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DIRECT SCATTERING OF AKNS SYSTEMS WITH L^2 POTENTIALS

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Abstract. In this article the Jost solutions of the AKNS system with suitably weighted L^2 potential are constructed as Hardy space perturbations of their space-infinity asymptotics. The reflection coefficients are proven to be L^2-functions when the transmission coefficients are L^∞-functions.

1. Introduction.

In this article we discuss direct scattering for the AKNS system [16, 1, 12, 6, 2, 3, 13]

$$iJ\frac{\partial X}{\partial x}(x,\lambda) - iQ(x)X(x,\lambda) = \lambda X(x,\lambda), \tag{1.1}$$

where

$$J = \begin{pmatrix} I_m & 0_{m\times n} \\ 0_{n\times m} & -I_n \end{pmatrix}, \qquad Q(x) = \begin{pmatrix} 0_{m\times m} & q(x) \\ r(x) & 0_{n\times n} \end{pmatrix}, \tag{1.2}$$

the potentials $q(x)$ and $r(x)$ have their entries in $L^2(\mathbb{R})$, λ is a spectral parameter, and I_p denotes the $p\times p$ unit matrix. In the *defocusing case* ($\sigma = -1$) and the *focusing case* ($\sigma = 1$) we have the symmetry relations $r(x) = \sigma q(x)^\dagger$ and hence $Q(x)^\dagger = \sigma Q(x)$, where the dagger denotes complex conjugate matrix transposition. Contrary to the usual situation in the literature, the potentials are not assumed to be L^1 (as in [2, 3, 13]) or to belong to the Schwartz class (as in [6]) but to satisfy

$$\int_{-\infty}^{\infty} dy\,|y-x|\,\|Q(y)\|^2 < +\infty, \qquad x\in\mathbb{R}. \tag{1.3}$$

The main application of the direct scattering theory of the Schrödinger equation on the line and the AKNS system is to solve the Cauchy problem of certain integrable nonlinear evolution equations by means of the inverse scattering transform (IST) method. This means that the time evolution of the potential is transformed, by means of the IST, into the elementary time evolution of the scattering data. This has led to an algorithm to solve the Korteweg-de Vries (KdV) equation by using the scattering theory of the Schrödinger equation on the line and an algorithm to solve the nonlinear matrix Schrödinger (NLS) equation by using that of the AKNS system. A natural question to answer is how to define a sufficiently extensive class of potentials and a sufficiently extensive class of scattering data such that there is a $1,1$-correspondence between potentials and scattering data by means of the IST (the so-called characterization problem).

Characterization of scattering data of linear differential systems similar to (1.1) has a long history. Melin [11] has characterized the scattering data of the Schrödinger equation on the line with real potentials $Q(x)$ such that $(1+|x|)Q(x)$ is integrable. Previous results for real potentials $Q(x)$ such that $(1+x^2)Q(x)$ is integrable, are due to Marchenko [10].

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For the AKNS system with L^1 potentials, Demontis and Van der Mee [5] have given a $1,1$-correspondence between L^1-potentials without spectral singularities and suitable scattering data. We refer to [15, 3, 4] for prior partial results. Unfortunately, these characterization results are not invariant under time evolution. In fact, time evolution according to the matrix NLS equation might lead to potentials and scattering data not belonging to the two classes in $1,1$-correspondence. Moreover, one cannot formulate even the simplest of the infinitely many conservation laws for every time dependent potential belonging to the class.

This article is meant as a contribution towards a characterization result, where a large class of potentials and a large class of scattering data are put in such a $1,1$-correspondence that either class is invariant under time evolution according to the matrix nonlinear Schrödinger (NLS) equation. Van der Mee [14] has given the following partial solution to the time-evolution invariant characterization problem:

a) Assuming a reflection coefficient to be continuous in $\lambda \in \mathbb{R}$, vanishing as $\lambda \to \pm\infty$, and L^2, plus reasonable bound state data, a unique L^2 potential can be constructed;

b) a dense linear subspace of potentials with entries in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is required to arrive at scattering data within the designated, time evolution invariant, class.

In this article we focuss on particular details of the construction: How to define Jost solutions and scattering coefficients for certain non-L^1-potentials, and how to prove the reflection coefficients to be L^2.

The AKNS operator $iJ\frac{d}{dx}I_{m+n} - iQ$ can be defined in a natural way on the orthogonal sum of $m+n$ copies of $L^2(\mathbb{R})$ for L^1_{loc} potentials [8]. In the defocusing case the AKNS operator has a unique selfadjoint extension. For L^2 potentials this operator has the same domain as the free AKNS operator $i\frac{d}{dx}J$ (namely, the direct sum of $m+n$ copies of the first Sobolev space) and has the real line as its essential spectrum [9].

Let us discuss the contents of the various sections. In Sec. 2 we write the Jost solutions in triangular representation form and iterate the resulting integral equations for the kernel functions in the L^2 norm. Under condition (1.3), this will lead to Jost solutions which are still analytic in the spectral variable in the upper or lower half-plane but are no longer continuous in the spectral variable when approaching the real line. Instead the Jost solutions will belong to suitable Hardy spaces of analytic functions. In Sec. 3 we construct the reflection and transmission coefficients as L^1_{loc} functions of $\lambda \in \mathbb{R}$. Assuming the absence of spectral singularities, the reflection coefficients are shown to have their entries in $L^2(\mathbb{R})$.

2. **Jost solutions.**

The Jost matrices $\Psi(\lambda,x)$ and $\Phi(\lambda,x)$ are those solutions to (1.1) which behave as $e^{-i\lambda xJ}[I_{m+n}+o(1)]$ as $x \to +\infty$ and $x \to -\infty$, respectively. They can be partitioned into Jost solutions as follows

$$\Psi(\lambda,x) = \begin{pmatrix} \overline{\psi}(\lambda,x) & \psi(\lambda,x) \end{pmatrix}, \qquad \Phi(\lambda,x) = \begin{pmatrix} \phi(\lambda,x) & \overline{\phi}(\lambda,x) \end{pmatrix},$$

where $\overline{\psi}(\lambda,x)$ and $\phi(\lambda,x)$ are $(m+n)\times m$ and $\psi(\lambda,x)$ and $\overline{\phi}(\lambda,x)$ are $(m+n)\times n$ matrices. For L^1 potentials and $\lambda \in \mathbb{R}$ their existence can be proved in the traditional way [2, 3, 13] by iterating Volterra integral equations. Here we prove their existence, for almost every $\lambda \in \mathbb{R}$, for potentials satisfying (1.3) by a different technique. For L^1_{loc}-potentials, Klaus [8] has constructed Jost solutions by using arbitrary bases of the linear spaces of AKNS solutions that are L^2 on either the left or the right half-line. In this article we do not pursue his construction but follow a more direct route to Jost solutions instead.

Writing the triangular representations

$$\Psi(\lambda,x) = \begin{pmatrix} \overline{\psi}(\lambda,x) & \psi(\lambda,x) \end{pmatrix} = e^{-i\lambda xJ} + \int_x^\infty dy\, \mathbb{K}(x,y)e^{-i\lambda yJ}, \tag{2.1a}$$

$$\Phi(\lambda,x) = \begin{pmatrix} \phi(\lambda,x) & \overline{\phi}(\lambda,x) \end{pmatrix} = e^{-i\lambda xJ} + \int_{-\infty}^x dy\, \mathbb{M}(x,y)e^{-i\lambda yJ}, \tag{2.1b}$$

where the kernel functions $\mathbb{K}(x,y)$ and $\mathbb{M}(x,y)$ can be decomposed as

$$\mathbb{K}(x,y) = \begin{pmatrix} \overline{K}^{\mathrm{up}}(x,y) & K^{\mathrm{up}}(x,y) \\ \overline{K}^{\mathrm{dn}}(x,y) & K^{\mathrm{dn}}(x,y) \end{pmatrix}, \ \mathbb{M}(x,y) = \begin{pmatrix} M^{\mathrm{up}}(x,y) & \overline{M}^{\mathrm{up}}(x,y) \\ M^{\mathrm{dn}}(x,y) & \overline{M}^{\mathrm{dn}}(x,y) \end{pmatrix},$$

we obtain the integral equations

$$\overline{K}^{\mathrm{up}}(x,y) = -\int_x^\infty dz\, q(z)\overline{K}^{\mathrm{dn}}(z, z+y-x), \tag{2.2a}$$

$$\overline{K}^{\mathrm{dn}}(x,y) = \tfrac{1}{2}r(\tfrac{1}{2}(x+y)) + \int_x^{\frac{1}{2}(x+y)} dz\, r(z)\overline{K}^{\mathrm{up}}(z, x+y-z), \tag{2.2b}$$

$$K^{\mathrm{up}}(x,y) = -\tfrac{1}{2}q(\tfrac{1}{2}(x+y)) - \int_x^{\frac{1}{2}(x+y)} dz\, q(z)K^{\mathrm{dn}}(z, x+y-z), \tag{2.2c}$$

$$K^{\mathrm{dn}}(x,y) = \int_x^\infty dz\, r(z)K^{\mathrm{up}}(z, z+y-x), \tag{2.2d}$$

as well as

$$M^{\mathrm{up}}(x,y) = \int_{-\infty}^x dz\, q(z)M^{\mathrm{dn}}(z, z+y-x), \tag{2.3a}$$

$$M^{\mathrm{dn}}(x,y) = -\tfrac{1}{2}r(\tfrac{1}{2}(x+y)) - \int_{\frac{1}{2}(x+y)}^x dz\, r(z)M^{\mathrm{up}}(z, x+y-z), \tag{2.3b}$$

$$\overline{M}^{\mathrm{up}}(x,y) = \tfrac{1}{2}q(\tfrac{1}{2}(x+y)) + \int_{\frac{1}{2}(x+y)}^x dz\, q(z)\overline{M}^{\mathrm{dn}}(z, x+y-z), \tag{2.3c}$$

$$\overline{M}^{\mathrm{dn}}(x,y) = -\int_{-\infty}^x dz\, r(z)\overline{M}^{\mathrm{up}}(z, z+y-x). \tag{2.3d}$$

For potentials with L^1 entries, (2.2) and (2.3) are easily shown to be uniquely solvable by iteration [2, 13], yielding

$$\int_x^\infty dy\, \|\mathbb{K}(x,y)\| + \int_{-\infty}^x dy\, \|\mathbb{M}(x,y)\| < +\infty,$$

uniformly in $x \in \mathbb{R}$.

We now establish the unique solvability of the integral equations (2.2) and (2.3) for potentials $Q(x)$ satisfying (1.3).

Theorem 2.1. *Let $\int_x^\infty dz\,(z-x)\|Q(z)\|^2$ converge for $x \geq x_0$. Then for $x \geq x_0$ the integral equations (2.2) have a unique solution $\mathbb{K}(x,y)$ such that $\int_x^\infty dy\, \|\mathbb{K}(x,y)\|^2$ and $\int_{x_0}^\infty dx \int_x^\infty dy\, \|\mathbb{K}(x,y)\|^2$ converge. Analogously, if $\int_{-\infty}^x dz\,(x-z)\|Q(z)\|^2$ converges for $x \leq x_0$, then for $x \leq x_0$ the integral equations (2.3) have a unique solution $\mathbb{M}(x,y)$ such that $\int_{-\infty}^x dy\, \|\mathbb{M}(x,y)\|^2$ and $\int_{-\infty}^{x_0} dx \int_{-\infty}^x dy\, \|\mathbb{M}(x,y)\|^2$ converge.*

Proof. We only prove the first statement. Estimating (2.2a) we get

$$\begin{aligned}\int_x^\infty dy\, \|\overline{K}^{\mathrm{up}}(x,y)\|^2 &\leq \int_x^\infty dy \left[\int_x^\infty dz\, \|q(z)\| \|\overline{K}^{\mathrm{dn}}(z, z+y-x)\|\right]^2 \\ &\leq \left(\int_x^\infty d\hat{z}\, \|q(\hat{z})\|^2\right) \int_x^\infty dz \int_z^\infty d\hat{y}\, \|\overline{K}^{\mathrm{dn}}(z,\hat{y})\|^2.\end{aligned}$$

Iterating (2.2b) we obtain

$$\begin{aligned}\int_x^\infty dy\,\|\overline{K}^{\mathrm{dn}}(x,y)-\tfrac{1}{2}r(\tfrac{1}{2}(x+y))\|^2 &\\ \le \int_x^\infty dy &\left[\int_x^{\frac{x+y}{2}} dz\,\|r(z)\|\|\overline{K}^{\mathrm{up}}(z,x+y-z)\|\right]^2\\ &= \left(\int_x^\infty d\hat{z}\,\|r(\hat{z})\|^2\right)\int_x^\infty dz\int_z^\infty d\hat{y}\,\|\overline{K}^{\mathrm{up}}(z,\hat{y})\|^2.\end{aligned}$$

Schematically, these two inequalities can be written as

$$\|\overline{K}^{\mathrm{up}}(x,\cdot)\|_2^2 \le \left(\int_x^\infty d\hat{z}\,\|q(\hat{z})\|^2\right)\int_x^\infty dz\,\|\overline{K}^{\mathrm{dn}}(z,\cdot)\|_2^2, \tag{2.4a}$$

$$\|\overline{K}^{\mathrm{dn}}(x,\cdot)-\tfrac{1}{2}r(\tfrac{1}{2}(x+\cdot))\|_2^2 \le \left(\int_x^\infty d\hat{z}\,\|r(\hat{z})\|^2\right)\int_x^\infty dz\,\|\overline{K}^{\mathrm{up}}(z,\cdot)\|_2^2. \tag{2.4b}$$

Taking the square root and applying the triangle inequality we obtain

$$\begin{aligned}\|\overline{K}^{\mathrm{up}}(x,\cdot)\|_2 &\le [Q](x)\left[\int_x^\infty dz\,\|\overline{K}^{\mathrm{dn}}(z,\cdot)\|_2^2\right]^{1/2},\\ \|\overline{K}^{\mathrm{dn}}(x,\cdot)\|_2 &\le [Q](x)\left\{\frac{1}{\sqrt{2}}+\left[\int_x^\infty dz\,\|\overline{K}^{\mathrm{up}}(z,\cdot)\|_2^2\right]^{1/2}\right\},\end{aligned}$$

where $[Q](x)=\left(\int_x^\infty d\hat{z}\,\|Q(\hat{z})\|^2\right)^{1/2}$. With some effort we get the estimate

$$\frac{\|\overline{K}^{\mathrm{up}}(x,\cdot)\|_2^2+\|\overline{K}^{\mathrm{dn}}(x,\cdot)\|_2^2}{[Q](x)^2}\le 1+2\int_x^\infty dz\,[Q](z)^2\,\frac{\|\overline{K}^{\mathrm{up}}(z,\cdot)\|_2^2+\|\overline{K}^{\mathrm{dn}}(z,\cdot)\|_2^2}{[Q](z)^2}.$$

Using Gronwall's inequality we get

$$\sqrt{\|\overline{K}^{\mathrm{up}}(x,\cdot)\|_2^2+\|\overline{K}^{\mathrm{dn}}(x,\cdot)\|_2^2}\le [Q](x)e^{\int_x^\infty dz\,(z-x)\|Q(z)\|^2},$$

where we have used that

$$\int_x^\infty dz\,[Q](z)^2=\int_x^\infty d\hat{z}\,(\hat{z}-x)\|Q(\hat{z})\|^2.$$

Next, put $B(x)=\int_x^\infty dz\,(z-x)\|Q(z)\|^2$. Then the above inequalities (2.4) imply that for $x_1\ge x_0$

$$\begin{aligned}\int_{x_1}^\infty dx\int_x^\infty dy\,\|\overline{K}^{\mathrm{up}}(x,y)\|^2 &\le B(x_1)\int_{x_1}^\infty dx\int_x^\infty dy\,\|\overline{K}^{\mathrm{dn}}(x,y)\|^2,\\ \int_{x_1}^\infty dx\int_x^\infty dy\,\|\overline{K}^{\mathrm{dn}}(x,y)\|^2 &\le B(x_1)\left[1+2\int_{x_1}^\infty dx\int_x^\infty dy\,\|\overline{K}^{\mathrm{up}}(x,y)\|^2\right],\end{aligned}$$

so that the left-hand sides are finite if $B(x_1)<2^{-1/2}$. Taking points $x_0=\xi_0<\xi_1<\ldots<\xi_m=x_1$ satisfying $\int_{\xi_{s-1}}^{\xi_s} dx\int_x^\infty dz\,\|Q(z)\|^2<2^{-1/2}$, we can apply the same argument on the successive intervals $[\xi_{s-1},\xi_s]$. In fact, from (2.4) we get

$$\int_{\xi_{s-1}}^{\xi_s} dx\,\|\overline{K}^{\mathrm{up}}(x,\cdot)\|_2^2 \le \left[\int_{\xi_{s-1}}^{\xi_s} dx \int_x^\infty dz\,\|Q(z)\|^2\right]\left(\int_{\xi_{s-1}}^{\xi_s}+\int_{\xi_s}^\infty\right) dx\,\|\overline{K}^{\mathrm{dn}}(x,\cdot)\|_2^2,$$

$$\int_{\xi_{s-1}}^{\xi_s} dx\,\|\overline{K}^{\mathrm{dn}}(x,\cdot)\|_2^2 \le 2\left[\int_{\xi_{s-1}}^{\xi_s} dx \int_x^\infty dz\,\|Q(z)\|^2\right]\left(\int_{\xi_{s-1}}^{\xi_s}+\int_{\xi_s}^\infty\right) dx\,\|\overline{K}^{\mathrm{up}}(x,\cdot)\|_2^2 + \int_{\xi_{s-1}}^{\xi_s} dz\,[Q](z)^2,$$

which proves the finiteness of $\int_{\xi_{s-1}}^{\xi_s} dx \int_x^\infty dy \left(\|\overline{K}^{\mathrm{up}}(x,y)\|^2+\|\overline{K}^{\mathrm{dn}}(x,y)\|^2\right)$. We have thus established the finiteness of $\int_{x_0}^\infty dx \int_x^\infty dy \left(\|\overline{K}^{\mathrm{up}}(x,y)\|^2+\|\overline{K}^{\mathrm{dn}}(x,y)\|^2\right)$.

The proofs for the other kernel functions are identical. □

Theorem 2.1 implies that, under the condition (1.3),

$$\int_{-\infty}^\infty d\lambda\,\|\Psi(\lambda,x)-e^{-i\lambda xJ}\|^2 = 2\pi\int_x^\infty dy\,\|\mathbb{K}(x,y)\|^2,$$
$$\int_{-\infty}^\infty d\lambda\,\|\Phi(\lambda,x)-e^{-i\lambda xJ}\|^2 = 2\pi\int_{-\infty}^x dy\,\|\mathbb{M}(x,y)\|^2,$$

where $x\in\mathbb{R}$.

We now introduce the Hardy spaces $H^2(\mathbb{C}^\pm)$ as the complex Hilbert spaces of those analytic functions $f(\lambda)$ in $\lambda\in\mathbb{C}^\pm$ such that

$$\|f\| = \left[\lim_{\varepsilon\to 0^+}\int_{-\infty}^\infty d\lambda\,|f(\lambda\pm i\varepsilon)|^2\right]^{1/2}$$

is finite. Then $H^2(\mathbb{C}^\pm)$ coincides with the image of the L^2 functions supported on $\mathbb{R}^\pm$ under Fourier transformation [7].

Corollary 2.2. *Suppose the potential $Q(x)$ satisfies (1.3). Then for each $x\in\mathbb{R}$ we have the following:*

a. $e^{i\lambda x}\overline{\psi}(\lambda,x)-\begin{pmatrix} I_m \\ 0_{n\times n}\end{pmatrix}$ *has its entries in $H^2(\mathbb{C}^-)$;*
b. $e^{-i\lambda x}\psi(\lambda,x)-\begin{pmatrix} 0_{m\times n} \\ I_n\end{pmatrix}$ *has its entries in $H^2(\mathbb{C}^+)$;*
c. $e^{i\lambda x}\phi(\lambda,x)-\begin{pmatrix} I_m \\ 0_{n\times n}\end{pmatrix}$ *has its entries in $H^2(\mathbb{C}^+)$;*
d. $e^{-i\lambda x}\overline{\phi}(\lambda,x)-\begin{pmatrix} 0_{m\times n} \\ I_n\end{pmatrix}$ *has its entries in $H^2(\mathbb{C}^-)$,*

where the Jost solutions are defined in terms of the kernel functions by (2.1).

Using $JQ(x)J=-Q(x)$ and assuming X^{-1} exists, Eq. (1.1) implies that

$$-i\frac{\partial Y}{\partial x}(\lambda,x)J + iY(\lambda,x)Q(x) = \lambda Y(\lambda,x), \tag{2.5}$$

where $Y=X^{-1}$. Then

$$\frac{\partial}{\partial x}(YX) = Y[-i\lambda JX+JQX]+[i\lambda YJ+YQJ]X = Y(JQ+QJ)X \tag{2.6}$$

vanishes for any solution X to (1.1) and Y to (2.5). Taking $Y = \Psi^{-1}$ or $Y = \Phi^{-1}$, we can derive the triangular representations

$$\Psi(\lambda,x)^{-1} = \begin{pmatrix} \breve{\psi}(\lambda,x) \\ \breve{\overline{\psi}}(\lambda,x) \end{pmatrix} = e^{i\lambda xJ} + \int_x^\infty dy\, e^{i\lambda yJ}\breve{\mathbb{K}}(y,x), \tag{2.7a}$$

$$\Phi(\lambda,x)^{-1} = \begin{pmatrix} \breve{\overline{\phi}}(\lambda,x) \\ \breve{\phi}(\lambda,x) \end{pmatrix} = e^{i\lambda xJ} + \int_{-\infty}^x dy\, e^{i\lambda yJ}\breve{\mathbb{M}}(y,x), \tag{2.7b}$$

where the kernel functions $\breve{\mathbb{K}}(y,x)$ and $\breve{\mathbb{M}}(y,x)$ can be decomposed as

$$\breve{\mathbb{K}}(y,x) = \begin{pmatrix} \breve{K}^{\mathrm{lt}}(y,x) & \breve{K}^{\mathrm{rt}}(y,x) \\ \breve{\overline{K}}^{\mathrm{lt}}(y,x) & \breve{\overline{K}}^{\mathrm{rt}}(y,x) \end{pmatrix}, \qquad \mathbb{M}(y,x) = \begin{pmatrix} \breve{\overline{M}}^{\mathrm{lt}}(y,x) & \breve{\overline{M}}^{\mathrm{rt}}(y,x) \\ \breve{M}^{\mathrm{lt}}(y,x) & \breve{M}^{\mathrm{rt}}(y,x) \end{pmatrix}.$$

For later use, we proceed as above and derive the analogs of (2.2) and (2.3) as well as the following results.

Theorem 2.3. *Let $\int_x^\infty dz\,(z-x)\|Q(z)\|^2$ converge for $x \geq x_0$. Then for $x \geq x_0$ there exists a unique kernel function $\breve{\mathbb{K}}(y,x)$ such that (2.7a) is satisfied and $\int_x^\infty dy\,\|\breve{\mathbb{K}}(y,x)\|^2$ and $\int_{x_0}^\infty dx \int_x^\infty dy\,\|\breve{\mathbb{K}}(y,x)\|^2$ converge. If $\int_{-\infty}^x dz\,(x-z)\|Q(z)\|^2$ converges for $x \leq x_0$, then for $x \leq x_0$ there exists a unique kernel function $\breve{\mathbb{M}}(y,x)$ satisfying (2.7b) such that $\int_{-\infty}^x dy\,\|\breve{\mathbb{M}}(y,x)\|^2$ and $\int_{-\infty}^{x_0} dx \int_{-\infty}^x dy\,\|\breve{\mathbb{M}}(y,x)\|^2$ converge.*

Corollary 2.4. *Suppose the potential $Q(x)$ satisfies (1.3). Then for each $x \in \mathbb{R}$ we have the following:*

a. $e^{-i\lambda x}\breve{\psi}(\lambda,x) - \begin{pmatrix} I_m & 0_{m\times n} \end{pmatrix}$ *has its entries in* $H^2(\mathbb{C}^+)$;
b. $e^{i\lambda x}\breve{\overline{\psi}}(\lambda,x) - \begin{pmatrix} 0_{n\times n} & I_n \end{pmatrix}$ *has its entries in* $H^2(\mathbb{C}^-)$;
c. $e^{-i\lambda x}\breve{\overline{\phi}}(\lambda,x) - \begin{pmatrix} I_m & 0_{m\times n} \end{pmatrix}$ *has its entries in* $H^2(\mathbb{C}^-)$,
d. $e^{i\lambda x}\breve{\phi}(\lambda,x) - \begin{pmatrix} 0_{n\times n} & I_n \end{pmatrix}$ *has its entries in* $H^2(\mathbb{C}^+)$,

where the inverses of the Jost solutions are defined in terms of the kernel functions by (2.7).

3. Scattering data.

For potentials satisfying (1.3), the transition coefficients

$$a_l(\lambda) = \begin{pmatrix} a_{l1}(\lambda) & a_{l2}(\lambda) \\ a_{l3}(\lambda) & a_{l4}(\lambda) \end{pmatrix}, \qquad a_r(\lambda) = \begin{pmatrix} a_{r1}(\lambda) & a_{r2}(\lambda) \\ a_{r3}(\lambda) & a_{r4}(\lambda) \end{pmatrix},$$

can be defined by

$$a_l(\lambda) = \Phi(\lambda,x)^{-1}\Psi(\lambda,x), \qquad a_r(\lambda) = \Psi(\lambda,x)^{-1}\Phi(\lambda,x), \tag{3.1}$$

as L^1_{loc} functions of $\lambda \in \mathbb{R}$ which do not depend on $x \in \mathbb{R}$. It is then easily verified that $a_{l1}(\lambda)$ and $a_{r4}(\lambda)$ extend to functions that are analytic in $\lambda \in \mathbb{C}^+$, whereas $a_{r1}(\lambda)$ and $a_{l4}(\lambda)$ extend to functions that are analytic in $\lambda \in \mathbb{C}^-$. In fact, $\Psi(x,\lambda)$ and $\Phi(x,\lambda)$ are both square matrix solutions of the same first order system (1.1) and hence one is obtained from the other by postmultiplication by a square matrix $a_{r/l}(\lambda)$ not depending on $x \in \mathbb{R}$.

In this paper we make the following no-spectral-singularity assumption:[1]

> *$\|a_{l1}(\lambda)\|$ (or $\|a_{r4}(\lambda)\|$) and $\|a_{r1}(\lambda)\|$ (or $\|a_{l4}(\lambda)\|$) are both almost everywhere positive and their reciprocals are essentially bounded in $\lambda \in \mathbb{R}$.*

[1]For L^1-potentials this assumption amounts to the absence of spectral singularities adopted in virtually all publications on the subject. Without such an assumption, no inverse scattering theory for (1.1) has ever been developed.

In the defocusing case, this assumption is always satisfied. Under the no-spectral-singularity assumption, for each $x \in \mathbb{R}$ the modified Jost matrices

$$F_+(\lambda,x) = \begin{pmatrix}\phi(\lambda,x) & \psi(\lambda,x)\end{pmatrix} = \Phi(\lambda,x)\boldsymbol{E}_+ + \Psi(\lambda,x)\boldsymbol{E}_-,$$
$$F_-(\lambda,x) = \begin{pmatrix}\overline{\psi}(\lambda,x) & \overline{\phi}(\lambda,x)\end{pmatrix} = \Phi(\lambda,x)\boldsymbol{E}_- + \Psi(\lambda,x)\boldsymbol{E}_+,$$

where $\boldsymbol{E}_+ = I_m \oplus 0_{n\times n}$ and $\boldsymbol{E}_- = 0_{m\times m} \oplus I_n$, extend to matrix functions analytic in $\lambda \in \mathbb{C}^\pm$. Moreover,

$$F_-(\lambda,x) = F_+(\lambda,x)\begin{pmatrix} T_r(\lambda) & -L(\lambda) \\ -R(\lambda) & T_l(\lambda)\end{pmatrix}, \tag{3.2}$$

where $T_{r/l}(\lambda)$ are *transmission coefficients* and $R(\lambda)$ and $L(\lambda)$ are *reflection coefficients.* The transmission coefficients are bounded in $\lambda \in \mathbb{R}$ and meromorphic in $\lambda \in \mathbb{C}^+$ with the same finitely many poles. It is easily shown [2, 13] that

$$R(\lambda) = -a_{l4}(\lambda)^{-1}a_{l3}(\lambda) = a_{r3}(\lambda)a_{r1}(\lambda)^{-1}, \tag{3.3a}$$
$$L(\lambda) = -a_{r1}(\lambda)^{-1}a_{r2}(\lambda) = a_{l2}(\lambda)a_{l4}(\lambda)^{-1}, \tag{3.3b}$$

where $R(\lambda)$ and $L(\lambda)$ have their entries in $L^1_{\text{loc}}(\mathbb{R})$.

Using (2.7) and (3.1) we obtain

$$a_r(\lambda) - I_{m+n} = \begin{pmatrix} \breve{\psi}(\lambda,x)\phi(\lambda,x) - I_m & \breve{\psi}(\lambda,x)\overline{\phi}(\lambda,x) \\ \breve{\overline{\psi}}(\lambda,x)\phi(\lambda,x) & \breve{\overline{\psi}}(\lambda,x)\overline{\phi}(\lambda,x) - I_n \end{pmatrix},$$

and similarly for $a_l(\lambda) - I_{m+n}$. Using Theorems 2.1 and 2.3 it is easily verified that the entries of $a_{r/l}(\lambda) - I_{m+n}$ belong to $L^2 + L^1$ and are in fact Fourier transforms of functions in $L^2 + C_0$. Thus, in the absence of spectral singularities, the reflection coefficients $R(\lambda)$ and $L(\lambda)$ belong to $L^2 + L^1$. We intend to do better than that, as indicated by the following theorem.

Theorem 3.1. *Under the no-spectral-singularity assumption and the hypothesis* (1.3)*, the reflection coefficients have their entries in* $L^2(\mathbb{R})$.

Proof. For L^1 potentials $Q(x)$ with L^1 entries, we have the following:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda\,[a_{l1}(\lambda) - I_m]e^{i\lambda y} = -\int_{-\infty}^{\infty} dz\, q(z)\overline{K}^{\text{dn}}(z,z+y), \tag{3.4a}$$
$$\frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda\, a_{l2}(\lambda)e^{-i\lambda y} = -\tfrac{1}{2}q(\tfrac{1}{2}y) - \int_{-\infty}^{\frac{y}{2}} dz\, q(z)K^{\text{dn}}(z,y-z), \tag{3.4b}$$
$$\frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda\, a_{l3}(\lambda)e^{i\lambda y} = +\tfrac{1}{2}r(\tfrac{1}{2}y) + \int_{-\infty}^{\frac{y}{2}} dz\, r(z)\overline{K}^{\text{up}}(z,y-z), \tag{3.4c}$$
$$\frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda\,[a_{l4}(\lambda) - I_n]e^{-i\lambda y} = +\int_{-\infty}^{\infty} dz\, r(z)K^{\text{up}}(z,z+y), \tag{3.4d}$$

as well as

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda\,[a_{r1}(\lambda) - I_m]e^{-i\lambda y} = +\int_{-\infty}^{\infty} dz\, q(z)M^{\text{dn}}(z,z-y), \tag{3.4e}$$
$$\frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda\, a_{r2}(\lambda)e^{-i\lambda y} = +\tfrac{1}{2}q(\tfrac{1}{2}y) + \int_{\frac{y}{2}}^{\infty} dz\, q(z)\overline{M}^{\text{dn}}(z,y-z), \tag{3.4f}$$
$$\frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda\, a_{r3}(\lambda)e^{i\lambda y} = -\tfrac{1}{2}r(\tfrac{1}{2}y) - \int_{\frac{y}{2}}^{\infty} dz\, r(z)M^{\text{up}}(z,y-z), \tag{3.4g}$$
$$\frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda\,[a_{r4}(\lambda) - I_n]e^{i\lambda y} = -\int_{-\infty}^{\infty} dz\, r(z)\overline{M}^{\text{up}}(z,z-y), \tag{3.4h}$$

where (3.4a), (3.4d), (3.4e), and (3.4h) have their entries in $L^1(\mathbb{R}^+)$ and the other four equations have their entries in $L^1(\mathbb{R})$. Let us now derive similar estimates under the hypothesis (1.3). Indeed, writing

$$A_l(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda\,[a_l(\lambda) - I_{m+n}]e^{i\lambda yJ},$$
$$A_r(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda\,e^{-i\lambda yJ}[a_r(\lambda) - I_{m+n}],$$

we can apply Theorem 2.1 to estimate

$$\int_{x_0}^{\infty} dy\,\|A_l(y)\|^2 \leq \|Q\|_2^2 \int_{x_0}^{\infty} dz\,\|\mathbb{K}(z,\cdot)\|_2^2 < +\infty,$$
$$\int_{-\infty}^{x_0} dy\,\|A_r(y)\|^2 \leq \|Q\|_2^2 \int_{-\infty}^{x_0} dz\,\|\mathbb{M}(z,\cdot)\|_2^2 < +\infty,$$

irrespective of the choice of $x_0 \in \mathbb{R}$. Using (3.3), we now easily prove that, under the no-spectral-singularity assumption, $R(\lambda)$ and $L(\lambda)$ are L^2-functions on each right half-line and each left half-line and hence on the full real line, as claimed. $\square$

In the defocusing case the scattering matrix

$$S(\lambda) = \begin{pmatrix} S_r(\lambda) & L(\lambda) \\ R(\lambda) & T_l(\lambda) \end{pmatrix} \tag{3.5}$$

is unitary for almost everywhere $\lambda \in \mathbb{R}$. Thus $R(\lambda)$ and $L(\lambda)$ are L^2- as well as L^∞-functions of $\lambda \in \mathbb{R}$. Moreover, the no-spectral-singularity assumption is always satisfied.

In the focusing case the scattering matrix given by (3.5) is J-unitary in the sense that $S(\lambda)JS(\lambda)^\dagger = J$ for almost every $\lambda \in \mathbb{R}$. Thus if the no-spectral-singularity assumption is satisfied, the reflection coefficients $R(\lambda)$ and $L(\lambda)$ are L^2- as well as L^∞-functions of $\lambda \in \mathbb{R}$.

4. **Conclusions.**

In this article we have made an interesting contribution towards a time evolution invariant characterization result for the AKNS system. In the defocusing and focusing cases, we have shown that the reflection coefficients are L^2 as well as L^∞ if the hypothesis (1.3) and the no-spectral-singularity assumption are satisfied. On the other hand, it is known [14] that in the defocusing and focusing cases the potential is L^2 if the reflection coefficients are L^2, are continuous, and vanish as $\lambda \to \pm\infty$. Close examination of the proofs contained in [14] reveals that the potential is L^2 if the reflection coefficients are L^2 as well as L^∞. The continuity of the reflection coefficients in $\lambda \in \mathbb{R}$ is only required to prove the compactness of the Marchenko integral operators in an L^2-setting, while this compactness property is not used to render the potential obtained by inverse scattering L^2.

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