Communications in Applied and Industrial Mathematics, ISSN 2038-0909, e-372
DOI: 10.1685/journal.caim.372

Exact solutions to the integrable discrete nonlinear Schrödinger equation under a quasiscalarity condition

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Communicated by Nicola Bellomo

Abstract

In this article we derive explicit solutions of the matrix integrable discrete nonlinear Schrödinger equation under a quasiscalarity condition by using the inverse scattering transform and the Marchenko method. The Marchenko equation is solved by separation of variables, where the Marchenko kernel is represented in the form

$$CA^{-(n+j+1)}e^{i\tau(A-A^{-1})^2}B,$$

(A, B, C) being a matrix triplet where A has only eigenvalues of modulus larger than one. The class of solutions obtained contains the N-soliton and breather solutions as special cases. Unitarity properties of the scattering matrix are derived.

Keywords: Integrable discrete nonlinear Schrödinger equation, Marchenko equation, inverse scattering transform.

AMS Subject Classification: 35Q55, 35Q58

1. Introduction.

In this article we derive explicit solutions of the system of *integrable discrete nonlinear Schrödinger (IDNLS) equations*

$$i\frac{d}{d\tau}\boldsymbol{u}_n = \boldsymbol{u}_{n+1} - 2\boldsymbol{u}_n + \boldsymbol{u}_{n-1} - \boldsymbol{u}_{n+1}\boldsymbol{w}_n\boldsymbol{u}_n - \boldsymbol{u}_n\boldsymbol{w}_n\boldsymbol{u}_{n-1}, \tag{1a}$$

$$-i\frac{d}{d\tau}\boldsymbol{w}_n = \boldsymbol{w}_{n+1} - 2\boldsymbol{w}_n + \boldsymbol{w}_{n-1} - \boldsymbol{w}_{n+1}\boldsymbol{u}_n\boldsymbol{w}_n - \boldsymbol{w}_n\boldsymbol{u}_n\boldsymbol{w}_{n-1}, \tag{1b}$$

where n is an integer labeling "position" and $\boldsymbol{u}_n$ and $\boldsymbol{w}_n$ are $N \times M$ and $M \times N$ matrix functions depending on "time" $\tau \in \mathbb{R}$. We assume that the potentials $\{\boldsymbol{u}_n\}_{n=-\infty}^{\infty}$ and $\{\boldsymbol{w}_n\}_{n=-\infty}^{\infty}$ satisfy the *quasiscalarity condition*

$$\boldsymbol{u}_n\boldsymbol{w}_n = \boldsymbol{w}_n\boldsymbol{u}_n = c_n I_N, \qquad n \in \mathbb{Z}, \tag{2}$$

Received 2011 05 02, Accepted 2011 05 20, Published 2011 08 19.

where $N = M$, I_N is the identity matrix of order N, and, for each $n \in \mathbb{Z}$, c_n is an unknown complex number distinct from 1. Under this condition the system of IDNLS equations can be written in the form

$$i\frac{d}{d\tau}\boldsymbol{u}_n = (1-c_n)\left[\boldsymbol{u}_{n+1}+\boldsymbol{u}_{n-1}\right] - 2\boldsymbol{u}_n, \tag{3a}$$

$$-i\frac{d}{d\tau}\boldsymbol{w}_n = (1-c_n)\left[\boldsymbol{w}_{n+1}+\boldsymbol{w}_{n-1}\right] - 2\boldsymbol{w}_n, \tag{3b}$$

where $\{\boldsymbol{u}_n\}_{n=-\infty}^{\infty}$ and $\{\boldsymbol{w}_n\}_{n=-\infty}^{\infty}$ are unknown sequences. In the scalar case ($N = M = 1$) the system of IDNLS equations (1) reduces to (2)+(3), because (2) is trivially satisfied. The *focusing case* occurs if $\boldsymbol{w}_n = -\boldsymbol{u}_n^\dagger$ for each integer n, where the dagger denotes conjugate matrix transposition; in this case $1 - c_n$ is a constant such that $1 - c_n > 0$.

The system of scalar ($N = M = 1$) IDNLS equations was first studied by Ablowitz and Ladik [1–3] by the inverse scattering transform (IST) method. The matrix equations (2)+(3) were studied in detail using the IST method by Ablowitz, Prinari, and Trubatch [4, Ch. 5] and by Tsuchida, Ujino, and Wadati [5]. The second set of authors has also derived the N-soliton and breather solutions to (2)-(3) in terms of solutions to $N \times N$ linear systems [5, Eq. (3.43)]. Breather solutions to (2)-(3) were also constructed by using the Hirota method [6].

When viewing the matrix IDNLS equation as a finite difference approximation of the matrix NLS equation, it has the same applications as the matrix NLS equation, namely electromagnetic wave propagation in nonlinear media [7,8], surface waves on sufficiently deep waters [7], and signal propagation in optical fibers [9,10]. Apart from that, the matrix IDNLS equation has applications to the dynamics of a discrete curve on an ultraspherical surface [11], the dynamics of triangulations of surfaces [12], and Hamiltonian flows [13,14].

The IST method associates (2)+(3) to the *discrete Zakharov-Shabat system*

$$\boldsymbol{v}_{n+1} = \begin{pmatrix} zI_N & \boldsymbol{u}_n \\ \boldsymbol{w}_n & z^{-1}I_N \end{pmatrix}\boldsymbol{v}_n, \tag{4}$$

where z is the (complex) spectral parameter and the potentials $\{\boldsymbol{u}_n\}_{n=-\infty}^{\infty}$ and $\{\boldsymbol{w}_n\}_{n=-\infty}^{\infty}$ satisfy the ℓ^1-condition

$$\sum_{n=-\infty}^{\infty}\{\|\boldsymbol{u}_n\| + \|\boldsymbol{w}_n\|\} < +\infty. \tag{5}$$

Here $\|\cdot\|$ denotes any matrix norm. The direct and inverse scattering of the discrete Zakharov-Shabat system (4) has been studied as early as in

1981 [15,16]. More complete accounts have been given in [5] and [4, Ch. 5]. In all of these sources it is assumed that the discrete eigenvalues of (10) are algebraically and geometrically simple.

Under the assumptions (2) and (5), the potential can be easily computed [4,5] from the unique solutions of the Marchenko equations

$$\bar{\boldsymbol{\kappa}}(n,j) + \begin{pmatrix} 0_{NN} \\ I_N \end{pmatrix} \boldsymbol{F}(n+j) + \sum_{j'=n+1}^{\infty} \boldsymbol{\kappa}(n,j')\boldsymbol{F}(j+j') = 0_{2N,N}, \tag{6a}$$

$$\boldsymbol{\kappa}(n,j) + \begin{pmatrix} I_N \\ 0_{NN} \end{pmatrix} \bar{\boldsymbol{F}}(n+j) + \sum_{j'=n+1}^{\infty} \bar{\boldsymbol{\kappa}}(n,j')\bar{\boldsymbol{F}}(j+j') = 0_{2N,N}, \tag{6b}$$

where $j \geq n+1$ and $\boldsymbol{F}$ and $\bar{\boldsymbol{F}}$ are the Marchenko kernels. In fact, we have

$$\boldsymbol{u}_n = -\boldsymbol{\kappa}^{(\mathrm{up})}(n,n+1), \qquad \boldsymbol{w}_n = -\bar{\boldsymbol{\kappa}}^{(\mathrm{dn})}(n,n+1), \tag{7}$$

where, for any $\boldsymbol{K}$ having $2N$ rows, $\boldsymbol{K}^{(\mathrm{up})}$ and $\boldsymbol{K}^{(\mathrm{dn})}$ denote the submatrices consisting of the first N rows and the last N rows, respectively.

If we allow the potentials to be time dependent in such a way that (2)+(3) are satisfied, then the time evolution of the scattering data is such that the Marchenko kernels $\boldsymbol{F}(n;\tau)$ and $\bar{\boldsymbol{F}}(n;\tau)$ satisfy the discrete evolution equations

$$i\frac{d}{d\tau}\boldsymbol{F}(n;\tau) = \boldsymbol{F}(n+2;\tau) - 2\boldsymbol{F}(n;\tau) + \boldsymbol{F}(n-2;\tau), \tag{8a}$$

$$-i\frac{d}{d\tau}\bar{\boldsymbol{F}}(n;\tau) = \bar{\boldsymbol{F}}(n+2;\tau) - 2\bar{\boldsymbol{F}}(n;\tau) + \bar{\boldsymbol{F}}(n-2;\tau). \tag{8b}$$

As can be verified by substitution, explicit solutions to (8) can be written as follows:

$$\boldsymbol{F}(n;\tau) = Ce^{i\tau(A-A^{-1})^2}A^{-(n+1)}B, \tag{9a}$$

$$\bar{\boldsymbol{F}}(n;\tau) = \bar{C}e^{-i\tau(\bar{A}-\bar{A}^{-1})^2}\bar{A}^{n-1}\bar{B}, \tag{9b}$$

where

(i) A, B, and C are complex $p\times p$, $p\times N$, and $N\times p$ matrices, respectively, and A is a matrix having only eigenvalues of modulus larger than one;

(ii) $\bar{A}$, $\bar{B}$, and $\bar{C}$ are complex $\bar{p}\times\bar{p}$, $\bar{p}\times N$, and $N\times\bar{p}$ matrices, respectively, and $\bar{A}$ is a nonsingular matrix which has only eigenvalues of modulus less than one.

The matrix functions (9) allow us to solve the time evolved Marchenko equations (6) explicitly in terms of the two matrix triplets by separation of variables and then, by using (7), to derive the explicit matrix IDNLS solutions $\boldsymbol{u}_n(\tau)$ and $\boldsymbol{w}_n(\tau)$.

The idea to solve a Marchenko system like (6) is not completely new, as representations of Marchenko kernels of the type

$$\boldsymbol{F}(x,\tau) = Ce^{-xA}e^{i\tau\varphi(A)}B,$$

where the time factor $e^{i\tau\varphi(A)}$ commutes with A, have been successfully used to find closed form solutions of integrable nonlinear evolution equations in terms of matrix exponentials and solutions of Lyapunov equations. We mention results for the KdV [17], NLS [18,19], and sine-Gordon equations [20]. We also note that similar results were obtained for pseudo-canonical systems [21] and the sine-Gordon [22] and Toda lattice equations [23] with the help of matrix or operator triplets, but without using Marchenko theory. In the KdV, NLS, and sine-Gordon cases, such explicit solutions provide a concise way to express closed form solutions, which can equivalently be expressed in terms of exponential, trigonometric, and polynomials of x and t by "unpacking" the matrix exponentials and matrix inverses appearing in these formulas. As the matrix size increases, the unpacked expressions become very long. However, such expressions can be evaluated explicitly for any matrix size either by hand or by using computer algebra such as Mathematica. One of the powerful features of our method stems from the fact that our explicit solution formulas are valid for any matrix size in the matrix exponentials. For discrete position variables (Toda lattice, matrix IDNLS) our method has similar advantages, where integer matrix powers take the place of matrix exponentials. In other available methods, exact solutions are attempted in terms of elementary functions without using matrix exponentials or integer matrix powers. Thus we cannot expect these methods to produce our solutions when the matrix size is large.

It appears [4,5] that the spectrum of the discrete matrix Zakharov-Shabat system (4) is invariant under the sign inversion $z \mapsto -z$ and that, as a result, the Marchenko kernels $\boldsymbol{F}(n;\tau)$ and $\bar{\boldsymbol{F}}(n;\tau)$ vanish if n is an even integer. Thus further constraints on the triplets (A,B,C) and $(\bar{A},\bar{B},\bar{C})$ are required to represent the Marchenko kernels in the form (9). In fact, the matrix triplets have to be decomposed as in (29) below in terms of matrix triplets $(\mathcal{A},\mathcal{B},\mathcal{C})$ and $(\bar{\mathcal{A}},\bar{\mathcal{B}},\bar{\mathcal{C}})$, where $\mathcal{A}$ and $\bar{\mathcal{A}}$ have half the matrix orders that A and $\bar{A}$ have.

As will be shown soon, our method allows us to state many results regarding the scattering theory of the discrete Zakharov-Shabat system without further ado. Most of the proofs can be found in [4, Ch. 5] in virtually

DOI: 10.1685/journal.caim.372

the same notations or in [5] in somewhat different notations. In contrast to [4,5], we introduce transmission coefficients, left reflection coefficients, and a second pair of Marchenko equations. We have given several details on the sign inversion symmetry reduction of the Marchenko equations to make the derivation of our solution formulas more transparent. We have also given new unitarity results for the scattering matrix.

Let us now discuss the contents of the various sections. In Section 2 we introduce Jost solutions and scattering coefficients along with their basic properties. We formulate the various analyticity properties by writing the Jost solutions and scattering coefficients as sums of absolutely convergent Fourier series. We also derive unitarity properties of the scattering matrix. In Section 3 we apply sign inversion symmetry to reduce the Marchenko equations (6) and discuss conjugation symmetry to get a further reduction specific to the focusing case. In Section 4 we write the matrix IDNLS solutions $\boldsymbol{u}_n(\tau)$ in terms of matrix triplets (A,B,C) and $(\bar{A},\bar{B},\bar{C})$, both without symmetries on the potential and in the focusing case. In Section 4.1 we present some illustrative examples.

Notice that overlined quantities constitute an established notation inherited from [4] which has nothing to do with complex conjugation. The complex conjugate of a complex number z is written as z^*, whereas the conjugate transpose of a matrix A is written as $A^\dagger$.

2. Jost solutions and scattering coefficients.

In this section we define the Jost solutions, the transition coefficients expressing their linear dependence, and the reflection and transmission coefficients. We essentially follow [4, Ch. 5], although, unlike the authors of [4], we emphasize continuity and analyticity properties as the natural consequence of dealing with sums of absolutely convergent Fourier series and define transmission coefficients and scattering matrices explicitly. We also derive novel unitarity properties of the scattering matrix.

Let us define the four *Jost solutions* $\phi_n(z)$, $\bar{\phi}_n(z)$, $\psi_n(z)$, and $\bar{\psi}_n(z)$ as those $2N\times N$ matrix solutions to (4) satisfying the asymptotic conditions

$$\begin{aligned}
&\phi_n(z)\sim z^n\begin{pmatrix} I_N\\ 0_{NN}\end{pmatrix}, \qquad \bar{\phi}_n(z)\sim z^{-n}\begin{pmatrix} 0_{NN}\\ I_N\end{pmatrix}, \qquad n\to-\infty,\\
&\psi_n(z)\sim z^{-n}\begin{pmatrix} 0_{NN}\\ I_N\end{pmatrix}, \quad \bar{\psi}_n(z)\sim z^{n}\begin{pmatrix} I_N\\ 0_{NN}\end{pmatrix}, \qquad n\to+\infty.
\end{aligned}$$

Since the discrete matrix Zakharov-Shabat system is a homogeneous first order difference equation, we can reduce any pair of $2N\times 2N$ matrix solutions to each other by postmultiplication by a matrix not depending on n.

We thus define the *transition coefficient matrices* $\boldsymbol{T}(z)$ and $\bar{\boldsymbol{T}}(z)$ by

$$\big(\phi_n(z)\ \bar{\phi}_n(z)\big) = \big(\bar{\psi}_n(z)\ \psi_n(z)\big)\,\boldsymbol{T}(z), \tag{10a}$$

$$\big(\bar{\psi}_n(z)\ \psi_n(z)\big) = \big(\phi_n(z)\ \bar{\phi}_n(z)\big)\,\bar{\boldsymbol{T}}(z), \tag{10b}$$

where $\boldsymbol{T}(z)$ and $\bar{\boldsymbol{T}}(z)$ are each other's inverses. Writing

$$\boldsymbol{T}(z) \stackrel{\text{def}}{=} \begin{pmatrix} \boldsymbol{a}(z) & \bar{\boldsymbol{b}}(z) \\ \boldsymbol{b}(z) & \bar{\boldsymbol{a}}(z) \end{pmatrix}, \qquad \bar{\boldsymbol{T}}(z) \stackrel{\text{def}}{=} \begin{pmatrix} \bar{\boldsymbol{c}}(z) & \boldsymbol{d}(z) \\ \bar{\boldsymbol{d}}(z) & \boldsymbol{c}(z) \end{pmatrix},$$

we obtain the $N \times N$ *transition coefficients* $\boldsymbol{a}(z)$ and $\bar{\boldsymbol{c}}(z)$, $\boldsymbol{b}(z)$ and $\bar{\boldsymbol{d}}(z)$, $\bar{\boldsymbol{a}}(z)$ and $\boldsymbol{c}(z)$, and $\bar{\boldsymbol{b}}(z)$ and $\boldsymbol{d}(z)$.

The Jost solutions can be represented as discrete Fourier transforms of the solutions of the Marchenko equations (21) below, as shown by the following

Theorem 2.1. *The Jost solutions can be represented as follows:*

$$\psi_n(z) = \sum_{j=n}^{\infty} z^{-j}\boldsymbol{K}(n,j), \quad \bar{\psi}_n(z) = \sum_{j=n}^{\infty} z^{j}\bar{\boldsymbol{K}}(n,j), \tag{11a}$$

$$\phi_n(z) = \sum_{j=-\infty}^{n} z^{j}\boldsymbol{L}(n,j), \quad \bar{\phi}_n(z) = \sum_{j=-\infty}^{n} z^{-j}\bar{\boldsymbol{L}}(n,j), \tag{11b}$$

where

$$\sum_{j=n}^{\infty}\left\{\|\boldsymbol{K}(n,j)\| + \|\bar{\boldsymbol{K}}(n,j)\|\right\} < +\infty, \quad \sum_{j=-\infty}^{n}\left\{\|\boldsymbol{L}(n,j)\| + \|\bar{\boldsymbol{L}}(n,j)\|\right\} < +\infty.$$

As a result, $z^n\psi_n(z)$ and $z^{-n}\phi_n(z)$ are continuous in $|z| \geq 1$, are analytic in $|z| > 1$, and tend to $\boldsymbol{K}(n,n)$ and $\boldsymbol{L}(n,n)$ as $|z| \to +\infty$. Similarly, $z^{-n}\bar{\psi}_n(z)$ and $z^n\bar{\phi}_n(z)$ are continuous in $|z| \leq 1$ and analytic in $|z| < 1$.

In the sequel we can restrict our discussion to quantities derived from the Jost solutions $\psi_n(z)$ and $\bar{\psi}_n(z)$. With some effort one may verify that

$$\boldsymbol{K}(n,n) = \begin{pmatrix} 0_{NN} \\ \Omega_n^{-1} I_N \end{pmatrix}, \qquad \bar{\boldsymbol{K}}(n,n) = \begin{pmatrix} \Omega_n^{-1} I_N \\ 0_{NN} \end{pmatrix}, \tag{12}$$

where

$$\Omega_n = \prod_{j=-\infty}^{n} (1 - c_j) \tag{13}$$

and the absolute convergence of the infinite product in (13) can be derived from the ℓ^2-condition

$$\sum_{n=-\infty}^{\infty} \left\{ \|\boldsymbol{u}_n\|^2 + \|\boldsymbol{w}_n\|^2 \right\} < +\infty \tag{14}$$

on the potentials $\{\boldsymbol{u}_n\}_{n=-\infty}^{\infty}$ and $\{\boldsymbol{w}_n\}_{n=-\infty}^{\infty}$. In the focusing case the absolute convergence of the infinite products in (13) is equivalent to (14). In the sequel we always assume our potentials to satisfy both of the sufficient conditions (5) and (14).

Using the analyticity properties of the Jost solutions we can write (10) as the Riemann-Hilbert problems

$$\begin{aligned} \left(\bar{\psi}_n(z)\ \bar{\phi}_n(z)\right) &= \left(\phi_n(z)\ \psi_n(z)\right) J\boldsymbol{S}(z)J, \qquad |z|=1, \\ \left(\phi_n(z)\ \psi_n(z)\right) &= \left(\bar{\psi}_n(z)\ \bar{\phi}_n(z)\right) J\bar{\boldsymbol{S}}(z)J, \qquad |z|=1, \end{aligned}$$

where $J = \left(\begin{smallmatrix} I_N & 0_{NN} \\ 0_{NN} & -I_N \end{smallmatrix}\right)$ and

$$\boldsymbol{S}(z) = \begin{pmatrix} \boldsymbol{t}_r(z) & \boldsymbol{\ell}(z) \\ \boldsymbol{\rho}(z) & \boldsymbol{t}_l(z) \end{pmatrix}, \qquad \bar{\boldsymbol{S}}(z) = \boldsymbol{S}(z)^{-1} = \begin{pmatrix} \bar{\boldsymbol{t}}_l(z) & \bar{\boldsymbol{\rho}}(z) \\ \bar{\boldsymbol{\ell}}(z) & \bar{\boldsymbol{t}}_r(z) \end{pmatrix}, \tag{15}$$

are called *scattering matrices*. The $N \times N$ matrices $\boldsymbol{t}_r(z)$, $\boldsymbol{t}_l(z)$, $\bar{\boldsymbol{t}}_l(z)$, and $\bar{\boldsymbol{t}}_r(z)$ are called *transmission coefficients*, while $\boldsymbol{\rho}(z)$, $\boldsymbol{\ell}(z)$, $\bar{\boldsymbol{\rho}}(z)$, and $\bar{\boldsymbol{\ell}}(z)$ are called *reflection coefficients*.

A complex number z of modulus 1 is called a *spectral singularity* if at least one of the "diagonal" transition coefficients $\boldsymbol{a}(z)$, $\bar{\boldsymbol{a}}(z)$, $\boldsymbol{c}(z)$, and $\bar{\boldsymbol{c}}(z)$ is nonsingular.

Theorem 2.2. *Suppose there are no spectral singularities. Then the following is true:*

- (i) *The reflection coefficients are continuous in $|z| = 1$ and are in fact sums of absolutely convergent Fourier series.*
- (ii) *The transmission coefficients $\boldsymbol{t}_r(z)$ and $\boldsymbol{t}_l(z)$ are continuous in $|z| \geq 1$, are meromorphic in $|z| > 1$ with at most finitely many poles, and tend to I_N and $\prod_{k=-\infty}^{\infty}(1-c_k)I_N$, respectively, as $|z| \to +\infty$. They are sums of absolutely convergent Fourier series.*
- (iii) *The transmission coefficients $\bar{\boldsymbol{t}}_r(z)$ and $\bar{\boldsymbol{t}}_l(z)$ are continuous in $|z| \leq 1$, are meromorphic in $|z| < 1$ with at most finitely many, nonzero, poles, and tend to I_N and $\prod_{k=-\infty}^{\infty}(1-c_k)I_N$, respectively, as $z \to 0$. They are sums of absolutely convergent Fourier series.*
- (iv) *The transmission coefficients $\boldsymbol{t}_l(z)$ and $\boldsymbol{t}_r(z)$ have the same poles and pole orders for $|z| > 1$, while $\bar{\boldsymbol{t}}_l(z)$ and $\bar{\boldsymbol{t}}_r(z)$ have the same poles and pole orders for $0 < |z| < 1$.*

Letting $\boldsymbol{v}_n(z)$ satisfy the integrable discrete Zakharov-Shabat system (4) and $\breve{\boldsymbol{v}}_n(z)$ its dual system

(16) $$\breve{\boldsymbol{v}}_{n+1}(z) = \breve{\boldsymbol{v}}_n(z) \begin{pmatrix} z^{-1}I_N & \boldsymbol{u}_n \\ \boldsymbol{w}_n & zI_N \end{pmatrix},$$

we easily derive that

$$\begin{aligned}\breve{\boldsymbol{v}}_{n+1}(z)J\boldsymbol{v}_{n+1}(z)J &= \breve{\boldsymbol{v}}_n(z) \begin{pmatrix} z^{-1}I_N & \boldsymbol{u}_n \\ \boldsymbol{w}_n & zI_N \end{pmatrix} \begin{pmatrix} zI_N & -\boldsymbol{u}_n \\ -\boldsymbol{w}_n & z^{-1}I_N \end{pmatrix} J\boldsymbol{v}_n(z)J \\ &= \breve{\boldsymbol{v}}_n(z) \begin{pmatrix} I_N - \boldsymbol{u}_n\boldsymbol{w}_n & 0_{NN} \\ 0_{NN} & I_N - \boldsymbol{w}_n\boldsymbol{u}_n \end{pmatrix} J\boldsymbol{v}_n(z)J \\ &= (1-c_n)\breve{\boldsymbol{v}}_n(z)J\boldsymbol{v}_n(z)J,\end{aligned}$$

where (2) has been used in an essential way. Putting

$$\omega_n^+ = \prod_{k=n}^{\infty}(1-c_k), \qquad \omega_n^- = \left[\prod_{k=-\infty}^{n-1}(1-c_k)\right]^{-1},$$

we see that

$$\omega_n^{\pm}\breve{\boldsymbol{v}}_n(z)J\boldsymbol{v}_n(z)J$$

does not depend on $n \in \mathbb{Z}$. On the other hand, in the focusing case any solution $\breve{\boldsymbol{v}}_n(z)$ leads to a solution $J\breve{\boldsymbol{v}}_n(1/z^*)^\dagger J$ of (4), whereas any solution $\boldsymbol{v}_n(z)$ of (4) leads to a solution $J\boldsymbol{v}_n(1/z^*)^\dagger J$ of (16). Putting

$$\omega \overset{\text{def}}{=} \prod_{k=-\infty}^{\infty}(1-c_k),$$

it can be shown that, in the focusing case where ω is an infinite product of positive numbers and hence positive, the matrix $\omega^{-1/2}\boldsymbol{T}(z)$ is unitary and the matrix $\boldsymbol{\omega}^{1/4}\boldsymbol{S}(z)\boldsymbol{\omega}^{-1/4}$, with $\boldsymbol{\omega} = \operatorname{diag}(\omega^{1/2}I_N, \omega^{-1/2}I_N)$, is J-unitary. In other words,

Theorem 2.3. *Suppose there are no spectral singularities. Then in the focusing case we have the following conjugation relations for* $|z| = 1$*:*

(17a) $$\boldsymbol{T}(z)^\dagger = \omega\boldsymbol{T}(z)^{-1},$$

(17b) $$\boldsymbol{S}(z)^\dagger J\boldsymbol{\omega}^{1/2}\boldsymbol{S}(z) = J\boldsymbol{\omega}^{-1/2},$$

Moreover,

(17c) $$\bar{\boldsymbol{S}}(z) = \boldsymbol{S}(z)^{-1} = \boldsymbol{\omega}^{1/2}J\boldsymbol{S}(z)^\dagger J\boldsymbol{\omega}^{1/2}, \qquad |z| = 1.$$

Proof. In the focusing case the square matrix of order $2N$

$$\omega_n^+ \boldsymbol{v}_n(1/z^*)^\dagger \boldsymbol{v}_n(z) = J\left(\omega_n^+ \breve{\boldsymbol{v}}_n(z) J \boldsymbol{v}_n(z)\right) J$$

does not depend on n. Taking $\boldsymbol{v}_n(z) = \left(\phi_n(z)\ \bar{\phi}_n(z)\right)$ and using the $n \to \pm\infty$ asymptotic relations we get

$$\boldsymbol{T}(z)^\dagger \boldsymbol{T}(z) = \omega I_{2N}, \qquad |z| = 1,$$

which implies (17a) as well as $\omega \boldsymbol{d}(z) = \boldsymbol{b}(z)^\dagger$.

Using the relationships

$$\begin{pmatrix} I_N & \bar{\boldsymbol{b}}(z) \\ 0_{NN} & \bar{\boldsymbol{a}}(z) \end{pmatrix} = \begin{pmatrix} \boldsymbol{a}(z) & 0_{NN} \\ \boldsymbol{b}(z) & I_N \end{pmatrix} J\boldsymbol{S}(z)J,$$
$$\begin{pmatrix} \bar{\boldsymbol{c}}(z) & 0_{NN} \\ \bar{\boldsymbol{d}}(z) & I_N \end{pmatrix} = \begin{pmatrix} I_N & \boldsymbol{d}(z) \\ 0_{NN} & \boldsymbol{c}(z) \end{pmatrix} J\bar{\boldsymbol{S}}(z)J$$

between transition coefficients and scattering matrices, we obtain after tedious calculations

$$J\boldsymbol{S}(z)^\dagger J \boldsymbol{\omega}^{1/2} \boldsymbol{S}(z) \boldsymbol{\omega}^{-1/2} = \boldsymbol{\omega}^{-1},$$

which implies (17b) and (17c). □

Using (15) we obtain the conjugation symmetry relations

(18a) $$\bar{\boldsymbol{\rho}}(z) = -\boldsymbol{\rho}(z)^\dagger, \qquad \bar{\boldsymbol{\ell}}(z) = -\boldsymbol{\ell}(z)^\dagger,$$

(18b) $$\bar{\boldsymbol{t}}_l(z) = \omega\, \boldsymbol{t}_r(z)^\dagger, \qquad \bar{\boldsymbol{t}}_r(z) = \frac{1}{\omega} \boldsymbol{t}_l(z)^\dagger.$$

3. Marchenko equations.

In this section we write down the Marchenko equations in terms of the scattering data. The Marchenko kernels are the sums of two contributions, one derived from the Fourier coefficients of a reflection coefficient and the other derived from the poles of a transmission coefficient and so-called norming constants. We again follow [4, Ch. 5], but we move beyond the rather restrictive assumption that the poles of the transmission coefficients are simple. In the second half of this section we shall exploit the invariance of the discrete Zakharov-Shabat spectrum under the sign inversion $\lambda \mapsto -\lambda$ to reduce the number of quantities to be computed by a factor of two.

Assuming there are no spectral singularities, we write the reflection coefficients as the absolutely convergent Fourier series

$$\text{(19a)} \qquad \boldsymbol{\rho}(z) = \sum_{s=-\infty}^{\infty} z^s \hat{\boldsymbol{\rho}}(s), \qquad \bar{\boldsymbol{\rho}}(z) = \sum_{s=-\infty}^{\infty} z^{-s} \hat{\bar{\boldsymbol{\rho}}}(s),$$

$$\text{(19b)} \qquad \bar{\boldsymbol{\ell}}(z) = \sum_{s=-\infty}^{\infty} z^s \hat{\bar{\boldsymbol{\ell}}}(s), \qquad \boldsymbol{\ell}(z) = \sum_{s=-\infty}^{\infty} z^{-s} \hat{\boldsymbol{\ell}}(s).$$

Under the condition that the poles of the transmission coefficients are all simple, we define the Marchenko kernels

$$\text{(20)} \qquad \boldsymbol{F}(j) = \hat{\boldsymbol{\rho}}(j) + \sum_k \zeta_k^{-(j+1)} \boldsymbol{C}_k, \quad \bar{\boldsymbol{F}}(j) = \hat{\bar{\boldsymbol{\rho}}}(j) - \sum_k \bar{\zeta}_k^{j-1} \bar{\boldsymbol{C}}_k.$$

Here ζ_k, with $|\zeta_k| > 1$, are the finitely many simple poles of $\boldsymbol{t}_r(z)$ and $\boldsymbol{t}_l(z)$, whereas $\bar{\zeta}_k$, with $0 < |\bar{\zeta}_k| < 1$, are the finitely many simple poles of $\bar{\boldsymbol{t}}_l(z)$ and $\bar{\boldsymbol{t}}_r(z)$. The quantities $\boldsymbol{C}_k$ and $\bar{\boldsymbol{C}}_k$ are called the *norming constants*. Using the Kronecker delta δ_{nj}, the Marchenko equations are then given by

$$\text{(21a)} \qquad \bar{\boldsymbol{K}}(n,j) = \begin{pmatrix} I_N \\ 0_{NN} \end{pmatrix} \delta_{nj} - \sum_{j'=n}^{\infty} \boldsymbol{K}(n,j') \boldsymbol{F}(j'+j),$$

$$\text{(21b)} \qquad \boldsymbol{K}(n,j) = \begin{pmatrix} 0_{NN} \\ I_N \end{pmatrix} \delta_{nj} - \sum_{j'=n}^{\infty} \bar{\boldsymbol{K}}(n,j') \bar{\boldsymbol{F}}(j'+j),$$

where $j \geq n$. The potentials can then be expressed in terms of the solutions to (21) as follows:

$$\text{(22a)} \qquad \boldsymbol{u}_n = -\boldsymbol{K}^{(\text{up})}(n,n+1) \boldsymbol{K}^{(\text{dn})}(n,n)^{-1},$$

$$\text{(22b)} \qquad \boldsymbol{w}_n = -\bar{\boldsymbol{K}}^{(\text{dn})}(n,n+1) \bar{\boldsymbol{K}}^{(\text{up})}(n,n)^{-1}.$$

In the focusing case the two systems of Marchenko equations are easily seen to be uniquely solvable [4].

If the transmission coefficients have multiple poles, the Marchenko equations (21) and the expressions (22) for the potentials in terms of their solutions do not change. The bound state terms in (20) become much more complicated, because each pole term gets replaced by a number of terms equal to the corresponding pole order [cf. (27) below].

It is easily verified that, for each solution $\boldsymbol{v}_n(z)$ of (4), also $\tilde{\boldsymbol{v}}_n(z) = (-1)^n J \boldsymbol{v}_n(-z)$ is a solution of (4). As a result, we get for the Jost functions

DOI: 10.1685/journal.caim.372

and transition coefficient matrices the sign inversion symmetries

$$\begin{aligned}\left(\bar{\psi}_n(-z)\ \psi_n(-z)\right) &= (-1)^n J\left(\bar{\psi}_n(z)\ \psi_n(z)\right) J,\\ \left(\phi_n(-z)\ \bar{\phi}_n(-z)\right) &= (-1)^n J\left(\phi_n(z)\ \bar{\phi}_n(z)\right) J,\\ \boldsymbol{T}(-z) = J\boldsymbol{T}(z)J, &\qquad \bar{\boldsymbol{T}}(-z) = J\bar{\boldsymbol{T}}(z)J,\end{aligned}$$

so that the transmission coefficients are even functions of z (and hence the discrete Zakharov-Shabat spectrum is invariant under sign inversion) and the reflection coefficients are odd functions of z. Therefore the functions $\hat{\boldsymbol{\rho}}(s)$, $\hat{\bar{\boldsymbol{\rho}}}(s)$, $\hat{\boldsymbol{\ell}}(s)$, and $\hat{\bar{\boldsymbol{\ell}}}(s)$ appearing in (19) vanish if s is even. Using (11) together with the sign inversion symmetry of the Jost functions, we get

$$\begin{aligned}\left(\bar{\boldsymbol{K}}(n,j)\ \boldsymbol{K}(n,j)\right) &= (-1)^{j-n} J\left(\bar{\boldsymbol{K}}(n,j)\ \boldsymbol{K}(n,j)\right) J,\\ \left(\boldsymbol{L}(n,j)\ \bar{\boldsymbol{L}}(n,j)\right) &= (-1)^{n-j} J\left(\boldsymbol{L}(n,j)\ \bar{\boldsymbol{L}}(n,j)\right) J.\end{aligned}$$

Therefore, $\boldsymbol{K}^{(\mathrm{up})}(n,j)$, $\bar{\boldsymbol{K}}^{(\mathrm{dn})}(n,j)$, $\boldsymbol{L}^{(\mathrm{dn})}(n,j)$, and $\bar{\boldsymbol{L}}^{(\mathrm{up})}(n,j)$ vanish if $j-n$ is even, while $\boldsymbol{K}^{(\mathrm{dn})}(n,j)$, $\bar{\boldsymbol{K}}^{(\mathrm{up})}(n,j)$, $\boldsymbol{L}^{(\mathrm{up})}(n,j)$, and $\bar{\boldsymbol{L}}^{(\mathrm{dn})}(n,j)$ vanish if $j-n$ is odd. From these symmetry properties we see that the Marchenko kernels $\boldsymbol{F}(s)$ and $\bar{\boldsymbol{F}}(s)$ vanish if s is even.

Breaking up the Marchenko equations (21) for quantities like $\boldsymbol{K}(n,j)$ into separate equations for quantities like $\boldsymbol{K}^{(\mathrm{up})}(n,j)$ and $\boldsymbol{K}^{(\mathrm{dn})}(n,j)$ and executing one iteration of each resulting coupled pair of equations in order to get them decoupled, we arrive at so-called uncoupled Marchenko equations whose Marchenko kernels have the form

$$\mathbb{K}(j,j') \stackrel{\text{def}}{=} \sum_{j''=n}^{\infty} \boldsymbol{F}_1(j'+j'')\boldsymbol{F}_2(j''+j).$$

These kernels $\mathbb{K}(j,j')$ vanish if one of j,j' is even and the other is odd. As a result, the uncoupled Marchenko equations can be decoupled further. Before doing so we first modify the Marchenko kernels $\mathbb{K}(j,j')$.

Using (13) to put

$$\boldsymbol{K}(n,j) = \Omega_n^{-1}\boldsymbol{\kappa}(n,j), \qquad \bar{\boldsymbol{K}}(n,j) = \Omega_n^{-1}\bar{\boldsymbol{\kappa}}(n,j), \tag{23}$$

we write (21) as follows:

$$\bar{\boldsymbol{\kappa}}(n,j) = -\begin{pmatrix}0_{NN}\\ I_N\end{pmatrix}\boldsymbol{F}(n+j) - \sum_{j'=n+1}^{\infty}\boldsymbol{\kappa}(n,j')\boldsymbol{F}(j'+j), \tag{24a}$$

$$\boldsymbol{\kappa}(n,j) = -\begin{pmatrix}I_N\\ 0_{NN}\end{pmatrix}\bar{\boldsymbol{F}}(n+j) - \sum_{j'=n+1}^{\infty}\bar{\boldsymbol{\kappa}}(n,j')\bar{\boldsymbol{F}}(j'+j), \tag{24b}$$

where $j \geq n+1$. Then the potentials are given by

$$\boldsymbol{u}_n = -\boldsymbol{\kappa}^{(\mathrm{up})}(n,n+1), \qquad \boldsymbol{w}_n = -\bar{\boldsymbol{\kappa}}^{(\mathrm{dn})}(n,n+1). \tag{25}$$

Let us decouple (24) further as follows:

(26a)

$$\bar{\boldsymbol{\kappa}}^{(\mathrm{up})}(n,n+2\sigma) = \sum_{\sigma''=0}^{\infty} \bar{\boldsymbol{F}}(2[n+\sigma'']+1)\boldsymbol{F}(2[n+\sigma''+\sigma]+1)$$
$$+ \sum_{\sigma'=1}^{\infty} \bar{\boldsymbol{\kappa}}^{(\mathrm{up})}(n,n+2\sigma') \sum_{\sigma''=0}^{\infty} \bar{\boldsymbol{F}}(2[n+\sigma'+\sigma'']+1)\boldsymbol{F}(2[n+\sigma''+\sigma]+1),$$

(26b)

$$\boldsymbol{\kappa}^{(\mathrm{up})}(n,n+2\sigma+1) = -\bar{\boldsymbol{F}}(2[n+\sigma]+1)$$
$$+ \sum_{\sigma'=0}^{\infty} \boldsymbol{\kappa}^{(\mathrm{up})}(n,n+2\sigma'+1) \sum_{\sigma''=1}^{\infty} \boldsymbol{F}(2[n+\sigma'+\sigma'']+1)\bar{\boldsymbol{F}}(2[n+\sigma''+\sigma]+1),$$

(26c)

$$\bar{\boldsymbol{\kappa}}^{(\mathrm{dn})}(n,n+2\sigma+1) = -\boldsymbol{F}(2[n+\sigma]+1)$$
$$+ \sum_{\sigma'=0}^{\infty} \bar{\boldsymbol{\kappa}}^{(\mathrm{dn})}(n,n+2\sigma'+1) \sum_{\sigma''=1}^{\infty} \bar{\boldsymbol{F}}(2[n+\sigma'+\sigma'']+1)\boldsymbol{F}(2[n+\sigma''+\sigma]+1),$$

(26d)

$$\boldsymbol{\kappa}^{(\mathrm{dn})}(n,n+2\sigma) = \sum_{\sigma''=0}^{\infty} \boldsymbol{F}(2[n+\sigma'']+1)\bar{\boldsymbol{F}}(2[n+\sigma''+\sigma]+1)$$
$$+ \sum_{\sigma'=1}^{\infty} \boldsymbol{\kappa}^{(\mathrm{dn})}(n,n+2\sigma') \sum_{\sigma''=0}^{\infty} \boldsymbol{F}(2[n+\sigma'+\sigma'']+1)\bar{\boldsymbol{F}}(2[n+\sigma''+\sigma]+1).$$

Equations (26a) and (26d) are valid for $\sigma \geq 1$, whereas (26b) and (26c) are valid for $\sigma \geq 0$. This distinction in the ranges of the summation index σ is to bear in mind when deriving exact solutions to (2)+(3).

4. IDNLS solutions in terms of matrix triplets.

In this section we write the solutions of the Marchenko equations in terms of suitable matrix triplets if the reflection coefficients vanish. Once the time evolution of the scattering data has been taken into account as well as the maximal reduction of the Marchenko equations, we quickly arrive at explicit IDNLS solutions.

DOI: 10.1685/journal.caim.372

Using two matrix triplets, we generalize the expressions (20) for the Marchenko kernels as follows:

$$\boldsymbol{F}(j) = \hat{\boldsymbol{\rho}}(j) + CA^{-(j+1)}B, \qquad \bar{\boldsymbol{F}}(j) = \hat{\bar{\boldsymbol{\rho}}}(j) + \bar{C}\bar{A}^{j-1}\bar{B}, \tag{27}$$

where the triplets (A, B, C) and $(\bar{A}, \bar{B}, \bar{C})$ have the following properties:

(i) A, B, and C are $p \times p$, $p \times N$, and $N \times p$ matrices, respectively, and A is a matrix having only eigenvalues of modulus larger than one,

(ii) $\bar{A}$, $\bar{B}$, and $\bar{C}$ are $\bar{p} \times \bar{p}$, $\bar{p} \times N$, and $N \times \bar{p}$ matrices, respectively, and $\bar{A}$ is a nonsingular matrix which has only eigenvalues of modulus less than one.

If the poles of the transmission coefficients are all simple, we can recover the original expressions (20) by taking

$$A = \operatorname{diag}(\zeta_1, \cdots, \zeta_p), \quad B = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, \qquad C = \begin{pmatrix} \boldsymbol{C}_1 \cdots \boldsymbol{C}_p \end{pmatrix}, \tag{28a}$$

$$\bar{A} = \operatorname{diag}(\bar{\zeta}_1, \cdots, \bar{\zeta}_{\bar{p}}), \quad \bar{B} = \begin{pmatrix} \bar{\boldsymbol{C}}_1 \\ \vdots \\ \bar{\boldsymbol{C}}_{\bar{p}} \end{pmatrix}, \quad \bar{C} = \begin{pmatrix} -1 \cdots -1 \end{pmatrix}, \tag{28b}$$

where the norming constants are encoded by C and $\bar{B}$. Here p is the number of poles of $\boldsymbol{t}_r(z)$ or $\boldsymbol{t}_l(z)$ and $\bar{p}$ is the number of poles of $\bar{\boldsymbol{t}}_l(z)$ or $\bar{\boldsymbol{t}}_r(z)$.

Because the Marchenko kernels $\boldsymbol{F}(s)$ and $\bar{\boldsymbol{F}}(s)$ vanish if s is even, we need to restrict the class of matrix triplets by writing

$$A = \begin{pmatrix} \mathcal{A} & 0 \\ 0 & -\mathcal{A} \end{pmatrix}, \quad B = \begin{pmatrix} \mathcal{B} \\ \mathcal{B} \end{pmatrix}, \quad C = \begin{pmatrix} \mathcal{C} \; \mathcal{C} \end{pmatrix}, \tag{29a}$$

$$\bar{A} = \begin{pmatrix} \bar{\mathcal{A}} & 0 \\ 0 & -\bar{\mathcal{A}} \end{pmatrix}, \quad \bar{B} = \begin{pmatrix} \bar{\mathcal{B}} \\ \bar{\mathcal{B}} \end{pmatrix}, \quad \bar{C} = \begin{pmatrix} \bar{\mathcal{C}} \; \bar{\mathcal{C}} \end{pmatrix}. \tag{29b}$$

If the transmission coefficients only have simple poles, then the triplets (28) can be made to correspond to (29) by properly ordering the poles, because norming constants corresponding to $\pm$ pairs of poles coincide. Consequently,

$$\boldsymbol{F}(j) = \hat{\boldsymbol{\rho}}(j) + [1 + (-1)^{j+1}]\mathcal{C}\mathcal{A}^{-(j+1)}\mathcal{B}, \tag{30a}$$

$$\bar{\boldsymbol{F}}(j) = \hat{\bar{\boldsymbol{\rho}}}(j) + [1 + (-1)^{j+1}]\bar{\mathcal{C}}\bar{\mathcal{A}}^{j-1}\bar{\mathcal{B}}. \tag{30b}$$

In the focusing case, we have the following conjugation symmetry relations for the Marchenko kernels:

$$\bar{\boldsymbol{F}}(j) = -\boldsymbol{F}(j)^\dagger. \tag{31}$$

Equation (31) leads to uniquely solvable Marchenko equations and focusing potentials. Relating the matrix triplets to each other in the following way:

$$\bar{\mathcal{A}} = \mathcal{A}^{\dagger^{-1}}, \qquad \bar{\mathcal{B}} = \mathcal{A}^{\dagger^{-1}}\mathcal{C}^\dagger, \qquad \bar{\mathcal{C}} = -\mathcal{B}^\dagger\mathcal{A}^{\dagger^{-1}}, \tag{32}$$

we get Marchenko kernels of the form (27) that satisfy the conjugation symmetry relations (31).

Returning to the general case, we now put

$$\mathcal{Q} = \sum_{\sigma=0}^{\infty} \bar{\mathcal{A}}^{2\sigma}\bar{\mathcal{B}}\mathcal{C}\mathcal{A}^{-2\sigma}, \qquad \mathcal{N} = \sum_{\sigma=0}^{\infty} \mathcal{A}^{-2\sigma}\mathcal{B}\bar{\mathcal{C}}\bar{\mathcal{A}}^{2\sigma}. \tag{33}$$

Because the spectral radii of $\mathcal{A}^{-1}$ and $\bar{\mathcal{A}}$ are strictly less than one, the series in (33) are absolutely convergent. It is immediate that $\mathcal{Q}$ and $\mathcal{N}$ are the unique solutions of the matrix equations (cf. [24, Thm. 18.1], using that $\mathcal{A}^2$ and $\bar{\mathcal{A}}^2$ do not have eigenvalues in common)

$$\mathcal{Q} - \bar{\mathcal{A}}^2\mathcal{Q}\mathcal{A}^{-2} = \bar{\mathcal{B}}\mathcal{C}, \qquad \mathcal{N} - \mathcal{A}^{-2}\mathcal{N}\bar{\mathcal{A}}^2 = \mathcal{B}\bar{\mathcal{C}}.$$

Suppose the Marchenko kernels are given by (30), where the reflection coefficients vanish. In other words, assume the Marchenko kernels to be given in terms of suitable matrix triplets. Then each of the four uncoupled Marchenko equations (26) can be solved by separation of variables, using well-known techniques detailed in [18,19]. The results will be listed in Proposition 4.1 and Theorems 4.1 and 4.2 below.

Proposition 4.1. *Suppose the Marchenko kernels are given by* (30)*, where the reflection coefficients vanish. Then the Marchenko equations* (26) *have the solutions*

$$\bar{\boldsymbol{\kappa}}^{(up)}(n, n+2\sigma) = 4\bar{\mathcal{C}}[I - 4\bar{\mathcal{A}}^{2n}\mathcal{Q}\mathcal{A}^{-2(n+2)}\mathcal{N}\bar{\mathcal{A}}^2]^{-1}\bar{\mathcal{A}}^{2n}\mathcal{Q}\mathcal{A}^{-2(n+\sigma+1)}\mathcal{B},$$
$$\boldsymbol{\kappa}^{(up)}(n, n+2\sigma+1) = -2\bar{\mathcal{C}}\bar{\mathcal{A}}^{-2}[I - 4\bar{\mathcal{A}}^{2(n+1)}\mathcal{Q}\mathcal{A}^{-2(n+2)}\mathcal{N}]^{-1}\bar{\mathcal{A}}^{2(n+\sigma+1)}\bar{\mathcal{B}},$$
$$\bar{\boldsymbol{\kappa}}^{(dn)}(n, n+2\sigma+1) = -2\mathcal{C}\mathcal{A}^2[I - 4\mathcal{A}^{-2(n+2)}\mathcal{N}\bar{\mathcal{A}}^{2(n+1)}\mathcal{Q}]^{-1}\mathcal{A}^{-2(n+\sigma+2)}\mathcal{B},$$
$$\boldsymbol{\kappa}^{(dn)}(n, n+2\sigma) = 4\mathcal{C}[I - 4\mathcal{A}^{-2(n+1)}\mathcal{N}\bar{\mathcal{A}}^{2(n+1)}\mathcal{Q}\mathcal{A}^{-2}]^{-1}\mathcal{A}^{-2(n+1)}\mathcal{N}\bar{\mathcal{A}}^{2(n+\sigma)}\bar{\mathcal{B}}.$$

provided the matrix inverses appearing in these expressions exist. In this case the potentials are given by

$$\boldsymbol{u}_n = 2\bar{\mathcal{C}}[\bar{\mathcal{A}}^{-2n} - 4\mathcal{Q}\mathcal{A}^{-2(n+2)}\mathcal{N}\bar{\mathcal{A}}^2]^{-1}\bar{\mathcal{B}},$$
$$\boldsymbol{w}_n = 2\mathcal{C}[\mathcal{A}^{2(n+1)} - 4\mathcal{N}\bar{\mathcal{A}}^{2(n+1)}\mathcal{Q}\mathcal{A}^{-2}]^{-1}\mathcal{B}.$$

DOI: 10.1685/journal.caim.372

Let us now take into account the time dependence of the scattering data. Then for odd j the Marchenko kernels (30) are to be modified as follows:

$$\text{(34a)}\quad \boldsymbol{F}(j;\tau) = \frac{1}{2\pi i}\oint dz\, \frac{\boldsymbol{\rho}(z)}{z^{j+1}} e^{i\tau(z-z^{-1})^2} + 2\mathcal{C}\mathcal{A}^{-(j+1)} e^{i\tau(\mathcal{A}-\mathcal{A}^{-1})^2}\mathcal{B},$$

$$\text{(34b)}\quad \bar{\boldsymbol{F}}(j;\tau) = \frac{1}{2\pi i}\oint dz\, z^{j-1}\bar{\boldsymbol{\rho}}(z) e^{-i\tau(z-z^{-1})^2} + 2\bar{\mathcal{C}} e^{-i\tau(\bar{\mathcal{A}}-\bar{\mathcal{A}}^{-1})^2}\bar{\mathcal{A}}^{j-1}\bar{\mathcal{B}},$$

where the contour integration is performed over the unit circle.

We now easily arrive at the following main theorem.

Theorem 4.1. *Suppose the Marchenko kernels are given by (34), where the reflection coefficients vanish. Then the integrable discrete nonlinear Schrödinger solutions are given by*

(35a)
$$\boldsymbol{u}_n(\tau) = 2\bar{\mathcal{C}}[\bar{\mathcal{A}}^{-2n} e^{i\tau(\bar{\mathcal{A}}-\bar{\mathcal{A}}^{-1})^2} - 4\mathcal{Q}\mathcal{A}^{-2(n+2)} e^{i\tau(\mathcal{A}-\mathcal{A}^{-1})^2}\mathcal{N}\bar{\mathcal{A}}^2]^{-1}\bar{\mathcal{B}},$$
(35b)
$$\boldsymbol{w}_n(\tau) = 2\mathcal{C}[\mathcal{A}^{2(n+1)} e^{-i\tau(\mathcal{A}-\mathcal{A}^{-1})^2} - 4\mathcal{N} e^{-i\tau(\bar{\mathcal{A}}-\bar{\mathcal{A}}^{-1})^2}\bar{\mathcal{A}}^{2(n+1)}\mathcal{Q}\mathcal{A}^{-2}]^{-1}\mathcal{B}.$$

Proof. It is sufficient to prove Theorem 4.1 for $\tau = 0$ and then to make the following changes in the final result:

$$\mathcal{B} \mapsto e^{i\tau(\mathcal{A}-\mathcal{A}^{-1})^2}\mathcal{B},\ \bar{\mathcal{C}} \mapsto \bar{\mathcal{C}} e^{-i\tau(\bar{\mathcal{A}}-\bar{\mathcal{A}}^{-1})^2},\ \mathcal{N} \mapsto e^{i\tau(\mathcal{A}-\mathcal{A}^{-1})^2}\mathcal{N} e^{-i\tau(\bar{\mathcal{A}}-\bar{\mathcal{A}}^{-1})^2},$$

whereas $\mathcal{A}$, $\mathcal{C}$, $\bar{\mathcal{B}}$, and $\mathcal{Q}$ remain unchanged. □

Let us now return to the focusing case, where we recall (32). Define the nonnegative selfadjoint matrices

$$\text{(36)}\qquad \boldsymbol{Q} = \sum_{\sigma=0}^{\infty} \mathcal{A}^{\dagger^{-2\sigma}}\mathcal{C}^\dagger\mathcal{C}\mathcal{A}^{-2\sigma}, \qquad \boldsymbol{N} = \sum_{\sigma=0}^{\infty} \mathcal{A}^{-2\sigma}\mathcal{B}\mathcal{B}^\dagger\mathcal{A}^{\dagger^{-2\sigma}},$$

so that

$$\mathcal{Q} = \mathcal{A}^{\dagger^{-1}}\boldsymbol{Q}, \qquad \mathcal{N} = -\boldsymbol{N}\mathcal{A}^{\dagger^{-1}}.$$

Then $\boldsymbol{Q}$ and $\boldsymbol{N}$ are the unique solutions of the Stein equations [24]

$$\text{(37)}\qquad \boldsymbol{Q} - \mathcal{A}^{\dagger^{-2}}\boldsymbol{Q}\mathcal{A}^{-2} = \mathcal{C}^\dagger\mathcal{C}, \qquad \boldsymbol{N} - \mathcal{A}^{-2}\boldsymbol{N}\mathcal{A}^{\dagger^{-2}} = \mathcal{B}\mathcal{B}^\dagger.$$

Using (31), (32), and (36) we now specialize Theorem 4.1 to the focusing case as follows.

Theorem 4.2 (Focusing case). *Suppose the Marchenko kernel is given by*
$$\boldsymbol{F}(2s+1)=-\bar{\boldsymbol{F}}(2s+1)^{\dagger}=2\mathcal{C}\mathcal{A}^{-2(s+1)}e^{i\tau(\mathcal{A}-\mathcal{A}^{-1})^2}\mathcal{B},$$
Then the potential is given by

(38)
$$\boldsymbol{u}_n(\tau)=-2\mathcal{B}^{\dagger}[\mathcal{A}^{\dagger 2(n+1)}e^{i\tau(\mathcal{A}^{\dagger}-\mathcal{A}^{\dagger -1})^2}+4\boldsymbol{Q}\mathcal{A}^{-2(n+2)}e^{i\tau(\mathcal{A}-\mathcal{A}^{-1})^2}\boldsymbol{N}\mathcal{A}^{\dagger -2}]^{-1}\mathcal{C}^{\dagger},$$

where the matrix inverses exist for every $n\in\mathbb{Z}$. Moreover,

(39)
$$\boldsymbol{w}_n(\tau)=-\boldsymbol{u}_n(\tau)^{\dagger},\qquad n\in\mathbb{Z}.$$

Proof. Let us define the additional triplet $(\bar{\mathcal{A}},\bar{\mathcal{B}},\bar{\mathcal{C}})$ by (32). Then (38) follows immediately from (35a) and (36). Using (35b), (32), and (36) we derive

(40)
$$\boldsymbol{w}_n(\tau)=2\mathcal{C}[\mathcal{A}^{2(n+1)}e^{-i\tau(\mathcal{A}-\mathcal{A}^{-1})^2}+4\boldsymbol{N}\mathcal{A}^{\dagger -2(n+2)}e^{-i\tau(\mathcal{A}^{\dagger}-\mathcal{A}^{\dagger -1})^2}\boldsymbol{Q}\mathcal{A}^{-2}]^{-1}\mathcal{B},$$

which implies (38). □

4.1. *Examples.*

In this section we work out some illustrative examples. First, we point out that certain choices of the triplets $(\mathcal{A},\mathcal{B},\mathcal{C})$ lead to certain IDNLS solutions. For instance, the one-soliton occurs if $p=1$ (i.e., if $\mathcal{A}$ is a scalar), the N-soliton soliton (as given by [5]) occurs if $\mathcal{B}$ and $\mathcal{C}$ do not have zero entries and $\mathcal{A}$ is a diagonal matrix containing diagonal elements α_j for which the numbers α_j^2 are distinct, and the breather solution occurs if $\mathcal{A}$ is 2×2 and has complex conjugate eigenvalues.

Example 4.1 (Two-soliton interaction). *Consider the focusing case*
$$\mathcal{A}=\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix},\quad \mathcal{B}=\begin{pmatrix}1\\ 1\end{pmatrix},\quad \mathcal{C}=\begin{pmatrix}c_1 & c_2\end{pmatrix},$$
where $|\alpha|>1$, $|\beta|>1$, $\alpha^2\neq\beta^2$, $c_1\neq 0$, and $c_2\neq 0$. For the moment we assume α, β, c_1, and c_2 to be complex numbers.

We have
$$\boldsymbol{Q}=\sum_{\sigma=0}^{\infty}\begin{pmatrix}|\alpha|^{-4\sigma}|c_1|^2 & (\alpha^*\beta)^{-2\sigma}c_1^*c_2\\ (\alpha\beta^*)^{-2\sigma}c_1c_2^* & |\beta|^{-4\sigma}|c_2|^2\end{pmatrix}=\begin{pmatrix}\dfrac{|c_1|^2}{1-|\alpha|^{-4}} & \dfrac{c_1^*c_2}{1-(\alpha^*\beta)^{-2}}\\ \dfrac{c_1c_2^*}{1-(\alpha\beta^*)^{-2}} & \dfrac{|c_2|^2}{1-|\beta|^{-4}}\end{pmatrix},$$

$$\boldsymbol{N} = \begin{pmatrix} \dfrac{1}{1-|\alpha|^{-4}} & \dfrac{1}{1-(\alpha\beta^*)^{-2}} \\ \dfrac{1}{1-(\alpha^*\beta)^{-2}} & \dfrac{1}{1-|\beta|^{-4}} \end{pmatrix}, \; e^{i\tau(\mathcal{A}-\mathcal{A}^{-1})^2} = \begin{pmatrix} e^{i\tau(\alpha-\alpha^{-1})^2} & 0 \\ 0 & e^{i\tau(\beta-\beta^{-1})^2} \end{pmatrix}.$$

According to (38) *we have* $\boldsymbol{u}_n(\tau) = -2\mathcal{B}^\dagger\Gamma^{-1}\mathcal{C}^\dagger$, *where*

$$\Gamma = \mathcal{A}^{\dagger\,2(n+1)} e^{i\tau(\mathcal{A}^\dagger - \mathcal{A}^{\dagger\,-1})^2} + 4\boldsymbol{Q}\mathcal{A}^{-2(n+2)} e^{i\tau(\mathcal{A}-\mathcal{A}^{-1})^2}\boldsymbol{N}\mathcal{A}^{\dagger\,-2}.$$

Let us now compute the four elements of $\Gamma - \mathcal{A}^{\dagger\,2(n+1)} e^{i\tau(\mathcal{A}^\dagger - \mathcal{A}^{\dagger\,-1})^2}$:

$$(1,1): \quad \frac{4(\alpha^*)^{-2}|c_1|^2\alpha^{-2(n+2)}e^{i\tau(\alpha-\alpha^{-1})^2}}{[1-|\alpha|^{-4}]^2} + \frac{4(\alpha^*)^{-2}c_1^*c_2\beta^{-2(n+2)}e^{i\tau(\beta-\beta^{-1})^2}}{|1-(\alpha^*\beta)^{-2}|^2},$$
$$(1,2): \quad \frac{4(\beta^*)^{-2}|c_1|^2\alpha^{-2(n+2)}e^{i\tau(\alpha-\alpha^{-1})^2}}{(1-|\alpha|^{-4})(1-(\alpha\beta^*)^{-2})} + \frac{4(\beta^*)^{-2}c_1^*c_2\beta^{-2(n+2)}e^{i\tau(\beta-\beta^{-1})^2}}{(1-|\beta|^{-4})(1-(\alpha^*\beta)^{-2})},$$
$$(2,1): \quad \frac{4(\alpha^*)^{-2}c_1c_2^*\alpha^{-2(n+2)}e^{i\tau(\alpha-\alpha^{-1})^2}}{(1-|\alpha|^{-4})(1-(\alpha\beta^*)^{-2})} + \frac{4(\alpha^*)^{-2}|c_2|^2\beta^{-2(n+2)}e^{i\tau(\beta-\beta^{-1})^2}}{(1-|\beta|^{-4})(1-(\alpha^*\beta)^{-2})},$$
$$(2,2): \quad \frac{4(\beta^*)^{-2}c_1c_2^*\alpha^{-2(n+2)}e^{i\tau(\alpha-\alpha^{-1})^2}}{|1-(\alpha\beta^*)^{-2}|^2} + \frac{4(\beta^*)^{-2}|c_2|^2\beta^{-2(n+2)}e^{i\tau(\beta-\beta^{-1})^2}}{[1-|\beta|^{-4}]^2},$$

where

$$\mathcal{A}^{\dagger\,2(n+1)} e^{i\tau(\mathcal{A}^\dagger - \mathcal{A}^{\dagger\,-1})^2} = diag\left[(\alpha^*)^{2(n+1)}e^{i\tau(\alpha^*-\alpha^{*\,-1})^2}, (\beta^*)^{2(n+1)}e^{i\tau(\beta^*-\beta^{*\,-1})^2}\right].$$

The determinant of Γ *contains 13 terms without cancellations if* α, β, c_1, *and* c_2 *are complex constants. This number can be reduced to* 10 *if* α, β, c_1, *and* c_2 *are real. We then get*

$$\det\Gamma = (\alpha\beta)^{2(n+1)}e^{i\tau(\alpha-\alpha^{-1})^2}e^{i\tau(\beta-\beta^{-1})^2} + \frac{4(\alpha\beta)^{-2}c_1c_2e^{2i\tau(\alpha-\alpha^{-1})^2}}{[1-(\alpha\beta)^{-2}]^2}$$
$$+ \frac{4\alpha^{2(n+1)}\beta^{-2(n+3)}c_2^2e^{i\tau(\alpha-\alpha^{-1})^2}e^{i\tau(\beta-\beta^{-1})^2}}{(1-\beta^{-4})^2}$$
$$+ \frac{4\alpha^{-2(n+3)}\beta^{2(n+1)}c_1^2e^{i\tau(\alpha-\alpha^{-1})^2}}{(1-\alpha^{-4})^2} + \frac{4(\alpha\beta)^{-2}c_1c_2e^{2i\tau(\beta-\beta^{-1})^2}}{(1-(\alpha\beta)^{-2})^2}$$
$$+ \frac{32(\alpha^{-2}-\beta^{-2})^2c_1^2c_2^2e^{i\tau(\alpha-\alpha^{-1})^2}e^{i\tau(\beta-\beta^{-1})^2}(\alpha\beta)^{-2(n+3)}}{(1-\alpha^{-4})^2(1-\beta^{-4})^2(1-(\alpha\beta)^{-2})^2}.$$

Writing cofac Γ *for the cofactor matrix of* Γ, *the IDNLS solution* $\boldsymbol{u}_n(\tau)$ *is the ratio of the numerator* $-2\mathcal{B}^\dagger[\text{cofac}\,\Gamma]\mathcal{C}^\dagger$ *consisting of 10 terms without simplifications (even if* α, β, c_1, *and* c_2 *are real) and a denominator of 10 (6 if* α, β, c_1, *and* c_2 *are real) terms.*

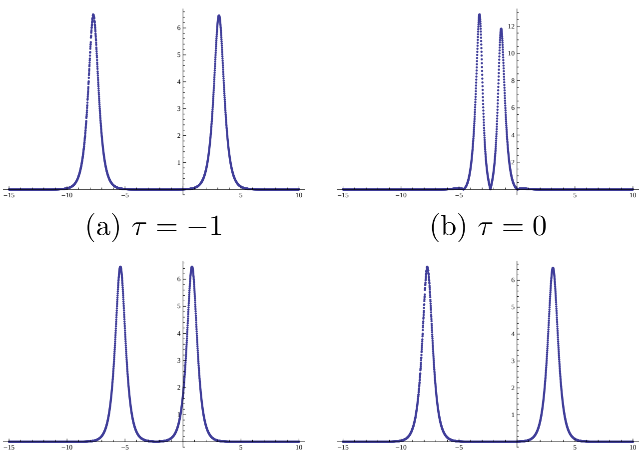
If the matrix triplets are more complicated, it becomes impractical to write down explicit IDNLS solutions by hand or by using computer algebra. For this reason we have applied Mathematica to write down the focusing IDNLS solutions (38), performing the following steps:

a. Input the matrix triplet $(\mathcal{A}, \mathcal{B}, \mathcal{C})$. All eigenvalues of $\mathcal{A}$ have modulus larger than one and $\mathcal{A}$ has no $\pm$ pair among its eigenvalues.
b. Compute the solutions $\mathcal{Q}$ and $\mathcal{N}$ of the Stein equations (37).
c. Use (38) to compute $u_n(\tau)$ for various values of τ and substitute the result in the focusing IDNLS equation (1), where $\boldsymbol{w}_n(\tau) = -\boldsymbol{u}_n(\tau)^\dagger$. Doing so, we have always found our solutions to satisfy (1) within an error of at most 10^{-10}.
d. Plot $|u_n(\tau)|$ for various values of τ.

Example 4.2 (Breather solutions). *Let us consider the focusing case, where* $\mathcal{A} = \begin{pmatrix} 3 & 2 \\ -2 & 3 \end{pmatrix}$, $\mathcal{B} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, *and* $C = \begin{pmatrix} 2 & 1 \end{pmatrix}$. *Solving the corresponding Stein equations we get*

$$\mathcal{Q} = \begin{pmatrix} \frac{4499}{1120} & \frac{953}{480} \\ \frac{953}{480} & \frac{3403}{3360} \end{pmatrix}, \mathcal{N} = \begin{pmatrix} \frac{561}{560} & \frac{239}{240} \\ \frac{239}{240} & \frac{1697}{1680} \end{pmatrix}.$$

Substituting the exact solution (38) into (1) we conclude that this equation is satisfied. In Figure 1 we have plotted $|u_n(\tau)|$ *for four different values of* τ.



(a) $\tau = -1$ (b) $\tau = 0$ (c) $\tau = 0,5$ (d) $\tau = 1$

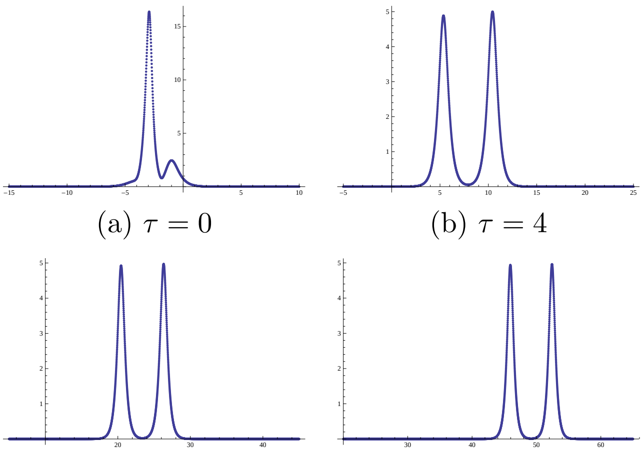
Figure 1. Breather solutions for four values of τ.

DOI: 10.1685/journal.caim.372

Example 4.3 (Double-pole solutions). *Consider the focusing case, where* $\mathcal{A} = \begin{pmatrix} 3-i & -1 \\ 0 & 3-i \end{pmatrix}$, $\mathcal{B} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, *and* $\mathcal{C} = \begin{pmatrix} 3 & 2 \end{pmatrix}$. *The corresponding Stein equations admit as solutions*

$$\mathcal{Q} = \begin{pmatrix} \frac{100}{11} & \frac{740}{121} + i\frac{20}{1089} \\ \frac{740}{121} - i\frac{20}{1089} & \frac{447560}{107811} \end{pmatrix}, \mathcal{N} = \begin{pmatrix} \frac{4040}{970299} & \frac{20}{3267} + i\frac{20}{9801} \\ \frac{20}{3267} - i\frac{20}{9801} & \frac{100}{99} \end{pmatrix}.$$

Substituting the exact solution (38) *into* (1) *we conclude that the IDNLS is satisfied. In Figure 2 we have plotted* $|u_n(\tau)|$ *for four different values of* τ.



(a) $\tau = 0$ (b) $\tau = 4$

(c) $\tau = 10$ (d) $\tau = 20$

Figure 2. Double pole solutions for four values of τ.

Acknowledgements.

This research was partially supported by INdAM, MIUR under PRIN grant No. 20083KLJEZ-003, and the Autonomous Region of Sardinia (RAS) under grant CRP3-138, L.R. 7/2007.

The first author (F.D.) is also supported by the Autonomous Region of Sardinia (RAS) under grant CRP3-138, L.R. 7/2007.

The authors are greatly indebted to Antonio Aricò for his assistance in developing the Mathematica code.

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