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Closed form solutions to the matrix sine-Gordon equation

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An explicit solution formula for the sine-Gordon equation in terms of matrix triplets (A, B, C), where A is $p \times p$, B is $p \times 1$ and C is $1 \times p$, is generalized by removing the limitations on the matrix sizes. The matrix sine-Gordon equation it satisfies is identified.

Keywords: sine-Gordon equation; N-soliton solutions; matrix triplet method.

1. Introduction

The (scalar) sine-Gordon equation appears in many interesting contexts, such as the description of surfaces of constant negative Gaussian curvature (Bour, 1862; Eisenhart, 1960), magnetic flux propagation in Josephson junctions (McLaughlin & Scott, 1978; Mineev & Shmidt, 1980) and propagation of deformations along the DNA double helix (Gaeta *et al.*, 1994; Lennholm & Hörnquist, 2003; Salerno, 1991; Yakushevich, 2004). Because of the importance of the applications in which the sine-Gordon equation arises (see Aktosun *et al.*, 2010 for details), it is not surprising that different approaches, such as the IST (Ablowitz *et al.*, 1973), and Darboux and Bäcklund transformations (Gu *et al.*, 2005; Rogers & Schief, 2002), were developed to get exact solutions. These exact solutions are written in terms of elementary functions and, in Lamb (1980) and Pöppe (1983), the reader can find many of the solutions coming from the methods cited above.

In Aktosun *et al.* (2010), we have presented a family of explicit solutions to the sine-Gordon equation

$$u_{xt} = \sin(u) \tag{1}$$

by writing the integral kernel of the Marchenko equation in the form

$$\Omega(x + y; t) = C\,\mathrm{e}^{-(x+y)A}\,\mathrm{e}^{-(t/2)A^{-1}}B,$$

where (A, B, C) is a suitable triplet of real matrices. The sine-Gordon solution is obtained by solving the Marchenko integral equation by separation of variables and integrating the resulting expression with respect to $x \in \mathcal{R}$. Various representations of the solutions were obtained which are in agreement with results by Schiebold (2002) who has used matrix triplets as well but not the Marchenko method. The methods used in Aktosun *et al.* (2010) and Schiebold (2002) lead to the usual (anti)kink, soliton–(anti)soliton interaction and breather solutions (Lamb, 1980), but also to a plethora of multipole soliton solutions.

Starting from the real matrices A, B and C of sizes $p \times p$, $p \times 1$ and $1 \times p$, respectively, where all of the eigenvalues of A have positive real parts, in Aktosun *et al.* (2010) we have derived the sine-Gordon solution formula

$$u(x,t) = -4\int_x^\infty \mathrm{d}y C[\mathrm{e}^{2xA+(1/2)tA^{-1}} + P\,\mathrm{e}^{-2xA-(1/2)tA^{-1}}P]^{-1}B, \tag{2}$$

where

$$P = \int_0^\infty \mathrm{d}s\,\mathrm{e}^{-sA}BC\,\mathrm{e}^{-sA}.$$

However, if we allow A, B and C to be $p \times p$, $p \times n$ and $m \times p$ matrices, where all of the eigenvalues of A have positive real parts, then (2) still makes sense. The question now is how to generalize the non-linear evolution equation (1) and to show that (2) satisfies this generalized sine-Gordon equation.

The paper is organized as follows: in Section 2, we give the AKNS pair which allows us to 'discover' the matrix version of the sine-Gordon equation. In Section 3, we introduce the scattering coefficients, state their time evolution law and write the Marchenko integral equation. Generalizing in an easy way the results obtained in Aktosun *et al.* (2010), we arrive at various equivalent formulas for the integrable matrix sine-Gordon equation. Finally, in Section 4, we discuss an interesting example.

2. AKNS pair

Consider the $(m+n) \times (m+n)$ matrices $X(x,\lambda;t)$ and $T(x,\lambda;t)$ given by

$$X = \begin{pmatrix} -i\lambda I_m & -\frac{1}{2}u_x \\ \frac{1}{2}u_x^\dagger & i\lambda I_n \end{pmatrix}, \quad T = \frac{i}{4\lambda}\begin{pmatrix} C & S \\ S_\bullet & -C_\bullet \end{pmatrix}, \tag{3}$$

where $u = u(x,t)$ is an $m \times n$ matrix function and the dagger denotes the matrix conjugate transpose. Then (X,T) is an AKNS pair for some non-linear evolution system to be specified below if

$$X_t + XT = T_x + TX. \tag{4}$$

Substitution of (3) into (4) yields

$$C_x = -\tfrac{1}{2}u_x S_\bullet - \tfrac{1}{2}S u_x^\dagger, \tag{5a}$$

$$(C_\bullet)_x = -\tfrac{1}{2}S_\bullet u_x - \tfrac{1}{2}u_x^\dagger S, \tag{5b}$$

$$S_x = 2i\lambda[u_{xt} - S] + \tfrac{1}{2}u_x C_\bullet + \tfrac{1}{2}C u_x, \tag{5c}$$

$$(S_\bullet)_x = -2i\lambda[u_{xt}^\dagger - S_\bullet] + \tfrac{1}{2}u_x^\dagger C + \tfrac{1}{2}C_\bullet u_x^\dagger. \tag{5d}$$

Letting (5c) and (5d) be true for arbitrary λ, we get the identities

$$S = u_{xt}, \quad S_\bullet = u_{xt}^\dagger, \tag{6}$$

which implies that $S_\bullet = S^\dagger$ and the two equations in (6) reduce to the single non-linear evolution system

$$u_{xt} = S \tag{7}$$

and its adjoint. Further, using (5a), (5b) and (6), we have

$$C^\dagger = C, \quad C_\bullet^\dagger = C_\bullet.$$

Note that differential system (5) can be written in the concise form

$$\begin{pmatrix} C_x & S_x \\ (S_\bullet)_x & -(C_\bullet)_x \end{pmatrix} = \frac{1}{2}\begin{pmatrix} C & S \\ S_\bullet & -C_\bullet \end{pmatrix}\begin{pmatrix} 0_{m\times m} & u_x \\ -u_x^\dagger & 0_{n\times n} \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 0_{m\times m} & u_x \\ -u_x^\dagger & 0_{n\times n} \end{pmatrix}\begin{pmatrix} C & S \\ S_\bullet & -C_\bullet \end{pmatrix}, \tag{8}$$

where the matrix with blocks C, S, $S_\bullet$ and $-C_\bullet$ is selfadjoint for each $x \in \mathcal{R}$. Being a commutator, the right-hand side of (8) has zero trace and hence

$$\mathrm{Tr}\, C - \mathrm{Tr}\, C_\bullet = \mathrm{Tr}\begin{pmatrix} C & S \\ S_\bullet & -C_\bullet \end{pmatrix} = \mathrm{Tr}\, J = m - n.$$

On the differential system (5), we impose both of the asymptotic conditions

$$C = I_m, \quad C_\bullet = I_n, \quad S = 0_{m\times n}, \quad S_\bullet = 0_{n\times m}, \tag{9}$$

as $x \to \pm\infty$.

In the scalar case ($m = n = 1$), for real-valued u and under condition (9), we obtain $C = C_\bullet = \cos(u)$ and $S = S_\bullet = \sin(u)$. This solution of differential system (8) satisfies both of the asymptotic conditions (9) whenever $u \to 2m_\pm\pi$ for some integers $m_\pm$ as $x \to \pm\infty$. Hence, in this case, the non-linear evolution equation is given by the sine-Gordon equation

$$u_{xt} = \sin u. \tag{10}$$

Let us now consider the matrix case. Introduce the $(m+n) \times 2$ matrix

$$\Pi = \begin{pmatrix} \pi_1^\dagger & 0_{m\times 1} \\ 0_{n\times 1} & \pi_4^\dagger \end{pmatrix},$$

where $\pi_1^\dagger$ and $\pi_4^\dagger$ are column vectors of length m and n and $\Pi^\dagger \Pi = I_2$. Thus, π_1 and π_4 are row vectors of euclidean norm 1 (i.e. $\pi_1\pi_1^\dagger = \pi_4\pi_4^\dagger = 1$). Define, for some real scalar function $v(x,t)$, the $m \times n$ matrix function

$$u(x,t) = \pi_1^\dagger v(x,t)\pi_4 \tag{11}$$

and construct the $(m+n) \times (m+n)$ matrix function

$$T = \frac{i}{4\lambda}\Pi\begin{pmatrix} \cos v & \sin v \\ \sin v & -\cos v \end{pmatrix}\Pi^\dagger = \frac{i}{4\lambda}\begin{pmatrix} \pi_1(\cos v)\pi_1^\dagger & \pi_1(\sin v)\pi_4^\dagger \\ \pi_4(\sin v)\pi_1^\dagger & -\pi_4(\cos v)\pi_4^\dagger \end{pmatrix}. \tag{12}$$

Then, the left-hand side of (8) can be written as

$$\Pi \begin{pmatrix} -v_x \sin v & v_x \cos v \\ v_x \cos v & v_x \sin v \end{pmatrix} \Pi^\dagger.$$

Taking into account that $\Pi^\dagger \Pi = I_2$, the right-hand side of (8) has the form

$$\frac{\Pi}{2} \begin{pmatrix} \cos v & \sin v \\ \sin v & -\cos v \end{pmatrix} \Pi^\dagger \Pi \begin{pmatrix} 0 & v_x \\ -v_x & 0 \end{pmatrix} \Pi^\dagger - \frac{\Pi}{2} \begin{pmatrix} 0 & v_x \\ -v_x & 0 \end{pmatrix} \Pi \Pi^\dagger \begin{pmatrix} \cos v & \sin v \\ \sin v & -\cos v \end{pmatrix} \Pi^\dagger.$$

Hence, (8) is satisfied, provided the following choices are made:

$$\begin{aligned} C &= \pi_1 (\cos v) \pi_1^\dagger = \cos((uu^\dagger)^{1/2}), \\ S &= \pi_1 (\sin v) \pi_4^\dagger = \operatorname{sinc}((uu^\dagger)^{1/2}) u, \\ S_\bullet &= \pi_4 (\sin v) \pi_1^\dagger = \operatorname{sinc}((u^\dagger u)^{1/2}) u^\dagger, \\ C_\bullet &= -\pi_4 (\cos v) \pi_4^\dagger = \cos((u^\dagger u)^{1/2}), \end{aligned}$$

where $(uu^\dagger)^{1/2}$ and $(u^\dagger u)^{1/2}$ are the non-negative hermitian square roots of $uu^\dagger$ and $u^\dagger u$, respectively, $\operatorname{sinc}(z) = \sin z / z$, and, for any square matrix w, $\cos(w)$ and $\operatorname{sinc}(w)$ are defined by means of their power series. Thus, the matrix function $u(x,t)$ defined by (11) satisfies the non-linear evolution equation

$$u_{xt} = \operatorname{sinc}((uu^\dagger)^{1/2}) u, \tag{13}$$

provided the real scalar function $v(x,t)$ satisfies the sine-Gordon equation (10) and $v(x,t) \to 2m_\pm \pi$ for some integers $m_\pm$ as $x \to \pm\infty$. Thus, the non-linear evolution equation (13) can be solved by using the IST method and we have found its AKNS pair.

More generally, we need to solve the differential system (8) under the simultaneous conditions (9) to find the non-linear evolution equation (7) that generalizes the sine-Gordon equation.

3. Solutions to the matrix sine-Gordon equation

Following Ablowitz *et al.* (2004, Section 4.2.4) and taking into account (9), we consider the differential systems

$$v_x = Xv, \qquad v_t = Tv. \tag{14}$$

Using that $u_x \to 0$ as $x \to \pm\infty$, the two differential systems reduce to

$$v_x = -i\lambda J v, \qquad v_t = \frac{i}{4\lambda} J v, \tag{15}$$

where $J = I_m \oplus (-I_n)$. Let us denote the Jost solutions of the matrix Zakharov–Shabat system $v_x = Xv$ by $\bar{\psi}$, ψ, ϕ and $\bar{\phi}$. Then

$$\begin{aligned} \begin{pmatrix} \bar{\psi}(\lambda, x; t) & \psi(\lambda, x, t) \end{pmatrix} &= \mathrm{e}^{-i\lambda J x} [I_{m+n} + o(1)], \quad x \to +\infty, \\ \begin{pmatrix} \phi(\lambda, x; t) & \bar{\phi}(\lambda, x; t) \end{pmatrix} &= \mathrm{e}^{-i\lambda J x} [I_{m+n} + o(1)], \quad x \to -\infty. \end{aligned}$$

Introduce the time-dependent functions

$$\bar{\Psi}(\lambda,x;t)=\mathrm{e}^{\mathrm{i}t/4\lambda}\bar{\psi}(\lambda,x;t),\quad \Psi(\lambda,x;t)=\mathrm{e}^{-\mathrm{i}t/4\lambda}\psi(\lambda,x;t),$$
$$\Phi(\lambda,x;t)=\mathrm{e}^{\mathrm{i}t/4\lambda}\phi(\lambda,x;t),\quad \bar{\Phi}(\lambda,x;t)=\mathrm{e}^{-\mathrm{i}t/4\lambda}\bar{\phi}(\lambda,x;t),$$

as solutions to (14). Then

$$\bar{\psi}_t=\left(T-\frac{i}{4\lambda}I_{m+n}\right)\bar{\psi},\quad \psi_t=\left(T+\frac{i}{4\lambda}I_{m+n}\right)\psi,$$
$$\phi_t=\left(T-\frac{i}{4\lambda}I_{m+n}\right)\phi,\quad \bar{\phi}_t=\left(T+\frac{i}{4\lambda}I_{m+n}\right)\bar{\phi}.$$

Using that

$$\phi(\lambda,x;t)=\psi(\lambda,x;t)a_{r3}(\lambda;t)+\bar{\psi}(\lambda,x;t)a_{r1}(\lambda;t),$$
$$\bar{\phi}(\lambda,x;t)=\psi(\lambda,x;t)a_{r4}(\lambda;t)+\bar{\psi}(\lambda,x;t)a_{r2}(\lambda;t),$$

as well as

$$\begin{pmatrix}\phi(\lambda,x;t) & \bar{\phi}(\lambda,x;t)\end{pmatrix}=\mathrm{e}^{-\mathrm{i}\lambda Jx}\begin{pmatrix}a_{r1}(\lambda;t) & a_{r2}(\lambda;t)\\ a_{r3}(\lambda;t) & a_{r4}(\lambda;t)\end{pmatrix}+o(1),\quad x\to+\infty,$$
$$\begin{pmatrix}\bar{\psi}(\lambda,x;t) & \psi(\lambda,x;t)\end{pmatrix}=\mathrm{e}^{-\mathrm{i}\lambda Jx}[I_{m+n}+o(1)],\quad x\to+\infty,$$

we obtain by taking the asymptotics as $x\to+\infty$

$$\partial_t a_{r1}(\lambda;t)=0_{m\times m},\quad \partial_t a_{r4}(\lambda;t)=0_{n\times n},$$
$$\partial_t a_{r3}(\lambda;t)=-\frac{i}{2\lambda}a_{r3}(\lambda;t),\quad \partial_t a_{r2}(\lambda;t)=\frac{i}{2\lambda}a_{r2}(\lambda;t).$$

Hence, $a_{r1}(\lambda;t)$ and $a_{r4}(\lambda;t)$ do not depend on t, whereas

$$a_{r2}(\lambda;t)=\mathrm{e}^{\mathrm{i}t/2\lambda}a_{r2}(\lambda;0),\quad a_{r3}(\lambda;t)=\mathrm{e}^{-\mathrm{i}t/2\lambda}a_{r3}(\lambda;0).$$

Consequently, the reflection coefficients defined by

$$R(\lambda;t)=a_{r3}(\lambda;t)a_{r1}(\lambda;t)^{-1},\quad L(\lambda;t)=-a_{r1}(\lambda;t)^{-1}a_{r2}(\lambda;t),$$

satisfy the time evolution identities

$$R(\lambda;t)=\mathrm{e}^{-\mathrm{i}t/2\lambda}R(\lambda;0),\quad L(\lambda;t)=\mathrm{e}^{\mathrm{i}t/2\lambda}L(\lambda;0).$$

Writing

$$\rho(y;t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}\lambda\,\mathrm{e}^{\mathrm{i}\lambda y}R(\lambda;t),\quad \ell(y;t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}\lambda\,\mathrm{e}^{-\mathrm{i}\lambda y}L(\lambda;t),$$

and differentiating with respect to y and t, we get the partial differential equations

$$\rho_{yt}=\tfrac{1}{2}\rho,\quad \ell_{yt}=-\tfrac{1}{2}\ell,$$

provided ρ, ℓ, ρ_t and ℓ_t are matrix functions having their entries in $L^1(\mathcal{R})$. Then the left and right Marchenko kernels are given by

$$\Omega_l(y;t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\lambda\, \mathrm{e}^{\mathrm{i}\lambda y} R(\lambda;t) + C_l\, \mathrm{e}^{-yA_l}\, \mathrm{e}^{-(1/2)tA_l^{-1}} B_l, \tag{16a}$$

$$\Omega_r(y;t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\lambda\, \mathrm{e}^{-\mathrm{i}\lambda y} L(\lambda;t) + C_r\, \mathrm{e}^{yA_r}\, \mathrm{e}^{-(1/2)tA_r^{-1}} B_r, \tag{16b}$$

where (A_l, B_l, C_l) and (A_r, B_r, C_r) are matrix triplets such that A_l and A_r only have eigenvalues with positive real parts. In this way, the two Marchenko kernels satisfy the partial differential equations

$$(\Omega_l)_{yt} = \tfrac{1}{2}\Omega_l, \quad (\Omega_r)_{yt} = -\tfrac{1}{2}\Omega_r.$$

Let us define, for each $(m+n)\times p$ matrix G, $G^{\mathrm{up}} = (I_m \quad 0_{m,n})G$ and $G^{\mathrm{dn}} = (0_{n,m} \quad I_n)G$. Now write

$$\left(\bar{\psi}(\lambda,x;t) \quad \psi(\lambda,x;t)\right) = \mathrm{e}^{-\mathrm{i}\lambda Jx} + \int_x^{\infty} \mathrm{d}y\, \left(\bar{K}(x,y;t) \quad K(x,y;t)\right) \mathrm{e}^{-\mathrm{i}\lambda Jy}.$$

Then, the left Marchenko equations are given by

$$K^{\mathrm{up}}(x,y;t) - \Omega_l(x+y;t)^\dagger + \int_x^{\infty} \mathrm{d}z \int_x^{\infty} \mathrm{d}v K^{\mathrm{up}}(x,v;t)\Omega_l(v+z;t)\Omega_l(z+y;t)^\dagger = 0_{m,n}, \tag{17a}$$

$$\bar{K}^{\mathrm{dn}}(x,y;t) + \Omega_l(x+y;t) + \int_x^{\infty} \mathrm{d}z \int_x^{\infty} \mathrm{d}v \bar{K}^{\mathrm{dn}}(x,v;t)\Omega_l(v+z;t)^\dagger\Omega_l(z+y;t) = 0_{n,m}, \tag{17b}$$

where $y > x$. Substituting, in the reflectionless case, the right-hand side of (16a) into (17), we can solve the Marchenko equations (17) explicitly by separation of variables. Mimicking the calculations detailed in Aktosun *et al.* (2010), we obtain

$$K^{\mathrm{up}}(x,y;t) = B_l^\dagger F(x;t)^{-1}\, \mathrm{e}^{-(y-x)A_l^\dagger} C_l^\dagger, \tag{18a}$$

$$\bar{K}^{\mathrm{dn}}(x,y;t) = -C_l E(x;t)^{-1}\, \mathrm{e}^{-(y-x)A_l} B_l, \tag{18b}$$

where

$$F(x;t) = \mathrm{e}^{t\beta^\dagger} + Q\,\mathrm{e}^{-t\beta}N, \quad E(x;t) = \mathrm{e}^{t\beta} + N\,\mathrm{e}^{-t\beta^\dagger}Q,$$

$$Q = \int_0^{\infty} \mathrm{d}s\, \mathrm{e}^{-sA_l^\dagger} C_l^\dagger C_l\, \mathrm{e}^{-sA_l}, \quad N = \int_0^{\infty} \mathrm{d}s\, \mathrm{e}^{-sA_l} B_l B_l^\dagger\, \mathrm{e}^{-sA_l^\dagger},$$

with the quantity β defined by

$$\beta = 2xA_l + \tfrac{1}{2}tA_l^{-1}.$$

It is worthwhile noting that Q and N are semi-definite selfadjoint. The Lyapunov solutions Q and N are invertible if and only A_l has the minimal matrix order among all triplets (A_l, B_l, C_l) having the same Marchenko kernel $\{C_l\,\mathrm{e}^{-xA_l}B_l : x \in \mathcal{R}\}$ (Aktosun *et al.*, 2010). Moreover, $F(x;t) = E(x;t)^\dagger$. It is well known (see, e.g. Ablowitz *et al.*, 2004) that the solutions of the non-linear evolution equation (7) can be found from the solutions of the Marchenko equations (17) as follows:

$$u(x,t) = -4\int_x^\infty \mathrm{d}s K^{\mathrm{up}}(s,s;t), \quad u(x,t)^\dagger = 4\int_x^\infty \mathrm{d}s \bar{K}^{\mathrm{dn}}(s,s;t).$$

We thus get the following solution formula:

$$u(x,t) = -4\int_x^\infty \mathrm{d}s B_l^\dagger F(s;t)^{-1} C_l^\dagger. \tag{19}$$

4. An example

Let us consider the following matrix triplet:

$$A_l = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad B_l = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad C_l = (1 \quad 0).$$

Taking into account that

$$Q = \int_0^\infty \mathrm{d}s\, \mathrm{e}^{-sA_l^\dagger} C_l^\dagger C_l\, \mathrm{e}^{-sA_l} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} \end{pmatrix},$$

$$N = \int_0^\infty \mathrm{d}s\, \mathrm{e}^{-sA_l} B_l B_l^\dagger\, \mathrm{e}^{-sA_l^\dagger} = \begin{pmatrix} \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{2} \end{pmatrix},$$

$$\mathrm{e}^{2xA_l^\dagger + (1/2)tA_l^{\dagger -1}} = \mathrm{e}^{2x+t/2} \begin{pmatrix} 1 & 0 \\ 2x - \frac{t}{2} & 1 \end{pmatrix},$$

$$\mathrm{e}^{-2xA_l - (1/2)tA_l^{-1}} = \mathrm{e}^{-2x-t/2} \begin{pmatrix} 1 & -2x + \frac{t}{2} \\ 0 & 1 \end{pmatrix},$$

we explicitly calculate

$$\begin{aligned} F(x;t) &= \mathrm{e}^{2xA_l^\dagger + (1/2)tA_l^{\dagger -1}} + Q\,\mathrm{e}^{-2xA_l - (1/2)tA_l^{-1}} N \\ &= \mathrm{e}^{2x+t/2} \begin{pmatrix} 1 & 0 \\ 2x - \frac{t}{2} & 1 \end{pmatrix} + \mathrm{e}^{-2x-t/2} \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} \end{pmatrix} \begin{pmatrix} 1 & -2x + \frac{t}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{2} \end{pmatrix} \\ &= \begin{pmatrix} \mathrm{e}^{2x+t/2} + \mathrm{e}^{-2x-t/2}\left[\frac{7}{16} + \frac{x}{4} - \frac{t}{16}\right] & \mathrm{e}^{-2x-t/2}\left[-\frac{1}{4} - \frac{x}{2} + \frac{t}{8}\right] \\ \mathrm{e}^{2x+t/2}\left[2x - \frac{t}{2}\right] + \mathrm{e}^{-2x-t/2}\left[-\frac{1}{4} - \frac{x}{8} + \frac{t}{32}\right] & \mathrm{e}^{2x+t/2} + \mathrm{e}^{-2x-t/2}\left[\frac{3}{16} + \frac{x}{4} - \frac{t}{16}\right] \end{pmatrix}. \end{aligned}$$

Therefore,

$$\det F(x;t) = \mathrm{e}^{4x+t} + \left[\frac{5}{8} + x - \frac{t}{4} + x^2 - \frac{xt}{2} + \frac{t^2}{16}\right] + \frac{5}{256}\,\mathrm{e}^{-4x-t}.$$

Using (19), we finally get

$$u_x(x,t) = \frac{4}{\det F(x;t)} \begin{pmatrix} \mathrm{e}^{2x+t/2} + \mathrm{e}^{-2x-t/2}\left[\frac{3}{16} + \frac{x}{4} - \frac{t}{16}\right] \\ \mathrm{e}^{2x+t/2}\left[-2x + \frac{t}{2}\right] + \mathrm{e}^{-2x-t/2}\left[\frac{1}{4} + \frac{x}{8} - \frac{t}{32}\right] \end{pmatrix}.$$

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