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Parameter estimation of monomial-exponential sums in one and two variables

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ABSTRACT

In this paper we propose a matrix-pencil method for the numerical identification of the parameters of monomial-exponential sums in one and two variables. While in the univariate case the proposed method is a variant of that developed by the authors in a preceding paper, the bivariate case is treated for the first time here. In the bivariate case, the method we propose, easily extendible to more variables, reduces the problem to a pair of univariate problems and subsequently to the solution of a linear system. As a result, the relative errors in the univariate and in the bivariate case are almost of the same order.

1. Introduction

In many problems concerning the applied sciences and engineering, it is important to identify the parameters and coefficients $\{n, \{f_j\}_{j=1}^n, \ \{c_j\}_{j=1}^n\}$ in the exponential sums

$$h(x) = \sum_{j=1}^{n} c_j e^{f_j x}, \tag{1}$$

where n is a positive integer, $\{c_j\}_{j=1}^n$ are complex or real coefficients and $\{f_j\}_{j=1}^n$ are distinct complex or real parameters, given a set of $2N$ $(2N \geqslant n)$ values of $h(x)$ in equidistant points of $\mathbb{R}$.

This problem arises, in particular, in the propagation of signals [1] [2], electromagnetics [3] [4] and high-resolution imaging of moving targets [5]. The two methods used most are Prony-like (or polynomial) methods and matrix-pencil methods. The first ones are based on the paper by de Prony [6] who was the first to investigate this problem, under the hypothesis that n is known and the data are exact. Several extensions and variants of this method have been proposed to consider the case where n is only approximately known or the data are affected by noise (see, for instance, [7, pp. 458–462], [8–14]). For the matrix-pencil methods, which have been proposed more recently (see, for instance, [15,16]), some attempts to recover the parameters in extended exponential polynomials of the type

$$g(x) = \sum_{j=1}^{n} c_j(x) e^{f_j x},$$

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where $c_j(x)$ is a polynomial, have been made in particular in [17,18] where no proof of unique reconstruction of the parameters from the data matrix has been given.

More recently, the authors have proposed a matrix-pencil method [19] to estimate the parameters of a monomial-exponential sum of the form

$$h(x) = \sum_{j=1}^{n} \sum_{s=0}^{m_j-1} c_{js} x^s e^{f_j x}, \tag{2}$$

where $\{c_{js}\}_{j=1,s=0}^{n,m_j-1}$ and $\{f_j\}_{j=1}^{n}$ are complex or real parameters and $\{m_j\}_{j=1}^{n}$ are positive integers. In the case $m_1 = m_2 = \cdots = m_n = 1$, the monomial exponential sum $h(x)$, of course, reduces to the exponential sum (1). More precisely, setting

$$M = m_1 + m_2 + \cdots + m_n,$$

the problem is to recover the $M + n$ parameters of h given $2N$ ($N \geqslant M$) observed data. In [19] the uniqueness of the recovery of parameters from the data matrix has been proved.

This problem is of primary interest, for instance, in the direct scattering problem concerning the solution of nonlinear partial differential equations (NPDEs) of integrable type [20] [21].

In this paper we propose a new technique to compute the eigenvalues of the matrix-pencil, that is to identify the parameters $\{f_j\}$ and the order $\{m_j\}$ of the monomials. Our numerical experiments (see Section 4) show that it is as effective as the two techniques proposed in [19], though its computational complexity is lower.

Furthermore we introduce a method to identify the parameters of the following bivariate monomial-exponential sums

$$h(x_1, x_2) = \sum_{j_1=1}^{n_1} \sum_{s_1=0}^{m_{1j_1}-1} \sum_{j_2=1}^{n_2} \sum_{s_2=0}^{m_{2j_2}-1} c_{(j_1,s_1),(j_2,s_2)} x_1^{s_1} e^{f_{1j_1} x_1} x_2^{s_2} e^{f_{2j_2} x_2}, \tag{3}$$

which of course reduces to bivariate exponential sums whenever $m_{1j_1} \equiv m_{2j_2} \equiv 1$ which is the case treated for instance in [22,23]. This method, which reduces the problem to a pair of univariate problems solvable by the method proposed in the univariate case, can easily be extended to more variables.

Let us now outline the organization of the paper. In Section 2 we illustrate our method in the one-variable case and in Section 3 we explain how to treat the bivariate case. Section 4 is devoted to the numerical results and Section 5 to the conclusions.

2. The numerical method for univariate sums

The numerical method we propose to recover the parameters of the monomial-exponential sum (2) reduces the non-linear approximation problem to two problems of linear algebra. The first one is a generalized eigenvalue problem, which allows us to recover n, f_j and m_j. The second one is the solution of a linear system with a Casorati matrix to compute the parameters c_{js}.

Firstly we note that, setting $z_j = e^{f_j} \neq 0$, we can rewrite the monomial exponential sum (2) as a monomial-power sum

$$h(x) = \sum_{j=1}^{n} \sum_{s=0}^{m_j-1} c_{js} x^s z_j^x. \tag{4}$$

For the sake of clarity let us assume initially that $2N$ sampled data with $N \geqslant M$, $M = m_1 + \cdots + m_n$,

$$h(k) = \sum_{j=1}^{n} \sum_{s=0}^{m_j-1} c_{js} k^s z_j^k, \quad 0^0 \equiv 1 \tag{5}$$

are given for the $2N$ integer values $k = k_0, k_0 + 1, \ldots, k_0 + 2N - 1$ with $k_0 \in \mathbb{N}^+ = \{0, 1, 2, \ldots, k_0, \ldots\}$. As we will show in Section 2.3, the problem can be treated as well whenever $h(x)$ is known in $2N$ equidistant points of any interval $[a, b]$. As generally happens in applications, we assume to know a reasonable overestimate $\widehat{M}$ of M and $N \geqslant \widehat{M}$. Under this hypothesis, preliminarily, we arrange the $2N$ given data in the following Hankel matrices of order $N \times \widehat{M}$

$$\mathbf{H}^0_{N\widehat{M}} = \begin{pmatrix} h(k_0) & h(k_0+1) & \ldots & h(k_0+\widehat{M}-1) \\ h(k_0+1) & h(k_0+2) & \ldots & h(k_0+\widehat{M}) \\ \vdots & \vdots & \vdots & \vdots \\ h(k_0+N-1) & h(k_0+N) & \ldots & h(k_0+\widehat{M}+N-2) \end{pmatrix} = \left[\mathbf{h}_0, \mathbf{h}_1, \ldots, \mathbf{h}_{\widehat{M}-1}\right], \tag{6}$$

$$\mathbf{H}^1_{N\widehat{M}} = \begin{pmatrix} h(k_0+1) & h(k_0+2) & \dots & h(k_0+\widehat{M}) \\ h(k_0+2) & h(k_0+3) & \dots & h(k_0+\widehat{M}+1) \\ \vdots & \vdots & \vdots & \vdots \\ h(k_0+N) & h(k_0+N+1) & \dots & h(k_0+\widehat{M}+N-1) \end{pmatrix} = \left[\mathbf{h}_1, \mathbf{h}_2, \dots, \mathbf{h}_{\widehat{M}}\right]. \tag{7}$$

Notice that $\mathbf{H}^1_{N\widehat{M}}$ is essentially a shift of $\mathbf{H}^0_{N\widehat{M}}$, as the first $\widehat{M}-1$ columns of $\mathbf{H}^1_{N\widehat{M}}$ coincide with the last $\widehat{M}-1$ columns of $\mathbf{H}^0_{N\widehat{M}}$, apart from the last entry.

Under the hypothesis that the sampled data are noiseless the following two properties are satisfied [19, Lemma 2.1]:

(a) The matrices (6) and (7) have rank M, that is

$$\operatorname{rank}\left(\mathbf{H}^0_{N\widehat{M}}\right) = \operatorname{rank}\left(\mathbf{H}^1_{N\widehat{M}}\right) = M. \tag{8}$$

(b) The following relation holds true

$$\mathbf{H}^1_{NM} = \mathbf{H}^0_{NM}\mathbf{C}_M(P) \tag{9}$$

where

$$\mathbf{C}_M(P) = \begin{pmatrix} 0 & 0 & \dots & 0 & -p_0 \\ 1 & 0 & \dots & 0 & -p_1 \\ \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & \dots & 1 & -p_{M-1} \end{pmatrix}$$

is the companion matrix of the associated Prony polynomial, i.e. of the monic polynomial of degree M

$$P(z) = \prod_{j=1}^{n}(z-z_j)^{m_j} = \sum_{k=0}^{M} p_k z^k, \quad p_M \equiv 1 \tag{10}$$

having z_j as the jth zero with multiplicity m_j. This polynomial is associated to the Hankel matrices (6) and (7) in the sense that it is straightforward to prove

$$\sum_{k=0}^{M-1} p_k \mathbf{h}_k + \mathbf{h}_M = \mathbf{0}. \tag{11}$$

Let us now recall the next theorem, proved in [19], which contains two results that are basic to our method.

Theorem 2.1. *The zeros of the Prony polynomial, with their multiplicities, are exactly the eigenvalues, with the same multiplicity, of the matrix-pencil*

$$\mathbf{H}_{MM}(z) = \left(\mathbf{H}^0_{NM}\right)^*\left(\mathbf{H}^1_{NM} - z\mathbf{H}^0_{NM}\right), \tag{12}$$

where the asterisk denotes the conjugate transpose.

Moreover, the coefficients c_{js} appearing in (2) are the solutions of the linear system

$$\mathbf{K}^0_{MM}\mathbf{c} = \mathbf{h}_0 \tag{13}$$

where $\mathbf{c} = [c_{1\,0}, \dots, c_{1\,m_1-1}, \dots, c_{n\,0}, \dots, c_{n\,m_n-1}]^T$, $\mathbf{h}_0 = [h(k_0), h(k_0+1), \dots, h(k_0+M-1)]^T$ *and* $\mathbf{K}^0_{MM}$ *is the Casorati matrix*

$$\mathbf{K}^0_{MM} = \begin{pmatrix} z_1^{k_0} & k_0 z_1^{k_0} & \dots & k_0^{m_1-1} z_1^{k_0} & \dots & z_n^{k_0} & k_0 z_n^{k_0} & \dots & k_0^{m_n-1} z_n^{k_0} \\ z_1^{k_1} & k_1 z_1^{k_1} & \dots & k_1^{m_1-1} z_1^{k_1} & \dots & z_n^{k_1} & k_1 z_n^{k_1} & \dots & k_1^{m_n-1} z_n^{k_1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ z_1^{k_{M-1}} & k_{M-1} z_1^{k_{M-1}} & \dots & k_{M-1}^{m_1-1} z_1^{k_{M-1}} & \dots & z_n^{k_{M-1}} & k_{M-1} z_n^{k_{M-1}} & \dots & k_{M-1}^{m_n-1} z_n^{k_{M-1}} \end{pmatrix}. \tag{14}$$

2.1. Computation of $\{n,\ z_j,\ m_j\}$

The starting point for the computation of the parameters $\{n,\ z_j,\ m_j\}$, is the factorization of the augmented Hankel matrix

$$\mathbf{H}_{N,M+1} = [\mathbf{h}_0, \mathbf{h}_1, \dots, \mathbf{h}_M] = \left[\mathbf{H}^0_{NM}, \mathbf{h}_M\right] = \left[\mathbf{h}_0, \mathbf{H}^1_{NM}\right], \tag{15}$$

by means of the QR decomposition, unlike in [19] where it is factorized by applying the SVD (Singular Value Decomposition) technique.

Proceeding in this way we obtain

$$\mathbf{H}_{N,M+1} = [\mathbf{q}_0, \mathbf{q}_1, \dots, \mathbf{q}_M] \begin{pmatrix} r_{11} & r_{12} & \dots & r_{1,M+1} \\ 0 & r_{22} & \dots & r_{2,M+1} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & r_{M+1,M+1} \end{pmatrix} = \mathbf{Q}_{N,M+1}\mathbf{R}_{M+1,M+1},$$

where $\mathbf{q}_i \in \mathbb{C}^N$ for each $i = 0, 1, 2, \dots, M$ with $(\mathbf{Q}_{N,M+1})^* \mathbf{Q}_{N,M+1} = \mathbf{I}_{M+1,M+1}$, (identity matrix of order $M+1$), and $\mathbf{R}_{M+1,M+1}$ is an upper triangular matrix of order $M+1$.

Hence, as

$$\mathbf{H}^0_{NM} = [\mathbf{h}_0, \mathbf{h}_1, \dots, \mathbf{h}_{M-1}],$$

we have

$$\mathbf{H}^0_{NM} = \mathbf{Q}^0_{NM}\mathbf{R}^0_{NM}, \tag{16}$$

where $\mathbf{Q}^0_{NM} = [\mathbf{q}_0, \mathbf{q}_1, \dots, \mathbf{q}_{M-1}]$ and $\mathbf{R}^0_{NM}$ is obtained from $\mathbf{R}_{M+1,M+1}$ by simply deleting its last row and its last column. Similarly

$$\mathbf{H}^1_{NM} = [\mathbf{h}_1, \mathbf{h}_2, \dots, \mathbf{h}_M] = \mathbf{Q}_{N,M+1}\widehat{\mathbf{R}}^1_{M+1,M},$$

where $\widehat{\mathbf{R}}^1_{M+1,M}$ is obtained from $\mathbf{R}_{M+1,M+1}$ by simply deleting its first column.

Furthermore, noting that

$$\left(\mathbf{H}^0_{NM}\right)^* \mathbf{H}^1_{NM} = \left(\mathbf{R}^0_{NM}\right)^* \left(\mathbf{Q}^0_{NM}\right)^* \mathbf{Q}_{N,M+1}\widehat{\mathbf{R}}^1_{M+1,M} = \left(\mathbf{R}^0_{NM}\right)^* \mathbf{R}^1_{MM},$$

where $\mathbf{R}^1_{MM}$ is obtained from $\widehat{\mathbf{R}}^1_{M+1,M}$ by simply ignoring its last row, that is from $\mathbf{R}_{M+1,M+1}$ by ignoring both its first column and its last row. Moreover, as

$$\left(\mathbf{H}^0_{NM}\right)^* \mathbf{H}^0_{NM} = \left(\mathbf{R}^0_{MM}\right)^* \left(\mathbf{Q}^0_{NM}\right)^* \mathbf{Q}^0_{NM}\mathbf{R}^0_{MM} = \left(\mathbf{R}^0_{MM}\right)^* \mathbf{R}^0_{MM}$$

and $\mathbf{R}^0_{MM}$ is not singular, the matrix pencil (12) can be written as follows:

$$\mathbf{H}_{MM}(z) = \left(\mathbf{R}^0_{MM}\right)^* \mathbf{R}^0_{MM}\left[\left(\mathbf{R}^0_{MM}\right)^{-1}\mathbf{R}^1_{MM} - z\mathbf{I}\right].$$

Furthermore, $\mathbf{R}^1_{MM}$ can be factorized as follows

$$\mathbf{R}^1_{MM} = \mathbf{R}^0_{MM}\mathbf{A}_{MM}$$

where

$$\mathbf{A}_{MM} = \begin{pmatrix} 0 & 0 & \dots & 0 & \alpha_0 \\ 1 & 0 & \dots & 0 & \alpha_1 \\ \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & \dots & 1 & \alpha_{M-1} \end{pmatrix}$$

is a companion matrix whose last column $\boldsymbol{\alpha} = [\alpha_0, \alpha_1, \dots, \alpha_{M-1}]^T$ is the solution of the system

$$\mathbf{R}^0_{MM}\boldsymbol{\alpha} = \mathbf{r}_{M+1} \tag{17}$$

where $\mathbf{r}_{M+1} = [r_{1,M+1}, r_{2,M+1}, \dots, r_{M,M+1}]^T$, that is the last column of $\mathbf{R}_{M+1,M+1}$ while ignoring its last element. As a result,

$$\mathbf{H}_{MM}(z) = \left(\mathbf{R}^0_{MM}\right)^* \mathbf{R}^0_{MM}(\mathbf{A}_{MM} - z\mathbf{I}_{MM}). \tag{18}$$

A comparison between $\mathbf{A}_{MM}$ and $\mathbf{C}_M(P)$ allows us to note that $p_j = -\alpha_j$, $j = 0, 1, \dots, M-1$.

As a result, the computation of $\{n,\ z_j,\ m_j\}$ and the coefficients of the Prony's polynomial, via QR factorization reduces to:

1. The QR factorization of the Hankel matrix $\mathbf{H}_{N,M+1}$.
2. The solution of the upper-triangular system (17).
3. The computation of the eigenvalues of the companion matrix $\mathbf{A}_{MM}$.

Note that the computational cost for the parameter identification is dominated by the cost of the QR algorithm which is $\mathcal{O}(N\widehat{M}^2)$. Hence, it is generally lower with respect to that based on the SVD algorithm, as in [19].

2.2. Computation of $\{c_{js}\}$

Once the parameters $\{n,\ z_j,\ m_j\}$ have been computed, we evaluate the coefficients c_{js}, given $h(k)$ in M distinct points $\{k_0, k_0+1, \ldots, k_0+M-1\}$. Indeed, we write down the Casorati matrix and then solve the linear system (13).

Although theoretically not necessary, our numerical experiments suggest to use more than $2M$ data. For this reason, whenever it is possible we prefer to use $2N$ $(N > M)$ sampled data and to compute the coefficients by solving, in the least squares sense, the overdetermined linear system

$$\mathbf{K}^0_{2N,M}\mathbf{c} = \mathbf{h}_0, \tag{19}$$

where $\mathbf{h}_0 = [h(k_0), h(k_0+1), \ldots, h(k_0+2N-1)]$ and $\mathbf{K}^0_{2N,M}$ is the Casorati matrix of order $2N \times M$ $(N > M)$, obtained as a natural extension of (14). As can be expected, this extension is increasingly important as the noise/signal ratio increases.

2.3. Sampling $h(x)$ in N points of an interval

Let us now explain how the method described in the previous paragraphs, with simple variants, can be applied as well in $2N$ $(N > \widehat{M})$ equidistant points of an interval $[a, b]$ instead of in $2N$ integer values. Under this hypothesis, setting

$$x_k = x_0 + k\delta, \quad k = 0, 1, \ldots, 2N, \quad \delta = \frac{b-a}{2N}$$

by (4) we have

$$h(k) \equiv h(x_k) = \sum_{j=1}^{n}\sum_{s=0}^{m_j-1} c_{js}(x_0+k\delta)^s z_j^{x_0+k\delta} = \sum_{j=1}^{n} z_j^{x_0}\left(\sum_{s=0}^{m_j-1} c_{js}\sum_{\ell=0}^{s}\binom{s}{\ell} x_0^{s-\ell}\delta^\ell k^\ell\right) z_j^{k\delta} = \sum_{j=1}^{n}\sum_{s=0}^{m_j-1} d_{js}k^s\xi_j^k, \tag{20}$$

where $\xi_j = z_j^\delta$ and $d_{js} = z_j^{x_0}\left[\sum_{t=s}^{m_j-1}\binom{t}{s}x_0^{t-s}c_{jt}\right]\delta^s$ with $\binom{0}{0} = 1$.

As a consequence, recalling that our matrix-pencil method applied to (20) allows us to recover the set of parameters $\{n, m_j, \xi_j, d_{js}\}$, we can estimate the parameters $\{z_j\}_{j=1}^n$ with their multiplicities and then the coefficients $\{c_{js}\}$ by means of the following backward recursion:

$$\begin{cases} d_{j,m_j-1} = z_j^{x_0}\binom{m_j-1}{m_j-1}c_{j,m_j-1}\,\delta^{m_j-1}, \\ d_{j,m_j-2} = z_j^{x_0}\left[\binom{m_j-2}{m_j-2}c_{j,m_j-2} + \binom{m_j-1}{m_j-2}x_0\,c_{j,m_j-1}\right]\delta^{m_j-2}, \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ d_{j1} = z_j^{x_0}\left[\binom{1}{1}c_{j,1} + \ldots + \binom{m_j-1}{1}x_0^{m_j-2}c_{j,m_j-1}\right]\delta \\ d_{j0} = z_j^{x_0}\left[c_{j0} + \binom{1}{0}x_0c_{j,1} + \ldots + \binom{m_j-1}{0}x_0^{m_j-1}c_{j,m_j-1}\right]. \end{cases}$$

This procedure based on the sampling of $h(x)$ in an interval can be effective, as happens in Example 3, also if the sum could be sampled in the integers. This occurs, in particular, when the sampling on N integers generates an extended Hankel matrix whose rows or columns are not scaled among them.

3. The numerical method for bivariate sums

In this section we generalize the technique developed in the previous section to the bivariate case, that is to the tensor product of two monomial-exponential sums

$$h(x_1, x_2) = \sum_{j_1=1}^{n_1}\sum_{s_1=0}^{m_{1j_1}-1}\sum_{j_2=1}^{n_2}\sum_{s_2=0}^{m_{2j_2}-1} c_{(j_1,s_1),(j_2,s_2)}\,x_1^{s_1}e^{f_{1j_1}x_1}\,x_2^{s_2}e^{f_{2j_2}x_2}, \tag{21}$$

which reduces to a bivariate sum of exponentials whenever $m_{1j_1} = m_{2j_2} = 1$. The problem consists of recovering the parameters and coefficients

$$\left\{n_1, n_2, m_{1j_1}, m_{2j_2}, f_{1j_1}, f_{2j_2}, c_{(j_1,s_1),(j_2,s_2)}\right\}$$

knowing $h(x_1, x_2)$ in a set of points $\{(x_{1\,k_1}, x_{2\,k_2})\}$ of a regular grid of a rectangle $[a_1, b_1] \times [a_2, b_2]$ where

$$x_{1k_1} = a_1 + k_1\delta_1, \quad k_1 = 0, 1, 2, \ldots, 2N_1, \quad \delta_1 = \frac{b_1 - a_1}{2N_1}, \quad N_1 > M_1 = m_{11} + \cdots + m_{1n_1},$$

$$x_{2k_2} = a_2 + k_2\delta_2, \quad k_2 = 0, 1, 2, \ldots, 2N_2, \quad \delta_2 = \frac{b_2 - a_2}{2N_2}, \quad N_2 > M_2 = m_{21} + \cdots + m_{2n_2}.$$

Following a "cascade-like" technique, fixing x_2 we put

$$h_{x_2}(x_1) = \sum_{j_1=1}^{n_1} \sum_{s_1=0}^{m_{1j_1}-1} a_{j_1,s_1}(x_2) x_1^{s_1} e^{f_{1j_1} x_1} \tag{22}$$

where x_2 is a parameter and x_1 is a variable that, for each fixed value x_{2k_2} of x_2, assumes the $2N_1$ values x_{1k_1}, $k_1 = 0, 1, 2, \ldots, 2N_1$.

Similarly, inverting the roles of x_1 (parameter) and x_2 (variable) we can write

$$h_{x_1}(x_2) = \sum_{j_2=1}^{n_2} \sum_{s_2=0}^{m_{2j_2}-1} b_{(j_2,s_2)}(x_1) x_2^{s_2} e^{f_{2j_2} x_2}, \tag{23}$$

where for each value x_{1k_1} of x_1, x_2 assumes the values x_{2k_2}, $k_2 = 0, 1, 2, \ldots, 2N_2$. Fixing x_2, $h(x_1, x_2)$ is a monomial-exponential sum whose parameters $\{n_1, m_{1j_1}, f_{1j_1}\}$ can be recovered by applying our matrix pencil method as $h(x_{1k_1}, x_{2k_2})$ is assumed to be known for $k_1 = 0, 1, 2, \ldots, 2N_1$ with $N_1 \geqslant M_1$. As the coefficients cannot be exactly the same for each value of x_2 we ignore them. Applying the same procedure to $h(x_1, x_2)$ with x_1 fixed, we can recover the parameters $\{n_2, m_{2j_2}, f_{2j_2}\}$.

At this point, having recovered all the parameters $\{n_1, n_2, m_{1j_1}, m_{2j_2}, f_{1j_1}, f_{2j_2}\}$, it remains to estimate $M_1 \times M_2$ coefficients $\{c_{(j_1,s_1),(j_2,s_2)}\}$, that is to solve a linear approximation problem. Indeed, we have to solve in the least squares sense the overdetermined linear system

$$\mathcal{F}\mathbf{c} = \mathbf{h}, \tag{24}$$

where the rows of $\mathcal{F}$ as well as the entries of $\mathbf{h}$ depend on the pair (k_1, k_2), while the columns of $\mathcal{F}$ as well as the entries of $\mathbf{c}$ depend on the pair (j_1, s_1), (j_2, s_2). For this reason the equations are sorted on the basis of the lexicographical order between k_1 and k_2, that is fixing $k_2 = 0, 1, 2, \ldots, 2N_2$ we put $k_1 = 0, 1, 2, \ldots, 2N_1$. Similarly, the columns of $\mathcal{F}$ as the entries of $\mathbf{c}$ are sorted on the basis of the lexicographical order between the pairs (j_1, s_1) and (j_2, s_2) k_1, which means that fixing $j_2 = 1, \ldots, n_2$ with $s_2 = 0, 1, \ldots, m_{2j_2} - 1$, we put $j_1 = 1, 2, \ldots, n_1$ and for each of them $s_1 = 0, 1, \ldots, m_{1j_1} - 1$.

This technique generalizes immediately to the case of more variables. If $h = (x_1, x_2, x_3)$, for example, fixing x_2 and x_3 we apply the univariate method to $h_{x_2,x_3}(x_1)$ to recover the parameters pertaining to x_1, then, fixing, x_1 and x_3 we apply the same method to compute the parameters pertaining x_2, as well as, fixing x_1 and x_2 we compute the parameters pertaining x_3. At this point it remains to solve a linear system for computing the coefficients.

4. Numerical results

In this section, to highlight the effectiveness of the proposed method, we illustrate the results of its applications to some examples in one and two variables. Concerning the univariate case, its effectiveness will be compared with that of the techniques proposed in [19]. More precisely, recalling that in [19] the factorization of the augmented Hankel matrix $\mathbf{H}_{N,\widehat{M}+1}$ is obtained by the SVD, while here it is obtained by the QR technique, and the simultaneously factorization of $\mathbf{H}^0_{N,\widehat{M}}$ and $\mathbf{H}^1_{N,\widehat{M}}$ in [19] is obtained by the GSVD (Generalized Singular Value Decomposition), to distinguish between them in the tables of the results we will write via QR, via SVD and via GSVD.

In the first two examples, which deal with the univariate case, we consider the noisy data

$$h(k) = h^*(k) + \delta e_k, \quad k = k_0, \ldots, k_0 + 2N - 1, \tag{25}$$

where $h(k) \equiv h(x_k)$ and $h^*(k) \equiv h^*(x_k)$ denote the noisy and exact values of the monomial-exponential sum in x_k, $e_k \in [0, 1]$ is a normally distributed random array and δ is the standard deviation of the sampled data. In each example we assume that only a reliable estimate $\widehat{M}$ of M is known. As in [19], to compute the relative errors of the parameters and coefficients, we adopt the error estimators

$$e(\mathbf{f}) = \max_{j=1,\ldots,n} \left|1 - \frac{f_j}{f_j^*}\right|, \quad e(\mathbf{c}) = \max_{\substack{j=1,\ldots,n \\ s=0,\ldots,m_j-1}} \left|1 - \frac{c_{js}}{c_{js}^*}\right| \tag{26}$$

where f_j^* and c_{js}^* denote the exact values of the parameters. Moreover, by using our estimates f_j and c_{js} of f_j^* and c_{js}^*, we evaluate the relative error of the monomial-exponential sum h as follows:

$$e(\mathbf{h}) = \max_{x \in X}\left|1 - \frac{h(x)}{h^*(x)}\right| \tag{27}$$

where h^* is the exact sum and $X = \{x_i = i\frac{b}{50},\ i = 1, \dots, 50\}$.

In other words we adopt the error estimators typical of a worst case analysis. Hence their values are expected to be larger than those obtained by using other estimators appearing in the literature. In [12], for example, instead of the expression for $e(\mathbf{f})$ in (26) the following estimator

$$e(\mathbf{f}) = \frac{\max_{j=1,\dots,n}\left|f_j - f_j^*\right|}{\max_{j=1,\dots,n}|f_j^*|} \tag{28}$$

has been adopted. The same consideration holds true for the error expression of $e(\mathbf{c})$ in (26) and for the expression for $e(\mathbf{h})$ in (27).

In the bivariate case, to highlight the dependence of our results on the level of the noise on the data, extending the above procedure, we assume

$$h(k_1, k_2) = h^*(k_1, k_2) + \delta(e_{k_1} + e_{k_2}), \tag{29}$$

where $h(k_1, k_2) \equiv h(x_{1\,k_1}, x_{2\,k_2})$ denotes the noisy data in $(x_{1\,k_1}, x_{2\,k_2})$, $h^*(k_1, k_2) \equiv h^*(x_{1\,k_1}, x_{2\,k_2})$ represents the exact one in $(x_{1\,k_1}, x_{2\,k_2})$ and δ, as well as e_{k_1} and e_{k_2}, are as before.

The expressions of the error estimators are the natural extensions to the bivariate case of the univariate expressions of the error estimators in (26) and (27), so that

$$e(\mathbf{f}) = \max\left\{\max_{j_1=1,\dots,n_1}\left|1 - \frac{f_{1j_1}}{f_{1j_1}^*}\right|, \max_{j_2=1,\dots,n_2}\left|1 - \frac{f_{2j_2}}{f_{2j_2}^*}\right|\right\},$$

$$e(\mathbf{c}) = \max_{(j_1,s_1),(j_2,s_2)}\left\{\left|1 - \frac{c_{(j_1,s_1)(j_2,s_2)}}{c_{(j_1,s_1)(j_2,s_2)}^*}\right| j_1 = 1, \dots, n_1,\ s_1 = 0, \dots, m_{j_1-1}\ j_2 = 1, \dots, n_2,\ s_2 = 0, \dots, m_{j_2-1}\right\},$$

where $f_{1j_1}^*$, $f_{2j_2}^*$ and $c_{(j_1,s_1)(j_2,s_2)}^*$ denote the exact values of the parameters and coefficients. Similarly, the relative error estimator of h in the rectangle $[a_1, b_1] \times [a_2, b_2]$ is

$$e(\mathbf{h}) = \max\left\{\left|1 - \frac{h(x_{1\,i_1}, x_{2\,i_2})}{h^*(x_{1\,i_1}, x_{2\,i_2})}\right|, x_{1\,i_1} = i_1\frac{b_1 - a_1}{50}, x_{2\,i_2} = i_2\frac{b_2 - a_2}{50},\ i_1 = i_2 = 0, 1, \dots, 50\right\}, \tag{30}$$

where $h^*(x_{1\,i_1}, x_{2\,i_2})$ denote the noisy and exact values of $h(x_1, x_2)$ in $(x_{1\,i_1}, x_{2\,i_2})$. Hence, also in the bivariate case the error estimators adopted are able to reveal the presence of a single point, among those sampled, in which the approximation is not satisfactory.

All computations were carried out in MATLAB version 8.1 (R2013a) 64-bit for Linux in double precision arithmetic.

Example 1 (*An application to NPDEs of integrable type*). As remarked in [19], an extensive area where effective methods for parameter identification in sums of monomial-exponential functions can be very useful is represented by the important class of non-linear partial differential equations (NPDEs) of integrable type [20,21].

In this context it is very important to identify the parameters $\{n, a_j, m_j\}$ and the coefficients $\{(\Gamma_\ell)_{js}, (\Gamma_r)_{js}\}$ of the monomial exponential sums

$$\Omega_\ell(x) = \sum_{j=1}^{n} e^{-a_j x}\sum_{s=0}^{m_j-1}(\Gamma_\ell)_{js}\frac{x^s}{s!}, \quad x \in \mathbb{R}^+, \tag{31}$$

$$\Omega_r(x) = \sum_{j=1}^{n} e^{a_j x}\sum_{s=0}^{m_j-1}(\Gamma_r)_{js}\frac{x^s}{s!}, \quad x \in \mathbb{R}^-, \tag{32}$$

where $0^0 \equiv 1$ and a_j are complex or real parameters with $Re(a_j) > 0$.

The application of our method to Ω_ℓ allows us to estimate $\{n, m_j, (\Gamma_\ell)_{js}\}$, knowing Ω_ℓ in $2N\,(N > \widehat{M})$ positive integer points, and then to recover $(\Gamma_r)_{js}$ by solving, in the least squares sense, a linear system of order $N \times M$, given Ω_r in $2N\ (N > M)$ negative integer nodes. The same results can of course be obtained as well by applying first the method to $\Omega_r(x)$ to identify $\{n, m_j, (\Gamma_r)_{js}\}$ and then to $\Omega_\ell(x)$ to identify $(\Gamma_\ell)_{js}$.

In Tables 1 and 2 we give the error estimates that we obtain in the identification of Ω_ℓ parameters and coefficients in the following two cases (representative of four-solitons with 4 simple bound states and with two double bound states):

(a) $n = 4, \quad m_1 = \ldots = m_4 = 1,$
$\mathbf{a} = \frac{1}{10}[1 + 7i, 1 - 7i, 1.4 + 4i, 1.4 - 4i]$ and $\boldsymbol{\Gamma}_\ell = [1 + i, 1 - i, 3 + i, 3 - i]$;
(b) $n = 2, \quad m_1 = m_2 = 2$
$\mathbf{a} = \frac{1}{10}[1 + 7i, 1 - 7i]$ and $\boldsymbol{\Gamma}_\ell = [1 + i, 1 - i, 2 + i, 2 - i]$.

Notice that $m_j = 1$ means that the jth bound state, identified by the parameter a_j, is simple, while $m_j = 2$ means that it is double. With reference to the Prony polynomial this fact implies that the jth zero is simple if m_j=1 and double whenever $m_j = 2$.

In both cases we considered [0, 5] as the interval of effective interest and then assumed $b = 5$.

The results reported in Tables 1 and 2, obtained with the QR technique, highlight that the identification of parameters and coefficients is satisfactory not only in the simpler case (a) but also in the presence of multiple bound states (case (b)) which represents a more difficult situation, as people working in the NPDEs area of integrable type well know. In fact, the results that we obtain are very good in the absence of noise and reliable in the presence of noise, although $\widehat{M}$ is a large overestimate of M. Furthermore Tables 1 and 2 make evident that good results can be obtained by taking a relatively small number of sampling data i.e. $N = 4M$. The results obtained via the SVD or via the GSVD have not been reported as they are nearly indistinguishable from those obtained via QR.

Example 2. Let $h(x)$ be the exponential sum (4), already considered in [13],

$$\mathbf{c} = e^{15i}\begin{bmatrix} 3.1 \\ 9.9 \\ 6.0 \\ 2.8 \\ 17 \end{bmatrix}, \quad \mathbf{z} = 2 * 10^{-5}\begin{bmatrix} -208 - 2\pi 1379i \\ -256 - 2\pi 685i \\ -197 - 2\pi 271i \\ -117 + 2\pi 353i \\ -808 + 2\pi 478i \end{bmatrix}, \tag{33}$$

where $n = 5$ and $m_j = 1$ for each j. The errors estimates, reported in Table 3, clearly show that the results obtained via QR, via SVD and via GSVD are very good. Moreover, being essentially the same, they validate each other. Notice that, as to be expected, the number of sampling points needed to obtain satisfactory results increases as the level of noise increases.

Example 3. Let us now consider the bivariate exponential-sum

$$h(x_1, x_2) = \sum_{j_1=1}^{2}\sum_{j_2=1}^{3} c_{j_1,j_2} e^{f_{1j_1}x_1 + f_{2j_2}x_2},$$

where

$$\mathbf{c} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}, \quad \mathbf{f_1} = [2, 4], \quad \mathbf{f_2} = [1, 3, 5]. \tag{34}$$

Proceeding as explained in Section 3, fixing x_2 we apply the univariate method to estimate the parameters $\{f_{1j_1}\}$ of the sum

Table 1
Numerical results for Example 1 (case (a)).

N	δ	$\widehat{M}$	$e(\mathbf{f})$	$e(\mathbf{c})$	$e(\mathbf{h})$
8	0	7	3.05e−14	9.32e−14	1.32e−13
16	0	7	2.25e−14	6.73e−14	5.68e−13
32	0	7	2.58e−14	8.75e−14	2.82e−13
8	10^{-9}	7	2.93e−09	1.44e−08	4.73e−08
16	10^{-9}	7	5.49e−10	3.93e−09	3.60e−09
32	10^{-9}	7	4.82e−10	2.79e−09	6.87e−08
8	10^{-7}	7	4.55e−07	2.02e−06	8.61e−07
16	10^{-7}	7	9.21e−08	4.93e−07	2.88e−06
32	10^{-7}	7	4.88e−08	3,33e−07	1.21e−06

Table 2
Numerical results for Example 1 (case (b)).

N	δ	$\widehat{M}$	$e(\mathbf{f})$	$e(\mathbf{c})$	$e(\mathbf{h})$
8	0	7	4.04e−08	4.76e−07	3.81e−07
16	0	7	7.46e−08	3.06e−06	1.20e−06
32	0	7	4.19e−08	1.35e−06	2.55e−07
8	10^{-9}	7	1.09e−05	1.98e−04	3.19e−05
16	10^{-9}	7	4.99e−06	1.82e−04	6.47e−05
32	10^{-9}	7	8.15e−06	2.90e−04	6.82e−05
8	10^{-7}	7	1.45e−04	1.73e−03	9.02e−04
16	10^{-7}	7	7.09e−05	2.65e−03	8.93e−04
32	10^{-7}	7	2.38e−05	7.79e−04	2.11e−04

Table 3
Numerical results for Example 2.

	N	δ	$\widehat{M}$	$e(\mathbf{f})$	$e(\mathbf{c})$	$e(\mathbf{h})$
via QR	10	0	5	2.09e−05	6.84e−05	2.00e−07
	50	0	5	7.34e−08	2.93e−07	3.09e−09
	100	0	5	2.04e−09	8.25e−09	1.00e−09
	10	10^{-11}	8	7.54e−03	2.50e−02	4.83e−05
	50	10^{-11}	8	2.00e−06	7.88e−06	1.46e−07
	100	10^{-11}	8	9.08e−08	5.60e−07	2.35e−08
	10	10^{-9}	8	2.33e−01	5.43e−01	1.68e−03
	50	10^{-9}	8	1.17e−04	4.78e−04	7.15e−06
	100	10^{-9}	8	7.18e−06	4.52e−05	1.90e−06
via SVD	10	0	5	2.38e−05	7.81e−05	2.21e−07
	50	0	5	7.39e−08	3.01e−07	4.46e−09
	100	0	5	1.23e−09	1.06e−08	8.25e−10
	10	10^{-11}	8	4.13e−03	1.40e−02	2.63e−05
	50	10^{-11}	8	1.00e−06	3.70e−06	9.56e−08
	100	10^{-11}	8	6.00e−08	3.86e−07	1.44e−08
	10	10^{-9}	8	1.24e−01	5.14e−01	8.74e−04
	50	10^{-9}	8	2.99e−05	1.30e−04	1.45e−06
	100	10^{-9}	8	1.89e−06	1.20e−05	2.74e−07
via GSVD	10	0	5	3.95e−05	1.31e−04	3.19e−07
	50	0	5	3.62e−08	1.52e−07	1.69e−09
	100	0	5	5.33e−09	3.42e−08	1.38e−09
	10	10^{-11}	8	4.36e−02	1.70e−01	1.13e−03
	50	10^{-11}	8	9.82e−06	5.01e−05	5.40e−08
	100	10^{-11}	8	2.14e−07	1.15e−06	7.92e−08
	10	10^{-9}	8	3.28e−02	1.31e−01	7.27e−04
	50	10^{-9}	8	1.44e−06	4.19e−06	5.42e−09
	100	10^{-9}	8	3.16e−07	1.65e−06	1.12e−07

$$h_{x_1}(x_2) = \sum_{j_1=1}^{2} a_{j_2}(x_2) e^{f_{1j_1} x_1}$$

and then, fixing x_1, the parameters $\{f_{2j_2}\}$ are estimated by applying the same method to the sum

$$h_{x_2}(x_1) = \sum_{j_2=1}^{3} b_{j_1}(x_1) e^{f_{2j_2} x_2}.$$

The coefficients are then estimated by solving in the least squares sense, the linear system (24), where

$$(\mathcal{F})_{ij} = e^{f_{1j_1} x_{1i} + f_{2j_2} x_{2i}}, \quad (\mathbf{c})_j = c_{j_1 j_2}, \quad (\mathbf{h})_i = h(x_{1i}, x_{2i})$$

with $i = i_1 + 1 + (2N_1)i_2,\ i_1 = 0, 1, \ldots, 2N_1$ for $i_2 = 0, 1, \ldots, 2N_2$ and $j = j_1 + 2(j_2 - 1),\ j_1 = 1, 2$ for $j_2 = 1, 2, 3$.

The results concerning this example are reported in Table 4.

Table 4
Numerical results for Example 3 with the QR technique.

N	δ	$\widehat{M}_1$	$\widehat{M}_2$	$e(\mathbf{f})$	$e(\mathbf{c})$	$e(\mathbf{h})$
8	0	5	5	1.68e−13	9.09e−13	1.46e−11
16	0	5	5	3.08e−12	4.35e−12	5.02e−11
8	10^{-9}	5	5	6.72e−10	6.43e−10	2.53e−09
16	10^{-9}	5	5	7.01e−10	8.84e−10	2.41e−09
8	10^{-7}	5	5	3.00e−08	3.33e−08	1.08e−07
16	10^{-7}	5	5	2.93e−08	8.18e−09	1.38e−08

Table 5
Numerical results for Example 4 with the QR technique.

N	δ	$\widehat{M}_1$	$\widehat{M}_2$	$e(\mathbf{f})$	$e(\mathbf{c})$	$e(\mathbf{h})$
16	0	6	6	1.69e−08	1.40e−05	2.83e−04
32	0	6	6	1.85e−08	5.97e−05	1.07e−03
64	0	6	6	1.88e−08	2.71e−04	6.80e−03
16	10^{-9}	6	6	6.54e−07	2.74e−04	2.21e−05
32	10^{-9}	6	6	4.11e−07	4.84e−04	5.57e−05
64	10^{-9}	6	6	2.09e−07	9.93e−04	1.07e−04
16	10^{-7}	6	6	6.80e−06	2.34e−03	2.11e−04
32	10^{-7}	6	6	4.79e−06	6.14e−03	7.04e−04
64	10^{-7}	6	6	2.12e−06	1.15e−02	1.59e−03

As could be expected, the exponential parameters being positive and not small enough, the method is more effective whenever we sample $h(x_1, x_2)$ in regular nodal points of an interval. The results reported in Table 4, in particular, have been obtained assuming $h(x_1, x_2)$ sampled in the square $[0, 2] \times [0, 2]$. The results obtained proceeding via QR are very good even though M_1 and M_2 are largely overestimated. Furthermore, our experiments show that they are indistinguishable from those obtained via SVD and GSVD, not reported here. The identification of the parameters, in this example, turns out to be relatively easy, provided the largest distance between the sampling points is not too large. This conclusion is not surprising as the monomial factors are not present, that is as the zeros of the corresponding Prony polynomials are simple.

Example 4. Let us consider the bivariate monomial-exponential sum

$$\begin{aligned} h(x_1, x_2) &= \sum_{j_1=1}^{2}\sum_{s_1=0}^{1}\sum_{j_2=1}^{2}\sum_{s_2=0}^{1} c_{(j_1,s_1),(j_2,s_2)}\, x_1^{s_1} e^{f_{1j_1}x_1} x_2^{s_2} e^{f_{2j_2}x_2} \\ &= e^{0.48\pi i x_1}\{(1+2x_2)e^{0.52\pi i x_2} + (3+4x_2)e^{0.80\pi i x_2}\} + x_1 e^{0.48\pi i x_1}\{(1.2+1.4x_2)e^{0.52\pi i x_2} + (1.6+1.8x_2)e^{0.80\pi i x_2}\} \\ &\quad + e^{2\pi i x_1}\{(2.2+2.4x_2)e^{0.52\pi i x_2} + (2.6+2.8x_2)e^{0.80\pi i x_2}\} \\ &\quad + x_1 e^{2\pi i \pi x_1}\{(3.2+3.4x_2)e^{0.52\pi i x_2} + (3.6+3.8x_2)e^{0.80\pi i x_2}\}. \end{aligned}$$

It represents a monomial-exponential sum of 16 terms as $M_1 = M_2 = 4$ since it has four double parameters for each variable. The numerical results are given in Table 5. Let us note that the identification of the parameters in this case is more complex than in the previous example since the zeros of the associated Prony polynomials are double unlike in the previous case in which they are simple, and the number of terms is also considerably higher. For this reason the results shown in Table 5 can be considered satisfactory, thought the errors are larger.

5. Conclusions

The contribution of this paper is twofold:

(1) the introduction in our matrix-pencil method of the QR technique for the factorization of the augmented Hankel matrix in place of the SVD technique or the simultaneous factorization of the two Hankel matrices $\mathbf{H}^0$ and $\mathbf{H}^1$ by the GSVD;
(2) the development of a cascade-like technique which reduces the identification of the parameters and coefficients of a multivariate sum to the solution of a sequence of univariate problems solvable by the matrix-pencil method developed in the univariate case and the recovery of the coefficients to the solution of a linear system. It is remarkable

to note that the errors in the identification of the parameters in the multivariate case are essentially equivalent to those in the univariate case.

As a consequence of the point (1) we have three different and equally reliable techniques which can be used for a mutual validation of the results in the most difficult situations, i.e. when some zeros z_j are multiple or very close to each other. Moreover, the point (2) states that for the first time an effective method has been proposed for the identification of the parameters and coefficients of multivariate monomial exponential sums.

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