Adding the equations we obtain the functional equation

$$a(z^{-1})\rho(z) = f(z)$$

where

$$a(z^{-1}) = \sum_{j\in\mathbb{Z}} a_j z^{-j}, \quad \rho(z) = \sum_{\ell \equiv i+j \in \mathbb{Z}} \rho_\ell z^\ell, \quad f(z) = \sum_{i\in\mathbb{Z}} f_i z^i.$$

Considering that, for h sufficiently small, $a(z^{-1})$ has no zeros on the unit circle, we can also write

$$\rho(z) = u(z^{-1}) f(z), \quad u(z^{-1}) \equiv a(z^{-1})^{-1}.$$

Notice that the coefficients of the Fourier series $u(z^{-1})$ decay exponentially, as the coefficients of $a(z^{-1})$ are zero for $|j|$ large enough.

Fixing then a positive integer M value, we approximate $u(z^{-1})$ by means of the Laurent polynomial of order $2M$

$$u_M(z^{-1}) = \sum_{r=-M}^{M} u_r z^{-r} \tag{26}$$

whose coefficients have to be computed. Increasing M, the approximating function $u_M(z^{-1})$ converges very fast to $u(z^{-1})$, because the coefficients of $a(z^{-1})^{-1}$ decay exponentially as $|j| \to \infty$.

Setting

$$\omega_k = e^{\mathbf{i}k\frac{2\pi}{2M+1}}, \quad k = 0, 1, ..., 2M$$

the $2M+1$ coefficients $\{u_r\}$ can be obtained by imposing that

$$u_M(\bar{\omega}_k) = \sum_{r=-M}^{M} u_r \bar{\omega}_k^r = a(\bar{\omega}_k)^{-1}, \quad k = 0, 1, ..., 2M. \tag{27}$$

In other words, the vector of coefficients $\{u_r\}$ of $u_M(z^{-1})$ defined in (26) is the unique solution of the linear system

$$\begin{pmatrix} \bar{\omega}_0^{-M} & \bar{\omega}_0^{-M+1} & \dots & \bar{\omega}_0^{M-1} & \bar{\omega}_0^{M} \\ \bar{\omega}_1^{-M} & \bar{\omega}_1^{-M+1} & \dots & \bar{\omega}_1^{M-1} & \bar{\omega}_1^{M} \\ \vdots & \vdots & \dots & \vdots & \vdots \\ \bar{\omega}_{2M-1}^{-M} & \bar{\omega}_{2M-1}^{-M+1} & \dots & \bar{\omega}_{2M-1}^{M-1} & \bar{\omega}_{2M-1}^{M} \\ \bar{\omega}_{2M}^{-M} & \bar{\omega}_{2M}^{-M+1} & \dots & \bar{\omega}_{2M}^{M-1} & \bar{\omega}_{2M}^{M} \end{pmatrix} \begin{pmatrix} u_{-M} \\ u_{-M+1} \\ \vdots \\ u_{M-1} \\ u_M \end{pmatrix} = \begin{pmatrix} a(\bar{\omega}_0)^{-1} \\ a(\bar{\omega}_1)^{-1} \\ \vdots \\ a(\bar{\omega}_{2M-1})^{-1} \\ a(\bar{\omega}_{2M})^{-1} \end{pmatrix}$$

whose matrix

$$(\mathbf{F}_{2M+1})_{i,j} = \bar{\omega}_i^j, \quad i = 0, 1, ..., 2M, \quad j = -M, -M+1, ..., M-1, M$$

is a Fourier matrix of order $2M+1$, which implies that [8]

$$(\mathbf{F}_{2M+1})^* \mathbf{F}_{2M+1} = (2M+1)\mathbf{I}_{2M+1},$$

where $\mathbf{I}_{2M+1}$ is the identity matrix of order $2M+1$.

As a result, setting

$$\mathbf{u}_M = (u_{-M}, u_{-M+1}, \dots, u_{M-1}, u_M)^T$$

and

$$\mathbf{b}_M = (a(\bar{\omega}_0)^{-1}, a(\bar{\omega}_1)^{-1}, \dots, a(\bar{\omega}_{2M-1})^{-1}, a(\bar{\omega}_{2M})^{-1})$$

the solution of system (27) is

$$\mathbf{u}_M = \frac{1}{2M+1} \mathbf{F}_{2M+1}^* \mathbf{b}_M.$$

The approximation of $\rho(z)$ of order $2(M+N)$ can then be obtained by means of the relation

$$\begin{aligned} \rho_M(z) = u_M(z^{-1}) f(z) = (u_{-M} z^M + u_{-M+1} z^{M-1} + \dots u_{M-1} z^{-M+1} + u_M z^{-M}) \\ \times (f_N z^N + f_{N-1} z^{N-1} + ... + f_{-N+1} z^{-N+1} + f_{-N} z^{-N}) \end{aligned}$$

from which we can deduce that

$$\begin{aligned} r_{2(M+N)}(z) &\equiv z^{M+N} \rho_M(z) = v_{2M}(z)\, w_{2N}(z) \\ &= (v_{2M} z^{2M} + v_{2M-1} z^{2M-1} + ... + v_1 z + v_0) \\ &\qquad \times (w_{2N} z^{2N} + w_{2N-1} z^{2N-1} + ... + w_1 z + w_0) \end{aligned}$$

is a polynomial of degree $2(M+N)$ where

$$\begin{cases} v_\ell = u_{M-\ell}, & \ell = 2M, 2M-1, ..., 1, 0 \\ w_r = f_{r-N}, & r = 2N, 2N-1, ..., 1, 0. \end{cases}$$

Consequently, setting

$$r_{2(M+N)}(z) = r_{2(M+N)} z^{2(M+N)} + r_{2(M+N)-1} z^{2(M+N)-1} + \dots + r_1 z + r_0,$$

the kth coefficient of $r_{2(M+N)}(z)$ can be obtained by convolving the coefficients of $v_{2M}(z)$ and $w_{2N}(z)$, that is by taking

$$r_k = \sum_{h=0}^{k} v_{k-h} w_h = \sum_{h=0}^{k} u_{M-k+h} f_{h-N}, \quad k = 2(M+N), 2(M+N)-1, ..., 1, 0.$$

As a result

$$\rho_M(z) = \sum_{i=-M-N}^{M+N} \rho_i z^i, \quad \text{with} \quad \rho_i = r_{i+N+M}.$$

Considering that $u_M(z^{-1}) \to u(z^{-1})$ as $M \to \infty$, we can claim that also $\rho_M(z) \to \rho(z)$ as $M \to \infty$. Hence, the remaining problem is to choose M appropriately.

Choosing M

A crucial problem is choosing the optimal value of M, as the effectiveness of the approximation of $u_M(z)$ increases with M up to a saturation value. To determine the best M we adopt the following test:

(a) fixing M, we consider $2M+1$ additional points to the $2M+1$ points of the unit circle that we used to identify $u_M(z)$;
(b) denoting these values by $z_\ell = e^{\mathbf{i}\ell\frac{2\pi}{2(2M+1)}}$, $\ell = 0, 1, ..., 4M+1$ we compute $u_M(z)$ in all these points and check if these values are close enough to $a(z_\ell^{-1})^{-1}$. To this end, taken $\epsilon > 0$, we consider M satisfactory if

$$\max_{\ell=0,1,...,4M+1} |u_M(z_\ell^{-1}) a(z_\ell^{-1}) - 1| \le \epsilon. \tag{28}$$

If the above relation is satisfied, we assume M to be appropriate. On the contrary, we double M and continue until it is satisfied. The smallest value of M that satisfies (28) represents our best choice.

We do not explain how to compute $\ell(\alpha)$, as the same technique, with minor changes, can be used to solve (23), that is to approximate $\ell(\alpha)$.

5. Numerical results

To assess the effectiveness of our method, we considered the following potential (truncated one soliton potential)

$$u_{0,\tau}(x) = \begin{cases} 0 & x < \tau \\ 2\mathbf{i}\eta e^{\mathbf{i}(2\xi x+\phi)} \operatorname{sech}(x_0 - 2\eta x) & x \ge \tau \end{cases} \tag{29}$$

with $\xi, \phi, x_0 \in \mathbb{R}$ and $0 \neq \eta \in \mathbb{R}$. In the absence of truncation, $u_{0,\tau}$ represents the soliton potential, already considered by several authors [9]. The analytical representation of the associated transmission coefficient $T(\lambda)$ as well as of the reflection coefficients associated to $u_{0,\tau}$ are known [10]. In fact, setting

$$a = \eta + \mathbf{i}\xi, \qquad \gamma(x) = \eta \tanh(x_0 - 2\eta x) + \mathbf{i}\xi \qquad \text{and} \qquad \Gamma_\ell = 2\mathbf{i}\eta e^{x_0 - \mathbf{i}\phi}$$

we have

$$T_\tau(\lambda) = \frac{\lambda + \mathbf{i}a^*}{\lambda - \mathbf{i}\gamma(\tau)},$$

$$R_\tau(\lambda) = \frac{\mathbf{i}}{2} u_0(\tau) \frac{e^{-2\mathbf{i}\lambda\tau}}{\lambda - \mathbf{i}\gamma(\tau)} T_\tau(\lambda) \quad \text{and} \quad L_\tau(\lambda) = -\frac{\mathbf{i}}{2} u_0^*(\tau) \frac{e^{2\mathbf{i}\lambda\tau}}{\lambda - \mathbf{i}\gamma(\tau)}.$$

Moreover, we also have an exact representation of the Marchenko kernels from the right and from the left $\Omega_{\ell,\tau}$ and $\Omega_{r,\tau}$, and of the Fourier transforms of the reflection coefficients $\rho_\tau(\alpha)$ and $\ell_\tau(\alpha)$ that depend on the initial peak point position $\mu_0 = \frac{x_0}{2\eta}$ with respect to the truncation τ [10]. Indeed, if $\tau < \mu_0$, for $\alpha < 2\tau$ we have

$$\rho_\tau(\alpha) = 0 \qquad \text{and} \qquad \ell_\tau(\alpha) = \frac{\Gamma_\ell^* e^{-2\tau(\gamma(\tau)+a^*)}}{1 + e^{2(x_0 - 2\eta\tau)}} e^{\alpha\gamma(\tau)}$$

while for $\alpha > 2\tau$

$$\rho_\tau(\alpha) = \Gamma_\ell e^{-a\alpha} - \frac{(\gamma(\tau)+a^*)^2 (1 + e^{2(x_0-2\eta\tau)})}{\Gamma_\ell^* e^{-2\tau(\gamma(\tau)+a^*)}} e^{-\gamma(\tau)\alpha} \qquad \text{and} \qquad \ell_\tau(\alpha) = 0.$$

For $\tau < \mu_0$, we also have

$$\Omega_{\ell,\tau}(\alpha) = \rho_\tau(\alpha) + \frac{(\gamma(\tau)+a^*)^2 (1 + e^{2(x_0-2\eta\tau)})}{\Gamma_\ell e^{-2\tau(\gamma(\tau)+a^*)}} e^{-\gamma(\tau)\alpha} \qquad \text{and} \qquad \Omega_{r,\tau}(\alpha) = \ell_\tau(\alpha) + \frac{\Gamma_\ell^* e^{-2\tau(\gamma(\tau)+a^*)}}{1 + e^{2(x_0-2\eta\tau)}} e^{\gamma(\tau)\alpha}.$$

The last two relations show that we have only one single bound state, according to the representation of $T_\tau(\lambda)$. If $\tau > \mu_0$, for $\alpha < 2\tau$

$$\rho_\tau(\alpha) = \frac{(\gamma(\tau)+\Gamma_\ell^*)^2(1+e^{2(x_0-2\eta\tau)})}{\Gamma_\ell^* e^{-2\tau(\gamma(\tau)+\Gamma_\ell^*)}} e^{-\gamma(\tau)\alpha}, \qquad \text{and} \qquad \ell_\tau(\alpha) = 0$$

whereas for $\alpha > 2\tau$

$$\rho_\tau(\alpha) = \Gamma_\ell e^{-a\alpha}, \qquad \text{and} \qquad \ell_\tau(\alpha) = \frac{\Gamma_\ell^* e^{-2\tau(\gamma(\tau)+a^*)}}{1+e^{2(x_0-2\eta\tau)}} e^{\alpha\gamma(\tau)}.$$

Moreover if $\tau > \mu_0$

$$\Omega_{\ell,\tau}(\alpha) = \rho_\tau(\alpha), \quad \Omega_{r,\tau}(\alpha) = \ell_\tau(\alpha),$$

as there are no discrete eigenvalues.

Let us now consider the initial potential $u_{0,\tau}$ defined in (29) with $x_0 = \phi = 0$, $\eta = 2$ so that $\mu_0 = 0$ and $\xi = 0.1$.

Fixed at first $\tau = -1$, we noted that the right-hand side appearing in (24) is

$$f(\alpha) \neq 0, \qquad \alpha \in [-2, 32].$$

We then fixed $N = 3000$ so that the distance between two consecutive collocation points is $h \simeq 10^{-3}$. In order to approximate ρ_τ by means of $\rho_{\tau,M}$, as specified in (28), we have to compute the optimal value of M and the coefficients $\{\rho_i\}$. Following the procedure "*Choosing M*", illustrated before, taken $\epsilon = 10^{-12}$, we obtain that $M = 3000$ is the best choice for M. Moreover, considering that the support of the scattering matrix $\mathbf{S}_\tau(\lambda)$ is essentially $[-16, 16]$ and using the superscript $\sim$ to denote the numerical approximation we verified that

$$\max_{\lambda\in[-16,16]} \|\boldsymbol{S}_\tau(\lambda) - \tilde{\boldsymbol{S}}_\tau(\lambda)\|_\infty = 1.25e-06.$$

As theoretically expected our method recognized the single bound state $\lambda = 2 + 0.1\mathbf{i}$ and the norming constant $\Gamma_\ell = 4\mathbf{i}$ with the relative errors

$$\frac{|\lambda - \tilde{\lambda}|}{|\lambda|} = 2.03e-08, \qquad \frac{|\Gamma_\ell - \tilde{\Gamma}_\ell|}{|\Gamma_\ell|} = 7.32e-06.$$

As a second example, maintaining the same values for ξ, ϕ, x_0 and η we consider $\tau = 1$ in the potential $u_{0,\tau}$ which implies that we do not have bound states.

In this case we obtain that, as in the previous one, the best choice of M is $M = 3000$. Using this value, our method recognizes that we have not bound states as

$$\max_{\alpha\in[0,\,16]} |\tilde{\Omega}_{\ell,\tau}(\alpha) - \tilde{\rho}_\tau(\alpha)| = 4.37e-07.$$

The scattering matrix is well approximated in this case too as

$$\max_{\lambda\in[-16,16]} \|\boldsymbol{S}_\tau(\lambda) - \tilde{\boldsymbol{S}}_\tau(\lambda)\|_\infty = 8.94e-07.$$

6. Conclusions

The method recently proposed by the authors [4] works well under the assumptions that the initial potential $u_0 \in C^0(\mathbb{R})$ but not in the presence of discontinuity jumps. The algorithm proposed in Section 4 for solving the structured Fredholm integral equations (22)–(23) allows us to overcome this restriction, as the numerical results reported in Section 5 make evident.

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