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AN ABSTRACT MODEL FOR STRONG EVAPORATION

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SUNTO. — Si presenta una teoria astratta per problemi stazionari unidimensionali per il semispazio. Sulla base di questa teoria vengono provati alcuni risultati di esistenza e non esistenza già noti per modelli di evaporazione.

I. - INTRODUCTION.

In the past few years Arthur and Cercignani [1] and Siewert and Thomas [2, 3] have analyzed kinetic equations obtained by linearizing a BGK model equation about a drift Maxwellian. Let us denote the drift velocity (in suitable units) by $d > 0$. If one neglects transverse effects, one finds for the deviation $f(x, c)$ from the drift Maxwellian the equation (cf. [1, 2])

$$(1)\qquad (c + d)\frac{\partial f}{\partial x}(x, c) + f(x, c) = \pi^{-1/2}\int_{-\infty}^{\infty}\left\{1 + 2cc' + 2\left(c^2 - \frac{1}{2}\right)\left(c'^2 - \frac{1}{2}\right)\right\} e^{-c'^2} f(x, c')\, dc'.$$

If both longitudinal and transverse effects are incorporated, one finds the coupled system of equations (cf. [3])

$$(2)\qquad (c + d)\frac{\partial}{\partial x}\begin{bmatrix} f_+(x, c) \\ f_-(x, c)\end{bmatrix} + \begin{bmatrix} f_+(x, c) \\ f_-(x, c)\end{bmatrix} = \pi^{-1/2}\int_{-\infty}^{\infty}\begin{bmatrix} 1 + 2cc' + \frac{2}{3}\left(c^2 - \frac{1}{2}\right)\left(c'^2 - \frac{1}{2}\right) & \frac{2}{3}\left(c^2 - \frac{1}{2}\right) \\ \frac{2}{3}\left(c'^2 - \frac{1}{2}\right) & \frac{2}{3}\end{bmatrix}$$

$$\times \begin{bmatrix} f_-(x, c') \\ f_+(x, c') \end{bmatrix} e^{-c'^2} dc' .$$

The velocity c runs from $-\infty$ to ∞ and the position variable x from 0 to ∞. The boundary conditions to Eq. (1) are of the type

$$(3) \qquad f(0, c) = \varphi(c) \quad (c > -d), \quad \lim_{x \to \infty} f(x, c) = 0 .$$

On Eq. (2) one imposes the analogous boundary conditions

$$(4) \qquad \begin{bmatrix} f_+(0, c) \\ f_-(0, c) \end{bmatrix} = \begin{bmatrix} \varphi_+(c) \\ \varphi_-(c) \end{bmatrix} (c > -d), \quad \lim_{x \to \infty} \begin{bmatrix} f_+(x, c) \\ f_-(x, c) \end{bmatrix} = 0 .$$

The functions φ, φ_+ and φ_- are of a special type (namely, collision invariants).

In order to apply Hilbert space methods to the above boundary value problems we introduce the Hilbert space $L_2(\mathbb{R}, d\sigma)$ of square integrable functions $h, k : \mathbb{R} \to \mathbb{C}$ with inner product

$$(5) \qquad (h, k) = \int_{-\infty}^{\infty} h(c)\, \overline{k(c)}\, d\sigma(c), \ \frac{d\sigma}{dc} = \pi^{-1/2} e^{-c^2}.$$

The boundary value problem (1)-(3) can now be reformulated as the vector-valued differential equation.

$$(6) \qquad (Tf)'(x) = -Af(x), \ 0 < x < \infty$$

$$(7) \qquad Q_+ f(0) = \varphi, \ \lim_{x \to \infty} \|f(x)\| = 0$$

on $L_2(\mathbb{R}, d\sigma)$, where T, A and Q_+ are defined by

$$(Th)(c) = (c + d)\, h(c), \quad (Ah)(c) = h(c) -$$

$$- \pi^{-1/2} \int_{-\infty}^{\infty} \left\{ 1 + 2cc' + 2\left(c^2 - \frac{1}{2}\right)\left(c'^2 - \frac{1}{2}\right) \right\} e^{-c'^2} h(c')\, dc'$$

$$(Q_+ h)(c) = h(c) \quad (c > -d), \quad (Q_+ h)(c) = 0 \quad (c < -d).$$

The boundary value problem (2)-(4) can also be restated as Eqs. (6)-(7), but now the relevant Hilbert space is $L_2(\mathbb{R}, d\sigma) \oplus L_2(\mathbb{R}, d\sigma)$ and T, A and Q_+ are given by

$$(Th)(c) = (c + d)\, h(c),$$

$$(Ah)(c) = h(c) -$$

$$- \pi^{-1/2} \int_{-\infty}^{\infty} \begin{bmatrix} 1 + 2\, cc' + \frac{2}{3}\left(c^2 - \frac{1}{2}\right)\left(c'^2 - \frac{1}{2}\right) & \frac{2}{3}\left(c^2 - \frac{1}{2}\right) \\ \frac{2}{3}\left(c'^2 - \frac{1}{2}\right) & \frac{2}{3} \end{bmatrix} h(c')\, e^{-c'^2}\, dc'$$

$$(Q_+ h)(c) = h(c) \quad (c > -d), \quad (Q_+ h)(c) = 0 \quad (c < -d),$$

where $h = (h_+, h_-)$ is a column vector.

Abstract boundary value problems of the form (6)-(7) have been investigated intensively since the rigorous study by Hangelbroek [4] of the neutron transport equation below criticality. Concrete as well as abstract versions abound. Beals [5] and Van der Mee [6] studies Eqs. (6)-(7) on the abstract Hilbert space H, where T is a bounded injective self-adjoint and A a positive bounded self-adjoint operator with closed range, while Q_+ is the orthogonal projection of H onto the maximal positive T— invariant subspace. Further studies were done in [7, 8, 9] for unbounded A and in [5, 9] for unbounded T. Recently the results of [6, 8] were extended in [10] to non-positive A.

In [1, 2, 3] results on the existence and non-existence of solutions of the strong evaporation problems (1)-(3) and (2)-(4) were obtained only after considerable calculations, where φ was assumed to be an arbitrary vector in Q_+ [Ker A] (see Section III). There appeared to be a critical drift velocity d_M, corresponding to the speed of sound of the vapor, with the following properties:

(a) for $d \geq d_M$ there do not exist non-trivial solutions;

(b) for $0 < d < d_M$ there exist unique values of density and temperature (and transverse momenta for the vector equation) at $x = 0$ for which a solution exists.

For $d \geq d_M$ the non-existence of stationary non-trivial solutions has to do with the onset of turbulence at Mach one (i.e., $d = d_M$). For problem (1)-(3) it was found that $d_M = \sqrt{\frac{3}{2}}$ (see [1, 2]), whereas $d_M = \sqrt{\frac{5}{6}}$ was found in problem (2)-(4). In the present article we shall avoid these calculations by deriving these results from the abstract theory of [5] and [8]. We shall not give detailed proofs but instead refer for details to the future paper [11].

II. - Abstract half-space models.

Let us solve the abstract half-space problem (6)-(7), where T is a (possibly unbounded) self-adjoint operator with zero null space, A a bounded positive operator with finite-dimensional null space and Q_+ the orthogonal projection onto the maximal positive T — invariant subspace. Let us assume $\operatorname{Ker} A = \{0\}$ first. Then the hypotheses of [5] are fulfilled and the operator $A^{-1}T$ is self-adjoint with respect to the Hangelbroek inner product (cf. [4])

$$(h, k)_A = (Ah, k); \quad h, k \in H .$$

Following Beals [5] we define H_T as the completion of the domain $D(T)$ of T with respect to the inner product

$$(h, k)_T = (|T| h, k) , \tag{8}$$

and H_K as the completion of the domain $D(A^{-1}T) = D(T)$ of $A^{-1}T$ with respect to the inner product

$$(h, k)_K = (|A^{-1}T| h, k)_A = (A|A^{-1}T| h, k) . \tag{9}$$

It can be shown (see [5]) that the inner products (8) and (9) are equivalent on $D(T)$. We may thus identify the completions H_T and H_K. We immediately see that Q_+ leaves invariant $D(T)$ and that the restriction of Q_+ to $D(T)$ extends to an orthogonal projection in H_T. Similarly, let P_+ be the $(.,.)_A$-orthogonal projection of H onto the maximal $(.,.)_A$-positive $A^{-1}T$-invariant subspace. Then P_+ leaves invariant $D(A^{-1}T)$ and the restriction of P_+ to $D(A^{-1}T)$ extends to an orthogonal projection in H_K. Now exploit that $H_T = H_K$. As Beals [5] has shown, the operator

$$V = Q_+ P_+ + (I - Q_+)(I - P_+)$$

is well-defined and invertible on $H_T = H_K$. The inverse

$$E = V^{-1} : H_T \to H_T$$

now maps $Q_+ [H_T]$ onto $P_+ [H_T]$ and the solution of Eqs. (6)-(7) is unique and has the form

(10) $$f(x) = e^{-xT^{-1}A}\, E\varphi,\ 0 < x < \infty\,.$$

The semigroup in this expression is well-defined, since $E\varphi \in P_+[H_T]$. One thus obtains solutions in the extension space H_T of $D(T)$, whenever $\varphi \in D(T)$.

As we shall see, the problems (1)-(3) and (2)-(4) give rise to complications stemming from the non-triviality of Ker A. Under a minor regularity assumption on T and A (see [9]), the zero root linear manifolds

$$Z_0(T^{-1}A) = \bigcap_{n=0}^{\infty} \operatorname{Ker}(T^{-1}A)^n,\quad Z_0(AT^{-1}) = \bigcap_{n=0}^{\infty} \operatorname{Ker}(AT^{-1})^n$$

have the decomposition properties

(11) $$Z_0(T^{-1}A) \oplus Z_0(AT^{-1})^{\perp} = H$$

(12) $$Z_0(AT^{-1}) \oplus Z_0(T^{-1}A)^{\perp} = H\,,$$

where the orthogonal complement refers to the original inner product of H (see [9]). One observes that T is an invertible operator from the finite-dimensional space $Z_0(T^{-1}A)$ onto $Z_0(AT^{-1})$, while A is an invertible operator from $Z_0(AT^{-1})^{\perp}$ onto $Z_0(T^{-1}A)^{\perp}$. If one chooses an invertible operator β on $Z_0(T^{-1}A)$ such that

$$(T\beta h, h) \geq 0,\ h \in Z_0(T^{-1}A)\,,$$

then the linear operator A_β on H defined by

(13) $$A_\beta h = \begin{cases} T\beta^{-1}h, & h \in Z_0(T^{-1}A) \\ Ah, & h \in Z_0(AT^{-1})^{\perp}\,, \end{cases}$$

is strictly positive self-adjoint on H, while

$$A_\beta^{-1}T = \beta \oplus (T^{-1}A|_{Z_0(AT^{-1})^{\perp}})^{-1}\,.$$

The use of β does not change the non-zero spectrum of $T^{-1}A$, but replaces the zero part by the eigenvalues of a non-singular matrix β.

Next we use the operator A_β to reduce Eqs. (6)-(7) to two subproblems. Write $f = f_0 + f_1$ for the solution, where f_0 has its values in $Z_0(T^{-1}A)$ and f_1 in $Z_0(AT^{-1})^{\perp}$. Then Eqs. (6)-(7) decompose as follows:

(14) $$(Tf_0)'(x) = -Af_0(x) \qquad (0 < x < \infty)$$

(15) $$(Tf_1)'(x) = -Af_1(x) \qquad (0 < x < \infty),$$

where $\|f_1(x)\| \to 0$ for $x \to \infty$. Now consider the dummy equation

(16) $$(Tg_0)'(x) = -A_\beta g_0(x) \qquad (0 < x < \infty).$$

Also notice that A_β and A coincide on $Z_0(AT^{-1})$ (cf. Eq. (13)), which implies

(17) $$(Tf_1)'(x) = -A_\beta f_1(x) \qquad (0 < x < \infty).$$

Write $g = g_0 + f_1$; then Eqs. (16) and (17) can be summarized as

(18) $$(Tg)'(x) = -A_\beta g(x) \qquad (0 < x < \infty),$$

where A_β is strictly positive self-adjoint on H. Invoking Beals' result [5], there exists an invertible operator E_β on H_T, which maps $Q_+[H_T]$ onto $P_+[H_T]$ with P_+ the $(.,.)_{A_\beta}$-orthogonal projection of $H_K(=H_T)$ onto the maximal $(.,.)_{A_\beta}$-positive A_β^{-1} T-invariant subspace. The solution of Eq. (18) which vanishes for $x \to \infty$ generally has the form

$$g(x) = e^{-xT^{-1}A_\beta}\, g(0), \qquad 0 < x < \infty,$$

where $g(0) \in P_+[H_T]$. Because $Z_0(T^{-1}A)$ has a finite dimension, Eq. (14) has the general solution

$$f_0(x) = e^{-xT^{-1}A} f_0(0) = \sum_{n=0} \frac{(-x)^n}{n!} (T^{-1}A)^n f_0(0).$$

Hence, solutions of Eqs. (6)-(7) all have the property $f_0(0) = 0$ and the form

(19) $$f(x) = e^{-xT^{-1}A} f(0) \qquad (0 < x < \infty),$$

where $f(0) \in PP_+[H_T]$ and P is the projection of H onto $Z_0(AT^{-1})^\perp$ along $Z_0(T^{-1}A)$ (continuously extended to H_T). If one only requires that $\|f(x)\| = 0(1)$ $(x \to \infty)$, then $f_0(x) \equiv f_0(0)$ and

(20) $$f(x) = e^{-xT^{-1}A} g(0) + f_0(0) \qquad (0 < x < \infty),$$

where $g(0) \in PP_+[H_T]$ and $f_0(0) \in \operatorname{Ker} A$.

Formulas (19) and (20) represent general solutions only. As a final step we have to fulfill the half-range boundary condition $Q_+ f(0) = \varphi$ in Eq. (7), which gives the necessary and sufficient condition

$$\varphi \in PP_+[H_T] + (I - Q_+)[H_T] \tag{21}$$

for the existence of solutions of Eqs. (6)-(7). Because $PP_+[H_T]$ and $(I - Q_+)[H_T]$ have zero intersection, such solutions must be unique. Using the invertibility of E_β on H_T one may reduce condition (21) to an analysis on the finite-dimensional space Ker A. This we shall treat in detail in [11]. Here we state the main result:

THEOREM. - Let us choose a basis $x_1, \ldots, x_l$ of Ker A of vectors satisfying $(Tx_i, x_j) = 0$ for $i \neq j$. Among the l numbers (Tx_k, x_k) let m_+, m_0 and m_- the number of positive, zero and negative ones. Then

(i) Eqs. (6)-(7) have at most one solution, but the linear set of all $\varphi \in Q_+[H_T]$ for which a solution exists has codimension $m_+ + m_0$ in $Q_+[H_T]$.

(ii) Eq. (6) with boundary conditions

$$Q_+ f(0) = \varphi, \|f(x)\| = 0(1) \qquad (x \to \infty) \tag{22}$$

has at least one solution for every $\varphi \in Q_+[H_T]$, but the linear set of all solutions of Eqs. (6)-(22) with $\varphi = 0$ has dimension m_-.

Thus Eqs. (6)-(7) have measure of non-completeness $m_+ + m_0$ and solutions are unique. On the contrary, Eqs. (6)-(22) have measure of non-uniqueness m_- and solutions always exist. For Ker $A = \{0\}$ one finds $m_+ = m_0 = m_- = 0$ and Eqs. (6)-(7) and Eqs. (6)-(22) both are uniquely solvable. For non-positive A an analogous but more complicated theorem holds true (see [10]).

III. - Application to strong evaporation.

Let us apply the theorem to Eqs. (1)-(3). One easily computes (see [11]; cf. [1, 2]) that

$$\text{Ker}\, A = \{\Delta\varrho + 2c(d_0 - d) + \left(c^2 - \frac{1}{2}\right)\Delta T/\Delta\varrho, d_0, \Delta T \text{ arbitrary}\}.$$

As the basis of Ker A appearing in the theorem we take

$$x_1(c) = 1, \qquad x_2(c) = c, \qquad x_3(c) = dc - c^2 .$$

The $l = 3$ numbers $(Tx_k, x_k) = \int_{\infty}^{\infty} c|x_k(c)|^2 \, d\sigma(c)$ appear to be

$$(Tx_1, x_1) = d, \; (Tx_2, x_2) = \frac{1}{2} d, \; (Tx_3, x_3) = \frac{1}{2} d \left(d^2 - \frac{3}{2}\right).$$

Hence,

$$m_+ = 2, \quad m_0 = 0, \quad m_- = 1 \qquad \text{for } 0 < d < \sqrt{\frac{3}{2}}$$

$$m_+ = 2, \quad m_0 = 1, \quad m_- = 0 \qquad \text{for } d = \sqrt{\frac{3}{2}}$$

$$m_+ = 3, \quad m_0 = 0, \quad m_- = 0 \qquad \text{for } d > \sqrt{\frac{3}{2}}.$$

Thus $m_+ + m_0 = 2$ for $0 < d < d_M$ and $m_+ + m_0 = 3$ for $d \geq d_M$ $\left(\text{with } d_M{}^2 = \frac{3}{2}\right)$, which are the measures of non-completeness for the solution of Eqs. (1)-(3). Because of conservation laws one usually imposes two constraints to the solution and chooses $\varphi \in Q_+[\operatorname{Ker} A]$. For all drift speeds $0 < d < d_M$ there exist unique $\Delta\varrho$ and ΔT to every d_0, for which Eqs. (6)-(7) with boundary value function

$$\varphi(c) = \Delta\varrho + 2c(d_0 - d) + \left(c^2 - \frac{1}{2}\right)\Delta T, \quad c > -d \tag{23}$$

have a (unique) solution. For $d \geq d_M$ there are no non-trivial solutions to Eqs. (6)-(7), where φ is given by Eq. (23). An analogous result holds for Eqs. (2)-(4), but now $d_M = \sqrt{5/6}$. We thus recovered the main results of [1, 2, 3] from an abstract theory of half-range boundary value problems without substantial calculation.

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SUMMARY. — An abstract theory of one-dimensional stationary half-space problems is presented. On the basis of this theory some previously known existence and nonexistence results for evaporation models are proved.

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