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NOVEL FORMULATION OF INVERSE SCATTERING AND CHARACTERIZATION OF SCATTERING DATA

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Abstract. In this article we formulate the direct and inverse scattering theory for the focusing matrix Zakharov-Shabat system as the construction of a 1,1-correspondence between focusing potentials with entries in $L^1(\mathbb{R})$ and Marchenko integral kernels, given the fact that these kernels encode the usual scattering data (one reflection coefficient, the discrete eigenvalues with positive imaginary part, and the corresponding norming constants) faithfully. In the reflectionless case, we solve the Marchenko equations explicitly using matrix triplets and obtain focusing matrix NLS solutions in closed form.

1. Direct and inverse scattering theory.

Consider the focusing matrix nonlinear Schrödinger (NLS) equation

$$iu_t + u_{xx} + 2uu^\dagger u = 0, \tag{1}$$

where $u = u(x,t)$ is an $m \times n$ matrix function depending on position $x \in \mathbb{R}$ and time $t \in \mathbb{R}$ and the dagger indicates the matrix conjugate transpose. By means of the inverse scattering transform (IST), (1) is associated with the focusing matrix Zakharov-Shabat problem

$$iJ\frac{\partial X}{\partial x} - V(x,t)X(\lambda,x;t) = \lambda X(\lambda,x;t), \tag{2}$$

where

$$J = \begin{pmatrix} I_m & 0_{m\times n} \\ 0_{n\times m} & -I_n \end{pmatrix}, \qquad V(x,t) = \begin{pmatrix} 0_{m\times m} & iu(x,t) \\ iu(x,t)^\dagger & 0_{n\times n} \end{pmatrix},$$

the potential $u(x,t)$ has its entries in $L^1(\mathbb{R};dx)$ for each $t \in \mathbb{R}$, and λ is a spectral parameter. For background material we refer to the standard sources (e.g., [2, 1, 11, 13]).

Let us introduce the $(m+n)\times m$ and $(m+n)\times n$ *Jost functions from the right* $\overline{\psi}(\lambda,x)$ and $\psi(\lambda,x)$, the $(m+n)\times m$ and $(m+n)\times n$ *Jost solutions from the left* $\phi(\lambda,x)$ and $\overline{\phi}(\lambda,x)$, and the $(m+n)\times(m+n)$ *Jost matrices* $\Psi(\lambda,x)$ and $\Phi(\lambda,x)$ from

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the right and the left as those solutions of (2) satisfying the asymptotic conditions

$$\Psi(\lambda,x)=\begin{pmatrix}\overline{\psi}(\lambda,x) & \psi(\lambda,x)\end{pmatrix}=\begin{cases}e^{-i\lambda Jx}[I_{m+n}+o(1)], & x\to+\infty,\\ e^{-i\lambda Jx}a_l(\lambda)+o(1), & x\to-\infty,\end{cases}$$

$$\Phi(\lambda,x)=\begin{pmatrix}\phi(\lambda,x) & \overline{\phi}(\lambda,x)\end{pmatrix}=\begin{cases}e^{-i\lambda Jx}[I_{m+n}+o(1)], & x\to-\infty,\\ e^{-i\lambda Jx}a_r(\lambda)+o(1), & x\to+\infty.\end{cases}$$

The system of equations (2) being first order implies that

$$\Phi(\lambda,x)=\Psi(\lambda,x)a_r(\lambda),\qquad \Psi(\lambda,x)=\Phi(\lambda,x)a_l(\lambda),$$

where $a_l(\lambda)$ and $a_r(\lambda)$ are called *transition coefficient matrices*. It is easily verified that, for each $(\lambda,x)\in\mathbb{R}^2$, the Jost matrices $\Psi(\lambda,x)$ and $\Phi(\lambda,x)$ are unitary and have determinant $e^{i\lambda(n-m)x}$. Further, for $\lambda\in\mathbb{R}$, $a_l(\lambda)$ and $a_r(\lambda)$ are unitary matrices with unit determinant, one is the inverse of the other. For later use we partition the transition coefficient matrices as follows:

$$a_l(\lambda)=\begin{pmatrix}a_{l1}(\lambda) & a_{l2}(\lambda)\\ a_{l3}(\lambda) & a_{l4}(\lambda)\end{pmatrix},\qquad a_r(\lambda)=\begin{pmatrix}a_{r1}(\lambda) & a_{r2}(\lambda)\\ a_{r3}(\lambda) & a_{r4}(\lambda)\end{pmatrix},$$

where $a_{l1}(\lambda)$ and $a_{r1}(\lambda)$ are $m\times m$ matrices.

Writing

$$\Psi(\lambda,x)=e^{-i\lambda Jx}+\int_x^\infty dy\,\begin{pmatrix}\overline{K}(x,y) & K(x,y)\end{pmatrix}e^{-i\lambda Jy},$$
$$\Phi(\lambda,x)=e^{-i\lambda Jx}+\int_{-\infty}^x dy\,\begin{pmatrix}M(x,y) & \overline{M}(x,y)\end{pmatrix}e^{-i\lambda Jy},$$

where

$$\int_x^\infty dy\,\left(\|\overline{K}(x,y)\|+\|K(x,y)\|\right)+\int_{-\infty}^x dy\,\left(\|M(x,y)\|+\|\overline{M}(x,y)\|\right)<+\infty,$$

and reshuffling the columns of the Jost matrices as to create square matrix functions

$$F_+(\lambda,x)=\begin{pmatrix}\phi(\lambda,x) & \psi(\lambda,x)\end{pmatrix},\qquad F_-(\lambda,x)=\begin{pmatrix}\overline{\psi}(\lambda,x) & \overline{\phi}(\lambda,x)\end{pmatrix},$$

the former analytic in $\lambda\in\mathbb{C}^+$ and the latter analytic in $\lambda\in\mathbb{C}^-$, we obtain the Riemann-Hilbert problem

$$F_-(\lambda,x)=F_+(\lambda,x)JS(\lambda)J, \tag{3}$$

where, for each $x\in\mathbb{R}$, $F_+(\lambda,x)$ is continuous in $\lambda\in\overline{\mathbb{C}^+}$ and analytic in $\lambda\in\mathbb{C}^+$, $F_-(\lambda,x)$ is continuous in $\lambda\in\overline{\mathbb{C}^-}$ and analytic in $\lambda\in\mathbb{C}^-$, and

$$S(\lambda)=\begin{pmatrix}T_r(\lambda) & L(\lambda)\\ R(\lambda) & T_l(\lambda)\end{pmatrix}$$

is the *scattering matrix*. Here $\mathbb{C}^+$ and $\mathbb{C}^-$ denote the upper and lower complex open half-planes. Moreover,

$$F_+(\lambda,x)e^{i\lambda Jx}=I_{m+n}+\int_0^\infty d\alpha\,e^{i\lambda\alpha}\big(M(x,x-\alpha)\;K(x,x+\alpha)\big),$$
$$F_-(\lambda,x)e^{i\lambda Jx}=I_{m+n}+\int_0^\infty d\alpha\,e^{-i\lambda\alpha}\big(\overline{K}(x,x+\alpha)\;\overline{M}(x,x-\alpha)\big),$$

so that $F_\pm(\lambda,x)e^{i\lambda Jx}\to I_{m+n}$ as $\lambda\to\infty$ from within $\overline{\mathbb{C}^\pm}$. Under the technical assumption that there are no spectral singularities (i.e., that, for each $\lambda\in\mathbb{R}$,

$\det a_{r1}(\lambda) = \det a_{l4}(\lambda) \neq 0$ and $\det a_{l1}(\lambda) = \det a_{r4}(\lambda) \neq 0$), the scattering matrix $S(\lambda)$ is J-unitary in the sense that

$$S(\lambda)JS(\lambda)^\dagger = S(\lambda)^\dagger JS(\lambda) = J.$$

Further, under this assumption we can relate the *transmission coefficients* $T_r(\lambda)$ and $T_l(\lambda)$ and the *reflection coefficients* $R(\lambda)$ and $L(\lambda)$ to the transition coefficients as follows:

$$\begin{aligned}
T_r(\lambda) &= a_{r1}(\lambda)^{-1}, \\
T_l(\lambda) &= a_{l4}(\lambda)^{-1}, \\
R(\lambda) &= -a_{l4}(\lambda)^{-1}a_{l3}(\lambda) = a_{r3}(\lambda)a_{r1}(\lambda)^{-1}, \\
L(\lambda) &= -a_{r1}(\lambda)^{-1}a_{r2}(\lambda) = a_{l2}(\lambda)a_{l4}(\lambda)^{-1}.
\end{aligned}$$

Traditionally the direct scattering problem is formulated as consisting of the determination, starting from the potential $u(x)$, of the following scattering data: one reflection coefficient ($R(\lambda)$ or $L(\lambda)$), the discrete eigenvalues in either $\mathbb{C}^+$ or $\mathbb{C}^-$, and the corresponding norming constants. Then the inverse scattering problem consists of the unique evaluation of the potential from these scattering data, either by solving one of the pairs of coupled Marchenko integral equations (4) below or by solving the Riemann-Hilbert problem (3). Most practitioners in the field conveniently assume that the poles of the transmission coefficients $T_r(\lambda)$ and $T_l(\lambda)$, which must necessarily occur at the discrete eigenvalues in $\mathbb{C}^+$, are simple. In that case there is one norming constant per discrete eigenvalue and the Marchenko integral equation is easily formulated. Only recently [8, 7] the modifications required in the case of multiple poles have been indicated.

In the focusing case the Marchenko integral equations are given by

$$\overline{K}(x,y) + \begin{pmatrix} 0_{m\times n} \\ I_n \end{pmatrix} \Omega_l(x+y) + \int_x^\infty dz\, K(x,z)\Omega_l(z+y) = 0_{(m+n)\times m}, \tag{4a}$$

$$K(x,y) - \begin{pmatrix} I_m \\ 0_{n\times m} \end{pmatrix} \Omega_l(x+y)^\dagger - \int_x^\infty dz\, \overline{K}(x,z)\Omega_l(z+y)^\dagger = 0_{(m+n)\times n}, \tag{4b}$$

$$M(x,y) - \begin{pmatrix} 0_{m\times n} \\ I_n \end{pmatrix} \Omega_r(x+y)^\dagger - \int_{-\infty}^x dz\, \overline{M}(x,z)\Omega_r(z+y)^\dagger = 0_{(m+n)\times m}, \tag{4c}$$

$$\overline{M}(x,y) + \begin{pmatrix} I_m \\ 0_{n\times m} \end{pmatrix} \Omega_r(x+y) + \int_{-\infty}^x dz\, M(x,z)\Omega_r(z+y) = 0_{(m+n)\times n}, \tag{4d}$$

where the Marchenko kernels $\Omega_r(x+y)$ and $\Omega_l(x+y)$ depend in a one-to-one way on the scattering data. The pairs of equations (4a)-(4b) and (4c)-(4d) are easily seen to be uniquely solvable and the potentials found from their solutions by means of the identities

$$u(x) = -2K^{\rm up}(x,x) = +2\overline{K}^{\rm dn}(x,x)^\dagger, \tag{5a}$$

$$u(x) = +2\overline{M}^{\rm up}(x,x) = -2M^{\rm dn}(x,x), \tag{5b}$$

have their entries in $L^1(\mathbb{R})$ [19, 8, 9]. Here $L^{\rm up} = \begin{pmatrix} I_m & 0_{m\times n}\end{pmatrix} L$ and $L^{\rm dn} = \begin{pmatrix} 0_{n\times m} & I_n\end{pmatrix} L$ for any matrix L having $m+n$ rows. Writing

$$R(\lambda) = \int_{-\infty}^\infty dy\, e^{-i\lambda y}\rho(y), \qquad L(\lambda) = \int_{-\infty}^\infty dy\, e^{i\lambda y}\ell(y), \tag{6}$$

for certain matrix functions $\rho(y)$ and $\ell(y)$ with entries in $L^1(\mathbb{R})$, we have

$$\Omega_l(y) = \rho(y), \qquad \Omega_r(y) = \ell(y),$$

provided the matrix Zakharov-Shabat system does not have any discrete eigenvalues. On the other hand, if the pole $\lambda_j \in \mathbb{C}^+$ of the transmission coefficients has order N_j, we need to introduce N_j norming constants from the left and N_j norming constant from the right corresponding to this pole, resulting in Marchenko kernels of the form [8, 9]

$$\Omega_l(y) = \rho(y) + \sum_j e^{i\lambda_j y} \sum_{s=0}^{N_j-1} \frac{y^s}{s!} N_{ljs}, \tag{7a}$$

$$\Omega_r(y) = \ell(y) + \sum_j e^{-i\lambda_j y} \sum_{s=0}^{N_j-1} \frac{y^s}{s!} N_{rjs}, \tag{7b}$$

where the summations involve only finitely many terms. Letting

$$\boldsymbol{N}_+ = \left\{ \sum_j e^{i\lambda_j y} \sum_{s=0}^{N_j-1} \frac{y^s}{s!} N_{js} : \{\lambda_j\} \subset \mathbb{C}^+ \text{ finite}, \{N_{js}\} \subset \mathbb{C} \right\},$$

$$\boldsymbol{N}_- = \left\{ \sum_j e^{i\zeta_j y} \sum_{s=0}^{N_j-1} \frac{y^s}{s!} N_{js} : \{\zeta_j\} \subset \mathbb{C}^- \text{ finite}, \{N_{js}\} \subset \mathbb{C} \right\},$$

we see that $\Omega_l(y)$ is an $n \times m$ matrix function having its entries in $L^1(\mathbb{R}) + \boldsymbol{N}_+$ and $\Omega_r(y)$ is an $m \times n$ matrix function having its entries in $L^1(\mathbb{R}) + \boldsymbol{N}_-$. Further, the decomposition of a function in $L^1(\mathbb{R}) + \boldsymbol{N}_\pm$ as the sum of a function in $L^1(\mathbb{R})$ and a function in $\boldsymbol{N}_\pm$ is unique, because only the zero function can belong simultaneously to $L^1(\mathbb{R})$ and to $\boldsymbol{N}_\pm$. In other words,

$$\Omega_l \in [L^1(\mathbb{R}) \oplus \boldsymbol{N}_+] \otimes \mathbb{C}^{n\times m}, \qquad \Omega_r \in [L^1(\mathbb{R}) \oplus \boldsymbol{N}_-] \otimes \mathbb{C}^{m\times n}.$$

The usual scattering data can be replaced by either Marchenko kernel $\Omega_l(y)$ or $\Omega_r(y)$. Since the pair of Marchenko equations (4a)-(4b) is uniquely solvable, with the help of (5) we easily get a unique potential having its entries in $L^1(\mathbb{R})$. On the other hand, starting from a focusing potential having its entries in $L^1(\mathbb{R})$ we get Marchenko kernels of the form (7), provided there are no spectral singularities. In other words, we have the following characterization result:

> There is a $1,1$-correspondence between focusing L^1-potentials WITHOUT spectral singularities and Marchenko kernels $\Omega_l(y)$ belonging to $[L^1(\mathbb{R}) \oplus \boldsymbol{N}_+] \otimes \mathbb{C}^{n\times m}$ (resp., $\Omega_r(y)$ belonging to $[L^1(\mathbb{R}) \oplus \boldsymbol{N}_-] \otimes \mathbb{C}^{m\times n}$).

If the potential $u(x,t)$ evolves in time as a solution of the focusing matrix NLS equation (1) and there are no spectral singularities, then the Marchenko kernels evolve in time in the following way [2, 11, 4]:

$$\Omega_l(y,t) = \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda\, R(\lambda)e^{i\lambda y+4i\lambda^2 t}}_{\stackrel{\text{def}}{=}\rho(y;t)} + \sum_j e^{i\lambda_j y}\sum_{s=0}^{N_j-1}\frac{y^s}{s!}N_{ljs}(t),$$

$$\Omega_r(y,t) = \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda\, L(\lambda)e^{-i\lambda y-4i\lambda^2 t}}_{\stackrel{\text{def}}{=}\ell(y;t)} + \sum_j e^{-i\lambda_j y}\sum_{s=0}^{N_j-1}\frac{y^s}{s!}N_{rjs}(t),$$

where the norming constants $N_{ljs}(t)$ and $N_{rjs}(t)$ evolve in time such that

$$(\Omega_l)_t + 4i(\Omega_l)_{yy} = 0, \qquad (\Omega_r)_t - 4i(\Omega_r)_{yy} = 0.$$

Within the constraints imposed by a scattering theory for L^1-potentials, time evolution of the scattering data requires a (presently unknown) restriction of the potentials to a class for which, for each $t \in \mathbb{R}$, $\rho(y;t)$ and $\ell(y;t)$ have their entries in $L^1(\mathbb{R})$.

2. **Focusing NLS solutions.** It is clear from (6) and (7) that the Marchenko kernel $\Omega_l(y)$ faithfully encodes the scattering data $\{R(\lambda), \lambda_j, \{N_{ljs}\}_{s=0}^{N_j-1}\}$ and the Marchenko kernel $\Omega_r(y)$ faithfully encodes the scattering data $\{L(\lambda), \lambda_j, \{N_{rjs}\}_{s=0}^{N_j-1}\}$. Instead of using norming constants it is convenient to represent the bound state terms in (7) as "weighting patterns" of autonomous linear systems. In other words, we write

$$\Omega_l(y) = \rho(y) + C_l e^{-yA_l}B_l,$$
$$\Omega_r(y) = \ell(y) + C_r e^{yA_r}B_r,$$

where (A_l, B_l, C_l) and (A_r, B_r, C_r) are two matrix triplets consisting of square matrices A_l and A_r (of orders p_l and p_r) having only eigenvalues with positive real parts and B_l, C_l, B_r, and C_r are rectangular matrices of respective sizes $p_l \times m$, $n \times p_l$, $p_r \times n$, and $m \times p_r$. When these triplets are both minimal [6] in the sense that the matrix orders p_l and p_r have been minimized without changing the Marchenko kernels, the matrix triplets are uniquely determined by the Marchenko kernels up to similarity: Two minimal triplets (A_l, B_l, C_l) and (A_l', B_l', C_l') satisfying

$$\Omega_l(y) - \rho(y) = C_l e^{-yA_l}B_l = C_l' e^{-yA_l'}B_l'$$

are connected by a unique similarity transformation S such that

$$A_l' = SA_lS^{-1}, \qquad B_l' = SB_l, \qquad C_l' = C_lS^{-1}.$$

The same thing is true for minimal matrix triplets appearing in (7b). For mimimal matrix triplets (A_l, B_l, C_l) yielding (7a) and (A_r, B_r, C_r) yielding (7b) the matrices A_l and A_r are necessarily similar, i.e., they have the same Jordan normal form. These two matrices are diagonalizable whenever the poles of the transmission coefficients $T_r(\lambda)$ and $T_l(\lambda)$ in $\mathbb{C}^+$ are all simple.

Matrix triplets allow one to express the time evolution of the Marchenko kernels in the following succinct form:

$$\Omega_l(y,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda\, R(\lambda)e^{i\lambda y+4i\lambda^2 t} + C_l e^{-yA_l}e^{-4itA_l^2}B_l, \tag{8a}$$

$$\Omega_r(y,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda\, L(\lambda)e^{-i\lambda y-4i\lambda^2 t} + C_r e^{yA_r}e^{4itA_r^2}B_r. \tag{8b}$$

These representations preclude the need of describing the time evolution of the norming constants (as done, for nonsimple poles, in [8, 7] and pioneered in [14, 15]). Nevertheless, the norming constants can easily be expressed in the matrix triplets. Let $\{P_{lj}\}$ be a finite set of projections commuting with A_l such that $(A_l + i\lambda_j I_{p_l})$ is nilpotent of order N_j and $\sum_j P_{lj} = I_{p_l}$. Similarly, let $\{P_{rj}\}$ be a finite set of projections commuting with A_r such that $(A_r + i\lambda_j I_{p_r})$ is nilpotent of order N_j and $\sum_j P_{rj} = I_{p_r}$. Then

$$\Omega_l(y) = \sum_j e^{i\lambda_j y}\sum_{s=0}^{N_j-1}\frac{y^s}{s!}\underbrace{(-1)^s C_l(A_l+i\lambda_j I_{p_l})^s P_{lj}B_l}_{=N_{ljs}},$$

$$\Omega_r(y) = \sum_j e^{i\lambda_j y}\sum_{s=0}^{N_j-1}\frac{y^s}{s!}\underbrace{C_r(A_r+i\lambda_j I_{p_r})^s P_{rj}B_r}_{=N_{rjs}}.$$

Matrix triplets also permit one to derive closed form solutions of the focusing matrix NLS equation if the reflection coefficients $R(\lambda)$ and $L(\lambda)$ vanish. Iterating the up components of the pair of equations (4a)-(4b) once we get

$$K^{\text{up}}(x,y;t) - \Omega_l(x+y;t)^\dagger + \int_x^\infty dz\, K^{\text{up}}(x,z;t)\int_x^\infty d\hat{z}\,\Omega_l(z+\hat{z};t)\Omega_l(\hat{z}+y;t)^\dagger = 0_{m\times n}.$$

Substituting (8a) with $R(\lambda)\equiv 0$ and solving the above integral equation by separation of variables, we obtain

$$K^{\text{up}}(x,y;t) = B_l^\dagger[e^{2xA_l^\dagger}e^{-4itA_l^{\dagger 2}} + Q_l e^{-2xA_l}e^{-4itA_l^2}N_l]^{-1}e^{-(y-x)A_l^\dagger}C_l^\dagger,$$

where

$$Q_l = \int_0^\infty dy\, e^{-yA_l^\dagger}C_l^\dagger C_l e^{-yA_l}, \qquad N_l = \int_0^\infty dy\, e^{-yA_l}B_l B_l^\dagger e^{-yA_l^\dagger},$$

are the unique solutions of the Lyapunov equations

$$A_l^\dagger Q_l + Q_l A_l = C_l^\dagger C_l, \qquad A_l N_l + N_l A_l^\dagger = B_l B_l^\dagger.$$

With the help of (5a) we finally arrive at the matrix NLS solution

$$u(x,t) = -2B_l^\dagger[e^{2xA_l^\dagger}e^{-4itA_l^{\dagger 2}} + Q_l e^{-2xA_l}e^{-4itA_l^2}N_l]^{-1}C_l^\dagger, \tag{9}$$

which is easily seen to decay exponentially as $x\to\pm\infty$ for fixed $t\in\mathbb{R}$. In the same way we compute

$$M^{\text{up}}(x,y;t) = -C_r[e^{-2xA_r}e^{-4itA_r^2} + N_r e^{2xA_r^\dagger}e^{-4itA_r^{\dagger 2}}Q_r]^{-1}e^{-(x-y)A_r}B_r,$$

where

$$Q_r = \int_{-\infty}^0 dy\, e^{yA_r^\dagger}C_r^\dagger C_r e^{yA_r}, \qquad N_r = \int_{-\infty}^0 dy\, e^{yA_r}B_r B_r^\dagger e^{yA_r^\dagger},$$

are the unique solutions of the Lyapunov equations

$$A_r^\dagger Q_r + Q_r A_r = C_r^\dagger C_r, \qquad A_r N_r + N_r A_r^\dagger = B_r B_r^\dagger.$$

Consequently, we get with the help of (5b)

$$u(x,t) = 2C_r[e^{-2xA_r} e^{-4itA_r^2} + N_r e^{2xA_r^\dagger} e^{-4itA_r^{\dagger 2}} Q_r]^{-1} B_r, \tag{10}$$

which is easily seen to decay exponentially as $x \to \pm\infty$ for fixed $t \in \mathbb{R}$. The expressions (9) and (10) remain solutions of the focusing matrix NLS equation (1) if the matrix triplets are selected in such a way that A_l and A_r do not have imaginary eigenvalues nor pairs of eigenvalues symmetrically located with respect to the imaginary axis, without requiring all of the eigenvalues of A_l and A_r to have a positive real part [4]. The solutions (9) and (10) expressed in terms of such more general matrix triplets can also be expressed in minimal matrix triplets for which the matrices A_l and A_r only have eigenvalues with positive real parts [3].

The above method to derive explicit solutions of nonlinear evolution equations has been applied to the KdV, (matrix) NLS, and sine-Gordon equations [8, 4, 5]. Matrix or operator triplets to evaluate such solutions, but without solving Marchenko equations, have been employed for pseudo-canonical systems [12], the sine-Gordon equation [17], the Toda lattice equations [16], and, more recently, for the matrix NLS equation [10, 18].

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