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CLOSED FORM SOLUTIONS OF INTEGRABLE NONLINEAR EVOLUTION EQUATIONS

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Dedicated to Petar Popivanov at the occasion of his 65th birthday

Abstract. In this article we obtain closed form solutions of integrable nonlinear evolution equations associated with the nonsymmetric matrix Zakharov-Shabat system by means of the inverse scattering transform. These solutions are parametrized by triplets of matrices. Alternatively, the time evolution of the Marchenko integral kernels and direct substitution are employed in deriving these solutions.

1. Introduction. The initial-value problem of integrable nonlinear evolution equations such as the Korteweg-de Vries (KdV), nonlinear Schrödinger (NLS), modified Korteweg-de Vries (mKdV), sine-Gordon (SG), integrable discrete nonlinear Schrödinger (IDNLS), and Toda lattice equations, can be solved by the inverse scattering transform (IST) method. This method was originally conceived to solve the initial-value problem of the KdV equation [20] and consists of associating the nonlinear evolution equation with a linear spectral problem — the Schrödinger equation on the line for the KdV equation, the Zakharov-Shabat

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system for the NLS, mKdV and SG equations, a discretized Zakharov-Shabat system for the IDNLS, and a linear Jacobi matrix system for the Toda lattice equation — containing its solution $u(x,t)$ as a potential and translating the process of solving the nonlinear evolution equation into the usually more elementary time evolution of the so-called scattering data. These scattering data are asymptotic properties of certain solutions of the associated linear spectral problem which, at any time t, can be put into 1,1-correspondence with the solution of the nonlinear evolution equation. For details we refer to [2, 4, 22, 19, 1, 3].

The transition from the initial solution $u(x,0)$ of the nonlinear evolution equation to the initial scattering data is called the direct scattering problem. After evolving the scattering data in time, we solve the inverse scattering problem of computing the solution $u(x,t)$ of the nonlinear evolution equation at time t from the scattering data. In the literature the scattering data are usually described in a rather messy way as a reflection coefficient, a finite number of so-called bound state poles, and a norming constant associated with each pole, but they can also be represented more conveniently as the integral kernel $\Omega(y+z;t)$ of the Marchenko equation [14]. This integral kernel can be formulated faithfully in terms of the scattering data and the solution of the corresponding integral equation leads directly to the solution of the nonlinear evolution equation. Schematically we can represent the IST in the following way:

$$\begin{array}{ccc} u(x,0) & \xrightarrow{\text{direct scattering}} & \Omega(y+z;0) \\ \Big\downarrow & & \Big\downarrow \\ u(x,t) & \xleftarrow{\text{inverse scattering}} & \Omega(y+z;t) \end{array}$$

where, at each time instant t, there exists a 1,1-correspondence between potentials $u(x,t)$ and Marchenko kernels $\Omega(y+z;t)$. In continuous-position problems, where the position variable $x\in\mathbb{R}$ and the nonlinear evolution equation is a PDE in $(x,t)\in\mathbb{R}^2$, the Marchenko equation is a linear integral equation whose kernel $\Omega(y+z;t)$ depends on the sum of its variables $y,z\in\mathbb{R}$. In discrete-position problems, where the position variable $n\in\mathbb{Z}$ and the nonlinear evolution equation is differential in t and of difference type in n, the Marchenko equation is a linear summation equation whose kernel $\Omega(n+m;t)$ depends on the sum of the variables $n,m\in\mathbb{Z}$.

In this article we consider nonlinear evolution equations, where the corresponding Marchenko equation can be solved by separation of variables and its kernel satisfies a simple linear pseudo-PDE [in the continuous-position case] or a simple linear pseudodifferential-difference equation [in the discrete-position case] with constant coefficients. In this case it is convenient to parametrize the Marchenko kernel in terms of a matrix triplet (A,B,C) in the following way:

$$\begin{cases} \Omega(y+z) = Ce^{-(y+z)A}B, & \text{continuous-position case,} \\ \Omega(n+m) = CA^{-(n+m)}B, & \text{discrete-position case.} \end{cases} \tag{1.1}$$

Here time evolution of the scattering data does not affect A at all, but it has its impact on one of B and C by replacing B by $H(t)B$ and leaving C unchanged [or by replacing C by $CH(t)$ and leaving B unchanged], where $H(t)$ is an invertible time factor commuting with A. As a net result, we obtain closed form solutions of the nonlinear evolution equations in terms of the matrix triplet (A,B,C). Symmetries in the integrable evolution equation correspond to symmetries in the matrix triplets.

In control theory [28, 10], expressions such as appearing in the right-hand sides of (1.1) arise as the integral/summation kernels of input-output maps of linear autonomous continuous-time and discrete-time systems governed by the matrices A (determining time evolution of the system state), B (mapping input to state), and C (mapping state to output). These representations of Marchenko kernels have been proven most useful in obtaining closed form solutions. This method has been applied to get closed form solutions of the KdV [9], NLS [13, 6], mKdV [12], SG [7], and IDNLS [15] equations.

Matrix and operator triplets have been applied to derive solutions of nonlinear evolution equations by using a continuous multiplicative functional to pass from a solution in a large algebra to the actual solution. In this way solutions have been found of the KdV [5], Toda lattice [23], and SG [24] equations. This procedure has also been cast in the framework of bidifferential calculus in [16, 18], leading to explicit solutions of the NLS [17] and Ernst equations [18].

Solutions of integrable nonlinear evolution equations written in terms of matrix triplets (A,B,C) contain virtually all of the known explicit solutions to these equations, such as N-soliton and breather solutions. Their amenability to computer algebra makes these expressions into excellent tools to obtain solutions in unpacked analytical or graphical form.

Instead of using two matrix triplets to parametrize the integral kernels of two coupled Marchenko equations, in this article we employ one matrix triplet, $(\boldsymbol{A},\boldsymbol{B},\boldsymbol{C})$, whose three matrices of increased size contain the six matrices of the two constituent triplets as their nontrivial blocks. In this way we economize when solving the Marchenko equations to derive explicit solutions of integrable equations, while treating the general case of not having adjoint symmetries at the same level as the focusing and defocusing cases. In this way we also greatly simplify the direct substitution of the putative solution into the integrable nonlinear equation. In this article we do not discuss the simplifications, based on the use of determinants and matrix traces, in the case of scalar integrable equations.

Let us now discuss the contents of the various sections. In Section 2 we summarize the direct and inverse scattering theory of the matrix Zakharov-Shabat system with solutions vanishing as $x \to \pm\infty$. Although the basic results can be found in various standard publications, the tendency to maximize using matrices has led to peculiar notations that require explanation. In this section we also solve the inverse scattering problem in closed form under the hypothesis that the reflection coefficients vanish. No symmetries of the potentials are assumed, although we always discuss the focusing case separately. In Section 3 we employ the time evolution of the Marchenko integral kernels to derive explicit solutions of certain integrable nonlinear evolution equations. Similar results are derived in Section 4 by substitution of the putative solution into the nonlinear evolution system. In this way we supply two independent proofs of the validity of certain solution formulas. Appendix A is devoted to the Volterra integral equations satisfied by the Fourier transforms of the Jost solutions.

We denote the open upper and lower half complex planes by $\mathbb{C}^+$ and $\mathbb{C}^-$ and their closures (each including the point at infinity) by $\overline{\mathbb{C}^+}$ and $\overline{\mathbb{C}^-}$, respectively. By $A^\dagger$ we denote the conjugate transpose of a matrix A and by z^* the complex conjugate of $z \in \mathbb{C}$.

2. Matrix Zakharov-Shabat systems. In this section we summarize the direct and inverse scattering theory of the matrix Zakharov-Shabat system [2, 22, 19, 3] and obtain closed form solutions of the inverse scattering problem. We do not necessarily make symmetry assumptions on the Zakharov-Shabat potentials.

2.1. Direct and inverse scattering theory. The matrix Zakharov-Shabat system is given by

$$iJ\frac{\partial X}{\partial x}(\lambda, x) - V(x)X(\lambda, x) = \lambda X(\lambda, x), \tag{2.1}$$

where

$$J = \begin{pmatrix} I_m & 0_{m\times n} \\ 0_{n\times m} & -I_n \end{pmatrix}, \qquad V(x) = \begin{pmatrix} 0_{m\times m} & iq(x) \\ ir(x) & 0_{n\times n} \end{pmatrix}, \tag{2.2}$$

the potentials $q(x)$ and $r(x)$ have their entries in $L^1(\mathbb{R})$, and λ is a spectral parameter. Defining the *Jost matrices* $\Psi(\lambda, x)$ and $\Phi(\lambda, x)$ as the unique square matrix solutions to (2.1) satisfying the asymptotic conditions

(2.3a) $$\Psi(\lambda, x) = e^{-i\lambda Jx}[I_{m+n} + o(1)], \qquad x \to +\infty,$$

(2.3b) $$\Phi(\lambda, x) = e^{-i\lambda Jx}[I_{m+n} + o(1)], \qquad x \to -\infty,$$

there exist so-called *transition coefficients* $a_l(\lambda)$ and $a_r(\lambda)$ such that

(2.4) $$\Phi(\lambda, x) = \Psi(\lambda, x)a_r(\lambda), \qquad \Psi(\lambda, x) = \Phi(\lambda, x)a_l(\lambda).$$

In that case

(2.5a) $$\Psi(\lambda, x) = e^{-i\lambda Jx}a_l(\lambda) + o(1), \qquad x \to -\infty,$$

(2.5b) $$\Phi(\lambda, x) = e^{-i\lambda Jx}a_r(\lambda) + o(1), \qquad x \to +\infty.$$

Writing

(2.6a) $$\Psi(\lambda, x) = e^{-i\lambda Jx} + \int_x^\infty dy\, \alpha_l(x, y)e^{-i\lambda Jy},$$

(2.6b) $$\Phi(\lambda, x) = e^{-i\lambda Jx} + \int_{-\infty}^x dy\, \alpha_r(x, y)e^{-i\lambda Jy},$$

we obtain the Marchenko integral equations

(2.7a) $$\alpha_l(x, y) + \omega_l(x + y) + \int_x^\infty dz\, \alpha_l(x, z)\omega_l(z + y) = 0_{(m+n)\times(m+n)},$$

(2.7b) $$\alpha_r(x, y) + \omega_r(x + y) + \int_{-\infty}^x dz\, \alpha_r(x, z)\omega_r(z + y) = 0_{(m+n)\times(m+n)},$$

where $\omega_l(x + y)$ and $\omega_r(x + y)$ are called the left and right *Marchenko kernels*, respectively. These kernels anticommute with J in the sense that

$$J\omega_l(y + z) = -\omega_l(y + z)J, \qquad J\omega_r(y + z) = -\omega_r(y + z)J.$$

The so-called *Jost kernels* $\alpha_l(x, y)$ and $\alpha_r(x, y)$ can be computed from the potentials by solving the respective Volterra integral equations (A.1) and (A.3) given in Appendix A. The potentials $q(x)$ and $r(x)$ are related to the Marchenko solutions $\alpha_l(x, y)$ and $\alpha_r(x, y)$ as follows [cf. (A.2) and (A.4)]:

(2.8a) $$\alpha_l(x, x) = -\frac{1}{2}\begin{pmatrix} \int_x^\infty dz\, q(z)r(z) & q(x) \\ -r(x) & \int_x^\infty dz\, r(z)q(z) \end{pmatrix},$$

(2.8b) $$\alpha_r(x, x) = -\frac{1}{2}\begin{pmatrix} \int_{-\infty}^x dz\, q(z)r(z) & -q(x) \\ r(x) & \int_{-\infty}^x dz\, r(z)q(z) \end{pmatrix}.$$

Even though the Marchenko kernels could and will themselves be considered as scattering data, we express them in the traditional scattering data prevailing in the literature. Writing

$$\Psi(\lambda,x)=\begin{pmatrix}\overline{\psi}(\lambda,x) & \psi(\lambda,x)\end{pmatrix}, \qquad \Phi(\lambda,x)=\begin{pmatrix}\phi(\lambda,x) & \overline{\phi}(\lambda,x)\end{pmatrix},$$

where $\overline{\psi}(\lambda,x)$ and $\phi(\lambda,x)$ are $(m+n)\times m$ matrices and $\psi(\lambda,x)$ and $\overline{\phi}(\lambda,x)$ are $(m+n)\times n$ matrices called *Jost functions*, it can be shown that, for each $x\in\mathbb{R}$, $e^{i\lambda x}\phi(\lambda,x)$ and $e^{-i\lambda x}\psi(\lambda,x)$ are analytic in $\lambda\in\mathbb{C}^+$ and continuous in $\lambda\in\overline{\mathbb{C}^+}$ and $e^{i\lambda x}\overline{\psi}(\lambda,x)$ and $e^{-i\lambda x}\overline{\phi}(\lambda,x)$ are analytic in $\mathbb{C}^-$ and continuous in $\overline{\mathbb{C}^-}$. We thus obtain the Riemann-Hilbert problem

$$\begin{pmatrix}\overline{\psi}(\lambda,x) & \overline{\phi}(\lambda,x)\end{pmatrix}=\begin{pmatrix}\phi(\lambda,x) & \psi(\lambda,x)\end{pmatrix}JS(\lambda)J, \tag{2.9}$$

where

$$S(\lambda)=\begin{pmatrix}T_r(\lambda) & L(\lambda)\\ R(\lambda) & T_l(\lambda)\end{pmatrix}, \qquad S(\lambda)^{-1}=\begin{pmatrix}\breve{T}_l(\lambda) & \breve{R}(\lambda)\\ \breve{L}(\lambda) & \breve{T}_r(\lambda)\end{pmatrix}$$

are called the *scattering matrix* and its inverse. Under the technical assumption of absence of spectral singularities, i.e., by assuming that the diagonal $m\times m$ and $n\times n$ blocks of the transition matrices are invertible matrices for $\lambda\in\mathbb{R}$, the *transmission coefficients* $T_l(\lambda)$ and $T_r(\lambda)$ have the same determinant, are meromorphic in $\mathbb{C}^+$ with finitely many poles in $\mathbb{C}^+$, and tend to the identity as $\lambda\to\infty$ from within $\overline{\mathbb{C}^+}$. Under this assumption, the *reflection coeffients* $R(\lambda)$ and $L(\lambda)$ and the dual reflection coefficients $\breve{R}(\lambda)$ and $\breve{L}(\lambda)$ can be written as

$$R(\lambda)=\int_{-\infty}^{\infty}dy\,e^{-i\lambda y}\rho(y), \qquad L(\lambda)=\int_{-\infty}^{\infty}dy\,e^{i\lambda y}\ell(y), \tag{2.10a}$$

$$\breve{R}(\lambda)=\int_{-\infty}^{\infty}dy\,e^{i\lambda y}\breve{\rho}(y), \qquad \breve{L}(\lambda)=\int_{-\infty}^{\infty}dy\,e^{-i\lambda y}\breve{\ell}(y), \tag{2.10b}$$

where the entries of $\rho(y)$, $\ell(y)$, $\breve{\rho}(y)$, and $\breve{\ell}(y)$ belong to $L^1(\mathbb{R})$.

If there are no nonreal eigenvalue parameters λ for which (2.1) has a nontrivial column vector solution with entries belonging to $L^2(\mathbb{R})$, then the Marchenko kernels are given by

$$\omega_l(x)=\begin{pmatrix}0_{m\times m} & \breve{\rho}(x)\\ \rho(x) & 0_{n\times n}\end{pmatrix}, \quad \omega_r(x)=\begin{pmatrix}0_{m\times m} & \ell(x)\\ \breve{\ell}(x) & 0_{n\times n}\end{pmatrix}.$$

In general and in the absence of spectral singularities, we add to the Marchenko kernels the bound state contributions

$$\delta_l(x) = \begin{pmatrix} 0_{m\times m} & \displaystyle\sum_{j=1}^{\breve{N}} \sum_{s=0}^{j} \frac{x^s}{s!} e^{-x\breve{a}_j} [\breve{C}_l]_{js} \\ \displaystyle\sum_{j=1}^{N} \sum_{s=0}^{j} \frac{x^s}{s!} e^{-xa_j} [C_l]_{js} & 0_{n\times n} \end{pmatrix},$$

$$\delta_r(x) = \begin{pmatrix} 0_{m\times m} & \displaystyle\sum_{j=1}^{N} \sum_{s=0}^{j} \frac{x^s}{s!} e^{xa_j} [C_r]_{js} \\ \displaystyle\sum_{j=1}^{\breve{N}} \sum_{s=0}^{j} \frac{x^s}{s!} e^{x\breve{a}_j} [\breve{C}_r]_{js} & 0_{n\times n} \end{pmatrix},$$

where $a_1, \ldots, a_N$ and $\breve{a}_1, \ldots, \breve{a}_{\breve{N}}$ are finite (and possibly empty) sets of distinct numbers with positive real parts. In fact, $ia_1, \ldots, ia_N, -i\breve{a}_1, \ldots, -i\breve{a}_{\breve{N}}$ are the so-called *bound states*, i.e., the eigenvalue parameters λ for which (2.1) has a nontrivial column vector solution with entries in $L^2(\mathbb{R})$. The matrices $[C_l]_{js}$, $[\breve{C}_l]_{js}$, $[C_r]_{js}$, and $[\breve{C}_r]_{js}$ are called *norming constants*. As a result, there exist matrix triplets (A_l, B_l, C_l), $(\breve{A}_l, \breve{B}_l, \breve{C}_l)$, (A_r, B_r, C_r), and $(\breve{A}_r, \breve{B}_r, \breve{C}_r)$, where A_l, $\breve{A}_l$, A_r, and $\breve{A}_r$ are square matrices having their eigenvalues in the open right half-plane, such that

$$\omega_l(x) = \begin{pmatrix} 0_{m\times m} & \breve{\rho}(x) + \breve{C}_l e^{-x\breve{A}_l} \breve{B}_l \\ \rho(x) + C_l e^{-xA_l} B_l & 0_{n\times n} \end{pmatrix}, \tag{2.11a}$$

$$\omega_r(x) = \begin{pmatrix} 0_{m\times m} & \ell(x) + C_r e^{xA_r} B_r \\ \breve{\ell}(x) + \breve{C}_r e^{x\breve{A}_r} \breve{B}_r & 0_{n\times n} \end{pmatrix}. \tag{2.11b}$$

In the *defocusing case*, where $r(x) = -q(x)^\dagger$, there are no bound states nor spectral singularities, the transmission coefficients $T_l(\lambda)$ and $T_r(\lambda)$ are analytic in $\mathbb{C}^+$, and the scattering matrix $S(\lambda)$ is unitary. The Fourier transformed reflection coefficients satisfy the symmetry relations

$$\breve{\rho}(x) = \rho(x)^\dagger, \quad \breve{\ell}(x) = \ell(x)^\dagger.$$

The Marchenko kernels satisfy

$$\omega_l(x) = \omega_l(x)^\dagger, \qquad \omega_r(x) = \omega_r(x)^\dagger.$$

The Marchenko equations are uniquely solvable [8, 11].

In the *focusing case*, where $r(x) = q(x)^\dagger$, bound states usually exist and spectral singularities may exist under exceptional circumstances. Then the scattering matrix $S(\lambda)$ is J-unitary in the sense that

$$S(\lambda)^{-1} = JS(\lambda)^\dagger J.$$

The Fourier transformed reflection coefficients satisfy the symmetry relations

$$\breve{\rho}(x) = -\rho(x)^\dagger, \quad \breve{\ell}(x) = -\ell(x)^\dagger.$$

The Marchenko kernels satisfy

$$\omega_l(x) = -J\omega_l(x)^\dagger J, \qquad \omega_r(x) = -J\omega_r(x)^\dagger J.$$

The matrix triplets can be chosen in such a way that

$$(\breve{A}_l, \breve{B}_l, \breve{C}_l) = (A_l^\dagger, C_l^\dagger, -B_l^\dagger), \qquad (\breve{A}_r, \breve{B}_r, \breve{C}_r) = (A_r^\dagger, C_r^\dagger, -B_r^\dagger). \tag{2.12}$$

The Marchenko equations are uniquely solvable [26, 27, 11].

2.2. Closed form expressions. Write

$$\omega_l(x) = \boldsymbol{C}_l e^{-x\boldsymbol{A}_l}\boldsymbol{B}_l = \begin{pmatrix} 0_{m\times p} & \breve{C}_l \\ C_l & 0_{n\times q} \end{pmatrix}\begin{pmatrix} e^{-xA_l} & 0_{p\times q} \\ 0_{q\times p} & e^{-x\breve{A}_l} \end{pmatrix}\begin{pmatrix} B_l & 0_{p\times n} \\ 0_{q\times m} & \breve{B}_l \end{pmatrix}, \tag{2.13a}$$

$$\omega_r(x) = \boldsymbol{C}_r e^{x\boldsymbol{A}_r}\boldsymbol{B}_r = \begin{pmatrix} C_r & 0_{m\times q} \\ 0_{n\times p} & \breve{C}_r \end{pmatrix}\begin{pmatrix} e^{xA_r} & 0_{p\times q} \\ 0_{q\times p} & e^{x\breve{A}_r} \end{pmatrix}\begin{pmatrix} 0_{m\times p} & B_r \\ \breve{B}_r & 0_{n\times q} \end{pmatrix}. \tag{2.13b}$$

Solving (2.7a) by separation of variables we get

$$\alpha_l(x,y) = -\left\{\boldsymbol{C}_l e^{-x\boldsymbol{A}_l} + \boldsymbol{F}_l(x)\right\} e^{-y\boldsymbol{A}_l}\boldsymbol{B}_l,$$

where

$$\boldsymbol{F}_l(x) = \int_x^\infty dz\, \alpha_l(x,z)\boldsymbol{C}_l e^{-z\boldsymbol{A}_l}.$$

We then easily obtain

$$\boldsymbol{F}_l(x) = -\left\{\boldsymbol{C}_l e^{-x\boldsymbol{A}_l} + \boldsymbol{F}_l(x)\right\} e^{-x\boldsymbol{A}_l}\boldsymbol{P}_l e^{-x\boldsymbol{A}_l},$$

where

$$\boldsymbol{P}_l = \int_0^\infty dy\, e^{-y\boldsymbol{A}_l}\boldsymbol{B}_l\boldsymbol{C}_l e^{-y\boldsymbol{A}_l} = \begin{pmatrix} 0_{p\times p} & -\boldsymbol{N}_l \\ \boldsymbol{Q}_l & 0_{q\times q} \end{pmatrix}. \tag{2.14}$$

Here

$$\boldsymbol{Q}_l=\int_0^\infty dy\, e^{-y\breve{A}_l}\breve{B}_lC_le^{-yA_l},\quad \boldsymbol{N}_l=-\int_0^\infty dy\, e^{-yA_l}B_l\breve{C}_le^{-y\breve{A}_l}. \tag{2.15}$$

As a result,

$$\begin{aligned}\boldsymbol{F}_l(x)&=-\boldsymbol{C}_le^{-2x\boldsymbol{A}_l}\boldsymbol{P}_le^{-x\boldsymbol{A}_l}\left[I_{p+q}+e^{-x\boldsymbol{A}_l}\boldsymbol{P}_le^{-x\boldsymbol{A}_l}\right]^{-1},\\ \alpha_l(x,y)&=-\boldsymbol{C}_le^{-x\boldsymbol{A}_l}\left[I_{p+q}+e^{-x\boldsymbol{A}_l}\boldsymbol{P}_le^{-x\boldsymbol{A}_l}\right]^{-1}e^{-y\boldsymbol{A}_l}\boldsymbol{B}_l,\end{aligned}$$

provided the inverse matrix exists. In that case the inverse is given by

$$\begin{pmatrix}\breve{\Gamma}_l(x)^{-1} & e^{-xA_l}\boldsymbol{N}_le^{-x\breve{A}_l}\Gamma_l(x)^{-1}\\ -e^{-x\breve{A}_l}\boldsymbol{Q}_le^{-xA_l}\breve{\Gamma}_l(x)^{-1} & \Gamma_l(x)^{-1}\end{pmatrix},$$

where

$$\begin{aligned}\Gamma_l(x)&=I_q+e^{-x\breve{A}_l}\boldsymbol{Q}_le^{-2xA_l}\boldsymbol{N}_le^{-x\breve{A}_l},\\ \breve{\Gamma}_l(x)&=I_p+e^{-xA_l}\boldsymbol{N}_le^{-2x\breve{A}_l}\boldsymbol{Q}_le^{-xA_l}.\end{aligned}$$

As a result, by using (2.8a) we get

$$\begin{aligned}&\begin{pmatrix}\int_x^\infty dz\, q(z)r(z) & q(x)\\ -r(x) & \int_x^\infty dz\, r(z)q(z)\end{pmatrix}\\ &=2\boldsymbol{C}_le^{-x\boldsymbol{A}_l}\left[I_{p+q}+e^{-x\boldsymbol{A}_l}\boldsymbol{P}_le^{-x\boldsymbol{A}_l}\right]^{-1}e^{-x\boldsymbol{A}_l}\boldsymbol{B}_l.\end{aligned}$$

It is easily verified that, for each $x\in\mathbb{R}$, the invertibility of $\Gamma_l(x)$ is equivalent to the invertibility of $\breve{\Gamma}_l(x)$. We now easily obtain

$$q(x)=2\breve{C}_le^{-x\breve{A}_l}\Gamma_l(x)^{-1}e^{-x\breve{A}_l}\breve{B}_l=2\breve{C}_l\left[e^{2x\breve{A}_l}+\boldsymbol{Q}_le^{-2xA_l}\boldsymbol{N}_l\right]^{-1}\breve{B}_l, \tag{2.16a}$$

$$r(x)=-2C_le^{-xA_l}\breve{\Gamma}_l(x)^{-1}e^{-xA_l}B_l=-2C_l\left[e^{2xA_l}+\boldsymbol{N}_le^{-2x\breve{A}_l}\boldsymbol{Q}_l\right]^{-1}B_l, \tag{2.16b}$$

$$\begin{aligned}\int_x^\infty dz\, q(z)r(z)&=-2\breve{C}_le^{-2x\breve{A}_l}\boldsymbol{Q}_le^{-xA_l}\breve{\Gamma}_l(x)^{-1}e^{-xA_l}B_l\\ &=-2\breve{C}_le^{-2x\breve{A}_l}\boldsymbol{Q}_l\left[e^{2xA_l}+\boldsymbol{N}_le^{-2x\breve{A}_l}\boldsymbol{Q}_l\right]^{-1}B_l,\end{aligned} \tag{2.16c}$$

$$\begin{aligned}\int_x^\infty dz\, r(z)q(z)&=2C_le^{2xA_l}\boldsymbol{N}_le^{-x\breve{A}_l}\Gamma_l(x)^{-1}e^{-x\breve{A}_l}\breve{B}_l\\ &=2C_le^{2xA_l}\boldsymbol{N}_l\left[e^{2x\breve{A}_l}-\boldsymbol{Q}_le^{-2xA_l}\boldsymbol{N}_l\right]^{-1}\breve{B}_l.\end{aligned} \tag{2.16d}$$

The inverse matrices exist for every $x \in \mathbb{R}$, with the possible exception of finitely many x-values where the corresponding Marchenko equation (2.7a) is not uniquely solvable. Moreover, the potentials $q(x)$ and $r(x)$ decay exponentially as $x \to \pm\infty$.

In the focusing case, where the matrix triplets are related as in (2.12), we obtain

$$q(x) = -2B_l^\dagger \left[e^{2xA_l^\dagger} + \boldsymbol{Q}_l e^{-2xA_l} \boldsymbol{N}_l \right]^{-1} C_l^\dagger, \tag{2.17a}$$

$$\int_x^\infty dz\, q(z)q(z)^\dagger = 2B_l^\dagger e^{-2xA_l^\dagger} \boldsymbol{Q}_l \left[e^{2xA_l} + \boldsymbol{N}_l e^{-2xA_l^\dagger} \boldsymbol{Q}_l \right]^{-1} B_l, \tag{2.17b}$$

$$\int_x^\infty dz\, q(z)^\dagger q(z) = 2C_l e^{2xA_l} \boldsymbol{N}_l \left[e^{2xA_l^\dagger} + \boldsymbol{Q}_l e^{-2xA_l} \boldsymbol{N}_l \right]^{-1} C_l^\dagger. \tag{2.17c}$$

where

$$\boldsymbol{Q}_l = \int_0^\infty dy\, e^{-yA_l^\dagger} C_l^\dagger C_l e^{-yA_l}, \quad \boldsymbol{N}_l = \int_0^\infty dy\, e^{-yA_l} B_l B_l^\dagger e^{-yA_l^\dagger},$$

are nonnegative definite hermitian matrices; these matrices are invertible iff the matrix triplet (A_l, B_l, C_l) is minimal in the sense that the order of the square matrix A_l is minimal among all triplets having the same Marchenko kernel $\Omega_l(y) = C_l e^{-yA_l} B_l$ (cf. [6]). The inverse matrices in (2.17) are easily seen to exist for each $x \in \mathbb{R}$, irrespective of whether the triplet is minimal.

3. Explicit solutions. In this section we find closed form solutions of the matrix NLS, sine-Gordon, matrix modified KdV equations, and more general equations. We do not necessarily make symmetry assumptions on the Zakharov-Shabat potentials and the nonlinear evolution equations. To obtain such explicit solutions, we modify the Marchenko kernels employed. This amounts to altering the matrix triplets $(\boldsymbol{A}_l, \boldsymbol{B}_l, \boldsymbol{C}_l)$ and $(\boldsymbol{A}_r, \boldsymbol{B}_r, \boldsymbol{C}_r)$ by inserting time factors.

In the literature we find the following three examples:

1. **matrix NLS**: The two Marchenko kernels $\omega_l(y;t) = -J\omega_l(y;t)J$ and $\omega_r(y;t) = -J\omega_r(y;t)J$ satisfy the PDE's

$$[\omega_l]_t + 4iJ[\omega_l]_{yy} = 0, \qquad [\omega_r]_t + 4iJ[\omega_r]_{yy} = 0.$$

We thus assume the reflectionless Marchenko kernels to have the form

$$\omega_l(y;t) = \boldsymbol{C}_l e^{-y\boldsymbol{A}_l} e^{-4itJ\boldsymbol{A}_l^2} \boldsymbol{B}_l, \qquad \omega_r(y;t) = \boldsymbol{C}_r e^{y\boldsymbol{A}_r} e^{+4itJ\boldsymbol{A}_r^2} \boldsymbol{B}_r.$$

In [11, 6], only the focusing case has been worked out.

2. **matrix mKdV**: The Marchenko kernels $\omega_l(y;t) = -J\omega_l(y;t)J$ and $\omega_r(y;t) = -J\omega_r(y;t)J$ satisfy the PDE's

$$[\omega_l]_t + 8J[\omega_l]_{yyy} = 0, \qquad [\omega_r]_t - 8J[\omega_r]_{yyy} = 0.$$

We thus assume the reflectionless Marchenko kernels to have the form

$$\omega_l(y;t) = \boldsymbol{C}_l e^{-y\boldsymbol{A}_l} e^{8tJ\boldsymbol{A}_l^3} \boldsymbol{B}_l, \qquad \omega_r(y;t) = \boldsymbol{C}_r e^{y\boldsymbol{A}_r} e^{8tJ\boldsymbol{A}_r^3} \boldsymbol{B}_r.$$

In [12], the matrix mKdV equation worked out is focusing and contains only real quantities; thus the triplets consist of real matrices only.

3. **sine-Gordon**: The Marchenko kernels $\omega_l(y;t) = -J\omega_l(y;t)J$ and $\omega_r(y;t) = -J\omega_r(y;t)J$ satisfy the PDE's

$$[\omega_l]_{yt} = \tfrac{1}{2}J\omega_l, \qquad [\omega_r]_{yt} = -\tfrac{1}{2}J\omega_r.$$

Since the sine-Gordon solutions are real and scalar, the two matrix triplets consist of real matrices only, $p = q = 1$, and

$$\omega_l(y;t) = \boldsymbol{C}_l e^{-y\boldsymbol{A}_l} e^{-\frac{1}{2}tJ\boldsymbol{A}_l^{-1}} \boldsymbol{B}_l, \qquad \omega_r(y;t) = \boldsymbol{C}_r e^{y\boldsymbol{A}_r} e^{-\frac{1}{2}tJ\boldsymbol{A}_r^{-1}} \boldsymbol{B}_r.$$

In the examples the time dependent Marchenko kernels have the form

$$\omega_l(y;t) = \boldsymbol{C}_l e^{-y\boldsymbol{A}_l} e^{-itJ\phi_l(iJ\boldsymbol{A}_l)} \boldsymbol{B}_l, \quad \omega_r(y;t) = \boldsymbol{C}_r e^{y\boldsymbol{A}_r} e^{-itJ\phi_r(iJ\boldsymbol{A}_r)} \boldsymbol{B}_r, \tag{3.1}$$

where $\phi_l(z)$ and $\phi_r(z)$ are analytic functions defined on a neighborhood of the eigenvalues of $iJ\boldsymbol{A}_l$ and $iJ\boldsymbol{A}_r$. We call

$$\boldsymbol{H}_l(t) = H_l(t) \oplus \breve{H}_l(t) = e^{-itJ\phi_l(iJ\boldsymbol{A}_l)} = e^{-it\phi_l(iA_l)} \oplus e^{it\phi_l(-i\breve{A}_l)}, \tag{3.2a}$$

$$\boldsymbol{H}_r(t) = H_r(t) \oplus \breve{H}_r(t) = e^{-itJ\phi_r(iJ\boldsymbol{A}_r)} = e^{-it\phi_r(iA_l)} \oplus e^{it\phi_r(-i\breve{A}_r)}, \tag{3.2b}$$

time factors. The time factors and the signature matrix J obviously commute with $\boldsymbol{A}_l$ and $\boldsymbol{A}_r$, respectively. In the focusing case, where the matrix triplets are related as in (2.12), the time factors are related as follows:

$$\breve{H}_l(t) = H_l(t)^\dagger, \qquad \breve{H}_r(t) = H_r(t)^\dagger. \tag{3.3}$$

In this case the analytic functions $\phi_l(z)$ and $\phi_r(z)$ are real-valued on the parts of the real line within their domains.

Let us now derive explicit solutions of the nonlinear evolution equations. To do so, we modify the matrix triplets as follows:

$$(\boldsymbol{A}_l, \boldsymbol{B}_l, \boldsymbol{C}_l) \mapsto (\boldsymbol{A}_l, \boldsymbol{H}_l(t)\boldsymbol{B}_l, \boldsymbol{C}_l), \tag{3.4a}$$

$$(\boldsymbol{A}_r, \boldsymbol{B}_r, \boldsymbol{C}_r) \mapsto (\boldsymbol{A}_r, \boldsymbol{H}_r(t)\boldsymbol{B}_r, \boldsymbol{C}_r), \tag{3.4b}$$

such that (3.1) remains true. Then $\boldsymbol{P}_l$ is replaced by $\boldsymbol{H}_l(t)\boldsymbol{P}_l$ [cf. (2.14)]. Moreover, $\boldsymbol{Q}_l$ and $\boldsymbol{N}_l$ are replaced by $\breve{H}_l(t)\boldsymbol{Q}_l$ and $H_l(t)\boldsymbol{N}_l$, respectively [cf. (2.15)]. We then replace Γ_l and $\breve{\Gamma}_l$ by

$$\Gamma_l(x;t) = I_q + e^{-x\breve{A}_l}\breve{H}_l(t)\boldsymbol{Q}_l e^{-2xA_l} H_l(t)\boldsymbol{N}_l e^{-x\breve{A}_l},$$
$$\breve{\Gamma}_l(x;t) = I_p + e^{-xA_l} H_l(t)\boldsymbol{N}_l e^{-2x\breve{A}_l}\breve{H}_l(t)\boldsymbol{Q}_l e^{-xA_l},$$

respectively. Using that $H_l(t)^{-1} = H_l(-t)$ and $\breve{H}_l(t)^{-1} = \breve{H}_l(-t)$, we obtain

$$\begin{aligned} q(x,t) &= 2\breve{C}_l e^{-x\breve{A}_l}\Gamma_l(x;t)^{-1}e^{-x\breve{A}_l}\breve{H}_l(t)\breve{B}_l \\ &= 2\breve{C}_l\left[\breve{H}_l(-t)e^{2x\breve{A}_l} + \boldsymbol{Q}_l e^{-2xA_l}H_l(t)\boldsymbol{N}_l\right]^{-1}\breve{B}_l, \end{aligned} \tag{3.5a}$$

$$\begin{aligned} r(x,t) &= -2C_l e^{-xA_l}\breve{\Gamma}_l(x;t)^{-1}e^{-xA_l}H_l(t)B_l \\ &= -2C_l\left[e^{2xA_l}H_l(-t) + \boldsymbol{N}_l e^{-2x\breve{A}_l}\breve{H}_l(t)\boldsymbol{Q}_l\right]^{-1}B_l. \end{aligned} \tag{3.5b}$$

The inverse matrices exist for every $(x,t) \in \mathbb{R}^2$, with the possible exception of finitely many x-values for each value of t, where the corresponding Marchenko equation (2.7a) is not uniquely solvable. Moreover, for every $t \in \mathbb{R}$ the potentials $q(x,t)$ and $r(x,t)$ decay exponentially as $x \to \pm\infty$. In the focusing case, where the matrix triplets are related as in (2.12) and the time factors as in (3.3), we obtain

$$q(x,t) = -2B_l^\dagger\left[H_l(-t)^\dagger e^{2xA_l^\dagger} + \boldsymbol{Q}_l e^{-2xA_l}H_l(t)\boldsymbol{N}_l\right]^{-1}C_l^\dagger, \tag{3.6}$$

where the inverse matrix exists for all $(x,t) \in \mathbb{R}^2$.

4. Direct substitution. Exact solutions of integrable nonlinear evolution equations associated with the matrix Zakharov-Shabat system by means of the IST can be obtained in the concise form (3.5) or (3.6), provided the time factor $\boldsymbol{H}_l(t)$ is known. Determining the time factor requires the time evolution of the scattering data. More precisely, it requires proving that the Marchenko kernel $\omega_l(z)$ satisfies the partial pseudodifferential equation

$$[\omega_l]_t + iJ\phi_l(iJ\tfrac{d}{dx})\omega_l = 0. \tag{4.1}$$

This can be achieved by finding the Lax pair for the nonlinear equation involving the matrix Zakharov-Shabat operator and deriving the time evolution of the scattering data as in [2, 4].

Another method to determine the time factor is direct substitution of the putative solution (3.5) or (3.6) into the nonlinear equation. This has been succesfully accomplished for the KdV equation [9], the focusing NLS equation [6], the matrix modified KdV equation [12], and the sine-Gordon equation [24]. A similar direct substitution method has been developed for the Toda lattice equation [23]. Using bidifferential calculus to generate solutions of suitable nonlinear equations, exact solutions of the matrix NLS, IDNLS, and the Ernst equations have been given [16, 17].

Let us start from matrix triplets satisfying (2.12). Departing from the triplet $(\boldsymbol{A},\boldsymbol{B},\boldsymbol{C})$ where $\boldsymbol{A}$ is an $s\times s$ matrix, we define

$$\boldsymbol{\Pi}=e^{-x\boldsymbol{A}}\boldsymbol{B}\boldsymbol{C}e^{-x\boldsymbol{A}}, \tag{4.2a}$$

$$\boldsymbol{\Gamma}=I_s+\boldsymbol{\Pi}^2, \tag{4.2b}$$

$$\boldsymbol{X}_n=\boldsymbol{A}^n+(-1)^n\boldsymbol{\Pi}\boldsymbol{A}^n\boldsymbol{\Pi}, \tag{4.2c}$$

where we do not express their dependence on $x\in\mathbb{R}$. Then

$$(\boldsymbol{A}\boldsymbol{\Pi}+\boldsymbol{\Pi}\boldsymbol{A})\boldsymbol{\Gamma}^{-1}(\boldsymbol{A}\boldsymbol{\Pi}+\boldsymbol{\Pi}\boldsymbol{A})=\boldsymbol{X}_2-\boldsymbol{X}_1\Gamma^{-1}\boldsymbol{X}_1, \tag{4.3}$$

provided $\boldsymbol{\Gamma}$ is invertible. Indeed, using that

$$\boldsymbol{\Pi}\,\boldsymbol{\Gamma}^{-1}=\boldsymbol{\Gamma}^{-1}\boldsymbol{\Pi},\qquad \boldsymbol{\Pi}^2\boldsymbol{\Gamma}^{-1}=\boldsymbol{\Gamma}^{-1}\boldsymbol{\Pi}^2=I_s-\boldsymbol{\Gamma}^{-1},$$

we get

$$\begin{aligned}(\boldsymbol{A}\boldsymbol{\Pi}+\boldsymbol{\Pi}\boldsymbol{A})\boldsymbol{\Gamma}^{-1}(\boldsymbol{A}\boldsymbol{\Pi}+\boldsymbol{\Pi}\boldsymbol{A})&=\boldsymbol{A}\boldsymbol{\Gamma}^{-1}\boldsymbol{\Pi}\boldsymbol{A}\boldsymbol{\Pi}+\boldsymbol{\Pi}\boldsymbol{A}\boldsymbol{\Pi}\boldsymbol{\Gamma}^{-1}\boldsymbol{A}\\&\quad+\boldsymbol{A}(I_s-\boldsymbol{\Gamma}^{-1})\boldsymbol{A}+\boldsymbol{\Pi}\boldsymbol{A}\boldsymbol{\Gamma}^{-1}\boldsymbol{A}\boldsymbol{\Pi}\\&=[\boldsymbol{A}^2+\boldsymbol{\Pi}\boldsymbol{A}^2\boldsymbol{\Pi}]-(\boldsymbol{A}-\boldsymbol{\Pi}\boldsymbol{A}\boldsymbol{\Pi})\boldsymbol{\Gamma}^{-1}(\boldsymbol{A}-\boldsymbol{\Pi}\boldsymbol{A}\boldsymbol{\Pi})\\&=\boldsymbol{X}_2-\boldsymbol{X}_1\boldsymbol{\Gamma}^{-1}\boldsymbol{X}_1,\end{aligned}$$

as claimed.

Proposition 4.1. *Put*

$$\boldsymbol{q}(x)=-2\boldsymbol{C}e^{-x\boldsymbol{A}}\boldsymbol{\Gamma}^{-1}e^{-x\boldsymbol{A}}\boldsymbol{B}. \tag{4.4}$$

Then

$$\boldsymbol{q}_x = 4\boldsymbol{C}e^{-x\boldsymbol{A}}\boldsymbol{\Gamma}^{-1}\boldsymbol{X}_1\boldsymbol{\Gamma}^{-1}e^{-x\boldsymbol{A}}\boldsymbol{B}, \tag{4.5a}$$

$$\boldsymbol{q}_{xx} + 2\boldsymbol{q}^3 = -8\boldsymbol{C}e^{-x\boldsymbol{A}}\boldsymbol{\Gamma}^{-1}\boldsymbol{X}_2\boldsymbol{\Gamma}^{-1}e^{-x\boldsymbol{A}}\boldsymbol{B}, \tag{4.5b}$$

$$\boldsymbol{q}_{xxx} + 3\boldsymbol{q}^2\boldsymbol{q}_x + 3\boldsymbol{q}_x\boldsymbol{q}^2 = 16\boldsymbol{C}e^{-x\boldsymbol{A}}\boldsymbol{\Gamma}^{-1}\boldsymbol{X}_3\boldsymbol{\Gamma}^{-1}e^{-x\boldsymbol{A}}\boldsymbol{B}. \tag{4.5c}$$

P r o o f. We easily compute that

$$\begin{aligned}(\boldsymbol{\Gamma}^{-1})_x &= -\boldsymbol{\Gamma}^{-1}(\boldsymbol{\Pi}_x\boldsymbol{\Pi} + \boldsymbol{\Pi}\boldsymbol{\Pi}_x)\boldsymbol{\Gamma}^{-1}\\ &= \boldsymbol{\Gamma}^{-1}([\boldsymbol{A}\boldsymbol{\Pi} + \boldsymbol{\Pi}\boldsymbol{A}]\boldsymbol{\Pi} + \boldsymbol{\Pi}[\boldsymbol{A}\boldsymbol{\Pi} + \boldsymbol{\Pi}\boldsymbol{A}])\boldsymbol{\Gamma}^{-1}\\ &= \boldsymbol{\Gamma}^{-1}(\boldsymbol{A}\boldsymbol{\Pi}^2 + 2\boldsymbol{\Pi}\boldsymbol{A}\boldsymbol{\Pi} + \boldsymbol{\Pi}^2\boldsymbol{A})\boldsymbol{\Gamma}^{-1}\\ &= \boldsymbol{\Gamma}^{-1}\boldsymbol{A}(I_s - \boldsymbol{\Gamma}^{-1}) + 2\boldsymbol{\Gamma}^{-1}\boldsymbol{\Pi}\boldsymbol{A}\boldsymbol{\Pi}\boldsymbol{\Gamma}^{-1} + (I_s - \boldsymbol{\Gamma}^{-1})\boldsymbol{A}\boldsymbol{\Gamma}^{-1}\\ &= \boldsymbol{\Gamma}^{-1}\boldsymbol{A} + \boldsymbol{A}\boldsymbol{\Gamma}^{-1} - 2\boldsymbol{\Gamma}^{-1}(\boldsymbol{A} - \boldsymbol{\Pi}\boldsymbol{A}\boldsymbol{\Pi})\boldsymbol{\Gamma}^{-1}\\ &= \boldsymbol{\Gamma}^{-1}\boldsymbol{A} + \boldsymbol{A}\boldsymbol{\Gamma}^{-1} - 2\boldsymbol{\Gamma}^{-1}\boldsymbol{X}_1\boldsymbol{\Gamma}^{-1},\end{aligned}$$

which implies that

$$\left(e^{-x\boldsymbol{A}}\boldsymbol{\Gamma}^{-1}e^{-x\boldsymbol{A}}\right)_x = -2e^{-x\boldsymbol{A}}\boldsymbol{\Gamma}^{-1}\boldsymbol{X}_1\boldsymbol{\Gamma}^{-1}e^{-x\boldsymbol{A}}. \tag{4.6}$$

Equations (4.4) and (4.6) imply (4.5a).

Next,

$$\begin{aligned}\left(\boldsymbol{\Gamma}^{-1}\boldsymbol{X}_1\boldsymbol{\Gamma}^{-1}\right)_x &= (\boldsymbol{\Gamma}^{-1})_x\boldsymbol{X}_1\boldsymbol{\Gamma}^{-1} + \boldsymbol{\Gamma}^{-1}\boldsymbol{X}_1(\boldsymbol{\Gamma}^{-1})_x - \boldsymbol{\Gamma}^{-1}[\boldsymbol{\Pi}_x\boldsymbol{A}\boldsymbol{\Pi} + \boldsymbol{\Pi}\boldsymbol{A}\boldsymbol{\Pi}_x]\boldsymbol{\Gamma}^{-1}\\ &= [\boldsymbol{\Gamma}^{-1}\boldsymbol{A} + \boldsymbol{A}\boldsymbol{\Gamma}^{-1}]\boldsymbol{X}_1\boldsymbol{\Gamma}^{-1} + \boldsymbol{\Gamma}^{-1}\boldsymbol{X}_1[\boldsymbol{\Gamma}^{-1}\boldsymbol{A} + \boldsymbol{A}\boldsymbol{\Gamma}^{-1}]\\ &\quad - 4\boldsymbol{\Gamma}^{-1}\boldsymbol{X}_1\boldsymbol{\Gamma}^{-1}\boldsymbol{X}_1\boldsymbol{\Gamma}^{-1}\\ &\quad + \boldsymbol{\Gamma}^{-1}[\boldsymbol{A}\boldsymbol{\Pi}\boldsymbol{A}\boldsymbol{\Pi} + 2\boldsymbol{\Pi}\boldsymbol{A}^2\boldsymbol{\Pi} + \boldsymbol{\Pi}\boldsymbol{A}\boldsymbol{\Pi}\boldsymbol{A}]\boldsymbol{\Gamma}^{-1}\\ &= \boldsymbol{A}\boldsymbol{\Gamma}^{-1}\boldsymbol{X}_1\boldsymbol{\Gamma}^{-1} + \boldsymbol{\Gamma}^{-1}\boldsymbol{X}_1\boldsymbol{\Gamma}^{-1}\boldsymbol{A} + 2\boldsymbol{\Gamma}^{-1}\{\boldsymbol{X}_2 - 2\boldsymbol{X}_1\boldsymbol{\Gamma}^{-1}\boldsymbol{X}_1\}\boldsymbol{\Gamma}^{-1}.\end{aligned}$$

Thus,

$$\boldsymbol{q}_{xx} = 8\boldsymbol{C}e^{-x\boldsymbol{A}}\boldsymbol{\Gamma}^{-1}\{\boldsymbol{X}_2 - 2\boldsymbol{X}_1\boldsymbol{\Gamma}^{-1}\boldsymbol{X}_1\}\boldsymbol{\Gamma}^{-1}e^{-x\boldsymbol{A}}\boldsymbol{B}. \tag{4.7}$$

Next, using (4.4) and (4.3), we obtain

$$\boldsymbol{q}^2 = 4\boldsymbol{C}e^{-x\boldsymbol{A}}\boldsymbol{\Gamma}^{-1}[\boldsymbol{A}\boldsymbol{\Pi} + \boldsymbol{\Pi}\boldsymbol{A}]\boldsymbol{\Gamma}^{-1}e^{-x\boldsymbol{A}}\boldsymbol{B}. \tag{4.8}$$

Using (4.4) once more we get

$$\begin{aligned}\boldsymbol{q}^3 &= -8\boldsymbol{C}e^{-x\boldsymbol{A}}\boldsymbol{\Gamma}^{-1}[\boldsymbol{A}\boldsymbol{\Pi} + \boldsymbol{\Pi}\boldsymbol{A}]\boldsymbol{\Gamma}^{-1}[\boldsymbol{A}\boldsymbol{\Pi} + \boldsymbol{\Pi}\boldsymbol{A}]\boldsymbol{\Gamma}^{-1}e^{-x\boldsymbol{A}}\boldsymbol{B}\\ &= -8\boldsymbol{C}e^{-x\boldsymbol{A}}\boldsymbol{\Gamma}^{-1}\{\boldsymbol{X}_2 - 2\boldsymbol{X}_1\boldsymbol{\Gamma}^{-1}\boldsymbol{X}_1\}\boldsymbol{\Gamma}^{-1}e^{-x\boldsymbol{A}}\boldsymbol{B},\end{aligned}$$

where (4.3) has been used. The last identity together with (4.7) imply (4.5b).

Combining (4.5a) and (4.8) and using (4.3) we get

$$\begin{aligned}
&\boldsymbol{q}^2\boldsymbol{q}_x+\boldsymbol{q}_x\boldsymbol{q}^2\\
&=16\boldsymbol{C}e^{-x\boldsymbol{A}}\boldsymbol{\Gamma}^{-1}[\boldsymbol{A\Pi}+\boldsymbol{\Pi A}]\boldsymbol{\Gamma}^{-1}[\boldsymbol{A\Pi}+\boldsymbol{\Pi A}]\boldsymbol{\Gamma}^{-1}\boldsymbol{X}_1\boldsymbol{\Gamma}^{-1}e^{-x\boldsymbol{A}}\boldsymbol{B}\\
&+16\boldsymbol{C}e^{-x\boldsymbol{A}}\boldsymbol{\Gamma}^{-1}\boldsymbol{X}_1\boldsymbol{\Gamma}^{-1}[\boldsymbol{A\Pi}+\boldsymbol{\Pi A}]\boldsymbol{\Gamma}^{-1}[\boldsymbol{A\Pi}+\boldsymbol{\Pi A}]\boldsymbol{\Gamma}^{-1}e^{-x\boldsymbol{A}}\boldsymbol{B}\\
&=16\boldsymbol{C}e^{-x\boldsymbol{A}}\boldsymbol{\Gamma}^{-1}\{\boldsymbol{X}_2-\boldsymbol{X}_1\boldsymbol{\Gamma}^{-1}\boldsymbol{X}_1\}\boldsymbol{\Gamma}^{-1}\boldsymbol{X}_1\boldsymbol{\Gamma}^{-1}e^{-x\boldsymbol{A}}\boldsymbol{B}\\
&+16\boldsymbol{C}e^{-x\boldsymbol{A}}\boldsymbol{\Gamma}^{-1}\boldsymbol{X}_1\boldsymbol{\Gamma}^{-1}\{\boldsymbol{X}_2-\boldsymbol{X}_1\boldsymbol{\Gamma}^{-1}\boldsymbol{X}_1\}\boldsymbol{\Gamma}^{-1}e^{-x\boldsymbol{A}}\boldsymbol{B}.
\end{aligned}$$

Next, using (4.6) and (4.2c) we get

$$\begin{aligned}
(\boldsymbol{\Gamma}^{-1}\boldsymbol{X}_2\boldsymbol{\Gamma}^{-1})_x&=(\boldsymbol{\Gamma}^{-1})_x\boldsymbol{X}_2\boldsymbol{\Gamma}^{-1}\\
&+\boldsymbol{\Gamma}^{-1}\boldsymbol{X}_2(\boldsymbol{\Gamma}^{-1})_x+\boldsymbol{\Gamma}^{-1}[\boldsymbol{\Pi}_x\boldsymbol{A}^2\boldsymbol{\Pi}+\boldsymbol{\Pi A}^2\boldsymbol{\Pi}_x]\boldsymbol{\Gamma}^{-1}\\
&=\boldsymbol{A\Gamma}^{-1}\boldsymbol{X}_2\boldsymbol{\Gamma}^{-1}+\boldsymbol{\Gamma}^{-1}\boldsymbol{X}_2\boldsymbol{\Gamma}^{-1}\boldsymbol{A}\\
&+2\,\boldsymbol{\Gamma}^{-1}\boldsymbol{A}^3\boldsymbol{\Gamma}^{-1}+\boldsymbol{\Gamma}^{-1}[\boldsymbol{\Pi A}^2\boldsymbol{\Pi A}+\boldsymbol{A\Pi A}^2\boldsymbol{\Pi}]\boldsymbol{\Gamma}^{-1}\\
&-2\,\boldsymbol{\Gamma}^{-1}\boldsymbol{X}_1\boldsymbol{\Gamma}^{-1}\boldsymbol{X}_2\boldsymbol{\Gamma}^{-1}-2\,\boldsymbol{\Gamma}^{-1}\boldsymbol{X}_2\boldsymbol{\Gamma}^{-1}\boldsymbol{X}_1\boldsymbol{\Gamma}^{-1}\\
&-\boldsymbol{\Gamma}^{-1}[\boldsymbol{A\Pi A}^2\boldsymbol{\Pi}+2\boldsymbol{\Pi A}^3\boldsymbol{\Pi}+\boldsymbol{\Pi A}^2\boldsymbol{\Pi A}]\boldsymbol{\Gamma}^{-1}\\
&=\boldsymbol{A\Gamma}^{-1}\boldsymbol{X}_2\boldsymbol{\Gamma}^{-1}+\boldsymbol{\Gamma}^{-1}\boldsymbol{X}_2\boldsymbol{\Gamma}^{-1}\boldsymbol{A}\\
&+2\,\boldsymbol{\Gamma}^{-1}\boldsymbol{X}_3\boldsymbol{\Gamma}^{-1}-2\,\boldsymbol{\Gamma}^{-1}\boldsymbol{X}_2\boldsymbol{\Gamma}^{-1}\boldsymbol{X}_1\boldsymbol{\Gamma}^{-1}-2\,\boldsymbol{\Gamma}^{-1}\boldsymbol{X}_1\boldsymbol{\Gamma}^{-1}\boldsymbol{X}_2\boldsymbol{\Gamma}^{-1}.
\end{aligned}$$

In a similar way we obtain with the help of (4.6)

$$\begin{aligned}
(\boldsymbol{\Gamma}^{-1}\boldsymbol{X}_1\boldsymbol{\Gamma}^{-1}\boldsymbol{X}_1\boldsymbol{\Gamma}^{-1})_x&=\boldsymbol{A\Gamma}^{-1}\boldsymbol{X}_1\boldsymbol{\Gamma}^{-1}\boldsymbol{X}_1\boldsymbol{\Gamma}^{-1}+\boldsymbol{\Gamma}^{-1}\boldsymbol{X}_1\boldsymbol{\Gamma}^{-1}\boldsymbol{X}_1\boldsymbol{\Gamma}^{-1}\boldsymbol{A}\\
&-6\boldsymbol{\Gamma}^{-1}\boldsymbol{X}_1\boldsymbol{\Gamma}^{-1}\boldsymbol{X}_1\boldsymbol{\Gamma}^{-1}\boldsymbol{X}_1\boldsymbol{\Gamma}^{-1}\\
&+2\boldsymbol{\Gamma}^{-1}\boldsymbol{X}_2\boldsymbol{\Gamma}^{-1}\boldsymbol{X}_1\boldsymbol{\Gamma}^{-1}+2\boldsymbol{\Gamma}^{-1}\boldsymbol{X}_1\boldsymbol{\Gamma}^{-1}\boldsymbol{X}_2\boldsymbol{\Gamma}^{-1}.
\end{aligned}$$

Using (4.7) we easily derive (4.5c). □

Let us introduce a function $\phi(z)$ which is analytic on a neighborhood of the eigenvalues of $i\boldsymbol{JA}$. Consider the time factor

$$\boldsymbol{H}(t)=e^{-it\boldsymbol{J}\phi(i\boldsymbol{JA})}, \tag{4.9}$$

where $\boldsymbol{J}$ is a signature matrix (i.e., $\boldsymbol{J}=\boldsymbol{J}^\dagger=\boldsymbol{J}^{-1}$) satisfying

$$\boldsymbol{JA}=\boldsymbol{AJ},\qquad \boldsymbol{JB}=\boldsymbol{BJ},\qquad \boldsymbol{JC}=-\boldsymbol{CJ}. \tag{4.10}$$

Let us now replace the triplet $(\boldsymbol{A},\boldsymbol{B},\boldsymbol{C})$ by

$$(\boldsymbol{A},\boldsymbol{H}(t)\boldsymbol{B},\boldsymbol{C}).$$

Then $\boldsymbol{\Pi}$ is to be replaced by $\boldsymbol{H}(t)\boldsymbol{\Pi}$, while $\boldsymbol{\Gamma}$ will depend on $(x,t)\in\mathbb{R}^2$. In that case (4.4) is to be replaced by

$$\boldsymbol{q}(x,t) = -2\boldsymbol{C}e^{-x\boldsymbol{A}}\boldsymbol{\Gamma}(x;t)^{-1}e^{-x\boldsymbol{A}}\boldsymbol{H}(t)\boldsymbol{B}. \tag{4.11}$$

We easily verify that

$$\begin{aligned} i\boldsymbol{J}\boldsymbol{q}_t &= -2\boldsymbol{J}\boldsymbol{C}e^{-x\boldsymbol{A}}\boldsymbol{\Gamma}^{-1}\left\{\boldsymbol{J}\phi(i\boldsymbol{J}\boldsymbol{A}) - \boldsymbol{\Pi}\boldsymbol{J}\phi(i\boldsymbol{J}\boldsymbol{A})\boldsymbol{\Pi}\right]\boldsymbol{\Gamma}^{-1}e^{-x\boldsymbol{A}}\boldsymbol{H}(t)\boldsymbol{B} \\ &= 2\boldsymbol{C}e^{-x\boldsymbol{A}}\boldsymbol{\Gamma}^{-1}\left\{\phi(i\boldsymbol{J}\boldsymbol{A}) + \boldsymbol{\Pi}\phi(i\boldsymbol{J}\boldsymbol{A})\boldsymbol{\Pi}\right]\boldsymbol{\Gamma}^{-1}e^{-x\boldsymbol{A}}\boldsymbol{H}(t)\boldsymbol{B}, \end{aligned} \tag{4.12}$$

where we have used (4.10), $\boldsymbol{J}\boldsymbol{\Pi} = -\boldsymbol{\Pi}\boldsymbol{J}$, and $\boldsymbol{J}\boldsymbol{\Gamma} = \boldsymbol{\Gamma}\boldsymbol{J}$.

Let us now apply (4.12) to some well-known evolution equations:

a. $\phi(z) = -\dfrac{2}{\xi}z$ for some $0 \neq \xi \in \mathbb{R}$:

$$\boldsymbol{q}_t = \frac{4}{\xi}\boldsymbol{C}e^{-x\boldsymbol{A}}\boldsymbol{\Gamma}^{-1}[\boldsymbol{A} - \boldsymbol{\Pi}\boldsymbol{A}\boldsymbol{\Pi}]\boldsymbol{\Gamma}^{-1}e^{-x\boldsymbol{A}}e^{\xi t\boldsymbol{A}}\boldsymbol{B} = \frac{1}{\xi}\boldsymbol{q}_x,$$

which has the travelling wave $\boldsymbol{q}(x,t) = \boldsymbol{F}(x-\xi t)$ as its general solution.

b. $\phi(z) = -4z^2$ (matrix NLS):

$$iJ\boldsymbol{q}_t = 8\boldsymbol{C}e^{-x\boldsymbol{A}}\boldsymbol{\Gamma}^{-1}[\boldsymbol{A}^2 + \boldsymbol{\Pi}\boldsymbol{A}^2\boldsymbol{\Pi}]\boldsymbol{\Gamma}^{-1}e^{-x\boldsymbol{A}}e^{-4it\boldsymbol{J}\boldsymbol{A}^2} = -\boldsymbol{q}_{xx} - 2\boldsymbol{q}^3.$$

c. $\phi(z) = -8z^3$ (matrix mKdV):

$$\boldsymbol{q}_t = 16\boldsymbol{C}e^{-x\boldsymbol{A}}\boldsymbol{\Gamma}^{-1}[\boldsymbol{A}^3 - \boldsymbol{\Pi}\boldsymbol{A}^3\boldsymbol{\Pi}]\boldsymbol{\Gamma}^{-1}e^{-x\boldsymbol{A}}e^{8t\boldsymbol{A}^3}\boldsymbol{B} = \boldsymbol{q}_{xxx} + 3\boldsymbol{q}^2\boldsymbol{q}_x + 3\boldsymbol{q}_x\boldsymbol{q}^2.$$

d. $\phi(z) = \alpha_0 + 2\alpha_1 z + 4\alpha_2 z^2 + 8\alpha_3 z^3$ for real coefficients $\alpha_0,\alpha_1,\alpha_2,\alpha_3$ (matrix Hirota model):

$$\begin{aligned} &i\boldsymbol{J}\boldsymbol{q}_t + \alpha_0\boldsymbol{q} + i\alpha_1\boldsymbol{J}\boldsymbol{q}_x - \alpha_2(\boldsymbol{q}_{xx} + 2\boldsymbol{q}^3) \\ &\qquad - i\alpha_3\boldsymbol{J}(\boldsymbol{q}_{xxx} + 3\boldsymbol{q}^2\boldsymbol{q}_x + 3\boldsymbol{q}_x\boldsymbol{q}^2) = 0_{(m+n)\times(m+n)}. \end{aligned}$$

Writing

$$\boldsymbol{q} = \begin{pmatrix} 0_{m\times m} & q \\ r & 0_{n\times n} \end{pmatrix}, \tag{4.13}$$

we can write the matrix NLS system in the form

$$iq_t + q_{xx} + 2qrq = 0_{m\times n}, \qquad -ir_t + r_{xx} + 2rqr = 0_{n\times m}, \tag{4.14a}$$

the matrix mKdV system in the form

(4.14b) $$q_t = q_{xxx} + 3qrq_x + 3q_xrq, \qquad r_t = r_{xxx} + 3rqr_x + 3r_xqr,$$

and the matrix Hirota system [21, 25] in the form

(4.14c)
$$iq_t + \alpha_0 q + i\alpha_1 q_x - \alpha_2(q_{xx} + 2qrq) - i\alpha_3(q_{xxx} + 3qrq_x + 3q_xrq) = 0_{m\times n},$$
(4.14d)
$$-ir_t + \alpha_0 r - i\alpha_1 r_x - \alpha_2(r_{xx} + 2rqr) + i\alpha_3(r_{xxx} + 3rqr_x + 3r_xqr) = 0_{n\times m}.$$

In the focusing case we have $r = q^\dagger$ and hence $\boldsymbol{q}^\dagger = \boldsymbol{q}$ in (4.13). Equations (4.14) then take the folowing form:

(4.15a) $$iq_t + q_{xx} + 2qq^\dagger q = 0_{m\times n},$$

(4.15b) $$q_t = q_{xxx} + 3qq^\dagger q_x + 3q_xq^\dagger q,$$

(4.15c) $$iq_t + \alpha_0 q + i\alpha_1 q_x - \alpha_2(q_{xx} + 2qq^\dagger q) - i\alpha_3(q_{xxx} + 3qq^\dagger q_x + 3q_xq^\dagger q) = 0_{m\times n},$$

respectively.

A. Volterra integral equations. Writing $\alpha_l(x,y)$ and $\alpha_r(x,y)$ in block form as

$$\alpha_l(x,y) = \begin{pmatrix} \overline{K}^{\rm up}(x,y) & K^{\rm up}(x,y) \\ \overline{K}^{\rm dn}(x,y) & K^{\rm dn}(x,y)\end{pmatrix}, \quad \alpha_r(x,y) = \begin{pmatrix} M^{\rm up}(x,y) & \overline{M}^{\rm up}(x,y) \\ M^{\rm dn}(x,y) & \overline{M}^{\rm dn}(x,y)\end{pmatrix}$$

we obtain from the usual Volterra integral equations for the Jost matrices Volterra integral equations for the blocks of $\alpha_l(x,y)$ and $\alpha_r(x,y)$. In fact, for the blocks of $\alpha_l(x,y)$ we get

(A.1a) $$\overline{K}^{\rm up}(x,y) = -\int_x^\infty dz\, q(z)\overline{K}^{\rm dn}(z,z+y-x),$$

(A.1b) $$\overline{K}^{\rm dn}(x,y) = \tfrac{1}{2}r(\tfrac{1}{2}(x+y)) + \int_x^{\frac{1}{2}(x+y)} dz\, r(z)\overline{K}^{\rm up}(z,x+y-z),$$

(A.1c) $$K^{\rm up}(x,y) = -\tfrac{1}{2}q(\tfrac{1}{2}(x+y)) - \int_x^{\frac{1}{2}(x+y)} dz\, q(z)K^{\rm dn}(z,x+y-z),$$

(A.1d) $$K^{\rm dn}(x,y) = \int_x^\infty dz\, r(z)K^{\rm up}(z,z+y-x).$$

As a result, we can express the potentials and their (partial) energy integrals in terms of the kernel functions from the right as follows:

(A.2a) $$q(x) = -2K^{\mathrm{up}}(x,x), \qquad r(x) = 2\overline{K}^{\mathrm{dn}}(x,x),$$

(A.2b) $$\int_x^\infty dz\, r(z)q(z) = -2K^{\mathrm{dn}}(x,x), \quad \int_x^\infty dz\, q(z)r(z) = -2\overline{K}^{\mathrm{up}}(x,x).$$

On the other hand, for the blocks of $\alpha_r(x,y)$ we get

(A.3a) $$M^{\mathrm{up}}(x,y) = \int_{-\infty}^x dz\, q(z)M^{\mathrm{dn}}(z, z+y-x),$$

(A.3b) $$M^{\mathrm{dn}}(x,y) = -\tfrac{1}{2}r(\tfrac{1}{2}(x+y)) - \int_{\frac{1}{2}(x+y)}^x dz\, r(z)M^{\mathrm{up}}(z, x+y-z),$$

(A.3c) $$\overline{M}^{\mathrm{up}}(x,y) = \tfrac{1}{2}q(\tfrac{1}{2}(x+y)) + \int_{\frac{1}{2}(x+y)}^x dz\, q(z)\overline{M}^{\mathrm{dn}}(z, x+y-z),$$

(A.3d) $$\overline{M}^{\mathrm{dn}}(x,y) = -\int_{-\infty}^x dz\, r(z)\overline{M}^{\mathrm{up}}(z, z+y-x).$$

As a result, we can express the potentials and their (partial) energy integrals in terms of the kernel functions from the left as follows:

(A.4a) $$q(x) = 2\overline{M}^{\mathrm{up}}(x,x), \qquad r(x) = -2M^{\mathrm{dn}}(x,x),$$

(A.4b) $$\int_{-\infty}^x dz\, r(z)q(z) = -2\overline{M}^{\mathrm{dn}}(x,x), \quad \int_{-\infty}^x dz\, q(z)r(z) = -2M^{\mathrm{up}}(x,x).$$

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