STATIONARY TRANSPORT PROCESSES WITH UNBOUNDED COLLISION OPERATORS

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ABSTRACT. An abstract Hilbert space equation is studied, which models many of the stationary, one-dimensional transport equations with partial-range boundary conditions. In particular, the collision term may be unbounded and nondissipative. A complete existence and uniqueness theory is presented.

1. INTRODUCTION

Since 1973 an extensive literature has been developed on the solution of time-independent one-dimensional linear transport and kinetic equations by mathematically rigorous methods. Particular equations for which half-space boundary-value problems have been solved describe such diverse processes as neutron transport with angularly-dependent cross-sections [1, 2], radiative transfer of unpolarized light and of polarized light with Rayleigh scattering [3–5], the BGK kinetic equations for mass and heat transfer [6–8], and phonon transport [9], among others. More recently, study has been directed to the abstract differential equation

$$(Tf)'(x) = -(Af)(x), \quad 0 < x < \infty \tag{1}$$

where T and A are self-adjoint operators on an abstract Hilbert space H, Ker $T = 0$, and with boundary conditions appropriate to the specification of a given incoming flux, either

$$(Q_+f)(0) = f_+, \lim_{x \to \infty} \|f(x)\| < \infty \tag{2a}$$

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or

$$(Q_+f)(0) = f_+, \quad \lim_{x \to \infty} \|f(x)\| = 0. \tag{2b}$$

Such an abstract equation encompasses all of the particular processes mentioned above. Here Q_+ is the maximum positive projection associated with the self-adjoint operator T. These studies have depended, in an essential way, on the boundedness and positivity of A (and usually of its inverse) [5, 10].

We announce an existence and uniqueness theory for the boundary-value problems (1) – (2) for T and A both possibly unbounded and A not necessarily positive. The only restrictions are: A Fredholm, the nonpositive part of A finite dimensional, and some minor domain requirements (but the case T, A, $(A \mid \operatorname{Ran} A)^{-1}$ all unbounded and A nonpositive is to be excluded). These are the first existence and uniqueness results for boundary-value problems of the sort (1) – (2) which include problems for which half-range completeness in the sense of Case [11] may fail (due to the unboundedness of $Q_\pm$ in H_K). Complete proofs will appear elsewhere.

2. HALF-RANGE EXPANSIONS

To better understand the implications for applications, it is convenient to consider separately three cases (always T, A self-adjoint, $\operatorname{Ker} T = 0$):

(i) A positive Fredholm, T bounded;

(ii) A positive Fredholm, T unbounded;

(iii) A Fredholm with finite-dimensional negative part, T bounded.

The case (i) is typical, for example, of sub-critical and critical neutron transport and radiative transfer, T being multiplication by an angle cosine. The case (ii) is typical of gas kinetics, involving an unbounded velocity coordinate, and (iii) is relevant to supercritical media [11, 12].

Let $K = T^{-1}A$. For λ an eigenvalue of K, denote by $Z_\lambda(K)$ the root linear manifold $Z_\lambda(K) = \{f \in H \mid (K - \lambda I)^n f = 0 \text{ for some } n \in \mathbb{Z}_+\}$. If A is positive and $B: Z_0(K) \to Z_0(K)$ is invertible, let $P: H \to Z_0(K^*)^\perp$ be the projection of H onto $Z_0(K^*)^\perp$ along $Z_0(K)$, and put $A_B = AB + TB^{-1}(I - P)$. Then B may be chosen in such a way that A_B will be a strictly positive operator, i.e., $A_B \geqslant 0$, $\operatorname{Ker} A_B = 0$. Introduce the Hilbert spaces $H_A = D(A_B^{1/2})$ with inner product $(x, y)_{A_B} = (A_B x, y)$, H_K the completion of $D(A_B^{-1}T)$ in H_A with inner product $(x, y)_{K_B} = (|A_B^{-1}T|x, y)_{A_B}$, and H_T the completion of $D(T)$ with inner product $(x, y)_T = (|T|x, y)$. The B's are suppressed in symbols for the spaces because of equivalence of norms.

If A is not positive, the definition of P is more complicated, and involves a search for maximal negative K invariant subspaces M_λ of $Z_\lambda(K)$ with respect to the indefinite metric $(x, y)_A = (Ax, y)$ defined on $D(A)$. Let N_λ denote the extension of M_λ to all vectors from Jordan chains of K intersecting M_λ and $Z(K)$ the direct sum of all $Z_\lambda(K)$ for λ a nonreal eigenvalue of K and of all N_λ for λ a nonzero (regular) critical point of K and of $Z_0(K)$. Then P is defined to be the projection of H onto $(TZ(K))^\perp$ along $Z(K)$, A_B as before, and again B may be chosen in such a way that A_B will be strictly positive.

The following simple lemma is immediate:

LEMMA 1. *If* (i), *then* $K_B = T^{-1}A_B$ *is essentially self-adjoint on* H_A.

If (ii) *and also*

(iia) $D(T) \cap D(A) \subset H$ *densely*, $Z_0(K) \subset D(T)$, *and* $KZ_0(K)$ *has a complement in* Ker A *that is nondegenerate with respect to the indefinite metric* $[x, y] = (Tx, y)$,

then K_B *is symmetric on* H_A. *If either* A *or* A^{-1} *is bounded, or if there exists a signature operator on* H ($J = J^*, J^2 = I$) *which commutes with* A *and anti-commutes with* T, *then* K_B *has self-adjoint extensions.*

If (iii) *and also*

(iiia) $Z_\lambda(T^{-1}A)$ *nondegenerate with respect to* $(,)_A$ *for all real eigenvalues* λ, *and* $\dim Z_0(T^{-1}A) < \infty$,

then K_B *is essentially self-adjoint on* H_A.

Note that the Fredholm condition on A guarantees K is densely defined, and the first part of (iia) guarantees it is closable. The conditions (iiia) assure H_A is a Pontrjagin space [13] (if A non-invertible) and eliminate irregular critical points [14] in the real spectrum of $\bar{K}^A$.

Let $P_\pm$ denote the maximal positive/negative projections associated with self-adjoint extensions K_B of $T^{-1}A_B$ on H_A. Let $Q_\pm$ denote the maximal positive/negative projections associated with the self-adjoint operator T on H. The projections $P_\pm$ and $Q_\pm$ extend to orthogonal projections on H_K and H_T, respectively, and P extends to a bounded projection on H_K.

For cases (i) and (ii), the solution of the half-space problems (1) – (2) is intimately connected to the invertibility of the (unbounded) operator $V: H_K \to H_T$ defined by $V = Q_+P_+ + Q_-P_-$, although it is not at all transparent that V is even well-defined. However, we have in these cases, and assuming in (ii) a self-adjoint extension of K_B is specified, the following lemma:

LEMMA 2. *Assuming* (i) *or* (ii) – (iia), *there exists a unique albedo operator* $E: H_T \to H_K$ *that is bounded, injective, and satisfies* $Q_\pm EQ_\pm = Q_\pm E$ *and* $P_\mp EQ_\pm = 0$ *on* $D(T)$. *Further,* E *is bounded as an operator* $E: H_T \to H_T$.

Lemma 2 is an operator theoretic formulation of the so-called 'half-range completeness theorems'. The proof of the Lemma follows from a detailed study of the symmetric quadratic form defined by $V = E^{-1}$. Earlier methods, both on specific applications and on the abstract problem, either were perturbative, e.g., A a compact perturbation of the identity, or depended on the equivalence of the norms in H_K and H_T. In these cases, $V: H_T \to H_T$ is bounded. In the general setting, the boundedness of V is lost, which, physically speaking, implies that not all outgoing fluxes result from the stationary problem, but only a dense subset of them.

3. UNIQUENESS AND EXISTENCE THEORY

The half-space problem to be solved is actually a weakened version of (1) – (2), in the sense that the solution is to be found in H_K, rather than the original space H. An exact statement of the problem is the following: given $f_+ \in \text{Ran}\, Q_+$, construct a continuous function f: $[0, \infty) \to H_K$ with both KPf and $(I - P)f$ differentiable on $(0, \infty)$, such that

$$\frac{\mathrm{d}}{\mathrm{d}x} f = -Kf \tag{1'}$$

on H_K, $f(0) \in H_T$, and

$$(Q_+ f)(0) = f_+, \quad \lim_{x \to \infty} \|(Pf)(x)\|_K < \infty, \quad \lim_{x \to \infty} \|((I-P)f)(x)\| < \infty \tag{2a'}$$

or

$$(Q_+ f)(0) = f_+, \quad \lim_{x \to \infty} \|(Pf)(x)\|_K = 0, \quad \lim_{x \to \infty} \|((I-P)f)(x)\| = 0. \tag{2b'}$$

THEOREM 1. *Assume* (i). *Then the half-space problem* (1′) – (2a′) *is solvable for every* $f_+ \in Q_+(H_T)$. *The measure of nonuniqueness* $\delta^+ = \dim[\mathrm{Ran}\, PP_+ \oplus \mathrm{Ran}\, Q_-] \cap \mathrm{Ker}\, A$ *is equal to the dimension of a maximal strictly negative subspace of* Ker A *with respect to the indefinite metric* [,] . *The half-space problem* (1′) – (2b′) *has always at most one solution. The measure of noncompleteness (nonexistence)* $\gamma_0^+ = \mathrm{codim}_{H_T}\, \mathrm{Ran}\, (PP_+ \oplus \mathrm{Ran}\, Q_-)$ *as* f_+ *ranges over* $Q_+(H_T)$ *is equal to the dimension of a maximal nonnegative subspace of* Ker A *with respect to* [,] .

THEOREM 2. *Assume* (ii) – (iia) *and a fixed self-adjoint extension of* K_B, *or equivalently, a fixed* $(,)_A$*-self-adjoint extension of* $T^{-1}A \,|\, Z_0(K^*)^\perp$. *Then all of the conclusions of Theorem* 1 *are valid.*

For case (iii), neither uniqueness nor existence for either of the problems (1′) – (2a′) or (1′) – (2b′) is assured. Define $\mathsf{M} = \{\oplus_{\lambda_1} Z_{\lambda_1}(K)\} \oplus \mathrm{Ker}\, A$, $\mathsf{M}_0 = \oplus_{\lambda_2} Z_{\lambda_2}(K)$, $\mathsf{N} = \{\oplus_{\lambda_3} Z_{\lambda_3}(K)\} \oplus K Z_0(K)$, $\mathsf{N}_0 = \oplus_{\lambda_4} Z_{\lambda_4}(K)$, where $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ run over, respectively, the closed right-half plane deleted of zero, the open right-half plane, the open left-half plane, the closed left-half plane.

THEOREM 3. *Assume* (iii) – (iiia). *Then the measure of nonuniqueness* δ^+ *for the solution of the half-space problem* (1′) – (2a′) *is equal to the dimension of a maximal strictly negative subspace of* M. *The measure of noncompleteness* γ^+ *is equal to the dimension of a maximal strictly positive subspace of* N. *The measure of nonuniqueness* δ_0^+ *for the solution of the half-space problem* (1′) – (2b′) *is equal to the dimension of a maximal strictly negative subspace of* M_0. *The measure of noncompleteness* γ_0^+ *is equal to the dimension of a maximal strictly positive subspace of* N_0. *Positivity/negativity here is with respect to the indefinite metric* [,] .

For most applications, a signature operator J is provided by the physical symmetries of the transport problem. Thus the self-adjoint extension of $T^{-1}A \,|\, Z_0(K^*)^\perp$ in Theorem 2 is provided uniquely. We note also that, by virtue of Theorem 3, the one-speed neutron transport equation relevant to supercritical media fails to have a uniquely solvable half-space problem.

4. DISCUSSION

The use of the finite-dimensional linear transformation B to eliminate Ker A appears first in [5].

The spaces H_T and H_K were introduced by Beals [10]. Strong solutions (in H) for A a compact perturbation of the identity have been studied by Hangelbroek [15] and by Van der Mee [5].

Noncompleteness and nonuniqueness results are important in physical applications. For example, the one-dimensional linear BGK model equation for strong evaporation gives a measure of noncompleteness 2 below Mach number 1 and 3 above Mach number 1, and the three-dimensional equation gives $\gamma_0^+ = 4$ below Mach number 1 and $\gamma_0^+ = 5$ above. However, conservation laws at the boundaries reduce the dimensionality by two for the one-dimensional model (conservation of mass and energy) and four for the three-dimensional model (conservation of mass, energy, and two momenta). This breakdown of existence at Mach number 1 for stationary solutions has been observed in numerical experiments, and was first obtained in the linear theory by Cercignani [16, 17].

The measures of nonuniqueness and noncompleteness δ^+, δ_0^+, γ^+, γ_0^+ are related to the sign characteristics [18] of the self-adjoint matrix $T^{-1}A\,|Z(K)$ with respect to the indefinite metric $[\,,\,]$. Thus it is possible to obtain explicit formulae for these measures in terms of the Jordan decomposition of this matrix.

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