Wave operators for the matrix Zakharov–Shabat system

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In this article, we prove the similarity (and, in the focusing case, the J-unitary equivalence) of the free Hamiltonian and the restriction of the full Hamiltonian to the maximal invariant subspace on which its spectrum is real for the matrix Zakharov–Shabat system under suitable conditions on the potentials. This restriction of the full Hamiltonian is shown to be a scalar-type spectral operator whose resolution of the identity is evaluated. In the focusing case, the restricted full Hamiltonian is an absolutely continuous, J-self-adjoint non-J-definitizable, operator allowing a spectral theorem without singular critical points. To illustrate the results, two examples are provided. © *2010 American Institute of Physics.* [doi:10.1063/1.3377048]

I. INTRODUCTION

Consider the matrix Zakharov–Shabat system,

$$iJX'(x,\lambda) - V(x)X(x,\lambda) = \lambda X(x,\lambda), \quad x \in \mathbb{R}, \tag{1.1}$$

where the prime denotes differentiation with respect to $x \in \mathbb{R}$, λ is a spectral parameter, and

$$J = \begin{pmatrix} I_m & 0_{m\times n} \\ 0_{n\times m} & -I_n \end{pmatrix}, \quad V(x) = \begin{pmatrix} 0_{m\times m} & iq(x) \\ ir(x) & 0_{n\times n} \end{pmatrix},$$

with the entries of $q(x)$ and $r(x)$ belonging to $L^1(\mathbb{R})$. Note that $r(x)=q(x)^\dagger$ in the focusing case and $r(x)=-q(x)^\dagger$ in the defocusing case, where the dagger denotes the matrix conjugate transpose. (The asterisk is used to denote the complex conjugate of a complex number.) For $\lambda \in \mathbb{R}$, the Jost solutions $\bar{\psi}$, ψ, ϕ, and $\bar{\phi}$ and the Jost matrices Ψ and Φ are defined by

$$\Psi(x,\lambda) = \begin{pmatrix} \bar{\psi}(x,\lambda) & \psi(x,\lambda) \end{pmatrix} = \begin{pmatrix} e^{-i\lambda x}I_m & 0_{m\times n} \\ 0_{n\times m} & e^{i\lambda x}I_n \end{pmatrix} + o(1) = e^{-i\lambda Jx} + o(1), \quad x \to +\infty, \tag{1.2}$$

$$\Phi(x,\lambda) = \begin{pmatrix} \phi(x,\lambda) & \bar{\phi}(x,\lambda) \end{pmatrix} = \begin{pmatrix} e^{-i\lambda x}I_m & 0_{m\times n} \\ 0_{n\times m} & e^{i\lambda x}I_n \end{pmatrix} + o(1) = e^{-i\lambda Jx} + o(1), \quad x \to -\infty. \tag{1.3}$$

We refer the reader to Refs. 2, 8, and 21 for a detailed study of the asymptotic and analyticity properties of the solutions to (1.1). Therefore, we will use some of those results here without proof.

Since the matrix Zakharov–Shabat system is first order, there exist, for each $\lambda \in \mathbb{R}$, the so-called transition matrices $A_l(\lambda)$ and $A_r(\lambda)$, such that

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$$\Psi(x,\lambda) = \Phi(x,\lambda)A_l(\lambda), \quad \Phi(x,\lambda) = \Psi(x,\lambda)A_r(\lambda),$$

hence $A_l(\lambda)$ and $A_r(\lambda)$ are each other's inverses. Then from (1.1) and (1.2) we obtain

$$\Psi(x,\lambda) = e^{-i\lambda Jx}(A_l(\lambda) + o(1)), \quad x \to -\infty, \tag{1.4}$$

$$\Phi(x,\lambda) = e^{-i\lambda Jx}(A_r(\lambda) + o(1)), \quad x \to +\infty. \tag{1.5}$$

In order to state the asymptotics for complex λ, we partition the transition matrices into blocks by putting [throughout we adopt the following partitioning for $(m+n)\times(m+n)$ matrices $G=\begin{pmatrix} G_1 & G_2 \\ G_3 & G_4 \end{pmatrix}$, where G_1 is $m\times m$, G_2 is $m\times n$, G_3 is $n\times m$, and G_4 is $n\times n$]

$$A_l(\lambda) = \begin{pmatrix} A_{l1}(\lambda) & A_{l2}(\lambda) \\ A_{l3}(\lambda) & A_{l4}(\lambda) \end{pmatrix}, \quad A_r(\lambda) = \begin{pmatrix} A_{r1}(\lambda) & A_{r2}(\lambda) \\ A_{r3}(\lambda) & A_{r4}(\lambda) \end{pmatrix},$$

where A_{l1} and A_{r1} are of size $m\times m$. The solutions $\phi(x,\lambda)$ and $\psi(x,\lambda)$ are analytic in $\lambda \in \mathbb{C}^+$ and continuous in $\lambda \in \overline{\mathbb{C}^+}$, and the same holds true for the matrices $A_{l4}(\lambda)$ and $A_{r1}(\lambda)$. Here $\mathbb{C}^+$ and $\mathbb{C}^-$ denote the upper and lower open complex half planes, respectively. Moreover, for $\lambda \in \overline{\mathbb{C}^+}$ we have

$$\phi(x,\lambda) = \begin{cases} e^{-i\lambda x}\begin{pmatrix} I_m + o(1) \\ o(1) \end{pmatrix}, & x \to -\infty \\ e^{-i\lambda x}\begin{pmatrix} A_{r1}(\lambda) + o(1) \\ o(1) \end{pmatrix}, & x \to +\infty, \end{cases} \tag{1.6a}$$

$$\psi(x,\lambda) = \begin{cases} e^{i\lambda x}\begin{pmatrix} o(1) \\ A_{l4}(\lambda) + o(1) \end{pmatrix}, & x \to -\infty \\ e^{i\lambda x}\begin{pmatrix} o(1) \\ I_n + o(1) \end{pmatrix}, & x \to +\infty. \end{cases} \tag{1.6b}$$

It follows from earlier work (Ref. 8, Theorem 3.16, and Ref. 21, Theorem 5.3) that for $\lambda \in \mathbb{C}^+$, $\det A_{r1}(\lambda)=0$ if and only if $\det A_{l4}(\lambda)=0$, which is true if and only if λ is an eigenvalue. If $\lambda \in \mathbb{R}$ and $\det A_{r1}(\lambda)=0$ [or, equivalently, $\det A_{l4}(\lambda)=0$], then we call λ a *spectral singularity*. For the focusing scalar $(m=n=1)$ Zakharov–Shabat system detailed results about spectral singularities were obtained in Ref. 22. Similarly, $\bar{\phi}(x,\lambda)$ and $\bar{\psi}(x,\lambda)$ are analytic in $\lambda \in \mathbb{C}^-$, continuous in $\lambda \in \overline{\mathbb{C}^-}$, and obey

$$\bar{\phi}(x,\lambda) = \begin{cases} e^{i\lambda x}\begin{pmatrix} o(1) \\ I_n + o(1) \end{pmatrix}, & x \to -\infty \\ e^{i\lambda x}\begin{pmatrix} o(1) \\ A_{r4}(\lambda) + o(1) \end{pmatrix}, & x \to +\infty, \end{cases} \tag{1.7a}$$

$$\bar{\psi}(x,\lambda) = \begin{cases} e^{-i\lambda x}\begin{pmatrix} A_{l1}(\lambda) + o(1) \\ o(1) \end{pmatrix}, & x \to -\infty \\ e^{-i\lambda x}\begin{pmatrix} I_m + o(1) \\ o(1) \end{pmatrix}, & x \to +\infty. \end{cases} \tag{1.7b}$$

For $\lambda \in \mathbb{C}^-$, $\det A_{r4}(\lambda)=0$ if and only if $\lambda \in \mathbb{R}$, which is true if and only if λ is an eigenvalue in the lower half plane.

We now define the modified Jost functions $F_+(x,\lambda)$ and $F_-(x,\lambda)$ as follows:

$$F_+(x,\lambda) = \begin{pmatrix} \phi(x,\lambda) & \psi(x,\lambda) \end{pmatrix}, \quad F_-(x,\lambda) = \begin{pmatrix} \bar{\psi}(x,\lambda) & \bar{\phi}(x,\lambda) \end{pmatrix}.$$

Then $F_\pm(x,\lambda)$ is continuous in $\lambda \in \overline{\mathbb{C}^\pm}$ and analytic in $\lambda \in \mathbb{C}^\pm$. We introduce the scattering matrix $S(\lambda)$ and its inverse $\breve{S}(\lambda)$ by

$$F_-(x,\lambda) = F_+(x,\lambda)JS(\lambda)J, \quad F_+(x,\lambda) = F_-(x,\lambda)J\breve{S}(\lambda)J, \tag{1.8}$$

where $\lambda \in \mathbb{R}$. Putting

$$S(\lambda) = \begin{pmatrix} T_r(\lambda) & L(\lambda) \\ R(\lambda) & T_l(\lambda) \end{pmatrix}, \quad \breve{S}(\lambda) = \begin{pmatrix} \breve{T}_l(\lambda) & \breve{R}(\lambda) \\ \breve{L}(\lambda) & \breve{T}_r(\lambda) \end{pmatrix},$$

and taking $x \to \pm\infty$ in the first of (1.8) we obtain the relations

$$T_r(\lambda) = A_{r1}(\lambda)^{-1}, \quad T_l(\lambda) = A_{l4}(\lambda)^{-1}, \tag{1.9}$$

$$L(\lambda) = -A_{r1}(\lambda)^{-1}A_{r2}(\lambda) = A_{l2}(\lambda)A_{l4}(\lambda)^{-1}, \tag{1.10}$$

$$R(\lambda) = A_{r3}(\lambda)A_{r1}(\lambda)^{-1} = -A_{l4}(\lambda)^{-1}A_{l3}(\lambda). \tag{1.11}$$

In the same way, the second of (1.8) yields

$$\breve{T}_r(\lambda) = A_{r4}(\lambda)^{-1}, \quad \breve{T}_l(\lambda) = A_{l1}(\lambda)^{-1}, \tag{1.12}$$

$$\breve{L}(\lambda) = -A_{r4}(\lambda)^{-1}A_{r3}(\lambda) = A_{l3}(\lambda)A_{l1}(\lambda)^{-1}, \tag{1.13}$$

$$\breve{R}(\lambda) = A_{r2}(\lambda)A_{r4}(\lambda)^{-1} = -A_{l1}(\lambda)^{-1}A_{l2}(\lambda). \tag{1.14}$$

In the focusing case $(V(x)^\dagger = -V(x))$, it is easily verified that

$$(Z(x,\lambda)^\dagger X(x,\lambda))' = 0 \tag{1.15}$$

for any two solutions of (1.1). In particular, this implies that $\Phi(x,\lambda)^\dagger\Phi(x,\lambda) = \Psi(x,\lambda)^\dagger\Psi(x,\lambda) = I$, where, throughout this paper, $I = I_{m+n}$. Hence $A_l(\lambda)$ and $A_r(\lambda)$ are unitary, and thus

$$A_r(\lambda) = A_l(\lambda)^{-1} = A_l(\lambda)^\dagger.$$

It further follows that (for $\lambda \in \mathbb{R}$)

$$T_r(\lambda) = \breve{T}_l(\lambda)^\dagger, \quad T_l(\lambda) = \breve{T}_r(\lambda)^\dagger, \quad L(\lambda) = -\breve{L}(\lambda)^\dagger, \quad R(\lambda) = -\breve{R}(\lambda)^\dagger, \tag{1.16}$$

which implies $JS(\lambda)^\dagger J = \breve{S}(\lambda)$. This means that $S(\lambda)$ is J-unitary. In the defocusing case $(V(x) = V(x)^\dagger)$, we have $(Z(x,\lambda)^\dagger JX(x,\lambda))' = 0$, and hence

$$A_r(\lambda) = A_l(\lambda)^{-1} = JA_l(\lambda)^\dagger J,$$

so $S(\lambda)$ is unitary. For later use, we note that from (1.2)–(1.5) and (1.8) we have, for $\lambda \in \mathbb{R}$,

$$F_+(x,\lambda) = \begin{cases} e^{-i\lambda Jx}\begin{pmatrix} A_{r1}(\lambda) & 0_{m\times n} \\ A_{r3}(\lambda) & I_n \end{pmatrix} + o(1), & x \to +\infty \\ e^{-i\lambda Jx}\begin{pmatrix} I_m & A_{l2}(\lambda) \\ 0_{n\times m} & A_{l4}(\lambda) \end{pmatrix} + o(1), & x \to -\infty, \end{cases} \tag{1.17a}$$

$$F_-(x,\lambda)=\begin{cases} e^{-i\lambda Jx}\begin{pmatrix} I_m & A_{r2}(\lambda) \\ 0_{n\times m} & A_{r4}(\lambda)\end{pmatrix}+o(1), & x\rightarrow+\infty \\ e^{-i\lambda Jx}\begin{pmatrix} A_{l1}(\lambda) & 0_{m\times n} \\ A_{l3}(\lambda) & I_n\end{pmatrix}+o(1), & x\rightarrow-\infty. \end{cases} \tag{1.17b}$$

(Matrix) Zakharov–Shabat systems occur as the linear counterparts of nonlinear evolution systems when we solve them by means of the inverse scattering transform. These nonlinear systems include the (matrix) nonlinear Schrödinger equation, the (matrix) modified Korteweg–de Vries equation, and the sine-Gordon equation.[1–3] They are of major interest in applications as diverse as fiber optics,[15,29] surface waves on deep waters,[34] plasma waves,[25] transmission lines,[35] dislocations in crystals,[14] and surfaces of constant mean curvature.[13] Many mathematicians, physicists, and engineers have contributed to the development of a comprehensive theory of matrix Zakharov–Shabat systems on the line (e.g., Refs. 2, 5, 8, 21, 23, and 31). For a comprehensive theory of the closely related canonical systems on finite intervals or on the half-line, we refer to Ref. 6 and references therein.

In this article we prove the existence of wave operators $W_\pm$ on the direct sum $\mathcal{H}_{m+n}$ of $m+n$ copies of $L^2(\mathbb{R})$ which intertwine between the free Hamiltonian $H_0=iJ(d/dx)$ and the full Hamiltonian $H=H_0-V=iJ(d/dx)-V$ in the sense that

$$W_\pm(\zeta-H_0)^{-1}=(\zeta-H)^{-1}P_{\mathrm{ac}}W_\pm,$$

where P_{ac} is the projection onto the maximal H-invariant subspace which annihilates the eigenvectors and generalized eigenvalues of H corresponding to its nonreal eigenvalues. This result will be obtained under the following natural hypotheses:

(a) there are no spectral singularities;
(b) the number of nonreal eigenvalues of H is finite.

Either condition is satisfied in the defocusing case and, in general, for potentials with sufficiently small L^1-norm. Also, condition (b) follows from condition (a), because if there were an infinite number of eigenvalues they would accumulate toward a point on the real axis [since they are all contained in a compact region (Ref. 21, Theorem 6.1)], which would necessarily be a spectral singularity. Hypothesis (a) implies that the reflection coefficients $R(\lambda)$ and $L(\lambda)$ and transmission coefficients $T_l(\lambda)$ and $T_r(\lambda)$ are continuous functions of $\lambda\in\mathbb{R}$.

In the defocusing case, the free and full Hamiltonians are both self-adjoint on $\mathcal{H}_{m+n}$ and the difference of their resolvents is a trace-class operator (Ref. 9, proof of Theorem 4). Standard time-dependent scattering theory[16,28,32,33] then implies the existence and asymptotic completeness of the wave operators,

$$W_\pm=\lim_{t\rightarrow\pm\infty}P_{\mathrm{ac}}e^{itH}e^{-itH_0}, \tag{1.18}$$

where P_{ac} is the orthogonal projection onto the absolutely continuous subspace of H and the limits are taken in the strong operator topology. As a result, either wave operator $W_\pm$ acts as a unitary equivalence between H_0 and the restriction of H to the range of P_{ac}, while $S=(W_+)^\dagger W_-$ is a unitary operator on $\mathcal{H}_{m+n}$, called the scattering operator. In the focusing case, the Hamiltonian H fails to be self-adjoint and hence traditional methods cannot be applied. For this reason we generalize to the present situation integral representations of wave operators involving limits of free and full Hamiltonian resolvents[33] when λ approaches the real line, thereby relying on the concept of H_0-smoothness introduced by Kato.[17] As in Ref. 17, we then prove the existence and asymptotic completeness of $W_\pm$, but not necessarily under small L^1-norm restrictions. As a result, we prove that, under conditions (a) and (b) above, iH generates a bounded strongly continuous group $\{e^{itH}\}_{t\in\mathbb{R}}$ on $\mathcal{H}_{m+n}$ and that $W_\pm$ can be written in the form (1.18).

In proving H_0-smoothness it is crucial to factorize the matrix potential as $V=W^{(2)}W^{(1)}$ and to show that $W^{(1)}(\zeta-H_0)^{-1}$ and $(\zeta-H_0)^{-1}W^{(2)}$, as well as the Birman–Schwinger-type operator $W^{(1)}$

$\times(\zeta-H_0)^{-1}W^{(2)}$, are Hilbert–Schmidt. By also using a Prüfer transformation argument, Klaus and Shaw[23] have shown the nonexistence of nonreal eigenvalues of the scalar ($m=n=1$) focusing Zakharov–Shabat system if $\|q\|_1 \leq \pi/2$, thus improving on previous nonoptimal bounds.[3,26] Analogous results were obtained for the focusing Manakov system ($m=1$ and $n=2$) in Ref. 20 and for the general system (1.1) in Ref. 21.

In the focusing case H_0 and H are J-self-adjoint on $\mathcal{H}_{m+n}$ but not J-definitizable, i.e., no nontrivial polynomial of H_0 and H is J-non-negative;[7,24] details are given in the Appendix. Thus, although H_0 (as a self-adjoint operator) allows for a spectral theorem, no such result is known to hold (or to follow directly from standard J-self-adjoint operator theory) for H. Nevertheless, under conditions (a) and (b) above, the unitary equivalence of H_0 and the restriction $\tilde{H}$ of H to the maximal invariant subspace where the spectrum is real imply that $\tilde{H}$ is a scalar-type spectral operator (Ref. 12, Chap. XVII, and Ref. 11, Pt. 4). In the Appendix we compute the resolvent and the resolution of the identity of H if conditions (a) and (b) are satisfied.

II. RELATIVE SMOOTHNESS

In this section we define relative smoothness (Ref. 17, Definition 1.2, and also Ref. 33, Chap. 4, where T_0-smoothness is only defined for self-adjoint T_0) and prove that $|V|^{1/2}$ and $|V|^{-1/2}V^\dagger$ are H_0-smooth.

Following Ref. 17, let T_0 be a closed and densely defined linear operator on a complex Hilbert space $\mathcal{H}$ whose spectrum is a subset of the real line, and let A be a closed and densely defined linear operator from $\mathcal{H}$ into the complex Hilbert space $\mathcal{H}'$. Then A is called T_0-*smooth* if the domain $\mathcal{D}(T_0)$ of T_0 is contained in the domain $\mathcal{D}(A)$ of A and

$$\sup_{\varepsilon>0}\int_{-\infty}^{\infty} d\lambda \|A(\lambda \pm i\varepsilon - T_0)^{-1}\phi\|^2 < +\infty, \quad \phi \in \mathcal{H}.$$

Then, for $\varepsilon>0$, the linear operators $L_\varepsilon^\pm:\mathcal{H}\to L^2(\mathbb{R};\mathcal{H}')$ defined by

$$(L_\varepsilon^\pm \phi)(\lambda) = A(\lambda \pm i\varepsilon - T_0)^{-1}\phi$$

are closed and hence bounded, as a result of the closed graph theorem. By the Banach–Steinhaus theorem, the operators $L_\varepsilon^\pm$ are uniformly bounded in $\varepsilon>0$. Hence

$$\left[\int_{-\infty}^{\infty} d\lambda (\|A(\lambda + i\varepsilon - T_0)^{-1}\phi\|^2 + \|A(\lambda - i\varepsilon - T_0)^{-1}\phi\|^2)\right]^{1/2} \leq 2\pi\|A\|_{T_0}\|\phi\|, \tag{2.1a}$$

where $\|A\|_{T_0}$ is the smallest possible constant for which (2.1a) holds for $\phi\in\mathcal{H}$ and $\varepsilon>0$. For each $\phi\in\mathcal{H}$ the vectors $A(\lambda\pm i\varepsilon-T_0)^{-1}\phi$ have nontangential a.e. limits as $\varepsilon\to 0^+$ which we denote by $A(\lambda\pm i0-T_0)^{-1}\phi$. Consequently,

$$\left[\int_{-\infty}^{\infty} d\lambda (\|A(\lambda + i0 - T_0)^{-1}\phi\|^2 + \|A(\lambda - i0 - T_0)^{-1}\phi\|^2)\right]^{1/2} \leq 2\pi\|A\|_{T_0}\|\phi\|, \tag{2.1b}$$

where $\phi\in\mathcal{H}$.

Now let A be T_0-smooth and B be $T_0^\dagger$-smooth. Then under the assumptions that

(i) $Q(\zeta) \overset{\text{def}}{=} A(\zeta-T_0)^{-1}B^\dagger$ is uniformly bounded in $\zeta\in\mathbb{C}\setminus\mathbb{R}$;
(ii) $I-Q(\zeta)$ is invertible for each $\zeta\in\mathbb{C}\setminus\mathbb{R}$;
(iii) $(I-Q(\zeta))^{-1}$ is uniformly bounded in $\zeta\in\mathbb{C}\setminus\mathbb{R}$,

there exists a closed and densely defined linear operator T on $\mathcal{H}$ without nonreal spectrum such that

$$(\zeta - T)^{-1} = (\zeta - T_0)^{-1} + [(\zeta - T_0)^{-1}B^\dagger](I - Q(\zeta))^{-1}[A(\zeta - T_0)^{-1}], \tag{2.2}$$

where $\zeta \in \mathbb{C}\setminus\mathbb{R}$. Then T can be viewed as an extension of $T_0+B^\dagger A$, T has only real spectrum, A is T-smooth, and B is $T^\dagger$-smooth. [in Ref. 17 it is assumed that $\|Q(\zeta)\|<1$ for $\zeta \in \mathbb{C}\setminus\mathbb{R}$ instead of (ii)–(iii), but such a reduction in generality is not required.]

In this paper we will use the following generalization of the conditions (i)–(iii):

(i') $Q(\zeta)=A(\zeta-T_0)^{-1}B^\dagger$ is uniformly bounded in $\zeta \in \mathbb{C}\setminus\mathbb{R}$;
(ii') $I-Q(\zeta)$ is invertible for each ζ on $0<|\mathrm{Im}\ \zeta|\le\delta$;
(iii') $(I-Q(\zeta))^{-1}$ is uniformly bounded in ζ on $0<|\mathrm{Im}\ \zeta|\le\delta$.

As we will argue below, assumptions (a) and (b) will guarantee that the norm limits $I-Q(\lambda\pm i0)$, $\lambda\in\mathbb{R}$, are also invertible and bounded. Then we can prove as in Ref. 17, cf. Theorem 1.5, that there exists a closed and densely defined linear operator T on $\mathcal{H}$ having its spectrum outside the two strips $0<|\mathrm{Im}\ \zeta|\le\delta$ where (2.2) is valid. Then T can be viewed as an extension of $T_0+B^\dagger A$ and A is T-smooth and B is $T^\dagger$-smooth in the sense that (2.1a) holds with the restriction $\varepsilon\le\delta$.

For the sake of convenience, we prove the following two lemmas.[8,9,23]

Lemma 2.1: Let $W(x)$ be an $(m+n)\times(m+n)$ matrix function having its entries in $L^2(\mathbb{R})$. Then for each nonreal ζ the operators $W(\zeta-H_0)^{-1}$ and $(\zeta-H_0)^{-1}W$ are Hilbert–Schmidt on $\mathcal{H}_{m+n}$.

Let us introduce the *Fourier transform map* $\mathbb{F}$ satisfying

$$\hat{\phi}(\xi) = (\mathbb{F}\phi)(\xi) = \int_{-\infty}^{\infty} dx e^{i\xi Jx}\phi(x), \quad \phi(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\xi e^{-i\xi Jx}(\mathbb{F}\phi)(\xi).$$

Proof of Lemma 2.1: We compute

$$(\mathbb{F}(\zeta - H_0)^{-1}W\mathbb{F}^{-1}\hat{\phi})(\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\eta \frac{\hat{W}(\xi-\eta)}{\zeta-\xi}\hat{\phi}(\eta). \tag{2.3}$$

It follows that $(\zeta-H_0)^{-1}W$ is Hilbert–Schmidt because

$$\|(\zeta - H_0)^{-1}W\|_{\mathrm{HS}} = \frac{1}{2\pi}\left[\int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta \frac{\|\hat{W}(\xi-\eta)\|_{\mathrm{HS}}^2}{|\zeta-\xi|^2}\right]^{1/2} = \frac{1}{2\pi}\left[\frac{\pi}{|\mathrm{Im}\ \zeta|}\int_{-\infty}^{\infty} d\eta \|\hat{W}(\eta)\|_{\mathrm{HS}}^2\right]^{1/2} < \infty.$$

Also $W(\zeta-H_0)^{-1}=[(\zeta^*-H_0)^{-1}W^\dagger]^\dagger$ is Hilbert–Schmidt. ∎

Lemma 2.2: Let $W^{(1)}(x)$ and $W^{(2)}(x)$ be $(m+n)\times(m+n)$ matrix functions having their entries in $L^2(\mathbb{R})$. Then, the operator-valued function $\zeta\mapsto W^{(1)}(\zeta-H_0)^{-1}W^{(2)}$ from $\mathbb{C}^+$ into the Hilbert–Schmidt operators is analytic and bounded on $\mathbb{C}^+$ and has a continuous continuation to $\overline{\mathbb{C}^+}$; the analogous result is true on $\mathbb{C}^-$.

Proof: It is easily verified that

$$[(\zeta - H_0)^{-1}\phi](x) = \begin{cases} \begin{pmatrix} i\int_x^{\infty} dy e^{i\zeta(y-x)}\phi^{\mathrm{up}}(y) \\ i\int_{-\infty}^{x} dy e^{i\zeta(x-y)}\phi^{\mathrm{dn}}(y) \end{pmatrix}, & \mathrm{Im}\ \zeta > 0 \\ \begin{pmatrix} -i\int_{-\infty}^{x} dy e^{-i\zeta(x-y)}\phi^{\mathrm{up}}(y) \\ -i\int_x^{\infty} dy e^{-i\zeta(y-x)}\phi^{\mathrm{dn}}(y) \end{pmatrix}, & \mathrm{Im}\ \zeta < 0, \end{cases} \tag{2.4}$$

where $\phi^{\mathrm{up}}(x)=(I_m\ 0_{m\times n})\phi(x)$, $\phi^{\mathrm{dn}}(x)=(0_{n\times m}\ I_n)\phi(x)$, and $\phi\in\mathcal{H}_{m+n}$. Partitioning $W^{(1)}(x)$ and $W^{(2)}(x)$ as follows:[2]

$$W^{(s)}(x)=\begin{pmatrix} W_1^{(s)}(x) & W_2^{(s)}(x) \\ W_3^{(s)}(x) & W_4^{(s)}(x)\end{pmatrix},\quad s=1,2,$$

we see that for Im $\zeta>0$ the operator $-iW^{(1)}(\zeta-H_0)^{-1}W^{(2)}$ is an $(m+n)\times(m+n)$ matrix of integral operators on $L^2(\mathbb{R})$ with matrix integral kernel,

$$\begin{aligned}&\begin{pmatrix} W_1^{(1)}(x) & W_2^{(1)}(x) \\ W_3^{(1)}(x) & W_4^{(1)}(x)\end{pmatrix}\begin{pmatrix} e^{i\zeta(y-x)}\Theta(y-x)I_m & 0_{m\times n} \\ 0_{n\times m} & e^{i\zeta(x-y)}\Theta(x-y)I_n\end{pmatrix}\begin{pmatrix} W_1^{(2)}(y) & W_2^{(2)}(y) \\ W_3^{(2)}(y) & W_4^{(2)}(y)\end{pmatrix}\\ &\quad=e^{i\zeta(y-x)}\Theta(y-x)\begin{pmatrix} W_1^{(1)}(x) \\ W_3^{(1)}(x)\end{pmatrix}(W_1^{(2)}(y)\quad W_2^{(2)}(y))\\ &\qquad+e^{i\zeta(x-y)}\Theta(x-y)\begin{pmatrix} W_2^{(1)}(x) \\ W_4^{(1)}(x)\end{pmatrix}(W_3^{(2)}(y)\quad W_4^{(2)}(y)),\end{aligned}\tag{2.5}$$

where $\Theta(t)$ denotes the Heaviside function. Therefore, its squared Hilbert–Schmidt norm is given by

$$\int_{-\infty}^{\infty}dx\int_x^{\infty}dy e^{-(y-x)\mathrm{Im}\,\zeta}\left\|\begin{pmatrix} W_1^{(1)}(x) \\ W_3^{(1)}(x)\end{pmatrix}(W_1^{(2)}(y)\quad W_2^{(2)}(y))\right\|_{\mathrm{HS}}^2$$
$$+\int_{-\infty}^{\infty}dx\int_{-\infty}^{x}dy e^{-(x-y)\mathrm{Im}\,\zeta}\left\|\begin{pmatrix} W_2^{(1)}(x) \\ W_4^{(1)}(x)\end{pmatrix}(W_3^{(2)}(y)\quad W_4^{(2)}(y))\right\|_{\mathrm{HS}}^2.$$

Here the first double integral is less than (since Im $\zeta>0$)

$$\int_{-\infty}^{\infty}dx\left\|\begin{pmatrix} W_1^{(1)}(x) \\ W_3^{(1)}(x)\end{pmatrix}\right\|_{\mathrm{HS}}^2\cdot\int_{-\infty}^{\infty}dy\|(W_1^{(2)}(y)\quad W_2^{(2)}(y))\|_{\mathrm{HS}}^2.$$

The second double integral can be estimated analogously. Therefore, $\zeta\mapsto W^{(1)}(\zeta-H_0)^{-1}W^{(2)}$ is bounded on $\mathbb{C}^+$. Analyticity is obvious and continuity down to the real line follows from Ref. 30, Theorem 2.21, since the requisite weak continuity is easily established. The proof for $\mathbb{C}^-$ is the same. ■

Using the two polar decompositions,

$$q(x)=U_q(x)[q(x)^\dagger q(x)]^{1/2},\quad r(x)=U_r(x)[r(x)^\dagger r(x)]^{1/2},$$

where $U_q(x)$ and $U_r(x)$ are partial isometries that are measurable in $x\in\mathbb{R}$, we get the polar decomposition $V(x)=U_V(x)|V(x)|$, where

$$U_V(x)=\begin{pmatrix} 0_{m\times m} & iU_q(x) \\ iU_r(x) & 0_{n\times n}\end{pmatrix},\quad |V(x)|=\begin{pmatrix} [r(x)^\dagger r(x)]^{1/2} & 0_{m\times n} \\ 0_{n\times m} & [q(x)^\dagger q(x)]^{1/2}\end{pmatrix}.$$

We now prove that $|V|^{1/2}$ and $|V|^{1/2}U_V^\dagger$ are H_0-smooth.

Theorem 2.3: *Let the entries of $V(x)$ belong to $L^1(\mathbb{R})$. Then $|V|^{1/2}$ and $|V|^{1/2}U_V^\dagger$ are H_0-smooth.*

Proof: It suffices to prove that W is H_0-smooth if W is an $(m+n)\times(m+n)$ matrix having its entries in $L^2(\mathbb{R})$. Using the factorization $W(x)=(W(x)/\|W(x)\|)\|W(x)\|$, where the first factor has norm ≤ 1 and $\|W(x)\|$ acts as a scalar multiplication operator, we obtain, for any $\phi\in\mathcal{H}_{m+n}$,

$$\int_{-\infty}^{\infty} d\lambda \|W(\lambda \pm i\varepsilon - H_0)^{-1}\phi\|_2^2 \leq \int_{-\infty}^{\infty} d\lambda \| \|W(x)\| [(\lambda \pm i\varepsilon - H_0)^{-1}\phi]^{\mathrm{up}}\|_2^2 + \int_{-\infty}^{\infty} d\lambda \| \|W(x)\| [(\lambda \pm i\varepsilon - H_0)^{-1}\phi]^{\mathrm{dn}}\|_2^2.$$

Now, by (2.4), Fubini's theorem, and Parseval's equation, we get for the (+) sign

$$\int_{-\infty}^{\infty} d\lambda \| \|W(x)\| [(\lambda + i\varepsilon - H_0)^{-1}\phi]^{\mathrm{up}}\|_2^2 = \int_{-\infty}^{\infty} dx \|W(x)\|^2 e^{2\varepsilon x} \int_{-\infty}^{\infty} d\lambda \left\| \int_x^{\infty} dy e^{(i\lambda - \varepsilon)y} \phi^{\mathrm{up}}(y) \right\|^2$$
$$= 2\pi \int_{-\infty}^{\infty} dx \|W(x)\|^2 e^{2\varepsilon x} \int_x^{\infty} dy e^{-2\varepsilon y} \|\phi^{\mathrm{up}}(y)\|^2 \leq 2\pi \|W\|_2^2 \|\phi^{\mathrm{up}}\|^2,$$

where $\|W\|_2^2 = \int_{-\infty}^{\infty} dx \|W(x)\|^2$. Similarly, we find

$$\int_{-\infty}^{\infty} d\lambda \| \|W(x)\| [(\lambda + i\varepsilon - H_0)^{-1}\phi]^{\mathrm{dn}}\|_2^2 \leq 2\pi \|W\|_2^2 \|\phi^{\mathrm{dn}}\|^2.$$

For the $(-)$ sign, we get the same results. Hence

$$\int_{-\infty}^{\infty} d\lambda \|W(\lambda \pm i\varepsilon - H_0)^{-1}\phi\|_2^2 \leq 2\pi \|W\|_2^2 \|\phi\|^2,$$

from which we see that W is H_0-smooth and, by (2.1a),

$$\|W\|_{H_0} \leq \frac{1}{\sqrt{\pi}} \|W\|_2.$$

■

The following result shows that the conditions (ii') and (iii') are valid for ζ up to the real axis.

Theorem 2.4: *Let the entries of $V(x)$ belong to $L^1(\mathbb{R})$. Then there is an $r_0 \geq 0$ such that $(I - Q(\zeta))^{-1}$ is uniformly norm bounded in ζ in $\{\zeta \in \overline{\mathbb{C}^+} : |\zeta| \geq r_0\}$. An analogous result holds in the lower half plane.*

Proof: It suffices to show that $\|Q(\zeta)^2\|_{\mathrm{HS}}$ goes to zero as $|\zeta| \to \infty$ in $\overline{\mathbb{C}^+}$ because then the claim follows from $(I - Q(\zeta))^{-1} = (I + Q(\zeta))(I - Q(\zeta)^2)^{-1}$. Note that from Lemma 2.2 we know that $Q(\zeta)$ is bounded.

To estimate $Q(\zeta)^2$ we first observe that $Q(\zeta)^2$ is a block diagonal operator with integral kernel $[Q(\zeta)^2](x,y) = \mathrm{diag}(R_1(x,y;\zeta), R_2(x,y;\zeta))$, where

$$R_1(x,y;\zeta) = (r(x)^\dagger r(x))^{1/4} H_1(x,y;\zeta) U_r(y) (r(y)^\dagger r(y))^{1/4}.$$

Here

$$H_1(x,y;\zeta) = \begin{cases} e^{i\zeta(x-y)} G_1(x;\zeta) & x > y \\ e^{i\zeta(y-x)} G_1(y;\zeta) & x < y, \end{cases}$$

$$G_1(x;\zeta) = e^{-2i\zeta x} \int_x^{\infty} e^{2i\zeta z} q(z) dz,$$

$$R_2(x,y;\zeta) = (q(x)^\dagger q(x))^{1/4} H_2(x,y;\zeta) U_q(y) (q(y)^\dagger q(y))^{1/4},$$

where

$$H_2(x,y;\zeta) = \begin{cases} e^{i\zeta(x-y)}G_2(y;\zeta), & x > y \\ e^{i\zeta(y-x)}G_2(x;\zeta), & x < y, \end{cases}$$

$$G_2(x;\zeta) = e^{2i\zeta x}\int_{-\infty}^{x} e^{-2i\zeta z}r(z)dz.$$

It suffices to estimate the norm of the integral operator $R_1(x,y;\zeta)$. To this end, we note that by an application of the Riemann–Lebesgue lemma (approximate r by a smooth function of compact support and integrate by parts), we have that

$$\sup_{x\in\mathbb{R}}\|G_1(x;\zeta)\| \to 0, \quad \sup_{x\in\mathbb{R}}\|G_2(x;\zeta)\| \to 0, \quad |\zeta| \to \infty.$$

Similar arguments have been used before to estimate the Jost solutions (Ref. 8, Sec. 3.2, and Ref. 19, Sec. 2) or to find bounds on the location of eigenvalues (Ref. 19, Theorem 3.1, and Ref. 21, Theorem 6.1). Put $M_1(\zeta)=\sup_{x\in\mathbb{R}}\|G_1(x;\zeta)\|$ and $\text{Im }\zeta=\tau$. Then we can estimate the matrix norm of the kernel $R_1(x,y;\zeta)$ by

$$\|R_1(x,y;\zeta)\| \le \begin{cases} M_1(\zeta)\|r(x)\|^{1/2}e^{-\tau(x-y)}\|r(y)\|^{1/2}, & x > y \\ M_1(\zeta)\|r(x)\|^{1/2}e^{-\tau(y-x)}\|r(y)\|^{1/2}, & x < y. \end{cases}$$

By Lemma 2.2, the right-hand side is a Hilbert–Schmidt kernel whose Hilbert–Schmidt norm is less than $(1/\sqrt{2})M_1(\zeta)\int_{-\infty}^{\infty}\|r(x)\|dx$. As a result, the norm of the kernel $R_1(x,y;\zeta)$ tends to zero as $|\zeta|\to\infty$. The same proof works for $R_2(x,y;\zeta)$ and for $\text{Im }\zeta<0$. ■

Putting $T_0=H_0$, $A=|V|^{1/2}$, $B=-|V|^{1/2}U_V^\dagger$, and hence $H=T=T_0+B^\dagger A$ (on suitable domains), we can now conclude that conditions (i')–(iii') are satisfied, and that $I-Q(\lambda)$ [$=W(\zeta)$; cf. (2.6a) below] is invertible for all ζ within some strip $0<|\text{Im }\zeta|\le\delta$. Moreover, $W(\zeta)$ remains invertible in the limit as ζ approaches the real axis from either above or below.

First, (i') is a direct consequence of Lemma 2.2. Then putting

$$W(\zeta) = I + |V|^{1/2}(\zeta - H_0)^{-1}U_V|V|^{1/2}, \tag{2.6a}$$

we get

$$(\zeta - H)^{-1} - (\zeta - H_0)^{-1} = -(\zeta - H_0)^{-1}U_V|V|^{1/2}W(\zeta)^{-1}|V|^{1/2}(\zeta - H_0)^{-1}. \tag{2.6b}$$

Since $|V|^{1/2}$ and $U_V|V|^{1/2}$ have their entries in $L^2(\mathbb{R})$, by Lemma 2.1, the operators $(\zeta-H_0)^{-1}U_V|V|^{1/2}$ and $|V|^{1/2}(\zeta-H_0)^{-1}$ are Hilbert–Schmidt for nonreal ζ. Therefore, $(\zeta-H)^{-1}-(\zeta-H_0)^{-1}$ is the product of two Hilbert–Schmidt operators and hence trace class. Moreover, according to Lemma 2.2, $W(\zeta)-I$ is Hilbert–Schmidt for every nonreal ζ as are the limits $W(\lambda\pm i0)-I$ for $\lambda\in\mathbb{R}$. Finally, (2.6) imply that ζ is a nonreal eigenvalue of H if and only if $W(\zeta)$ is noninvertible. This result extends to spectral singularities: $\lambda\in\mathbb{R}$ is a spectral singularity if and only if $W(\lambda+i0)$ is noninvertible, which is true if and only if $W(\lambda-i0)$ is noninvertible. The proof given in Ref. 19, Lemma 4.4, extends to the general matrix case. Recall that by our definition of spectral singularity given below (1.6) together with (1.9), λ is a singularity of $T_l(\zeta)$ and $T_r(\zeta)$. Consequently, by (1.16), it is also a singularity of $\breve{T}_l(\zeta)$ and $\breve{T}_r(\zeta)$, which explains why $W(\lambda\pm i0)$ are both noninvertible. Since spectral singularities are ruled out by assumption (a) and eigenvalues cannot accumulate toward the real axis by assumption (b), there must be a strip of some width $\delta>0$ where (ii') and (iii') are true; the uniform boundedness in (iii') follows from Theorem 2.4.

Let us apply Lemma 2.2 and, in particular, (2.5) to the case $W^{(1)}(x)=|V(x)|^{1/2}$ and $W^{(2)}(y)=U_V(y)|V(y)|^{1/2}$, where $\zeta=\sigma+i\tau$ with $\sigma\in\mathbb{R}$ and $\tau>0$, and write (2.5) in the form (cf. Ref. 23, Theorem 4.2)

$$-\begin{pmatrix} e^{-i\sigma x}I_m & 0_{m\times n} \\ 0_{n\times m} & e^{i\sigma x}I_n \end{pmatrix}\begin{pmatrix} 0_{m\times m} & w_{12}(x,y) \\ w_{21}(x,y) & 0_{n\times n} \end{pmatrix}\begin{pmatrix} e^{-i\sigma y}I_m & 0_{m\times n} \\ 0_{n\times m} & e^{i\sigma y}I_n \end{pmatrix}, \tag{2.7}$$

where the outer matrix factors are premultiplications by unitary matrix functions which do not affect the operator or Hilbert–Schmidt norm of the integral operator, and

$$w_{12}(x,y) = e^{-\tau(y-x)}\Theta(y-x)[r(x)^\dagger r(x)]^{1/4}U_q(y)[q(y)^\dagger q(y)]^{1/4},$$

$$w_{21}(x,y) = e^{-\tau(x-y)}\Theta(x-y)[q(x)^\dagger q(x)]^{1/4}U_r(y)[r(y)^\dagger r(y)]^{1/4}.$$

Therefore, the Hilbert–Schmidt norm of the integral operator with kernel $w_{12}(x,y)$ obeys

$$\begin{aligned}\|w_{12}\|_{\mathrm{HS}}^2 &= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\ \mathrm{tr}[w_{12}(x,y)^\dagger w_{12}(x,y)] \le \int_{-\infty}^{\infty} dx \int_{x}^{\infty} dy e^{-2\tau(y-x)}\|r(x)\|_{\mathrm{HS}}\|q(y)\|_{\mathrm{HS}} \\ &\le \frac{1}{2}\left[\int_{-\infty}^{\infty} dx\ \max(\|q(x)\|_{\mathrm{HS}},\|r(x)\|_{\mathrm{HS}})\right]^2. \end{aligned}\tag{2.8}$$

A similar estimate holds for $\tau=\mathrm{Im}\ \zeta<0$. Estimate (2.8) is also valid for w_{21}. Since the operator norm of the middle factor in (2.7) is equal to $\max(\|w_{12}\|,\|w_{21}\|)$, it is bounded by the square root of the right-hand side of (2.8). As a result, if the right-hand side of (2.8) is less than 1, that is,

$$\int_{-\infty}^{\infty} dx\ \max(\|q(x)\|_{\mathrm{HS}},\|r(x)\|_{\mathrm{HS}}) < \sqrt{2},$$

then the Hamiltonian H has no nonreal eigenvalues and no spectral singularities, and therefore the transmission and reflection coefficients are continuous on the real line. A drawback of this result is that it is not optimal in the sense that the constant $\sqrt{2}$ can, in fact, be replaced by a larger constant, namely, $\pi/2$, and instead of the Hilbert–Schmidt norms of $q(x)$ and $r(x)$ we can take their uniform matrix norms. This result is proven in Ref. 21 by using a different factorization for W. For the convenience of the reader we rederive it here using the factorization in (2.7). Let

$$m(x) = \max(\|q(x)\|,\|r(x)\|)$$

($\|\cdot\|$ denotes the uniform matrix norm). Then $w_{12}(x,y)$ satisfies the estimate (for $\tau\ge 0$)

$$\|w_{12}(x,y)\| \le e^{-\tau(y-x)}\Theta(y-x)\|r(x)\|^{1/2}\|q(y)\|^{1/2} \le e^{-\tau(y-x)}\Theta(y-x)m(x)^{1/2}m(y)^{1/2} \overset{\mathrm{def}}{=} \beta(x,y) \tag{2.9}$$

and $m\in L^1(\mathbb{R})$. Now the norm of the operator associated with the kernel $\beta(x,y)$ is less than $(2/\pi)\|m\|_1$ [where $\|m\|_1=\int_{\mathbb{R}}\|m(x)\|dx$] if $\tau>0$, and it becomes equal to this value in the limit as $\tau\to 0$. This result follows from Ref. 19, cf. (4.10), and Ref. 23, proof of Theorem 4.2. Alternatively, it suffices to note that for $\tau>0$, $\beta\beta^\dagger$ has a non-negative symmetric kernel which is also positivity improving on the essential support of $m(x)$. When $\tau=0$, $\beta\beta^\dagger$ has eigenvalue η_0^2, where $\eta_0=(2/\pi)\|m\|_1$, with corresponding non-negative eigenfunction $m(x)^{1/2}\cos((1/\eta_0)\int_{-\infty}^{x}m(y)dy)$. Hence, in view of the variational characterization of the largest eigenvalue and Ref. 32, Theorem 10.32, we conclude that

$$\|w_{12}\| \le \frac{2}{\pi}\|m\|_1,$$

with strict inequality holding when $\tau>0$. The same bound holds for $\|w_{21}\|$.

III. WAVE OPERATORS: STATIONARY THEORY

In this section we prove the existence and asymptotic completeness of the wave operators $W_\pm$ if the matrix Zakharov–Shabat Hamiltonian does not have spectral singularities and therefore has only finitely many nonreal eigenvalues. We follow the proof given in Ref. 17. In the defocusing case we can rely on Ref. 33, Theorem IV 6.2, or on Pearson's theorem (Ref. 32, Satz 22.19) to arrive at the same result.

Suppose the Hamiltonian H satisfies the conditions (a) and (b) in Sec. I. Let P_{ac} denote the projection commuting with H onto the maximal H-invariant subspace of $\mathcal{H}_{m+n}$ that does not contain any eigenvectors and generalized eigenvectors corresponding to nonreal eigenvalues. Put

$$W_\pm = P_{\mathrm{ac}} + X_\pm, \quad Z_\pm = P_{\mathrm{ac}} + Y_\pm,$$

where

$$\langle X_\pm \phi, \psi\rangle = \pm \frac{1}{2\pi i}\int_{-\infty}^{\infty} d\lambda \langle |V|^{1/2}(\lambda \pm i0 - H_0)^{-1}\phi, |V|^{1/2}U_V^\dagger[(\lambda \mp i0 - H)^{-1}P_{\mathrm{ac}}]^\dagger \psi\rangle, \tag{3.1a}$$

$$\langle Y_\pm \phi, \psi\rangle = \mp \frac{1}{2\pi i}\int_{-\infty}^{\infty} d\lambda \langle |V|^{1/2}(\lambda \pm i0 - H)^{-1}P_{\mathrm{ac}}\phi, |V|^{1/2}U_V^\dagger[(\lambda \mp i0 - H_0)^{-1}]^\dagger \psi\rangle. \tag{3.1b}$$

Either integral converges absolutely for $\phi, \psi \in \mathcal{H}$, since $|V|^{1/2}$ and $|V|^{1/2}U_V^\dagger$ are H_0-smooth, $|V|^{1/2}$ is H-smooth, and $|V|^{1/2}U_V^\dagger$ is $H^\dagger$-smooth. We also adopt from now on the convention that the inner product in $\mathcal{H}$ is conjugate linear in the second factor. This will be needed for computations involving contour integrations.

In the absence of nonreal eigenvalues of H, the following result is immediate from Ref. 17, cf. Lemma 2.4 and Lemma 2.5, with the help of Theorem 2.3. We will prove Theorem 3.1 by similar methods.

Theorem 3.1: *Suppose H has no spectral singularities. Then the wave operators $W_\pm$ and $Z_\pm$ given by (3.1) are well defined and satisfy*

$$W_\pm Z_\pm = P_{\mathrm{ac}}, \quad Z_\pm W_\pm = I_{\mathcal{H}}, \tag{3.2a}$$

$$W_\pm(\zeta - H_0)^{-1} = (\zeta - H)^{-1}P_{\mathrm{ac}}W_\pm, \tag{3.2b}$$

$$Z_\pm(\zeta - H)^{-1}P_{\mathrm{ac}} = (\zeta - H_0)^{-1}Z_\pm. \tag{3.2c}$$

Proof: The assumption is equivalent to the statement that $W(\lambda \pm i0)$ is invertible for each $\lambda \in \mathbb{R}$. By using Theorem 2.4 and (2.2) we also conclude that under the assumption of the theorem, $|V|^{1/2}$ is H-smooth and $|V|^{1/2}U_V^\dagger$ is $H^\dagger$-smooth.

We first prove (3.2b) and then (3.2a). Note that then (3.2c) follows by multiplying (3.2b) on the left and right by $Z_\pm$.

(1) Put $A = |V|^{1/2}$ and $B = -|V|^{1/2}U_V^\dagger$. By replacing the vector ϕ by $(\zeta - H_0)^{-1}\phi$ with $\mathrm{Im}\,\zeta > 0$ in (3.1a), using

$$(\lambda + i0 - H_0)^{-1}(\zeta - H_0)^{-1} = \frac{(\zeta - H_0)^{-1} - (\lambda + i0 - H_0)^{-1}}{\lambda - \zeta}, \tag{3.3}$$

and applying Cauchy's theorem to prove that

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} d\lambda \frac{\langle A(\zeta - H_0)^{-1}\phi, B[(\lambda - i0 - H)^{-1}P_{\rm ac}]^\dagger \psi\rangle}{\lambda - \zeta} = 0, \tag{3.4}$$

we get

$$\langle X_+(\zeta - H_0)^{-1}\phi,\psi\rangle = \frac{1}{2\pi i}\int_{-\infty}^{\infty} d\lambda \frac{\langle A(\lambda + i0 - H_0)^{-1}\phi, B[(\lambda - i0 - H)^{-1}P_{\rm ac}]^\dagger \psi\rangle}{\lambda - \zeta}. \tag{3.5}$$

The step involving Cauchy's theorem is justified by choosing a large semicircle in the lower half plane to obtain a closed contour and noting that $(\lambda - i\varepsilon - H)^{-1}P_{\rm ac}$ has no singularities in $\mathbb{C}^-$ and that by Theorem 2.4 and (2.2) the contribution from the semicircle goes to zero as its radius tends to infinity. Alternatively, we can use the fact that since B is $H^\dagger$-smooth, the numerator underneath the integral sign in (3.4) is the boundary value of an element in the Hardy space $H^2(\mathbb{C}^-)$, whereas $\lambda \mapsto (\lambda - \zeta^*)^{-1}$ belongs to $H^2(\mathbb{C}^+)$ because $\operatorname{Im} \zeta^* < 0$. Since, the boundary values of functions in $H^2(\mathbb{C}^+)$ and $H^2(\mathbb{C}^-)$ are orthogonal, the integral in (3.4) is zero.

On the other hand, using (3.1a) and (3.3), with $i0 - H_0$ replaced by $-i0 - H$, we get

$$\begin{aligned}
&\langle X_+\phi, [(\zeta - H)^{-1}P_{\rm ac}]^\dagger\psi\rangle \\
&\quad = -\frac{1}{2\pi i}\int_{-\infty}^{\infty} d\lambda \langle A(\lambda + i0 - H_0)^{-1}\phi, B[(\lambda - i0 - H)^{-1}P_{\rm ac}]^\dagger[(\zeta - H)^{-1}P_{\rm ac}]^\dagger\psi\rangle \\
&\quad = \frac{1}{2\pi i}\int_{-\infty}^{\infty} d\lambda \frac{\langle A(\lambda + i0 - H_0)^{-1}\phi, B[(\lambda - i0 - H)^{-1}P_{\rm ac}]^\dagger\psi\rangle}{\lambda - \zeta} \\
&\quad - \frac{1}{2\pi i}\int_{-\infty}^{\infty} d\lambda \frac{\langle A(\lambda + i0 - H_0)^{-1}\phi, B[(\zeta - H)^{-1}P_{\rm ac}]^\dagger\psi\rangle}{\lambda - \zeta}.
\end{aligned}$$

Using (3.5) and applying Cauchy's integral formula to the last integral on the right-hand side (closing the contour in the upper half plane), we get

$$\begin{aligned}
\langle X_+\phi, [(\zeta - H)^{-1}P_{\rm ac}]^\dagger\psi\rangle - \langle X_+(\zeta - H_0)^{-1}\phi,\psi\rangle &= -\langle A(\zeta - H_0)^{-1}\phi, B[(\zeta - H)^{-1}P_{\rm ac}]^\dagger\psi\rangle = -\langle[(\zeta \\
&\quad - H)^{-1}P_{\rm ac}B^\dagger]A(\zeta - H_0)^{-1}\phi,\psi\rangle = -\langle P_{\rm ac}(\zeta \\
&\quad - H)^{-1}\phi,\psi\rangle + \langle P_{\rm ac}(\zeta - H_0)^{-1}\phi,\psi\rangle,
\end{aligned}$$

which implies that

$$(\zeta - H)^{-1}P_{\rm ac}X_+ - X_+(\zeta - H_0)^{-1} = -(\zeta - H)^{-1}P_{\rm ac} + P_{\rm ac}(\zeta - H_0)^{-1}.$$

Using $W_+ = P_{\rm ac} + X_+$ we get one of (3.2b). The other one is proven likewise.

(2) In (3.1b) with the plus sign we replace ψ by $X_+^\dagger\psi$. We get

$$\begin{aligned}
\langle Y_+\phi, X_+^\dagger\psi\rangle &= \lim_{\varepsilon\to 0^+}\frac{1}{2\pi i}\int_{-\infty}^{\infty} d\mu \langle A(\mu + i\varepsilon - H)^{-1}P_{\rm ac}\phi, B[(\mu - i\varepsilon - H_0)^{-1}]^\dagger X_+^\dagger\psi\rangle \\
&= \lim_{\varepsilon\to 0^+}\frac{1}{2\pi i}\int_{-\infty}^{\infty} d\mu \langle X_+[(\mu - i\varepsilon - H_0)^{-1}B^\dagger]A(\mu + i\varepsilon - H)^{-1}P_{\rm ac}\phi,\psi\rangle \\
&= \lim_{\varepsilon\to 0^+}\frac{1}{2\pi i}\int_{-\infty}^{\infty} d\mu \frac{-1}{2\pi i}\int_{-\infty}^{\infty} d\lambda \langle F(\lambda,\mu,\varepsilon)\phi, [B(\lambda - i0 - H)^{-1}P_{\rm ac}]^\dagger\psi\rangle,
\end{aligned}$$

where

$$F(\lambda,\mu,\varepsilon) = A(\lambda + i0 - H_0)^{-1}[(\mu - i\varepsilon - H_0)^{-1}B^\dagger]A(\mu + i\varepsilon - H)^{-1}P_{\mathrm{ac}}.$$

Now

$$\begin{aligned}A(\lambda + i0 - H_0)^{-1}[(\mu - i\varepsilon - H_0)^{-1}B^\dagger]w &= A[(\lambda + i0 - H_0)^{-1}(\mu - i\varepsilon - H_0)^{-1}B^\dagger]w \\ &= \frac{A[(\mu - i\varepsilon - H_0)^{-1}B^\dagger]w - A[(\lambda + i0 - H_0)^{-1}B^\dagger]w}{\lambda - \mu + i\varepsilon} \\ &= \frac{Q(\mu - i\varepsilon)w - Q(\lambda + i0)w}{\lambda - \mu + i\varepsilon},\end{aligned}$$

where $Q(\zeta)=A(\zeta-H_0)^{-1}B^\dagger$. Therefore, apart from terms vanishing as $\varepsilon\to 0^+$, we have

$$\begin{aligned}\langle Y_+\phi, X_+^\dagger\psi\rangle \simeq{}& \frac{1}{4\pi^2}\int\int d\mu d\lambda \frac{\langle Q(\mu - i\varepsilon)A(\mu + i\varepsilon - H)^{-1}P_{\mathrm{ac}}\phi, B[(\lambda - i0 - H)^{-1}P_{\mathrm{ac}}]^\dagger\psi\rangle}{\lambda - \mu + i\varepsilon} \\ &- \frac{1}{4\pi^2}\int\int d\mu d\lambda \frac{\langle Q(\lambda + i0)A(\mu + i\varepsilon - H)^{-1}P_{\mathrm{ac}}\phi, B[(\lambda - i0 - H)^{-1}P_{\mathrm{ac}}]^\dagger\psi\rangle}{\lambda - \mu + i\varepsilon},\end{aligned}$$

where the integrals are defined with respect to $\lambda,\mu\in\mathbb{R}$. In the first term we evaluate the integral with respect to λ, and in the second term we change the order of integration and evaluate the integral with respect to μ. As a result, apart from terms vanishing as $\varepsilon\to 0^+$ we have

$$\begin{aligned}\langle Y_+\phi, X_+^\dagger\psi\rangle \simeq{}& \frac{1}{2\pi i}\int_{-\infty}^{\infty} d\mu \langle Q(\mu - i\varepsilon)A(\mu + i\varepsilon - H)^{-1}P_{\mathrm{ac}}\phi, B[(\mu - i\varepsilon - H)^{-1}P_{\mathrm{ac}}]^\dagger\psi\rangle \\ &- \frac{1}{2\pi i}\int_{-\infty}^{\infty} d\lambda \langle Q(\lambda + i0)A(\lambda + 2i\varepsilon - H)^{-1}P_{\mathrm{ac}}\phi, B[(\lambda - i0 - H)^{-1}P_{\mathrm{ac}}]^\dagger\psi\rangle.\end{aligned}$$

Taking $\varepsilon\to 0^+$ under the integral signs we get

$$\begin{aligned}\langle Y_+\phi, X_+^\dagger\psi\rangle ={}& \frac{1}{2\pi i}\int_{-\infty}^{\infty} d\mu \langle Q(\mu - i0)A(\mu + i0 - H)^{-1}P_{\mathrm{ac}}\phi, B[(\mu - i0 - H)^{-1}P_{\mathrm{ac}}]^\dagger\psi\rangle \\ &- \frac{1}{2\pi i}\int_{-\infty}^{\infty} d\lambda \langle Q(\lambda + i0)A(\lambda + i0 - H)^{-1}P_{\mathrm{ac}}\phi, B[(\lambda - i0 - H)^{-1}P_{\mathrm{ac}}]^\dagger\psi\rangle.\end{aligned}$$

Using (2.2) we get

$$\begin{aligned}\langle Y_+\phi, X_+^\dagger\psi\rangle ={}& \frac{1}{2\pi i}\int_{-\infty}^{\infty} d\lambda \\ &\times \langle [A(\lambda + i0 - H_0)^{-1} - A(\lambda + i0 - H)^{-1}]P_{\mathrm{ac}}\phi, B[(\lambda - i0 - H)^{-1}P_{\mathrm{ac}}]^\dagger\psi\rangle \\ &+ \frac{1}{2\pi i}\int_{-\infty}^{\infty} d\mu \\ &\times \langle A(\mu + i0 - H)^{-1}P_{\mathrm{ac}}\phi, [-B(\mu + i0 - H_0^\dagger)^{-1}P_{\mathrm{ac}}^\dagger + B[(\mu - i0 - H)^{-1}P_{\mathrm{ac}}]^\dagger]\psi\rangle \\ ={}& \frac{1}{2\pi i}\int_{-\infty}^{\infty} d\lambda [\langle A(\lambda + i0 - H_0)^{-1}P_{\mathrm{ac}}\phi, B[(\lambda - i0 - H)^{-1}P_{\mathrm{ac}}]^\dagger\psi\rangle \\ &- \langle A(\lambda + i0 - H)^{-1}P_{\mathrm{ac}}\phi, B(\lambda + i0 - H_0)^{-1}P_{\mathrm{ac}}^\dagger\psi\rangle] \\ ={}& -\langle X_+P_{\mathrm{ac}}\phi, \psi\rangle - \langle Y_+\phi, P_{\mathrm{ac}}^\dagger\psi\rangle.\end{aligned}$$

Therefore, $X_+Y_+ + X_+ + Y_+ = 0$, which implies

$$W_+Z_+ = (P_{\mathrm{ac}} + X_+)(P_{\mathrm{ac}} + Y_+) = P_{\mathrm{ac}}.$$

In the same way we prove that $Z_+W_+ = I_{\mathcal{H}}$ (alternatively, following Ref. 17, p. 267, we can replace V by κV and use analyticity in κ to prove that $Z_\pm : \mathrm{Im}\, P_{\mathrm{ac}} \to \mathcal{H}_{m+n}$ and $W_\pm : \mathcal{H}_{m+n} \to \mathrm{Im}\, P_{\mathrm{ac}}$ are each other's inverses) as claimed. ∎

We note that in the defocusing case, where H is self-adjoint and has only real spectrum, the assumption of Theorem 3.1 implies that the spectrum of H is absolutely continuous (Ref. 18, Theorem 2.4), which was proven before in Ref. 9 for potentials whose entries $w(x)$ satisfy $\int_{-\infty}^{\infty} dx (1+|x|)|w(x)| < +\infty$.

IV. WAVE OPERATORS: TIME-DEPENDENT THEORY

In this section we write the wave operators obtained under the conditions of Theorem 3.5 in the more familiar time-dependent form.

Suppose T_0 is a closed and densely defined linear operator on a complex Hilbert space $\mathcal{H}$ with only real spectrum and assume that iT_0 generates a bounded strongly continuous group on $\mathcal{H}$. Writing

$$-i\int_0^\infty dt e^{i\lambda t} e^{-\varepsilon t} e^{-itT_0} x = (\lambda + i\varepsilon - T_0)^{-1} x, \tag{4.1a}$$

$$+i\int_{-\infty}^0 dt e^{i\lambda t} e^{\varepsilon t} e^{-itT_0} x = (\lambda - i\varepsilon - T_0)^{-1} x, \tag{4.1b}$$

where $\varepsilon > 0$, $x \in \mathcal{H}$, and $\lambda \in \mathbb{R}$, we see that a closed linear operator from $\mathcal{H}$ into the complex Hilbert space $\mathcal{H}'$ is T_0-smooth if and only if

$$\left[\int_{-\infty}^{\infty} dt \|A e^{-itT_0} x\|^2\right]^{1/2} \leq \mathrm{const}\,.\,\|x\|, \quad x \in \mathcal{H},$$

where $\|A\|_{T_0}$ is the smallest such constant. For self-adjoint T_0, various characterizations of T_0-smoothness and $\|A\|_{T_0}$ were given in Ref. 18 (also Ref. 33, Theorem 4.3.1).

Lemma 4.1: Suppose there are no spectral singularities. Then $-iH$ generates a bounded strongly continuous group on $\mathcal{H}_{m+n}$.

Proof: Under the hypotheses of this lemma, $|V|^{1/2}U_V^\dagger$ is H_0-smooth and $|V|^{1/2}$ is H-smooth. We may then apply (4.1a) in the form

$$-i\int_0^\infty dt e^{i\lambda t} e^{-\varepsilon t} |V|^{1/2} e^{-itH}\phi = |V|^{1/2}(\lambda + i\varepsilon - H)^{-1}\phi$$

to define the vector-valued L^2-function $|V|^{1/2}e^{-itH}\phi$ of $t \in \mathbb{R}$ for each $\phi \in \mathcal{H}_{m+n}$. Likewise we define the vector-valued L^2-function $|V|^{1/2}U_V^\dagger e^{itH_0}\psi$ for each $\psi \in \mathcal{H}_{m+n}$. We now define for $\phi, \psi \in \mathcal{H}_{m+n}$

$$\langle e^{-itH}\phi, \psi\rangle \overset{\mathrm{def}}{=} \langle e^{-itH_0}\phi, \psi\rangle - i\int_0^t ds \langle |V|^{1/2} e^{-isH}\phi, |V|^{1/2} U_V^\dagger e^{i(t-s)H_0}\psi\rangle.$$

As a result, the operator e^{-itH} so defined depends continuously on $t \in \mathbb{R}$ in the weak operator topology and is uniformly bounded in $t \in \mathbb{R}$. Taking the Fourier transform we get the resolvent identity involving H_0 and H. This in turn implies that e^{-itH} has the group property. Using Ref. 27, Theorem II 1.3, we see that $\{e^{-itH}\}_{t\in\mathbb{R}^+}$ is a bounded strongly continuous semigroup. In the same way we prove that $\{e^{itH}\}_{t\in\mathbb{R}^+}$ is a bounded strongly continuous semigroup. ∎

Theorem 4.7: *Suppose there are no spectral singularities. Then the wave operators $W_\pm$ and $Z_\pm$ satisfy*

$$W_{\pm} = \lim_{t\to\pm\infty} P_{\mathrm{ac}} e^{itH} e^{-itH_0}, \tag{4.2a}$$

$$Z_{\pm} = \lim_{t\to\pm\infty} e^{itH_0} e^{-itH} P_{\mathrm{ac}}, \tag{4.2b}$$

where the limits are taken in the strong operator topology on $\mathcal{H}_{m+n}$.

Proof: According to Lemma 4.1, we may assume that iH (and hence also $-iH^{\dagger}$) generates a bounded strongly continuous group. Following the derivation of Cook's lemma (Ref. 32, Theorem 11.7), we have for $\phi, \psi \in \mathcal{H}$

$$\frac{d}{dt}\langle P_{\mathrm{ac}} e^{itH} e^{-itH_0}\phi, \psi\rangle = -i\langle |V|^{1/2} e^{-itH_0}\phi, |V|^{1/2} U_V^{\dagger} e^{-itH^{\dagger}} P_{\mathrm{ac}}^{\dagger}\psi\rangle, \tag{4.3}$$

where the H_0-smoothness of $|V|^{1/2}$ and the $H^{\dagger}$-smoothness of $|V|^{1/2}U_V^{\dagger}$ imply that the right-hand side belongs to $L^1(\mathbb{R};dt)$ for each $\phi, \psi \in \mathcal{H}$. Integrating (4.3) over (t,∞) or $(-\infty,t)$, we see that $\langle P_{\mathrm{ac}} e^{itH} e^{-itH_0}\phi, \psi\rangle$ is absolutely continuous in $t\in\mathbb{R}$ and has finite limits $\langle \tilde{\Omega}_{\pm}\phi, \psi\rangle$ as $t\to\pm\infty$, which proves the existence of the limits in (4.2a) in the strong operator topology (Ref. 27, Theorem II 1.3). In fact,

$$\langle \tilde{\Omega}_{\pm}\phi, \psi\rangle = \langle \phi, \psi\rangle \mp i\int_{\mathbb{R}^{\pm}} ds\langle |V|^{1/2} e^{-isH_0}\phi, |V|^{1/2} U_V^{\dagger} e^{-isH^{\dagger}} P_{\mathrm{ac}}^{\dagger}\psi\rangle.$$

Using (4.1) we get

$$\begin{aligned}\langle \tilde{\Omega}_{\pm}\phi, \psi\rangle &= \langle \phi, \psi\rangle \mp \frac{1}{2\pi i}\int_{-\infty}^{\infty} d\lambda\langle |V|^{1/2}(\lambda \pm i0 - H_0)^{-1}\phi, |V|^{1/2} U_V^{\dagger}[(\lambda \mp i0 - H)^{-1} P_{\mathrm{ac}}]^{\dagger}\psi\rangle \\ &= \langle W_{\pm}\phi, \psi\rangle,\end{aligned}$$

which proves (4.2a). Equation (4.2b) is proven in the same way. ∎

V. EXAMPLES

In this section we discuss two examples which shed some light on what happens when spectral singularities or eigenvalues are present.

Example 5.1: In the first example H is neither of the focusing nor defocusing type. Let $m = n = 1$,

$$q(x) = \sigma e^{-x}\Theta(x), \quad r(x) = \mu e^{x}\Theta(-x),$$

where μ, $\sigma \geq 0$. Thus (1.1) reads (using the notation introduced in the proof of Lemma 2.2)

$$(\psi^{\mathrm{up}})' = -i\lambda\psi^{\mathrm{up}} + \sigma e^{-x}\Theta(x)\psi^{\mathrm{dn}},$$

$$(\psi^{\mathrm{dn}})' = -\mu e^{x}\Theta(-x)\psi^{\mathrm{up}} + i\lambda\psi^{\mathrm{dn}}.$$

A straightforward calculation of the Jost functions yields

$$F_+(\lambda,x)=\begin{cases} e^{-i\lambda Jx}\begin{pmatrix} 1+\dfrac{\mu\sigma[e^{(2i\lambda-1)x}-1]}{(1-2i\lambda)^2} & \dfrac{\sigma e^{(2i\lambda-1)x}}{2i\lambda-1} \\ \dfrac{\mu}{2i\lambda-1} & 1 \end{pmatrix}, & x>0 \\ e^{-i\lambda Jx}\begin{pmatrix} 1 & \dfrac{\sigma}{2i\lambda-1} \\ \dfrac{\mu e^{(1-2i\lambda)x}}{2i\lambda-1} & 1+\dfrac{\mu\sigma[e^{(1-2i\lambda)x}-1]}{(1-2i\lambda)^2} \end{pmatrix}, & x<0, \end{cases}$$

$$F_-(\lambda,x)=\begin{cases} e^{-i\lambda Jx}\begin{pmatrix} 1 & \dfrac{\sigma[e^{(2i\lambda-1)x}-1]}{2i\lambda-1} \\ 0 & 1 \end{pmatrix}, & x>0 \\ e^{-i\lambda Jx}\begin{pmatrix} 1 & 0 \\ \dfrac{\mu[e^{(1-2i\lambda)x}-1]}{2i\lambda-1} & 1 \end{pmatrix}, & x<0. \end{cases}$$

Using (1.8) gives

$$T_l(\lambda)=T_r(\lambda)=\frac{(1-2i\lambda)^2}{(1+\sqrt{\mu\sigma}-2i\lambda)(1-\sqrt{\mu\sigma}-2i\lambda)},\quad \lambda\in\overline{\mathbb{C}^+},$$

$$\breve{T}_l(\lambda)=\breve{T}_r(\lambda)=1,\quad \lambda\in\overline{\mathbb{C}^-},$$

and for $\lambda\in\mathbb{R}$,

$$L(\lambda)=\frac{\sigma(2i\lambda-1)}{(1+\sqrt{\mu\sigma}-2i\lambda)(1-\sqrt{\mu\sigma}-2i\lambda)},$$

$$R(\lambda)=\frac{\mu(2i\lambda-1)}{(1+\sqrt{\mu\sigma}-2i\lambda)(1-\sqrt{\mu\sigma}-2i\lambda)},$$

$$\breve{L}(\lambda)=\frac{\mu}{1-2i\lambda},\quad \breve{R}(\lambda)=\frac{\sigma}{1-2i\lambda}.$$

Set $a=\sqrt{\mu\sigma}$. We immediately see by looking at the transmission coefficients that there is exactly one purely imaginary eigenvalue $\lambda_0=(i/2)(a-1)$ in the upper half plane, provided that $a>1$. The corresponding eigenvector is

$$\phi(\lambda_0,x)=\begin{cases} \begin{pmatrix} e^{-(1/2)(a+1)x} \\ -\sqrt{\mu/\sigma}e^{(1/2)(1-a)x} \end{pmatrix}, & x>0 \\ \begin{pmatrix} e^{(1/2)(a-1)x} \\ -\sqrt{\mu/\sigma}e^{(1/2)(a+1)x} \end{pmatrix}, & x<0. \end{cases}$$

If $a<1$, then there are no eigenvalues and, if $a=1$, then $\lambda=0$ is a spectral singularity. For $a<1$, the wave operators $Z_\pm$ (and $W_\pm$) exist and $P_{ac}=I_2$. In fact, we can compute $Z_\pm$ explicitly. To avoid lengthy expressions, we pick a particular vector $\psi_0(x)$ and compute $Z_+\psi_0$. We set $\psi_0(x)=(\psi_0^{up}(x),\psi_0^{dn}(x))^T$ and choose $\psi_0^{up}(x)=e^{-x}\Theta(x)$, $\psi_0^{dn}(x)=0$. We do not restrict a for the moment and first proceed to determine $\psi(x,t)\overset{\text{def}}{=}[e^{-itH}\psi_0](x)$ for $t\geq 0$ by solving the underlying partial differential equation (PDE), $\psi_t=-iH\psi$, which reads

$$\psi_t^{\mathrm{up}}=\psi_x^{\mathrm{up}}-\sigma e^{-x}\Theta(x)\psi^{\mathrm{dn}},$$

$$\psi_t^{\mathrm{dn}}=-\mu e^{x}\Theta(-x)\psi^{\mathrm{up}}-\psi_x^{\mathrm{dn}},$$

with initial condition $\psi(x,0)=\psi_0(x)$. When $a>1$, the fact that $P_{\mathrm{ac}}\neq I_2$ will be taken into account later. With the help of the Laplace transform method we find

$$\psi^{\mathrm{up}}(x,t)=\left(\frac{e^{-t-x}}{1-a^2}+\frac{ae^{-(1/2)(1-a)(t+x)}}{2(1+a)}-\frac{ae^{-(1/2)(1+a)(t+x)}}{2(1-a)}\right)\Theta(t+x),\quad x<0,$$

$$\psi^{\mathrm{up}}(x,t)=e^{-x-t}+\frac{ae^{-x}}{2(a^2-1)}((1+a)e^{-(1/2)(1+a)(t-x)}-2ae^{-t+x}+(a-1)e^{(1/2)(a-1)(t-x)})\Theta(t-x),\quad x>0,$$

$$\psi^{\mathrm{dn}}(x,t)=\frac{-\mu e^{x}}{2(a^2-1)}(-(1+a)e^{-(1/2)(1+a)(t+x)}+2e^{-t-x}+(-1+a)e^{-(1/2)(1-a)(t+x)})\Theta(t+x),\quad x<0,$$

$$\psi^{\mathrm{dn}}(x,t)=\frac{-\mu}{2(a^2-1)}(-(1+a)e^{-(1/2)(1+a)(t-x)}+2e^{-t+x}+(-1+a)e^{-(1/2)(1-a)(t-x)})\Theta(t-x),\quad x>0.$$

When $a=1$, we have to take the limit as $a\to 1$ in the above expressions (see below). Using the fact that the free time evolution is given by $[e^{itH_0}\chi](x)=(\chi^{\mathrm{up}}(x-t),\chi^{\mathrm{dn}}(x+t))$, we find, by taking the (pointwise) limit as $t\to+\infty$,

$$(Z_+\psi_0)(x)=\frac{1}{2(1-a^2)}\begin{pmatrix}[2e^{-x}+a(1-a)e^{-(1/2)(1-a)x}-a(1+a)e^{-(1/2)(1+a)x}]\Theta(x)\\ \mu[2e^{x}-(1-a)e^{(1/2)(1-a)x}-(1+a)e^{(1/2)(1+a)x}]\Theta(-x)\end{pmatrix}. \tag{5.1}$$

If $a<1$, then the components of the right-hand side of (5.1) are in $L^2(\mathbb{R})$ and (5.1) agrees with the strong limit according to (4.2b). For example, consider $\psi^{\mathrm{up}}(x,t)$ for $x>0$. Then $\psi^{\mathrm{up}}(x-t,t)$ contains the term

$$\frac{ae^{-x+t}}{2(a+1)}e^{(1/2)(a-1)(2t-x)}=\frac{a}{2(a+1)}e^{at}e^{(-1/2)(a+1)x},\quad t\leq x\leq 2t.$$

Over the given interval, the L^2 norm of this term is $O(e^{-(1-a)t/2})$. Hence it goes to zero precisely because $a<1$. The other terms can be dealt with similarly.

For $a=1$ we have

$$\psi^{\mathrm{up}}(x,t)=\frac{e^{-t-x}}{4}(3+e^{t+x}-t-x)\Theta(t+x),\quad x<0,$$

$$\psi^{\mathrm{up}}(x,t)=\frac{e^{-t-x}}{4}(4+e^{t}-e^{x}(1+t-x))\Theta(t-x),\quad x>0,$$

$$\psi^{\mathrm{dn}}(x,t)=\frac{\mu e^{-t}}{4}(1-e^{t+x}-t-x)\Theta(t+x),\quad x<0,$$

$$\psi^{\mathrm{dn}}(x,t) = \frac{-\mu e^{-t}}{4}(e^t + e^x(-1+t-x))\Theta(t-x), \quad x>0.$$

Now, if we apply e^{itH_0} to this vector and let $t\to+\infty$, we do get a pointwise (for each x) limit, namely,

$$(Z_+\psi_0)(x) = \begin{pmatrix} (1/4)e^{-x}(3+e^x-x)\Theta(x) \\ (\mu/4)(-1+e^x+xe^x)\Theta(-x) \end{pmatrix}.$$

We see that the right-hand side is bounded but not in L^2. It follows that the wave operators do not exist in either the strong or the weak topology of L^2.

Let us also take a closer look at the case $a>1$. Using a contour integral we compute for the eigenprojection P_{λ_0} acting on ψ_0:

$$(P_{\lambda_0}\psi_0)(x) = \frac{1}{2(a+1)}\cdot\begin{cases} \begin{pmatrix} ae^{-(a+1)x/2} \\ -\mu e^{-(a-1)x/2} \end{pmatrix}, & x>0 \\ \begin{pmatrix} ae^{(a-1)x/2} \\ -\mu e^{(a+1)x/2} \end{pmatrix}, & x<0. \end{cases}$$

Then $P_{\mathrm{ac}}=I_2-P_{\lambda_0}$. Using this in (4.2b) and solving the corresponding PDE by means of a Laplace transform, which is now more involved since the initial condition does not vanish on $x<0$, we obtain

$$(Z_+\psi_0)(x) = \frac{1}{2(a^2-1)}\cdot\begin{cases} \begin{pmatrix} -2e^{-x}+a(a+1)e^{-(a+1)x/2} \\ \mu(a-1)e^{-(a-1)x/2} \end{pmatrix}, & x>0 \\ \begin{pmatrix} a(a-1)e^{(a-1)x/2} \\ \mu[-2e^x+(a+1)e^{(a+1)x/2}] \end{pmatrix}, & x<0. \end{cases}$$

As it should be, the components of the right-hand side are in $L^2(\mathbb{R})$.

Example 5.2: This example concerns the focusing case. Let $n=m=1$, $q(x)=\mu>0$ if $0\le x\le 1$, $q(x)=0$ otherwise, and $r(x)=q(x)$. Let $\psi_0(x)$ denote the vector with components

$$\psi_0^{\mathrm{up}}(x)=1, \quad x\in[0,1], \quad \psi_0^{\mathrm{up}}(x)=0, \quad x\notin[0,1], \quad \psi_0^{\mathrm{dn}}(x)=0.$$

For this potential, H has no eigenvalues so long as $\mu\le\pi/2$ but there is a spectral singularity at $\lambda=0$ when $\mu=\pi/2$. Again, we use a Laplace transform to compute $\psi_t(x,t)=[e^{-itH}\psi_0](x)$. To this end we have to solve

$$\psi_t^{\mathrm{up}}=\psi_x^{\mathrm{up}}, \quad \psi_t^{\mathrm{dn}}=-\psi_x^{\mathrm{dn}}, \quad x\notin[0,1],$$

$$\psi_t^{\mathrm{up}}=\psi_x^{\mathrm{up}}-\mu\psi^{\mathrm{dn}}, \quad \psi_t^{\mathrm{dn}}=-\psi_x^{\mathrm{dn}}-\mu\psi^{\mathrm{up}}, \quad x\in[0,1],$$

with $\psi(x,0)=\psi_0(x)$. Let $\hat\psi(x,s)=(\hat\psi^{\mathrm{up}}(x,s),\hat\psi^{\mathrm{dn}}(x,s))^T$ denote the Laplace transform of $\psi(x,t)$. Let

$$D(s)=w(s)\cosh[\omega(s)]+s\sinh[\omega(s)],$$

where $\omega(s)=\sqrt{s^2-\mu^2}$. Then

$$\hat\psi(x,s)=\begin{pmatrix} \dfrac{s}{\omega(s)^2}-\dfrac{s\omega(s)\cosh[\omega(s)x]+\mu^2\sinh[\omega(s)(1-x)]+s^2\sinh[\omega(s)x]}{\omega(s)^2D(s)} \\ -\dfrac{\mu}{\omega(s)^2}+\dfrac{\mu(\omega(s)\cosh[\omega(s)(1-x)]+s\sinh[\omega(s)(1-x)]+s\sinh[\omega(s)x])}{\omega(s)^2D(s)} \end{pmatrix}$$

for $0\le x\le 1$, and

$$\hat{\psi}(x,s) = \begin{pmatrix} \hat{\psi}_{\mathrm{up}}(0,s)e^{sx} \\ 0 \end{pmatrix}, \quad x<0, \quad \hat{\psi}(x,s) = \begin{pmatrix} 0 \\ \hat{\psi}_{\mathrm{dn}}(1,s)e^{-sx} \end{pmatrix}, \quad x>1,$$

where $\hat{\psi}_1(0,s)$ and $\hat{\psi}_2(1,s)$ can be obtained from the equation for $0 \leq x \leq 1$. As a function of s, $\hat{\psi}(x,s)$ is meromorphic for every x, despite appearances to the contrary. For example, $s=\mu$ is a removable singularity. Note that if we replace $\omega(s)$ by $-\omega(s)$, the function $\hat{\psi}(x,s)$ is unchanged, since $D(s)$ is odd in $w(s)$, so there are only even powers in the Taylor expansion in powers of $w(s)$ of $\hat{\psi}(x,s)$. A calculation also shows that $D(s)=e^s A_{r1}(is)\omega(s)$, where $A_{r1}(is)$ is given by (1.6a). This holds for $\mathrm{Re}\, s \geq 0$. A zero s_1 of $D(s)$ in the right half plane corresponds to an eigenvalue $\lambda_1 = is_1$ of H. Since the eigenvalues of H are all purely imaginary, the zeros of $D(s)$ in the right half plane must lie on the real axis. If $\mu \leq \pi/2$, then there are no eigenvalues and so $D(s)$ has no zeros in the right half plane. But if $\mu=\pi/2$, then $D(0)=0$ and $\hat{\psi}(x,s)$ has a pole of order of 1 at $s=0$. There are no zeros on the imaginary axis (except for 0), and there exists a $\delta>0$, such that there are no zeros in $\{s:\mathrm{Re}\, s \geq -\delta\}\setminus\{0\}$. There is an infinite sequence of zeros in the second quadrant, together with the complex conjugate values in the third quadrant, which, for large absolute values, are located at

$$s_n = \pi n i - \ln(n) + \ln(\mu/(2\pi)) - i\pi/2 + O(\ln(n)/n), \quad n \to +\infty.$$

In every infinite strip $a \leq \mathrm{Re}\ \mathrm{s} \leq \mathrm{b}$ the function $\hat{\psi}^{\mathrm{up}}(x,s)$ is of order $O(s^{-1})$ as $|s| \to +\infty$ uniformly in the strip and uniformly in $0 \leq x \leq 1$, and, similarly, $\hat{\psi}^{\mathrm{dn}}(x,s)=O(s^{-2})$. Moreover, in view of the oscillatory behavior of $\hat{\psi}^{\mathrm{up}}(x,s)$ as $\mathrm{Im}\, s \to +\infty$, a closer inspection shows that the integrals

$$\int_{-\infty}^{\infty} e^{ity}\hat{\psi}^{\mathrm{up}}(x,-\delta+iy)dy$$

converge uniformly for $t \geq T>0$ (any $T>0$) and for $0 \leq x \leq 1$ (cf. Ref. 10, p. 237). It follows, using the Riemann–Lebesgue lemma, that

$$\frac{1}{2\pi i}\int_{-\delta-i\infty}^{-\delta+i\infty} e^{st}\hat{\psi}^{\mathrm{up}}(x,s)ds = o(e^{-\delta t}), \quad t \to +\infty,$$

uniformly in $0 \leq x \leq 1$. The same result holds for $\psi^{\mathrm{dn}}(x,s)$ but the justification is easier since it is $O(s^{-2})$. As a consequence, we can apply standard results on the inverse Laplace transform and conclude that (see Ref. 10, Chap. 35)

$$\lim_{s\to 0} s\hat{\psi}(x,s) = \lim_{t\to+\infty} \psi(x,t) = \begin{pmatrix} \cos(\pi x/2) \\ -\sin(\pi x/2) \end{pmatrix}, \quad 0 \leq x \leq 1,$$

uniformly in $0 \leq x \leq 1$, and

$$\lim_{s\to 0} s\hat{\psi}(x,s) = \lim_{t\to+\infty} \psi(x,t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad x<0,$$

$$\lim_{s\to 0} s\hat{\psi}(x,s) = \lim_{t\to+\infty} \psi(x,t) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \quad x>1.$$

Suppose $0 \leq x \leq 1$ and apply the free time evolution to $\psi(x,t)$. This yields, for the upper component, $\psi^{\mathrm{up}}(x-t,t)$, for $t \leq x \leq t+1$. For large t this is close to $\cos(\pi(x-t)/2)$ (with an exponentially small error). Hence,

$$\int_t^{t+1} |\psi^{\mathrm{up}}(x-t,t)|^2 dx \not\to 0, \quad t \to +\infty,$$

showing that $Z_+\psi_0$ does not exist as a strong limit according to (1.18). However, this piece goes to zero weakly. If we consider $x<0$, then $\psi^{\mathrm{up}}(x,t)=\psi^{\mathrm{up}}(0,t+x)$ for $t+x\geq 0$ and zero otherwise. Hence, $\psi^{\mathrm{up}}(x-t,t)=\psi^{\mathrm{dn}}(0,x)$ for $x-t<0$ and $t+(x-t)=x>0$, i.e., on $0<x<t$. Since $\psi^{\mathrm{dn}}(0,x)\to 1$ as $x\to+\infty$, the limit as $t\to+\infty$ of $\psi^{\mathrm{up}}(x,t)$ does not exist in the strong or weak sense; it only exists in the pointwise sense. An analogous result holds for the lower component.

VI. CONCLUSIONS

Under the conditions (a) and (b) we have proved that the restriction $\tilde{H}$ of the full Hamiltonian H to the maximal invariant subspace, where its spectrum is real, is similar to the free Hamiltonian H_0, while H has at most finitely many nonreal eigenvalues with eigenvectors and generalized eigenvectors living in a finite-dimensional subspace. In the focusing case, where H_0 and H are J-self-adjoint, the free Hamiltonian and the thus restricted full Hamiltonian $\tilde{H}$ are J-unitarily equivalent. Since H_0 is self-adjoint, J-self-adjoint but not J-definitizable, and absolutely continuous with (uniform) spectral multiplicity $m+n$, we can draw the following conclusions.

(1) H is a spectral operator and its restriction $\tilde{H}$ to the maximal invariant subspace where its spectrum is real and is scalar-type spectral.
(2) $\tilde{H}$ is absolutely continuous.
(3) $\tilde{H}$ has a (uniform) spectral multiplicity of $m+n$.
(4) $\tilde{H}$ is J-self-adjoint but not J-definitizable.

A direct proof of these facts, which does not rely on wave operators and Kato smoothness, will be given in Appendix. As a result, in the focusing case a spectral theorem applies to H, where the resolution of the identity does not have the so-called singular critical points (Ref. 6, p. 211).[24]

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APPENDIX: RESOLVENT OF THE HAMILTONIAN

In this section we derive an expression for the resolvent of the matrix Zakharov–Shabat Hamiltonian H in terms of either modified Jost functions or Jost solutions. We will use this expression to answer the four assertions made in Sec. VI. Since the argument involves taking the limit $\operatorname{Im}\zeta\to 0$, assumptions (a) and (b) from Sec. I will be crucial. In other words, the derivation requires that $\det A_{l1}(\lambda)=\det A_{r4}(\lambda)$ and $\det A_{r1}(\lambda)=\det A_{l4}(\lambda)$ do not vanish for $\lambda\in\mathbb{R}$, or, equivalently, that the reflection and transmission coefficients are continuous in $\lambda\in\mathbb{R}$.

Let us partition $\Psi(x,\zeta)^{-1}$ and $\Phi(x,\zeta)^{-1}$ as follows:

$$\Psi(x,\zeta)^{-1} = \begin{pmatrix} \breve{\psi}(x,\zeta) \\ \breve{\bar{\psi}}(x,\zeta) \end{pmatrix}, \quad \Phi(x,\zeta)^{-1} = \begin{pmatrix} \breve{\bar{\phi}}(x,\zeta) \\ \breve{\phi}(x,\zeta) \end{pmatrix},$$

where $x,\zeta\in\mathbb{R}$. Here $\breve{\psi}(x,\zeta)$ and $\breve{\bar{\phi}}(x,\zeta)$ are $m\times(m+n)$ matrices and $\breve{\bar{\psi}}(x,\zeta)$ and $\breve{\phi}(x,\zeta)$ are $n\times(m+n)$ matrices.

Theorem A.1: If $\zeta\in\mathbb{C}\setminus\mathbb{R}$ is not an eigenvalue and $\varphi\in\mathcal{H}_{m+n}$ we have

$$[(\zeta - H)^{-1}\varphi](x) = \int_{-\infty}^{\infty} dy\mathcal{G}(x,y;\zeta)\varphi(y), \tag{A1}$$

where

$$\mathcal{G}(x,y;\zeta) = \begin{cases} -i\phi(x,\zeta)T_r(\zeta)\breve{\psi}(y,\zeta)J, & \zeta \in \mathbb{C}^+, \quad y > x \\ +i\psi(x,\zeta)T_l(\zeta)\breve{\phi}(y,\zeta)J, & \zeta \in \mathbb{C}^+, \quad y < x \\ +i\bar{\psi}(x,\zeta)\breve{T}_l(\zeta)\breve{\bar{\phi}}(y,\zeta)J, & \zeta \in \mathbb{C}^-, \quad y < x \\ -i\bar{\phi}(x,\zeta)\breve{T}_r(\zeta)\breve{\bar{\psi}}(y,\zeta)J, & \zeta \in \mathbb{C}^-, \quad y > x. \end{cases} \tag{A2}$$

The spectral projections of H onto the generalized eigenspaces of H can be obtained by computing the residues of $(\zeta - H)^{-1}$ at the discrete eigenvalues.

Proof: The general assumption means that the modified Jost matrices $F_\pm(x,\zeta)$ are invertible for each $\zeta \in \mathbb{R}$ and for each $\zeta \in \mathbb{C}^\pm$ that is not an eigenvalue of H. For given $\varphi \in \mathcal{H}_{m+n}$ we now seek $\chi \in \mathcal{H}_{m+n}$, such that $(\zeta - H)\chi = \varphi$, or

$$\zeta\chi(x,\zeta) - iJ\frac{\partial\chi}{\partial x}(x,\zeta) + V(x)\chi(x,\zeta) = \varphi(x).$$

Writing $\chi(x,\zeta) = F_\pm(x,\zeta)\chi_\pm(x,\zeta)$, we get

$$\Bigg[\underbrace{\zeta F_\pm(x,\zeta) - i\zeta J\frac{\partial F_\pm}{\partial x}(x,\zeta) + V(x)F_\pm(x,\zeta)}_{\text{vanishes}}\Bigg]\chi_\pm(x,\zeta) - iJF_\pm(x,\zeta)\frac{\partial\chi_\pm}{\partial x}(x,\zeta) = \varphi(x),$$

whence

$$\frac{\partial\chi_\pm}{\partial x}(x,\zeta) = iF_\pm(x,\zeta)^{-1}J\varphi(x).$$

Using (1.6) we see that

$$\chi_+^{\mathrm{up}}(x,\zeta) = -i\int_x^{\infty} dy\begin{pmatrix} I_m & 0_{m\times n}\end{pmatrix}F_+(y,\zeta)^{-1}J\varphi(y),$$

$$\chi_+^{\mathrm{dn}}(x,\zeta) = +i\int_{-\infty}^{x} dy\begin{pmatrix} 0_{n\times m} & I_n\end{pmatrix}F_+(y,\zeta)^{-1}J\varphi(y)$$

for $\zeta \in \mathbb{C}^+$ not an eigenvalue, and

$$\chi_-^{\mathrm{up}}(x,\zeta) = +i\int_{-\infty}^{x} dy\begin{pmatrix} I_m & 0_{m\times n}\end{pmatrix}F_-(y,\zeta)^{-1}J\varphi(y),$$

$$\chi_-^{\mathrm{dn}}(x,\zeta) = -i\int_x^{\infty} dy\begin{pmatrix} 0_{n\times m} & I_n\end{pmatrix}F_-(y,\zeta)^{-1}J\varphi(y)$$

for $\zeta \in \mathbb{C}^-$ not an eigenvalue. We may obviously write

$$\chi_{\pm}(x,\zeta) = \int_{-\infty}^{\infty} dy \mathcal{G}_0(x,y;\zeta)\varphi(y),$$

where

$$\mathcal{G}_0(x,y;\zeta) = \begin{cases} -\dfrac{i}{2}(I+J)[JF_+(y,\zeta)^{-1}J], & \zeta \in \mathbb{C}^+, \quad y > x \\ -\dfrac{i}{2}(I-J)[JF_+(y,\zeta)^{-1}J], & \zeta \in \mathbb{C}^+, \quad y < x \\ +\dfrac{i}{2}(I+J)[JF_-(y,\zeta)^{-1}J], & \zeta \in \mathbb{C}^-, \quad y < x \\ +\dfrac{i}{2}(I-J)[JF_-(y,\zeta)^{-1}J], & \zeta \in \mathbb{C}^-, \quad y > x, \end{cases}$$

which implies (A1), where

$$\mathcal{G}(x,y;\zeta) = \begin{cases} -iF_+(x,\zeta)\frac{1}{2}(I+J)[JF_+(y,\zeta)^{-1}J], & \zeta \in \mathbb{C}^+, \quad y > x \\ -iF_+(x,\zeta)\frac{1}{2}(I-J)[JF_+(y,\zeta)^{-1}J], & \zeta \in \mathbb{C}^+, \quad y < x \\ +iF_-(x,\zeta)\frac{1}{2}(I+J)[JF_-(y,\zeta)^{-1}J], & \zeta \in \mathbb{C}^-, \quad y < x \\ +iF_-(x,\zeta)\frac{1}{2}(I-J)[JF_-(y,\zeta)^{-1}J], & \zeta \in \mathbb{C}^-, \quad y > x. \end{cases} \tag{A3}$$

We now use the relations

$$F_+(x,\zeta) = \Psi(x,\zeta)\begin{pmatrix} A_{r1}(\zeta) & 0_{m\times n} \\ A_{r3}(\zeta) & I_{n\times n} \end{pmatrix} = \Phi(x,\zeta)\begin{pmatrix} I_m & A_{l2}(\zeta) \\ 0_{n\times m} & A_{l4}(\zeta) \end{pmatrix},$$

$$F_-(x,\zeta) = \Psi(x,\zeta)\begin{pmatrix} I_m & A_{r2}(\zeta) \\ 0_{n\times m} & A_{r4}(\zeta) \end{pmatrix} = \Phi(x,\zeta)\begin{pmatrix} A_{l1}(\zeta) & 0_{m\times n} \\ A_{l3}(\zeta) & I_n \end{pmatrix}$$

to write

$$JF_+(x,\zeta)^{-1}J = \begin{pmatrix} T_r(\zeta) & 0_{m\times n} \\ R(\zeta) & I_n \end{pmatrix} J\Psi(x,\zeta)^{-1}J = \begin{pmatrix} I_m & L(\zeta) \\ 0_{n\times m} & T_l(\zeta) \end{pmatrix} J\Phi(x,\zeta)^{-1}J,$$

$$JF_-(x,\zeta)^{-1}J = \begin{pmatrix} I_m & \breve{R}(\zeta) \\ 0_{n\times m} & \breve{T}_r(\zeta) \end{pmatrix} J\Psi(x,\zeta)^{-1}J = \begin{pmatrix} \breve{T}_l(\zeta) & 0_{m\times n} \\ \breve{L}(\zeta) & I_n \end{pmatrix} J\Phi(x,\zeta)^{-1}J.$$

Inserting these expressions in (A3) and using $\frac{1}{2}(I+J)=I_m\oplus 0_{n\times n}$ and $\frac{1}{2}(I-J)J=0_{m\times m}\oplus(-I_n)$, we obtain

$$
\mathcal{G}(x,y;\zeta)=\begin{cases}
-i(\phi(x,\zeta)\quad 0_{(m+n)\times n})\begin{pmatrix} T_r(\zeta) & 0_{m\times n} \\ 0_{n\times m} & 0_{n\times n}\end{pmatrix}\begin{pmatrix} \breve{\psi}(y,\zeta) \\ 0_{n\times(m+n)}\end{pmatrix}J \\
\qquad \zeta\in\mathbb{C}^+, y>x, \\
+i(0_{(m+n)\times m}\quad \psi(x,\zeta))\begin{pmatrix} 0_{m\times m} & 0_{m\times n} \\ 0_{n\times m} & T_l(\zeta)\end{pmatrix}\begin{pmatrix} 0_{m\times(m+n)} \\ \breve{\phi}(y,\zeta)\end{pmatrix}J \\
\qquad \zeta\in\mathbb{C}^+, y<x, \\
+i(\bar{\psi}(x,\zeta)\quad 0_{(m+n)\times n})\begin{pmatrix} \breve{T}_l(\zeta) & 0_{m\times n} \\ 0_{n\times m} & 0_{n\times n}\end{pmatrix}\begin{pmatrix} \breve{\bar{\phi}}(y,\zeta) \\ 0_{n\times(m+n)}\end{pmatrix}J \\
\qquad \zeta\in\mathbb{C}^-, y<x, \\
-i(0_{(m+n)\times m}\quad \bar{\phi}(x,\zeta))\begin{pmatrix} 0_{m\times m} & 0_{m\times n} \\ 0_{n\times m} & \breve{T}_r(\zeta)\end{pmatrix}\begin{pmatrix} 0_{m\times(m+n)} \\ \breve{\bar{\psi}}(y,\zeta)\end{pmatrix}J, \\
\qquad \zeta\in\mathbb{C}^-, y>x,
\end{cases}
$$

which implies (A2). ■

We remark that the above expressions for the Green's function agree with those derived in a different way[21] for potentials V whose entries are only in $L^1_{\text{loc}}(\mathbb{R})$. We can establish the connection between the two as follows. It suffices to consider the case $\operatorname{Im}\zeta>0$. We will use the symbol # to designate quantities that are associated with the adjoint matrix Zakharov–Shabat system, that is, (1.1) with potential $V^\#(x)=V(x)^\dagger$. Let $\bar{\psi}^\#(x,\zeta^*)$ and $\bar{\phi}^\#(x,\zeta^*)$ denote the solutions defined by (1.7a) and (1.7b) with corresponding matrices $A^\#_{r4}(\zeta^*)$ and $A^\#_{l1}(\zeta^*)$, respectively. In Ref. 21, Sec. 4, two matrices, called $F(x,\zeta)=(F_1(x,\zeta)\;\; F_2(x,\zeta))=(\psi(x,\zeta)\;\;\phi(x,\zeta))$ and $\hat{F}(x,\zeta^*)=(\hat{F}_1(x,\zeta^*)\;\;\hat{F}_2(x,\zeta^*))=(\bar{\phi}^\#(x,\zeta^*)\;\;\bar{\psi}^\#(x,\zeta^*))$ were introduced, and it was noted that

$$JF(x,\zeta)\begin{pmatrix} -T_l(\zeta) & 0_{n\times m} \\ 0_{m\times n} & T_r(\zeta)\end{pmatrix}\hat{F}(x,\zeta^*)^\dagger=I.$$

Since

$$F(x,\zeta)=F_+(x,\zeta)\begin{pmatrix} 0_{m\times n} & I_m \\ I_n & 0_{n\times m}\end{pmatrix},$$

we get

$$JF_+(x,\zeta)^{-1}J=\begin{pmatrix} 0_{n\times m} & T_r(\zeta) \\ T_l(\zeta) & 0_{m\times n}\end{pmatrix}\hat{F}(x,\zeta^*)^\dagger.$$

Inserting this in the first of (A3) yields

$$\mathcal{G}(x,y;\zeta)=-i\phi(x,\zeta)T_r(\zeta)\bar{\psi}^\#(x,\zeta^*)^\dagger,\quad \zeta\in\mathbb{C}^+,\quad y>x,$$

which is easily seen to be equal to the second equation in Ref. 21, Eq. (4.10), if we note that $K_2(\zeta)^{-1}=T_r(\zeta)$, $\phi(x,\zeta)=F_2(x,\zeta)$, $\bar{\psi}^\#(x,\zeta^*)=\hat{F}_2(x,\zeta^*)$, and that the resolvent studied there is $(H-\zeta)^{-1}$ while here it is $(\zeta-H)^{-1}$.

The expressions for $\mathcal{G}(x,y;\zeta)$ have finite limits as ζ approaches the real line from above or from below. Using (1.6) in (A3) we can write $\mathcal{G}(x,y;\zeta)$ as follows:

$$\mathcal{G}(x,y;\zeta+i0)=\begin{cases}-i\Psi(x,\zeta)\begin{pmatrix}I_m & 0_{m\times n}\\ R(\zeta) & 0_{n\times n}\end{pmatrix}[J\Psi(y,\zeta)^{-1}J], & y>x\\ -i\Psi(x,\zeta)\begin{pmatrix}0_{m\times m} & 0_{m\times n}\\ R(\zeta) & I_n\end{pmatrix}[J\Psi(y,\zeta)^{-1}J], & y<x,\end{cases}$$

$$\mathcal{G}(x,y;\zeta-i0)=\begin{cases}+i\Psi(x,\zeta)\begin{pmatrix}I_m & \breve{R}(\zeta)\\ 0_{n\times m} & 0_{n\times n}\end{pmatrix}[J\Psi(y,\zeta)^{-1}J], & y<x\\ +i\Psi(x,\zeta)\begin{pmatrix}0_{m\times m} & \breve{R}(\zeta)\\ 0_{n\times m} & I_n\end{pmatrix}[J\Psi(y,\zeta)^{-1}J], & y>x,\end{cases}$$

where $\zeta\in\mathbb{R}$. Notice that we only need to know the Jost matrix from the left and its inverse for $\zeta\in\mathbb{R}$. Alternatively, using the Jost matrix from the right and its inverse we can write

$$\mathcal{G}(x,y;\zeta+i0)=\begin{cases}-i\Phi(x,\zeta)\begin{pmatrix}I_m & L(\zeta)\\ 0_{n\times m} & 0_{n\times n}\end{pmatrix}[J\Phi(y,\zeta)^{-1}J], & y>x\\ -i\Phi(x,\zeta)\begin{pmatrix}0_{m\times m} & L(\zeta)\\ 0_{n\times m} & I_n\end{pmatrix}[J\Phi(y,\zeta)^{-1}J], & y<x,\end{cases}$$

$$\mathcal{G}(x,y;\zeta-i0)=\begin{cases}+i\Phi(x,\zeta)\begin{pmatrix}I_m & 0_{m\times n}\\ \breve{L}(\zeta) & 0_{n\times n}\end{pmatrix}[J\Phi(y,\zeta)^{-1}J], & y<x\\ +i\Phi(x,\zeta)\begin{pmatrix}0_{m\times m} & 0_{m\times n}\\ \breve{L}(\zeta) & I_n\end{pmatrix}[J\Phi(y,\zeta)^{-1}J], & y>x.\end{cases}$$

Irrespective of the sign of $(x-y)$ we then get for $\zeta\in\mathbb{R}$

$$\begin{aligned}\frac{1}{2\pi i}\lim_{\varepsilon\to0^+}&\{\mathcal{G}(x,y;\zeta-i\varepsilon)-\mathcal{G}(x,y;\zeta+i\varepsilon)\}\\ &=\frac{1}{2\pi}\Psi(x,\zeta)\begin{pmatrix}I_m & \breve{R}(\zeta)\\ R(\zeta) & I_n\end{pmatrix}[J\Psi(y,\zeta)^{-1}J]\\ &=\frac{1}{2\pi}\Phi(x,\zeta)\begin{pmatrix}I_m & L(\zeta)\\ \breve{L}(\zeta) & I_n\end{pmatrix}[J\Phi(y,\zeta)^{-1}J].\end{aligned}\tag{A4}$$

We can now justify the first two assertions of Sec. VI. For $\alpha<\beta$ we easily obtain from (A4) that H is a scalar-type spectral operator with resolution of the identity given by

$$\sigma((\alpha,\beta);x,y)\overset{\text{def}}{=}\lim_{\varepsilon\to0^+}\frac{1}{2\pi i}\int_\alpha^\beta d\zeta(\mathcal{G}(x,y;\zeta-i\varepsilon)-\mathcal{G}(x,y;\zeta+i\varepsilon)).$$

Since this expression is continuously differentiable with respect to α and β, the spectral measure σ is absolutely continuous with Radon–Nikodym derivative,

$$\sigma(d\zeta;x,y)/d\zeta=\frac{1}{2\pi}\Psi(x,\zeta)\begin{pmatrix}I_m & \breve{R}(\zeta)\\ R(\zeta) & I_n\end{pmatrix}J\Psi(y,\zeta)^{-1}J.\tag{A5}$$

Thus,[11,12] under conditions (a) and (b) from Sec. I, the restriction $\tilde{H}$ of H to the invariant subspace where the spectrum is real is given by $\int t\sigma(dt)$ (restricted to the domain of $\tilde{H}$), where $\sigma(dt)$ is the integral operator with kernel (A5). In the focusing case the projections $\sigma((\alpha,\beta))$ given by (A5) are all J-self-adjoint because in this case $\breve{R}(\zeta)=-R(\zeta)^\dagger$ and $\Psi(x,\zeta)^\dagger=\Psi(x,\zeta)^{-1}$ for $\zeta\in\mathbb{R}$ by (1.15).

The third statement made in Sec. VI regarding the spectral multiplicity of $\tilde{H}$ follows from the fact that by Theorem 3.1 the free Hamiltonian H_0 is similar to the restriction $\tilde{H}$ of H on the absolutely continuous subspace. Since by a Fourier transform, H_0 is unitarily equivalent to the direct sum of $m+n$ copies of the multiplication operator by the independent variable on $L^2(\mathbb{R})$, the claim follows Ref. 4, Sec. 72.

To establish the fourth assertion, we note that in the focusing case the projections $\sigma((\alpha,\beta))$ given by (A5) are all J-self-adjoint because in this case $\breve{R}(\zeta)=-R(\zeta)^\dagger$ and $\Psi(x,\zeta)^\dagger=\Psi(x,\zeta)^{-1}$ for $\zeta\in\mathbb{R}$ by (1.15). Moreover, the Radon–Nikodym derivative (A5) of the spectral measure σ is an invertible matrix of order $m+n$ whose norm and that of its inverse are bounded away from zero and infinity uniformly in $(x,\zeta)\in\mathbb{R}^2$. Indeed, this follows from the inversion formula,

$$\begin{pmatrix} I_m & \breve{R}(\zeta) \\ R(\zeta) & I_n \end{pmatrix}^{-1} = \begin{pmatrix} (I_m+R(\zeta)^\dagger R(\zeta))^{-1} & R(\zeta)^\dagger(I_n+R(\zeta)R(\zeta)^\dagger)^{-1} \\ -R(\zeta)(I_m+R(\zeta)^\dagger R(\zeta))^{-1} & (I_n+R(\zeta)R(\zeta)^\dagger)^{-1} \end{pmatrix},$$

in combination with the continuity of $R(\zeta)$ and the fact that $\|R(\zeta)\|\to 0$ as $\zeta\to\pm\infty$. Thus, $\tilde{H}$ is similar to the operator of multiplication by the independent variable on $\mathcal{H}_{m+n}$ endowed with the scalar product,

$$(f,g)=\int_{-\infty}^{\infty}\langle\sigma(d\zeta)f(\zeta),g(\zeta)\rangle=\int_{-\infty}^{\infty}d\zeta\langle\sigma'(\zeta)f(\zeta),g(\zeta)\rangle,$$

where the brackets denote the usual scalar product in $\mathbb{C}^{m+n}$ that is antilinear in the vector on the right. The operator $\tilde{H}$ is J-self-adjoint with respect to the indefinite scalar product,

$$(f,g)_J=\int_{-\infty}^{\infty}\langle J\sigma(d\zeta)f(\zeta),g(\zeta)\rangle=\int_{-\infty}^{\infty}d\zeta\langle J\sigma'(\zeta)f(\zeta),g(\zeta)\rangle.$$

Finally, $\tilde{H}$ is not J-definitizable in the sense that there exists no nontrivial polynomial p such that $p(\tilde{H})$ is J-positive, i.e., in the sense that $\langle Jp(\tilde{H})h,h\rangle$ is non-negative for every h in the range of P_{ac}. Indeed, if this were the case, then by J-unitary equivalence between $\tilde{H}$ and $\tilde{H}_0$, the multiplication operator by the independent variable on $\mathcal{H}_{m+n}$, there would exist a polynomial p, such that

$$\langle p(\tilde{H}_0)h,h\rangle=\sum_{j=1}^{m}\int_{-\infty}^{\infty}d\zeta p(\zeta)|h_j(\zeta)|^2-\sum_{j=m+1}^{m+n}\int_{-\infty}^{\infty}d\zeta p(\zeta)|h_j(\zeta)|^2$$

is non-negative for each $h=\{h_j\}_{j=1}^{m+n}$ in $\mathcal{H}_{m+n}$, which is obviously not the case.

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