Acta Appl Math (2014) 131:29–47
DOI 10.1007/s10440-013-9848-x

Characterization of Scattering Data for the AKNS System

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Received: 6 March 2013 / Accepted: 19 September 2013 / Published online: 28 September 2013

Abstract We characterize the scattering data of the AKNS system with vanishing boundary conditions. We prove a 1, 1-correspondence between L^1-potentials without spectral singularities and Marchenko integral kernels which are sums of an L^1 function (having a reflection coefficient as its Fourier transform) and a finite exponential sum encoding bound states and norming constants. We give characterization results in the focusing and defocusing cases separately.

Keywords AKNS system · Characterization problem · Scattering data · Zakharov-Shabat system · Inverse scattering transform

1 Introduction

The direct and inverse scattering theory of the AKNS system [1, 3, 4, 7, 11, 21, 25] is a powerful tool in solving the initial-value problem of matrix generalizations of the nonlinear Schrödinger (NLS) equation. By means of the inverse scattering transform (IST) [3, 11, 17, 21, 28], it allows one to convert this initial-value problem into the elementary time evolution of the scattering data. Varying the time evolution of the scattering data, other nonlinear integrable systems, such as the modified Korteweg-de Vries (mKdV) [27], sine-Gordon [2, 29], Hirota [12], and Sasa-Satsuma equations [22] can be solved. An important ingredient in providing a mathematical justification of the IST is to solve the characterization problem of establishing a 1, 1-correspondence between a sufficiently extensive class of coefficients ("potentials") of the AKNS system and a sufficiently extensive class of scattering data.

In this article we study the characterization problem for the AKNS system

$$iJ\frac{\partial X}{\partial x}(\lambda, x) - V(x)X(\lambda, x) = \lambda X(\lambda, x), \tag{1.1}$$

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where

$$J = \begin{pmatrix} I_m & 0_{m\times n} \\ 0_{n\times m} & -I_n \end{pmatrix}, \qquad V(x) = \begin{pmatrix} 0_{m\times m} & iq(x) \\ ir(x) & 0_{n\times n} \end{pmatrix}, \tag{1.2}$$

the potentials $q(x)$ and $r(x)$ have their entries in $L^1(\mathbb{R})$, and λ is a spectral parameter. In the *defocusing case* we have $r(x) = -q(x)^\dagger$ and hence $V(x)^\dagger = V(x)$; in the *focusing case* $r(x) = q(x)^\dagger$ and hence $V(x)^\dagger = -V(x)$. Here and from now on, daggers denote the complex conjugate transpose of a matrix and asterisks the complex conjugate of a scalar. Although in the literature one can find a fairly complete direct and inverse scattering theory of the AKNS system [1, 3, 4, 11, 21], the characterization problem of establishing a 1, 1-correspondence between a sufficiently extensive class of potential pairs $\{q(x), r(x)\}$ and the scattering data remains unsolved, even in the defocusing and focusing cases.

In most of the literature, the scattering data for the AKNS system are formulated in terms of one $n \times m$ or $m \times n$ reflection coefficient, the bound state poles, and the corresponding $n \times m$ or $m \times n$ bound state norming constants. It is usually assumed that the transmission coefficients only have simple poles, so that the complication of having to deal with non-diagonal Jordan structure at the discrete eigenvalues can be avoided. It is then easily seen that the integral kernels of the Marchenko integral equations solving the inverse scattering problem codify the scattering data in a unique way, which is also the case if the transmission coefficients have multiple poles (see [9] for more details). We therefore seek a 1, 1-correspondence between the potential pairs $\{q(x), r(x)\}$ having their entries in $L^1(\mathbb{R})$ and a suitable class of Marchenko integral kernel pairs $\{\Omega(x + y), \breve{\Omega}(x + y)\}$. In the focusing and defocusing cases, where the potentials and the Marchenko kernels satisfy the symmetry relations

$$\begin{cases} r(x) = +q(x)^\dagger, \;\; \breve{\Omega}(x+y) = -\Omega(x+y)^\dagger, & \text{focusing case,} \\ r(x) = -q(x)^\dagger, \;\; \breve{\Omega}(x+y) = +\Omega(x+y)^\dagger, & \text{defocusing case,} \end{cases} \tag{1.3}$$

we seek a 1, 1-correspondence between potentials $q(x)$ having their entries in $L^1(\mathbb{R})$ and suitable Marchenko kernels $\Omega(x + y)$.

To be able to formulate a Marchenko theory in the first place, we need to assume that there are no spectral singularities. Here by a *spectral singularity* we mean a point $\lambda \in \mathbb{R}$ where at least one of the two transmission coefficients is discontinuous. In the absence of spectral singularities, there is a finite number of isolated (and necessarily nonreal) eigenvalues of (1.1), the so-called bound state poles, all of which have finite algebraic multiplicity. Although in principle a spectral singularity is a property of the scattering data, we shall often use the terminology "potential (pair) without spectral singularities" instead of the rather cumbersome "potential (pair) leading to scattering data without spectral singularities."

In the absence of spectral singularities the Marchenko equations have the following form [3, 4, 8, 11]:

$$\overline{K}(x,y) + \begin{pmatrix} 0_{m\times n} \\ I_n \end{pmatrix} \Omega(x+y) + \int_x^\infty dz\, K(x,z)\Omega(z+y) = 0_{(m+n)\times m}, \tag{1.4a}$$

$$K(x,y) + \begin{pmatrix} I_m \\ 0_{n\times m} \end{pmatrix} \breve{\Omega}(x+y) + \int_x^\infty dz\, \overline{K}(x,z)\breve{\Omega}(z+y) = 0_{(m+n)\times n}, \tag{1.4b}$$

where $y \geq x$. Once these equations have been solved, the potential pair is obtained by using the identities

$$q(x) = -2\begin{pmatrix} I_m & 0_{m\times n} \end{pmatrix} K(x,x), \qquad r(x) = 2\begin{pmatrix} 0_{n\times m} & I_n \end{pmatrix} \overline{K}(x,x). \tag{1.5}$$

Either Marchenko kernel can be written as the sum of the Fourier transform of a reflection coefficient and bound state terms. In fact, we write

$$\Omega(x+y)=\rho(x+y)+Ce^{-(x+y)A}B, \tag{1.6a}$$

$$\breve{\Omega}(x+y)=\breve{\rho}(x+y)+\breve{C}e^{-(x+y)\breve{A}}\breve{B}, \tag{1.6b}$$

where $\rho(x)$ and $\breve{\rho}(x)$ have their entries in $L^1(\mathbb{R})$ and (A,B,C) and $(\breve{A},\breve{B},\breve{C})$ are triplets of size compatible matrices such that A and $\breve{A}$ only have eigenvalues with positive real parts. By expanding the matrix exponentials for matrices A which are not necessarily diagonalizable, the bound state terms can be written in the form (see [8] for details)

$$Ce^{-xA}B=\sum_{j=1}^{N}\sum_{l=0}^{\nu_j-1}\boldsymbol{C}_{j,l}\frac{x^l}{l!}e^{-\kappa_j x},$$

where $\kappa_1,\ldots,\kappa_N$ are distinct numbers with positive real parts, ν_j are the orders of the poles of the transmission coefficient at the discrete eigenvalues $i\kappa_j$, and $\boldsymbol{C}_{j,l}$ are the so-called norming constants. In the literature it is customary to employ as scattering data the reflection coefficients

$$R(\lambda)=\int_{-\infty}^{\infty}dy\,e^{-i\lambda y}\rho(y),\qquad \breve{R}(\lambda)=\int_{-\infty}^{\infty}dy\,e^{i\lambda y}\breve{\rho}(y), \tag{1.7}$$

together with the bound state poles $i\kappa_1,\ldots,i\kappa_N$, and the norming constants, but we shall employ the more convenient Marchenko kernels (1.6a), (1.6b) instead.

Analogously, in the absence of spectral singularities another pair of Marchenko equations has the following form:

$$M(x,y)+\begin{pmatrix}0_{m\times n}\\ I_n\end{pmatrix}\breve{\Xi}(x+y)+\int_{-\infty}^{x}dz\,\overline{M}(x,z)\breve{\Xi}(x+y)=0_{(m+n)\times m}, \tag{1.8a}$$

$$\overline{M}(x,y)+\begin{pmatrix}I_m\\ 0_{n\times m}\end{pmatrix}\Xi(x+y)+\int_{-\infty}^{x}dz\,M(x,z)\Xi(z+y)=0_{(m+n)\times n}, \tag{1.8b}$$

where $y\leq x$. Once these equations have been solved, the potential pair is obtained using the identities

$$q(x)=2\begin{pmatrix}I_m & 0_{m\times n}\end{pmatrix}\overline{M}(x,x),\qquad r(x)=-2\begin{pmatrix}0_{n\times m} & I_n\end{pmatrix}M(x,x). \tag{1.9}$$

Either Marchenko kernel can be written as

$$\Xi(x+y)=\ell(x+y)+\mathcal{C}e^{(x+y)\mathcal{A}}\mathcal{B}, \tag{1.10a}$$

$$\breve{\Xi}(x+y)=\breve{\ell}(x+y)+\breve{\mathcal{C}}e^{(x+y)\breve{\mathcal{A}}}\breve{\mathcal{B}}, \tag{1.10b}$$

where $\ell(x)$ and $\breve{\ell}(x)$ have their entries in $L^1(\mathbb{R})$ and $(\mathcal{A},\mathcal{B},\mathcal{C})$ and $(\breve{\mathcal{A}},\breve{\mathcal{B}},\breve{\mathcal{C}})$ are triplets of size compatible matrices such that $\mathcal{A}$ and $\breve{\mathcal{A}}$ only have eigenvalues with positive real parts. For later use, we define the reflection coefficients

$$L(\lambda)=\int_{-\infty}^{\infty}dy\,e^{i\lambda y}\ell(y),\qquad \breve{L}(\lambda)=\int_{-\infty}^{\infty}dy\,e^{-i\lambda y}\breve{\ell}(y). \tag{1.11}$$

An analogous characterization problem can be studied for the Schrödinger equation on the line. This characterization problem has been solved by Marchenko [18, Sect. 3.5] for potentials in $L^1(\mathbb{R}; (1+x^2)dx)$. A full characterization for Faddeev class potentials was given by Melin [20] using microlocal analysis. For the AKNS system (1.1) there are only partial results. In [6] a unique defocusing L^1-potential was reconstructed under the assumption that the reflection coefficient $R(\lambda)$ satisfies

$$\sup_{\lambda\in\mathbb{R}} \left\| R(\lambda) \right\| < 1 \tag{1.12}$$

and is the Fourier transform of an L^1-function $\rho(y)$ which satisfies the estimate $\int_x^\infty dy\, \|\rho(y)\|^2 < +\infty$ for each $x \in \mathbb{R}$. The same result, but without the contractivity condition (1.12), was obtained in [24] for focusing L^1-potentials without bound states. In [26] a focusing L^1-potential having transmission coefficients with simple poles was reconstructed by assuming that the scalar function

$$\tilde{\Omega}(y) = \operatorname*{ess\,sup}_{z\geq y} \left\| \Omega(z) \right\|$$

has a right L^1-tail. In [19] a characterization result for AKNS systems on the half-line was given, yielding a focusing potential with an L^1-tail (cf. [19, Eq. (1.14)]) instead of a potential with entries in $L^1(\mathbb{R}^+)$.

In this article we prove the existence of a 1, 1-correspondence between

(1) potential pairs $\{q(x), r(x)\}$ with L^1 entries and without spectral singularities, and
(2) Marchenko kernel pairs $\{\Omega(x+y), \breve{\Omega}(x+y)\}$ of the form (1.6a), (1.6b), where the reflection coefficients $R(\lambda)$ and $\breve{R}(\lambda)$ are Fourier transforms of L^1-functions.

In the focusing case the result boils down to a 1, 1-correspondence between L^1 potentials $q(x)$ without spectral singularities and Marchenko kernels $\Omega(x+y)$ of the form (1.6a), (1.6b), where $\rho(x)$ has L^1-entries. In the defocusing case, where there are neither bound states nor spectral singularities, the result will be a 1, 1-correspondence between L^1-potentials $q(x)$ and reflection coefficients $R(\lambda)$ that are Fourier transforms of L^1-functions and satisfy (1.12). Similar characterization results will relate potentials with L^1 entries and Marchenko kernel pairs $\{\Xi(x+y), \breve{\Xi}(x+y)\}$.

Let us describe the contents of this article. In Sect. 2 we present the essential direct and inverse scattering theory of the AKNS system (1.1). In this presentation we maximize the use of matrices for the sake of conciseness. In Sect. 3 we prove characterization results in terms of right or left scattering data. In Sect. 4 we state the main characterization results, those without symmetries, those in the focusing case, and those in the defocusing case. In the final Sect. 5 we pose the open problem of having a characterization result which is invariant under the time evolution of the scattering data. The three appendices are devoted to the compactness properties of the Marchenko integral operators on various function spaces, to the uniform limit of a sequence of Volterra operators being a Volterra operator, and to the power compactness of a certain integral operator. In this way most of the functional analysis applied has been relegated to the appendices.

Let us introduce some notations. We denote the upper and lower open complex half-planes by $\mathbb{C}^+$ and $\mathbb{C}^-$ and the corresponding closed half-planes by $\overline{\mathbb{C}^\pm} = \mathbb{C}^\pm \cup \mathbb{R}$. We partition matrices M having $m+n$ rows as follows:

$$M^{\text{up}} = \begin{pmatrix} I_m & 0_{m\times n} \end{pmatrix} M, \qquad M^{\text{dn}} = \begin{pmatrix} 0_{n\times m} & I_n \end{pmatrix} M,$$

where $0_{p\times q}$ denotes the $p \times q$ matrix having only zero elements. By I_p we denote the identity matrix of order p. A generic identity operator is written as I. The range and null space of a linear operator T will be denoted by $\operatorname{Im} T$ and $\operatorname{Ker} T$, respectively.

2 Preliminaries

In this section we present some well-known results from the scattering theory of the AKNS system (1.1), since they will be instrumental in proving the characterization results in Sects. 3 and 4. The proofs can be found in [4, 7, 11], albeit often in different notations.

1. Jost Functions and Transition Coefficients Let us define the $(m+n)\times m$ and $(m+n)\times n$ *Jost functions from the right* $\overline{\psi}(\lambda, x)$ and $\psi(\lambda, x)$, the $(m+n)\times m$ and $(m+n)\times n$ *Jost functions from the left* $\phi(\lambda, x)$ and $\overline{\phi}(\lambda, x)$, and the $(m+n)\times(m+n)$ *Jost matrices* $\Psi(\lambda, x)$ and $\Phi(\lambda, x)$ from the right and the left as those solutions to the AKNS system (1.1) satisfying the asymptotic conditions

$$\Psi(\lambda,x) = \begin{pmatrix}\overline{\psi}(\lambda,x) & \psi(\lambda,x)\end{pmatrix} = \begin{cases} e^{-i\lambda Jx}[I_{m+n}+o(1)], & x\to+\infty,\\ e^{-i\lambda Jx}a_l(\lambda)+o(1), & x\to-\infty,\end{cases} \tag{2.1a}$$

$$\Phi(\lambda,x) = \begin{pmatrix}\phi(\lambda,x) & \overline{\phi}(\lambda,x)\end{pmatrix} = \begin{cases} e^{-i\lambda Jx}[I_{m+n}+o(1)], & x\to-\infty,\\ e^{-i\lambda Jx}a_r(\lambda)+o(1), & x\to+\infty.\end{cases} \tag{2.1b}$$

Then the system of (1.1) being first order implies

$$\Phi(\lambda,x) = \Psi(\lambda,x)a_r(\lambda), \qquad \Psi(\lambda,x) = \Phi(\lambda,x)a_l(\lambda). \tag{2.2}$$

We shall call $a_l(\lambda)$ and $a_r(\lambda)$ *transition matrices* from the left and the right, respectively, to distinguish them from the scattering matrices.

2. Volterra Integral Equations and Analyticity Writing the AKNS system (1.1) in the form

$$\frac{\partial}{\partial y}\left(e^{-i\lambda J(x-y)}X(\lambda,y)\right) = -iJe^{-i\lambda J(x-y)}V(y)X(\lambda,y),$$

we get

$$\Psi(\lambda,x) = e^{-i\lambda Jx} + iJ\int_x^\infty dy\, e^{i\lambda J(y-x)}V(y)\Psi(\lambda,y), \tag{2.3a}$$

$$\Phi(\lambda,x) = e^{-i\lambda Jx} - iJ\int_{-\infty}^x dy\, e^{-i\lambda J(x-y)}V(y)\Phi(\lambda,y). \tag{2.3b}$$

The Volterra integral equations (2.3a), (2.3b) can be used to prove the existence and uniqueness of the solutions $\Psi(\lambda, x)$ and $\Phi(\lambda, x)$ to (1.1) that satisfy the asymptotic conditions (2.1a), (2.1b). In the well-known proof it is used in an essential way that the spectral parameter λ is real and the entries of the potential $V(x)$ belong to $L^1(\mathbb{R})$.

Write

$$\Psi(\lambda,x) = e^{-i\lambda Jx} + \int_x^\infty dy\, \alpha_l(x,y)e^{-i\lambda Jy}, \tag{2.4a}$$

$$\Phi(\lambda, x) = e^{-i\lambda Jx} + \int_{-\infty}^{x} dy\, \alpha_r(x, y) e^{-i\lambda Jy}, \tag{2.4b}$$

where

$$\alpha_l(x, y) = \begin{pmatrix} \overline{K}(x, y) & K(x, y) \end{pmatrix}, \qquad \alpha_r(x, y) = \begin{pmatrix} M(x, y) & \overline{M}(x, y) \end{pmatrix}.$$

Using (2.4a), (2.4b) to convert (2.3a), (2.3b) into equations for the blocks of $\alpha_l(x, y)$ and $\alpha_r(x, y)$, we obtain the four pairs of coupled Volterra integral equations

$$\overline{K}^{\mathrm{up}}(x, y) = -\int_{x}^{\infty} dz\, q(z) \overline{K}^{\mathrm{dn}}(z, z + y - x), \tag{2.5a}$$

$$\overline{K}^{\mathrm{dn}}(x, y) = \frac{1}{2} r\left(\frac{1}{2}(x + y)\right) + \int_{x}^{\frac{1}{2}(x+y)} dz\, r(z) \overline{K}^{\mathrm{up}}(z, x + y - z), \tag{2.5b}$$

$$K^{\mathrm{up}}(x, y) = -\frac{1}{2} q\left(\frac{1}{2}(x + y)\right) - \int_{x}^{\frac{1}{2}(x+y)} dz\, q(z) K^{\mathrm{dn}}(z, x + y - z), \tag{2.5c}$$

$$K^{\mathrm{dn}}(x, y) = \int_{x}^{\infty} dz\, r(z) K^{\mathrm{up}}(z, z + y - x), \tag{2.5d}$$

and

$$M^{\mathrm{up}}(x, y) = \int_{-\infty}^{x} dz\, q(z) M^{\mathrm{dn}}(z, z + y - x), \tag{2.6a}$$

$$M^{\mathrm{dn}}(x, y) = -\frac{1}{2} r\left(\frac{1}{2}(x + y)\right) - \int_{\frac{1}{2}(x+y)}^{x} dz\, r(z) M^{\mathrm{up}}(z, x + y - z), \tag{2.6b}$$

$$\overline{M}^{\mathrm{up}}(x, y) = \frac{1}{2} q\left(\frac{1}{2}(x + y)\right) + \int_{\frac{1}{2}(x+y)}^{x} dz\, q(z) \overline{M}^{\mathrm{dn}}(z, x + y - z), \tag{2.6c}$$

$$\overline{M}^{\mathrm{dn}}(x, y) = -\int_{-\infty}^{x} dz\, r(z) \overline{M}^{\mathrm{up}}(z, z + y - x). \tag{2.6d}$$

Equations (1.5) and (1.9) are immediate from (2.5a)–(2.5d) and (2.6a)–(2.6d), respectively. Indeed, using (2.5c) we get for $\alpha > 0$

$$\begin{aligned}
&\int_{-\infty}^{\infty} dx \left\| K^{\mathrm{up}}(x, x + \alpha) + \frac{1}{2} q\left(x + \frac{1}{2}\alpha\right) \right\| \\
&\quad \le \int_{-\infty}^{\infty} dx \int_{x}^{x+\frac{1}{2}\alpha} dz\, \|q(z)\| \left\| K^{\mathrm{dn}}(z, 2x + \alpha - z) \right\| \\
&\quad = \int_{-\infty}^{\infty} dz\, \|q(z)\| \int_{z-\frac{1}{2}\alpha}^{z} dx\, \left\| K^{\mathrm{dn}}(z, 2x + \alpha - z) \right\| \\
&\quad = \int_{-\infty}^{\infty} dz\, \|q(z)\| \int_{z}^{z+\alpha} dw\, \left\| K^{\mathrm{dn}}(z, w) \right\|,
\end{aligned}$$

where the last member vanishes as $\alpha \to 0^+$, because of the convergence of the integral $\int_z^\infty dw\, \|K^{\mathrm{up}}(z, w)\|$. Thus the first of (1.5) can be justified in the L^1 sense. Similar justifications can be given for the second of (1.5) and for either identity (1.9).

Defining

$$\mu_+(K;z)=\int_z^\infty dw\,\|K(z,w)\|,\qquad \mu_-(M;z)=\int_{-\infty}^z dw\,\|M(z,w)\|,$$

for blocks of $\alpha_l(x,y)$ and $\alpha_r(x,y)$, respectively, we can derive straightforward estimates for the solutions of the Volterra systems (2.5a), (2.5b) and (2.6a), (2.6b) which allow us, after applying Gronwall's inequality [4, 7], to prove the unique solvability of these equations. As a result, we obtain

$$\operatorname*{ess\,sup}_{x\in\mathbb{R}}\left(\int_x^\infty dx\,\|\alpha_l(x,y)\|+\int_{-\infty}^x dy\,\|\alpha_r(x,y)\|\right)<+\infty. \tag{2.7}$$

It is now easily verified that for each $x\in\mathbb{R}$ the Jost functions $\psi(\lambda,x)$ and $\phi(\lambda,x)$ are continuous in $\lambda\in\overline{\mathbb{C}^+}$, are analytic in $\lambda\in\mathbb{C}^+$, and converge as $|\lambda|\to+\infty$ from within $\overline{\mathbb{C}^+}$. Analogously, for each $x\in\mathbb{R}$ the Jost functions $\overline{\psi}(\lambda,x)$ and $\overline{\phi}(\lambda,x)$ are continuous in $\lambda\in\overline{\mathbb{C}^-}$, are analytic in $\lambda\in\mathbb{C}^-$, and converge as $|\lambda|\to+\infty$ from within $\overline{\mathbb{C}^-}$.

Taking the limit of (2.3a) as $x\to+\infty$ and the limit of (2.3b) as $x\to-\infty$ and substituting (2.4a) and (2.4b), respectively, we arrive at Fourier representations for (the blocks of) the transition matrices $a_l(\lambda)$ and $a_r(\lambda)$. In fact,

$$a_l(\lambda)=I_{m+n}+\int_{-\infty}^\infty dy\,\beta_l(y)e^{+i\lambda Jy},$$

$$a_r(\lambda)=I_{m+n}+\int_{-\infty}^\infty dy\,\beta_r(y)e^{-i\lambda Jy},$$

where the entries of $\beta_l(x)$ and $\beta_r(x)$ belong to $L^1(\mathbb{R})$. The diagonal blocks of $\beta_l(x)$ and $\beta_r(x)$ are supported on the positive half-line, whereas their off-diagonal blocks are usually supported on the whole real line. Using the block representations

$$a_l(\lambda)=\begin{pmatrix}a_{l1}(\lambda)&a_{l2}(\lambda)\\a_{l3}(\lambda)&a_{l4}(\lambda)\end{pmatrix}=I_{m+n}+\begin{pmatrix}\int_0^\infty dy\,e^{i\lambda y}\beta_{l1}(y)&\int_{-\infty}^\infty dy\,e^{-i\lambda y}\beta_{l2}(y)\\\int_{-\infty}^\infty dy\,e^{i\lambda y}\beta_{l3}(y)&\int_0^\infty dy\,e^{-i\lambda y}\beta_{l4}(y)\end{pmatrix},$$

$$a_r(\lambda)=\begin{pmatrix}a_{r1}(\lambda)&a_{r2}(\lambda)\\a_{r3}(\lambda)&a_{r4}(\lambda)\end{pmatrix}=I_{m+n}+\begin{pmatrix}\int_0^\infty dy\,e^{-i\lambda y}\beta_{r1}(y)&\int_{-\infty}^\infty dy\,e^{i\lambda y}\beta_{r2}(y)\\\int_{-\infty}^\infty dy\,e^{-i\lambda y}\beta_{r3}(y)&\int_0^\infty dy\,e^{i\lambda y}\beta_{r4}(y)\end{pmatrix},$$

we see that $a_{l1}(\lambda)$ and $a_{r4}(\lambda)$ are continuous in $\lambda\in\overline{\mathbb{C}^+}$, are analytic in $\lambda\in\mathbb{C}^+$, and tend to the identity matrix as $|\lambda|\to+\infty$ from within $\overline{\mathbb{C}^+}$. In the same way we see that $a_{r1}(\lambda)$ and $a_{l4}(\lambda)$ are continuous in $\lambda\in\overline{\mathbb{C}^-}$, are analytic in $\lambda\in\mathbb{C}^-$, and tend to the identity matrix as $|\lambda|\to+\infty$ from within $\overline{\mathbb{C}^-}$. The remaining blocks $a_{l2}(\lambda)$, $a_{l3}(\lambda)$, $a_{r2}(\lambda)$, and $a_{r3}(\lambda)$ are continuous in $\lambda\in\mathbb{R}$ and vanish as $\lambda\to\pm\infty$.

3. Reflection and Transmission Coefficients The above analyticity properties imply that for each $x\in\mathbb{R}$ the *modified Jost matrices* $F_\pm(\lambda,x)$ defined by

$$F_+(\lambda,x)=\begin{pmatrix}\phi(\lambda,x)&\psi(\lambda,x)\end{pmatrix},\qquad F_-(\lambda,x)=\begin{pmatrix}\overline{\psi}(\lambda,x)&\overline{\phi}(\lambda,x)\end{pmatrix}, \tag{2.8}$$

are continuous in $\lambda\in\overline{\mathbb{C}^\pm}$, are analytic in $\mathbb{C}^\pm$, and converge as $|\lambda|\to+\infty$ from within $\overline{\mathbb{C}^\pm}$. The two modified Jost matrices are related as follows:

$$F_-(\lambda,x)=F_+(\lambda,x)JS(\lambda)J,\qquad F_+(\lambda,x)=F_-(\lambda,x)J\breve{S}(\lambda)J, \tag{2.9}$$

where the scattering matrices $S(\lambda)$ and $\breve{S}(\lambda)$ are each other's inverses. By writing them as the block matrices

$$S(\lambda)=\begin{pmatrix}T_r(\lambda) & L(\lambda)\\ R(\lambda) & T_l(\lambda)\end{pmatrix},\qquad \breve{S}(\lambda)=\begin{pmatrix}\breve{T}_l(\lambda) & \breve{R}(\lambda)\\ \breve{L}(\lambda) & \breve{T}_r(\lambda)\end{pmatrix},$$

we obtain the reflection coefficients $R(\lambda)$ and $\breve{R}(\lambda)$ from the right, the reflection coefficients $L(\lambda)$ and $\breve{L}(\lambda)$ from the left, and the transmission coefficients $T_l(\lambda)$ and $T_r(\lambda)$ meromorphic in $\lambda\in\mathbb{C}^+$, and the transmission coefficients $\breve{T}_l(\lambda)$ and $\breve{T}_r(\lambda)$ meromorphic in $\lambda\in\mathbb{C}^-$.

We recall that $\lambda\in\mathbb{R}$ is a *spectral singularity* if at least one of the diagonal blocks $a_{l1}(\lambda)$, $a_{l4}(\lambda)$, $a_{r1}(\lambda)$, and $a_{r4}(\lambda)$ of the transition matrices is a singular matrix. The points $\lambda\in\mathbb{C}^+$, where $a_{l1}(\lambda)$ and $a_{r4}(\lambda)$ are singular matrices, are exactly the isolated eigenvalues of the system (1.1) in $\mathbb{C}^+$. On the other hand, the points $\lambda\in\mathbb{C}^-$, where $a_{r1}(\lambda)$ and $a_{l4}(\lambda)$ are singular matrices, are exactly the isolated eigenvalues of (1.1) in $\mathbb{C}^-$. We note that, in general, $\det a_{l1}(\lambda)=\det a_{r4}(\lambda)$ and $\det a_{r1}(\lambda)=\det a_{l4}(\lambda)$. In the defocusing case there do not exist spectral singularities of (1.1). There do not exist neither spectral singularities as well if

$$\max\left(\int_{-\infty}^{\infty}dx\,\|q(x)\|,\int_{-\infty}^{\infty}dx\,\|r(x)\|\right)<\frac{\pi}{2},\tag{2.10}$$

irrespective of the symmetries on the potential pair [13, 14].

If there are no spectral singularities, the scattering matrices are continuous functions of $\lambda\in\mathbb{R}$ which tend to I_{m+n} as $\lambda\to\pm\infty$. Further, in that case the number of isolated eigenvalues of the system (1.1) is finite. Moreover, the reflection coefficients can be written in the form (1.7) and (1.11), where the entries of $\rho(y)$, $\breve{\rho}(y)$, $\ell(y)$, and $\breve{\ell}(y)$ belong to $L^1(\mathbb{R})$.

4. Marchenko Equations Suppose there are no spectral singularities. Then the kernel functions $K(x,y)$ and $\overline{K}(x,y)$ satisfy the Marchenko integral equations (1.4a), (1.4b), where the Marchenko kernels can be expressed in the reflection coefficients from the right, the isolated eigenvalues in $\mathbb{C}^+$, and the norming constants as in (1.6a), (1.6b). Once the Marchenko equations have been solved, the potential pair $\{q(x),r(x)\}$ follows by applying (1.5). On the other hand, the kernel functions $M(x,y)$ and $\overline{M}(x,y)$ satisfy the Marchenko integral equations (1.8a), (1.8b) where the Marchenko kernels can be expressed in the reflection coefficients from the left, the isolated eigenvalues in $\mathbb{C}^-$, and the norming constants as in (1.10a), (1.10b). Once the Marchenko equations have been solved, the potential pair $\{q(x),r(x)\}$ follows by applying (1.9). The potential pair $\{q(x),r(x)\}$ can alternatively be constructed from the solution to the Marchenko equations (1.4a), (1.4b) (by using (1.5)) or from the solution to the Marchenko equations (1.8a), (1.8b) (by using (1.9)).

It is well-known that the Marchenko equations are uniquely solvable in any L^p setting ($1\le p\le+\infty$), provided we are in the defocusing case [6, 7, 26] or in the focusing case [7, 24, 26].

3 Characterization Results

In this section we derive the characterization results pertaining to half-line data. First we prove the existence of a right (resp., left) L^1-tail of the potentials under the assumption that the Marchenko kernels $\{\Omega(x+y),\breve{\Omega}(x+y)\}$ (resp., $\{\Xi(x+y),\breve{\Xi}(x+y)\}$) have right (resp., left) L^1-tails. We then go on to prove a general characterization theorem based on half-line data.

3.1 Constructing Potentials with an Integrable Tail

In this subsection we prove that the potential pair $\{q(x), r(x)\}$ has a right or left integrable tail when evaluating it from the Marchenko kernels by solving the Marchenko equations and equating the arguments of the solutions.

Proposition 3.1 (Right L^1-tail) *Suppose the Marchenko equations* (1.4a), (1.4b) *are uniquely solvable for $x \geq x_0$ and have Marchenko kernels of the form* (1.6a), (1.6b), *where $\rho(x)$ and $\breve{\rho}(x)$ have their entries in $L^1(\mathbb{R})$ and the matrices A and $\breve{A}$ only have eigenvalues with positive real parts. Then there exists $x_1 \geq x_0$ such that the potential pair $\{q(x), r(x)\}$ obtained by solving the Marchenko equations* (1.4a), (1.4b) *and applying* (1.5) *satisfies*

$$\int_{x_1}^{\infty} dx \big(\|q(x)\| + \|r(x)\|\big) < +\infty.$$

Proof Suppose that the Marchenko equations (1.4a), (1.4b) are uniquely solvable for $x \geq x_0$. Then, according to Proposition A.1, we obtain, for $x \geq x_0$, matrix functions $\overline{K}(x, y)$ and $K(x, y)$ supported on $y \geq x$ such that

$$\operatorname*{ess\,sup}_{x \geq x_0} \int_x^{\infty} dy \big(\|\overline{K}(x, y)\| + \|K(x, y)\|\big) < +\infty. \tag{3.1}$$

Moreover, since the norm of the Marchenko integral operators involved in (1.4a), (1.4b) vanishes as $x \to +\infty$ [cf. Appendix B, last paragraph], the essential supremum of the integral appearing in (3.1) vanishes as $x_0 \to +\infty$. Writing (2.5b) as an integral equation to evaluate the potential $r(x)$ from known $\overline{K}(x, y)$, we obtain the straightforward estimate

$$\int_x^{\infty} dy \, \|r(y)\| \leq \mu_+\big(\overline{K}^{\mathrm{dn}}; x\big) + \int_x^{\infty} dz \, \|r(z)\| \mu_+\big(\overline{K}^{\mathrm{up}}; z\big).$$

Since $\operatorname{ess\,sup}_{z \geq x} \mu_+(\overline{K}^{\mathrm{up}}; z)$ vanishes as $x \to +\infty$, we can choose $x_1 \geq x_0$ such that

$$\operatorname*{ess\,sup}_{x \geq x_1} \mu_+\big(\overline{K}^{\mathrm{up}}; x\big) < 1.$$

Thus, by solving (2.5b) for $r(x)$ for $x \geq x_1$, we see that $\int_{x_1}^{\infty} dy \, \|r(y)\| < +\infty$, as claimed. Likewise, by considering (2.5c) as an integral equation for $q(x)$, we prove that $\int_{x_1}^{\infty} dy \, \|q(y)\| < +\infty$ for a large enough x_1. □

In the same way we prove

Proposition 3.2 (Left L^1-tail) *Suppose the Marchenko equations* (1.8a), (1.8b) *are uniquely solvable for $x \leq x_0$ and have Marchenko kernels of the form* (1.10a), (1.10b), *where $\ell(x)$ and $\breve{\ell}(x)$ have their entries in $L^1(\mathbb{R})$ and the matrices $\mathcal{A}$ and $\breve{\mathcal{A}}$ only have eigenvalues with positive real parts. Then there exists $x_1 \leq x_0$ such that the potential pair $\{q(x), r(x)\}$ obtained by solving the Marchenko equations* (1.4a), (1.4b) *and applying* (1.9) *satisfies*

$$\int_{-\infty}^{x_1} dx \big(\|q(x)\| + \|r(x)\|\big) < +\infty.$$

3.2 Characterization Based on Half-Line Data

In this subsection we obtain characterization results based on half-line data.

Theorem 3.3 *Suppose the Marchenko equations* (1.4a), (1.4b) *are uniquely solvable for* $x \geq x_0$ *and have Marchenko kernels of the form* (1.6a), (1.6b), *where* $\rho(x)$ *and* $\breve{\rho}(x)$ *have their entries in* $L^1(\mathbb{R})$ *and the matrices* A *and* $\breve{A}$ *only have eigenvalues with positive real parts. Then the potential pair* $\{q(x), r(x)\}$ *obtained by solving the Marchenko equations* (1.4a), (1.4b) *and applying* (1.5) *satisfies*

$$\int_{x_0}^{\infty} dx\big(\|q(x)\| + \|r(x)\|\big) < +\infty. \tag{3.2}$$

Proof In view of Proposition 3.1, it suffices to prove that

$$\int_{x_0}^{x_1} dy\big(\|q(y)\| + \|r(y)\|\big) < +\infty$$

for any $x_1 > x_0$. Writing $w = \frac{1}{2}(x + y)$ in (2.5b) and rearranging terms we get

$$r(w) = 2\overline{K}^{\mathrm{dn}}(x, 2w - x) - 2\int_x^w dz\, r(z)\overline{K}^{\mathrm{up}}(z, 2w - z). \tag{3.3}$$

We intend to study (3.3) as an integral equation for $r(w)$ in the function space $L^1((x_0, x_1); \mathbb{C}^{n\times m})$.

In this proof we will be dealing with integral operators on vector function spaces or, said otherwise, matrices of integral operators on (scalar) function spaces. Using the norm $\|\{x_{j,l}\}\| = \sum_{j,l} |x_{j,l}|$ for complex matrices when dealing with integral operators on L^1 vector function spaces and the norm $\|\{x_{j,l}\}\| = \max_{j,l} |x_{j,l}|$ for complex matrices when dealing with integral operators in L^∞ vector function spaces or in spaces of continuous matrix functions, the usual exact expressions for the operator norms of an integral operator on an L^1 space or an L^∞ space can be applied.

Suppose first that $\overline{K}^{\mathrm{up}}(z, 2w - z)$ is a bounded continuous function in (z, w), where $x_0 \leq z \leq w \leq x_1$. Then $\|\overline{K}^{\mathrm{up}}(z, 2w - z)\| \leq \kappa$ for all such (z, w). Then it is easily verified that the iterates of the integral operator

$$(\mathcal{L}r)(w) = 2\int_x^w dz\, r(z)\overline{K}^{\mathrm{up}}(z, 2w - z) \tag{3.4}$$

satisfy

$$\int_x^{x_1} dw\, \big\|\big(\mathcal{L}^m r\big)(w)\big\| \leq \frac{(x_1 - x)^m \kappa^m}{m!} \int_x^{x_1} dz\, \big\|r(z)\big\|.$$

As a result,

$$\big\|\mathcal{L}^m\big\|^{1/m} \leq \frac{(x_1 - x_0)\kappa}{\sqrt[m]{m!}},$$

which vanishes as $m \to +\infty$. Consequently, $\mathcal{L}$ has a zero spectral radius and therefore $\int_{x_0}^{x_1} dy\, \|r(y)\| < +\infty$.

Let us discuss the general case. The norm of the integral operator $\mathcal{L}$ defined by (3.4) is given by

$$\operatorname*{ess\,sup}_{x_0\le z\le x_1} 2\int_z^{x_1} dw\, \big\|\overline{K}^{\text{up}}(z,2w-z)\big\| = \operatorname*{ess\,sup}_{x_0\le z\le x_1} \int_{2x_0-z}^{2x_1-z} dy\, \big\|\overline{K}^{\text{up}}(z,y)\big\|,$$

which is bounded above by $\operatorname{ess\,sup}_{z\ge x_0} \mu_+(\overline{K}^{\text{up}}; z)$. We now approximate the matrix function $\overline{K}^{\text{up}}(z,2w-z)$ by a sequence of essentially bounded matrix functions $\overline{K}_n^{\text{up}}(z,2w-z)$ such that

$$\lim_{n\to+\infty} \operatorname*{ess\,sup}_{x_0\le z\le x_1} 2\int_z^{x_1} dw\, \big\|\overline{K}^{\text{up}}(z,2w-z) - \overline{K}_n^{\text{up}}(z,2w-z)\big\| = 0,$$

where the details of the approximation will be given below. When doing so, we approximate the integral operator $\mathcal{L}$ defined by (3.4) by a sequence of integral operators $\mathcal{L}_n$ of the same type such that $\|\mathcal{L}_n - \mathcal{L}\| \to 0$ as $n \to +\infty$ in the operator norm on $L^1((x_0,x_1);\mathbb{C}^{n\times m})$. However, each approximating operator $\mathcal{L}_n$ has a zero spectral radius and has a compact operator as its square. Using Theorem B.2 we conclude that $\mathcal{L}$ has a zero spectral radius and hence that $\int_{x_0}^{x_1} dy\, \|r(y)\| < +\infty$. In the same way, using (2.5c) as an integral equation to compute $q(x)$, we prove that $\int_{x_0}^{x_1} dy\, \|q(y)\| < +\infty$.

It remains to give the details of the approximation. Let us approximate the matrix functions $\rho(x)$ and $\breve{\rho}(x)$ appearing in (1.6a), (1.6b) in the L^1 norm by continuous matrix functions which vanish as $x \to \pm\infty$, without changing the triplets (A,B,C) and $(\breve{A},\breve{B},\breve{C})$. By Proposition A.1 applied to $E = C_0$, we would obtain uniquely solvable Marchenko equations of the form (1.4a), (1.4b) whose solutions $\overline{K}(x,y)$ and $K(x,y)$ would be continuous in $y \in [x,+\infty)$ and vanish as $y \to +\infty$, with a supremum norm which is uniformly bounded in $x \in [x_0,+\infty)$. The corresponding integral operators $\mathcal{L}$ defined as in (3.4) would then converge to the given $\mathcal{L}$ in the operator norm on $L^1((x_0,x_1);\mathbb{C}^{n\times m})$ and have bounded continuous integral kernels. By virtue of Proposition C.1, the approximating operators $\mathcal{L}$ would then have compact operators as their squares and have a zero spectral radius. According to Corollary B.3, also the original $\mathcal{L}$ has a zero spectral radius. As a result, (3.3) has a unique solution $r(w)$ in $L^1((x_0,x_1);\mathbb{C}^{n\times m})$, as claimed. This completes the proof. □

In the same way we prove

Theorem 3.4 *Suppose the Marchenko equations* (1.8a), (1.8b) *are uniquely solvable for* $x \le x_0$ *and have Marchenko kernels of the form* (1.10a), (1.10b), *where* $\ell(x)$ *and* $\breve{\ell}(x)$ *have their entries in* $L^1(\mathbb{R})$ *and the matrices* $\mathcal{A}$ *and* $\breve{\mathcal{A}}$ *only have eigenvalues with positive real parts. Then the potential pair* $\{q(x),r(x)\}$ *obtained by solving the Marchenko equations* (1.4a), (1.4b) *and applying* (1.9) *satisfies*

$$\int_{-\infty}^{x_0} dx\big(\|q(x)\| + \|r(x)\|\big) < +\infty. \tag{3.5}$$

4 Main Characterization Theorems

In [5] a theory of Darboux transformation is developed which departs from the Darboux transformations of the Marchenko [or Gelfand-Levitan] integral kernels and arrives at the

Darboux transformations of the Jost solutions and the potentials. When applying this article to the Marchenko integral equations for the AKNS system, it has been shown that a potential pair $\{q(x), r(x)\}$ satisfying (3.2) and without spectral singularities lead to Marchenko equations (1.4a), (1.4b) which are uniquely solvable for $x \geq x_0$. In fact, the *existence* of a solution to these Marchenko equations implies its uniqueness. The structure (1.6a), (1.6b) of the Marchenko kernels is a well-known result (see e.g. [4, 7]). Analogously, when applying this article to the Marchenko integral equations (1.8a), (1.8b), it has been shown that a potential pair $\{q(x), r(x)\}$ satisfying (3.5) and without spectral singularities lead to Marchenko equations (1.8a), (1.8b) which are uniquely solvable for $x \leq x_0$. The structure (1.10a), (1.10b) of the Marchenko kernels is a well-known result [4, 7].

We now combine the contents of the above paragraph and Theorems 3.3 and 3.4 to arrive at the following three characterization results: the first without symmetries on the potentials, the second for the focusing case, and third for the defocusing case.

Theorem 4.1 (Without symmetries) *Let $x_0 \in \mathbb{R}$. Then there exists a $1,1$-correspondence between* (i) *potential pairs $\{q(x), r(x)\}$ with entries in $L^1(\mathbb{R})$ and without spectral singularities and* (ii) *scattering data of the type*

$$\left\{\Omega(x+y), \breve{\Omega}(x+y)\right\}_{x,y\geq x_0} \quad \text{and} \quad \left\{\Xi(x+y), \breve{\Xi}(x+y)\right\}_{x,y\leq x_0},$$

where

a. *For $x, y \geq x_0$, the Marchenko kernels $\Omega(x+y)$ and $\breve{\Omega}(x+y)$ have the form* (1.6a), (1.6b), *where $\rho(x)$ and $\breve{\rho}(x)$ have their entries in $L^1(\mathbb{R})$ and the triplets (A, B, C) and $(\breve{A}, \breve{B}, \breve{C})$ of size compatible matrices are such that A and $\breve{A}$ have only eigenvalues with positive real parts*;
b. *For $x \geq x_0$ the Marchenko integral equations* (1.4a), (1.4b) *are uniquely solvable*;
c. *For $x, y \leq x_0$, the Marchenko kernels $\Xi(x+y)$ and $\breve{\Xi}(x+y)$ have the form* (1.10a), (1.10b), *where $\ell(x)$ and $\breve{\ell}(x)$ have their entries in $L^1(\mathbb{R})$ and the triplets $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ and $(\breve{\mathcal{A}}, \breve{\mathcal{B}}, \breve{\mathcal{C}})$ of size compatible matrices are such that $\mathcal{A}$ and $\breve{\mathcal{A}}$ have only eigenvalues with positive real parts*;
d. *For $x \leq x_0$ the Marchenko integral equations* (1.8a), (1.8b) *are uniquely solvable.*

In the focusing case, the symmetry relations (1.3) lead to the following simplification of Theorem 4.1. We recall that in the focusing case the Marchenko equations are always uniquely solvable [7, 24, 26].

Theorem 4.2 (Focusing case) *Let $x_0 \in \mathbb{R}$. Then there exists a $1,1$-correspondence between* (i) *potentials $q(x)$ with entries in $L^1(\mathbb{R})$ and without spectral singularities and* (ii) *scattering data of the type*

$$\Omega(x+y) \text{ for } x, y \geq x_0 \quad \text{and} \quad \Xi(x+y) \text{ for } x, y \leq x_0,$$

where

a. *For $x, y \geq x_0$, the Marchenko kernel $\Omega(x+y)$ has the form*

$$\Omega(x+y) = \rho(x+y) + Ce^{-(x+y)A}B,$$

where $\rho(x)$ has its entries in $L^1(\mathbb{R})$ and the triplet (A, B, C) of size compatible matrices is such that A has only eigenvalues with positive real parts;

b. *For $x, y \le x_0$, the Marchenko kernel $\Xi(x+y)$ has the form*

$$\Xi(x+y) = \ell(x+y) + \mathcal{C}e^{(x+y)\mathcal{A}}\mathcal{B},$$

where $\ell(x)$ has its entries in $L^1(\mathbb{R})$ and the triplet $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ of size compatible matrices is such that $\mathcal{A}$ has only eigenvalues with positive real parts.

In the defocusing case the nonexistence of spectral singularities and the absence of the matrix triplets pertaining to the bound states greatly simplify the above results. In this case the scattering matrix is uniquely solvable and the reflection coefficients $R(\lambda)$ and $L(\lambda)$ satisfy

$$\sup_{\lambda\in\mathbb{R}} \|R(\lambda)\| = \sup_{\lambda\in\mathbb{R}} \|L(\lambda)\| < 1,$$

while the transmission coefficients $T_r(\lambda)$ and $T_l(\lambda)$ are continuous in $\lambda \in \overline{\mathbb{C}^+}$, are analytic in $\lambda \in \mathbb{C}^+$, and tend to the identity matrix as $|\lambda| \to +\infty$ from within $\overline{\mathbb{C}^+}$.

Theorem 4.3 (Defocusing case) *There is a* 1, 1*-correspondence between* (i) *potentials $q(x)$ with entries in $L^1(\mathbb{R})$ and* (ii) *reflection coefficients from the right $R(\lambda)$ satisfying*

$$\sup_{\lambda\in\mathbb{R}} \|R(\lambda)\| < 1, \qquad R(\lambda) = \int_{-\infty}^{\infty} dx\, e^{-i\lambda x}\rho(x),$$

where $\rho(x)$ has its entries in $L^1(\mathbb{R})$. Similarly, there is a 1, 1*-correspondence between* (i) *potentials $q(x)$ with entries in $L^1(\mathbb{R})$ and* (ii) *reflection coefficients from the left $L(\lambda)$ satisfying*

$$\sup_{\lambda\in\mathbb{R}} \|L(\lambda)\| < 1, \qquad L(\lambda) = \int_{-\infty}^{\infty} dx\, e^{i\lambda x}\ell(x),$$

where $\ell(x)$ has its entries in $L^1(\mathbb{R})$.

5 Conclusions

We emphasize that the characterization results for (a) the Schrödinger equation on the line treated in [18, 20] and (b) the characterization results for the AKNS system discussed above *do not involve the time variable in any respect*. They regard 1, 1-correspondences between a class of potentials and a class of scattering data. As a matter of fact, time evolution of the Schrödinger potential according to the KdV equation could make the scattering data leave the class indicated in [18, 20]. Analogously, time evolution of the matrix Zakharov-Shabat potential (pair) according to the matrix NLS equation could cause the corresponding scattering data to leave the class indicated in Theorems 4.1–4.3.

Let us explain in more detail why the characterization results obtained so far are not time-evolution-proof. Here we make the following two points:

(1) Even though the reflection coefficient $R(\lambda)$ is the Fourier transform of the L^1-function $\rho(y)$ [cf. (1.5)], for the NLS time evolved reflection coefficient $R(\lambda)e^{4i\lambda^2 t}$ we do not necessarily have

$$R(\lambda)e^{4i\lambda^2 t} = \int_{-\infty}^{\infty} dy\, e^{-i\lambda y}\rho(y;t),$$

where for each $t > 0$ the function $\rho(y;t)$ has its entries in $L^1(\mathbb{R})$.

(2) We cannot exclude the occurrence of spectral singularities at some time $t > 0$ in situations where they do not exist at $t = 0$. For instance, the initial potential (pair) could satisfy (2.10) and hence not have spectral singularities, whereas at some $t > 0$ the condition (2.10) could be violated and a spectral singularity could occur.

It would therefore be desirable to solve a time-evolution-proof version of the characterization problem in which both the class of potentials and the class of scattering data remain invariant under the time evolution according to the IST scheme of e.g. the matrix NLS equation. At present no such time-evolution-proof characterization is known, although it is implied in the literature [11] that Schwarz class AKNS potentials correspond to Schwarz class scattering data and that this remains the case under time evolution according to the NLS system. An additional complication is that the solution of such a time-evolution-proof characterization problem may depend on the nonlinear integrable evolution equation solved by using the IST method.

Acknowledgements The research leading to this article was supported by INdAM, by MIUR under PRIN grant No. 20083KLJEZ–003, and by the Autonomous Region of Sardinia (RAS) under grant No. CRP3–138.

Appendix A: Analysis of Marchenko Operators

The boundedness and compactness properties of Marchenko operators on various function spaces, as given before in [6, 7, 26], are immediate from Proposition A.1 below.

Proposition A.1 *Suppose $\omega \in L^1(2x, +\infty)$ and let the space identifier E stand for one of L^p $(1 \le p \le +\infty)$, BC, or C_0, or AC. Then the Marchenko integral operator $K_\omega^{[x]}$ defined by*

$$\big(K_\omega^{[x]} f\big)(y) = \int_x^\infty dz\, \omega(y+z) f(z)$$

is compact on $E[x]$ and its nonzero spectrum and the Jordan structure of each nonzero eigenvalue do not depend on the choice of function space.

Proof It is easy to bound the operator norm of $K_\omega^{[x]}$ on $L^1(x, +\infty)$ and $L^\infty(x, +\infty)$ above by $\int_{2x}^\infty dz\, |\omega(z)|$. By the M. Riesz interpolation theorem [16], we get the same norm upper bound of $K_\omega^{[x]}$ on $L^p(x, +\infty)$. Next, for each $f \in L^\infty(x, +\infty)$ we have

$$\begin{aligned}\big|\big(K_\omega^{[x]} f\big)(y_1) - \big(K_\omega^{[x]} f\big)(y_2)\big| &\le \int_x^\infty dz\, \big|\omega(y_1+z) - \omega(y_2+z)\big|\, \big|f(z)\big| \\ &\le \|f\|_\infty \int_x^\infty dz\, \big|\omega(y_1+z) - \omega(y_2+z)\big|;\end{aligned}$$

a dominated convergence argument implies that $K_\omega^{[x]}$ maps $L^\infty(x, +\infty)$ into $C_0[x, +\infty)$.

Next, consider the operator norm of $K_\omega^{[x]}$ on $AC[x]$. Indeed, for bounded absolutely continuous $f : [x, +\infty) \to \mathbb{C}$ such that the a.e. existing derivative $f' \in L^1(x, +\infty)$, we obviously have $K_\omega^{[x]} f \in BC[x]$ and $K_\omega^{[x]} f' \in L^1[x]$. Also, for a.e. x we have

$$\big(K_\omega^{x]} f\big)'(x) = -\omega(2x) f(x) + \int_x^\infty dy\, \omega'(x+y) f(y)$$

$$= -\omega(2x)f(x) + \left[\omega(x+y)f(y)\right]_{y=x}^{\infty} - \int_x^\infty dy\, \omega(x+y)f'(y)$$

$$= -2\omega(2x)f(x) - \int_x^\infty dy\, \omega(x+y)f'(y).$$

Hence

$$\|K_\omega^{[x]} f'\|_{L^1[x]} \le \left(\int_{2x}^\infty dy\, \|\omega(y)\|\right)\left\{\|f\|_{BC[x]} + \|f'\|_{L^1[x]}\right\},$$

which proves the boundedness of $K_\omega^{[x]}$ on $AC[x]$.

It is easily verified that

$$\int_x^\infty dy \int_x^\infty dz\, \left|\omega(y+z)\right|^2 = \int_{2x}^\infty ds\, (s-2x)\left|\omega(s)\right|^2. \tag{A.1}$$

So, if this integral is finite, the operator $K_\omega^{[x]}$ is Hilbert-Schmidt and hence compact on $L^2(x,+\infty)$. By approximating $K_\omega^{[x]}$ with arbitrary L^1 kernels by such Hilbert-Schmidt operators, we prove that $K_\omega^{[x]}$ is a compact operator on $L^2(x,+\infty)$. By compact interpolation [15], we then prove that, for $1 < p < +\infty$, $K_\omega^{[x]}$ is compact on $L^p(x,+\infty)$.

Next, it is obvious that $K_\omega^{[x]}$ has separated variables if

$$\omega(y+z) = Ce^{-(y+z)A}B \tag{A.2}$$

for some complex $p \times p$ matrix A having only eigenvalues with positive real part, some complex $p \times 1$ matrix B, and some complex $1 \times p$ matrix C. Thus in this case $K_\omega^{[x]}$ is a compact operator on $L^p(x,+\infty)$ $(1 \le p < +\infty)$ and, because the Banach dual of $K_\omega^{[x]}$ is $K_\omega^{[x]}$ itself, also on $L^\infty(x,+\infty)$; by taking restrictions, it follows that $K_\omega^{[x]}$ is compact on $BC[x,+\infty)$ and $C_0[x,+\infty)$ as well. Since functions of the type (A.2) are dense in $L^1(2x,+\infty)$ [23, Sect. 7.3.2], a simple approximation argument yields that, for general L^1 kernels, $K_\omega^{[x]}$ is a compact operator on any of the above $E[x]$.

The final statement of the proposition can be based on the following:

> Let E_1 and E_2 be two complex Banach spaces, where E_2 is continuously and densely imbedded in E_1. Suppose F is a Fredholm operator of index zero on E_1 such that $F[E_2] \subset E_2$. Suppose also that the restriction of F to E_2 is a Fredholm operator of index zero on E_2. Then these two operators F are both invertible or both noninvertible.

For $m = 1, 2, 3, \ldots$ and $\lambda \in \mathbb{C}$, this property can now be applied to $F = [I - \lambda K_\omega^{[x]}]^m$, where the class of space identifiers contains L^p $(1 \le p < +\infty)$, BC, C_0, and the intersection of any two such spaces endowed with the sum of the norms. These operators F are obviously Fredholm of index zero on $E[x]$ for E as in the statement of this proposition, but also on the intersection of any two such spaces. To extend this result to $E = L^\infty$ we use the fact that F maps L^∞ continuously into C_0. $\square$

To compare $K_\omega^{[x]}$ for various $x \in \mathbb{R}$ we apply shift operators to define the Marchenko operators on $E[0]$. In that case $K_\omega^{[x]}$ is to be replaced by

$$\left(\tilde{K}_\omega^{[x]} f\right)(y) = \int_0^\infty dz\, \omega(2x+y+z)f(z),$$

but the translational similarity applied to modify the operators does not affect their boundedness and compactness properties nor does it affect their spectra. It is now quite obvious that $\tilde{K}_\omega^{[x]}$ depends on $x \in \mathbb{R}$ in the operator norm (on each $E[0]$) and vanishes in the operator norm as $x \to +\infty$.

Appendix B: Limits of Volterra Operators

In this appendix we prove that the uniform limit of a sequence of Volterra operators (i.e., compact operators having a zero spectral radius) is a Volterra operator.

We need the following lemma. Here we recall that a projection is a bounded linear operator P satisfying $P^2 = P$.

Lemma B.1 *If P and Q are projections defined on a complex Banach space X satisfying $\|P - Q\| < 1$, then* Im P *and* Im Q *have the same dimension.*

Proof Put

$$V = PQ + (I - P)(I - Q), \qquad W = QP + (I - Q)(I - P).$$

Then

$$\begin{aligned}
I - VW &= I - PQP - (I - P)(I - Q)(I - P) \\
&= I - PQP - (I - Q - P + QP) + (P - PQ - P + PQP) \\
&= Q + P - QP - PQ = (P - Q)^2, \\
I - WV &= I - QPQ - (I - Q)(I - P)(I - Q) \\
&= I - QPQ - (I - Q - P + PQ) + (Q - QP - Q + QPQ) \\
&= Q + P - QP - PQ = (P - Q)^2.
\end{aligned}$$

Thus under the hypothesis of the lemma we have

$$\max\bigl(\|I - V\|, \|I - W\|\bigr) \le \|P - Q\|^2 < 1,$$

which proves that V and W are invertible. Since V obviously maps $\operatorname{Im} Q = \operatorname{Ker}(I - Q)$ into $\operatorname{Im} P = \operatorname{Ker}(I - P)$ and $\operatorname{Ker} Q = \operatorname{Im}(I - Q)$ into $\operatorname{Ker} P = \operatorname{Im}(I - P)$, it is clear that V is an invertible linear operator mapping the range of Q onto the range of P. Consequently, the ranges of P and Q have the same dimension. □

The main result of this appendix is the following.

Theorem B.2 *The limit of a sequence of Volterra operators with respect to the operator norm is a Volterra operator.*

Proof Let $\{K_n\}_{n=1}^\infty$ be a sequence of Volterra operators, defined on a complex Banach space X, such that $\|K_n - K\| \to 0$ as $n \to +\infty$ for some bounded linear operator K on X. Then K is the uniform limit of compact operators and hence a compact operator [10, Lemma VI.5.3]. Suppose K has a positive spectral radius. Then K has at least one nonzero isolated eigenvalue λ of finite algebraic multiplicity. Assuming the deleted closed disk $\{z \in \mathbb{C} : |z-\lambda| \le \varepsilon\} \setminus \{\lambda\}$ to be free of eigenvalues of K, put

$$\delta = \max_{|z-\lambda|=\varepsilon} \left\| (z-K)^{-1} \right\|.$$

Letting P stand for the projection $P = \frac{1}{2\pi i} \oint_{|z-\lambda|=\varepsilon} dz\,(z-K)^{-1}$ and choosing n_0 such that $\|K_n - K\| < [\delta(1+\varepsilon\delta)]^{-1}$ for $n \ge n_0$, we obtain

$$\begin{aligned} &\left\| \frac{1}{2\pi i} \oint_{|z-\lambda|=\varepsilon} dz\,(z-K_n)^{-1} - \frac{1}{2\pi i} \oint_{|z-\lambda|=\varepsilon} dz\,(z-K)^{-1} \right\| \\ &\quad < \frac{1}{2\pi} 2\pi\varepsilon \max_{|z-\lambda|=\varepsilon} \left\| (z-K_n)^{-1} - (z-K)^{-1} \right\| \\ &\quad < \frac{1}{2\pi} 2\pi\varepsilon \frac{\delta^2 \|K_n - K\|}{1 - \delta\|K_n - K\|} < 1. \end{aligned}$$

Thus for $n \ge n_0$ the projection $P_n = \frac{1}{2\pi i} \oint_{|z-\lambda|=\varepsilon} dz\,(z-K_n)^{-1}$ commutes with K_n and satisfies $\|P_n - P\| < 1$. According to Lemma B.1, for $n \ge n_0$ the projection P_n has the algebraic multiplicity of the eigenvalue λ of K as its rank. As a result, for $n \ge n_0$ each K_n has at least one eigenvalue in the open disk $\{z \in \mathbb{C} : |z-\lambda| < \varepsilon\}$, which is a contradiction. Consequently, K has a zero spectral radius. □

The proof of the following corollary is now immediate.

Corollary B.3 *Let m be a positive integer and let $\{K_n\}_{n=1}^{\infty}$ be a sequence of linear operators having compact m-th powers and zero spectral radii such that $\|K_n - K\| \to 0$ as $n \to +\infty$ for some bounded linear operator K. Then K has a zero spectral radius.*

Proof Apply Theorem B.2 to the operators K_n^m approximating K^m. □

Appendix C: Power Compactness of an Integral Operator

In this appendix we prove the following elementary result.

Proposition C.1 *Suppose the function $K(x,y)$ is continuous for $x_0 \le x \le y \le x_1$. Then the integral operator $\mathcal{L}$ defined on $L^1(x_0,x_1)$ by*

$$(\mathcal{L}\psi)(x) = \int_x^{x_1} dy\, K(x,y)\psi(y)$$

has a compact operator as its square.

Proof It is well-known [10, Theorem IV.8.5] that $L^\infty(x_0,x_1)$ is the dual Banach space of $L^1(x_0,x_1)$ in the sense that the bounded linear functionals on $L^1(x_0,x_1)$ can be isometrically identified with the elements of $L^\infty(x_0,x_1)$. In that case the dual operator $\mathcal{L}'$ is defined on $L^\infty(x_0,x_1)$ by

$$\big(\mathcal{L}'\varphi\big)(x) = \int_{x_0}^{x} dy\, K(y,x)\varphi(y).$$

Also, $\mathcal{L}'$ maps $L^\infty(x_0, x_1)$ into its closed subspace $C[x_0, x_1]$ of continuous functions. By the Ascoli-Arzelà theorem [10, Theorem IV.6.7], the operator $\mathcal{L}'$ is compact when defined on $C[x_0, x_1]$. Using the diagram of bounded operators

$$L^\infty(x_0, x_1) \xrightarrow{\mathcal{L}'} C[x_0, x_1] \xrightarrow{\mathcal{L}'} C[x_0, x_1] \xrightarrow{\text{inclusion}} L^\infty(x_0, x_1),$$

we immediately see that $[\mathcal{L}']^2$ is a compact operator on $L^\infty(x_0, x_1)$ [10, Theorem VI.5.4]. But then its adjoint $\mathcal{L}^2$ is a compact operator on $L^1(x_0, x_1)$ (cf. [10, Theorem VI.5.2]), as claimed. □

References

1. Ablowitz, M.J., Clarkson, P.A.: Solitons, Nonlinear Evolution Equations and Inverse Scattering. Cambridge University Press, Cambridge (1991)
2. Ablowitz, M.J., Kaup, D.J., Newell, A.C., Segur, H.: Method for solving the sine-Gordon equation. Phys. Rev. Lett. **30**, 1262–1264 (1973)
3. Ablowitz, M.J., Kaup, D.J., Newell, A.C., Segur, H.: The inverse scattering transform—Fourier analysis for nonlinear problems. Stud. Appl. Math. **53**, 249–315 (1974)
4. Ablowitz, M.J., Prinari, B., Trubatch, A.D.: Discrete and Continuous Nonlinear Schrödinger Systems. Cambridge University Press, Cambridge (2004)
5. Aktosun, T., van der Mee, C.: A unified approach to Darboux transformations. Inverse Probl. **25**, 105003 (2009). 22 pp.
6. Aktosun, T., Klaus, M., van der Mee, C.: Direct and inverse scattering for selfadjoint Hamiltonian systems on the line. Integral Equ. Oper. Theory **38**, 129–171 (2000)
7. Demontis, F.: Matrix Zakharov-Shabat System and Inverse Scattering Transform. Lambert Academic Publishing, Saarbrücken (2012)
8. Demontis, F., van der Mee, C.: Marchenko equations and norming constants of the matrix Zakharov-Shabat system. Oper. Matrices **2**, 79–113 (2008)
9. Demontis, F., van der Mee, C.: Novel formulation of inverse scattering and characterization of scattering data. Discrete Contin. Dyn. Syst., Suppl. **2011**, 343–350 (2011). In: W. Feng, Z. Feng, M. Grasselli, A. Ibragimov, X. Lu, S. Siegmund, J. Voigt (eds.) Dynamical Systems and Differential Equations, DCDS Supplement 2011, Proceedings of the 8th AIMS International Conference, Dresden, Germany. American Institute of Mathematical Sciences, Springfield (2011)
10. Dunford, N., Schwartz, J.T.: Linear Operators. Part I: General Theory. Wiley-Interscience, New York (1958)
11. Faddeev, L.D., Takhtajan, L.A.: Hamiltonian Methods in the Theory of Solitons. Classics in Mathematics. Springer, New York (1987)
12. Hirota, R.: Exact envelope-soliton solutions of a nonlinear wave equation. J. Math. Phys. **14**, 805–809 (1973)
13. Klaus, M., Shaw, J.K.: On the eigenvalues of Zakharov-Shabat systems. SIAM J. Math. Anal. **34**, 759–773 (2003)
14. Klaus, M., van der Mee, C.: Wave operators for the matrix Zakharov-Shabat system. J. Math. Phys. **51**, 053503 (2010). 26 pp.
15. Krasnoselskii, M.A.: On a theorem of M. Riesz. Sov. Math. Dokl. **1**, 229–231 (1960); [Dokl. Akad. Nauk SSSR **131**, 246–248 (1960), Russian]
16. Krein, S.G., Petunin, Yu.I., Semenov, E.M.: Interpolation of Linear Operators. Transl. Math. Monographs, vol. 54. Am. Math. Soc., Providence (1982)
17. Manakov, S.V.: On the theory of two-dimensional stationary self-focusing of electromagnetic waves. Sov. Phys. JETP **38**, 248–253 (1974); [Zh. Eksp. Teor. Fiz. **65**, 505–516 (1973), Russian]
18. Marchenko, V.A.: Sturm-Liouville Operators and Applications. OT, vol. 22. Birkhäuser, Basel (1986); [Naukova Dumka, Kiev (1977), in Russian]
19. Melik-Adamyan, F.È.: On a class of canonical differential operators. Sov. J. Contemp. Math. Anal. **24**, 48–69 (1989); [Izv. Akad. Nauk Armyan. SSR Ser. Mat. **24**, 570–592, 620 (1989), Russian]
20. Melin, A.: Operator methods for inverse scattering on the real line. Commun. Partial Differ. Equ. **10**, 677–766 (1985)
21. Novikov, S.P., Manakov, S.V., Pitaevskii, L.B., Zakharov, V.E.: Theory of Solitons. The Inverse Scattering Method. Plenum, New York (1984)

22. Sasa, N., Satsuma, J.: New-type of soliton solutions for a higher-order nonlinear Schrödinger equation. J. Phys. Soc. Jpn. **60**, 409–417 (1991)
23. Sz.-Nagy, B.: Introduction to Real Functions and Orthogonal Expansions. Oxford Univ. Press, New York (1965)
24. van der Mee, C.: Direct and inverse scattering for skewselfadjoint Hamiltonian systems. In: Ball, J.A., Helton, J.W., Klaus, M., Rodman, L. (eds.) Current Trends in Operator Theory and Its Applications. OT, vol. 149, pp. 407–439 Birkhäuser, Basel (2004)
25. van der Mee, C.: Nonlinear Evolution Models of Integrable Type. SIMAI e-Lecture Notes, vol. 11. SIMAI, Torino (2013)
26. Villarroel, J., Ablowitz, M.J., Prinari, B.: Solvability of the direct and inverse problems for the nonlinear Schrödinger equation. Acta Appl. Math. **87**, 245–280 (2005)
27. Wadati, M.: The modified Korteweg-de Vries equation. J. Phys. Soc. Jpn. **34**, 1289–1296 (1973)
28. Zakharov, V.E., Shabat, A.B.: Exact theory of two-dimensional self-focusing and one dimensional self-modulation of waves in nonlinear media. Sov. Phys. JETP **34**, 62–69 (1972); [Zh. Eksp. Teor. Fiz. **61**, 118–134 (1971), Russian]
29. Zakharov, V.E., Takhtadzhyan, L.A., Faddeev, L.D.: Complete description of solutions of the sine-Gordon equation. Sov. Phys. Dokl. **19**, 824–826 (1975); [Dokl. Akad. Nauk SSSR **219**(6), 1334–1337 (1974), Russian]