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Integral Equations and Operator Theory

Scattering Operators for Matrix Zakharov-Shabat Systems

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Abstract. In this article the scattering matrix pertaining to the defocusing matrix Zakharov-Shabat system on the line is related to the scattering operator arising from time-dependent scattering theory. Further, the scattering data allowing for a unique retrieval of the potential in the defocusing matrix Zakharov-Shabat system are characterized.

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1. Introduction

Consider the matrix Zakharov-Shabat system

$$-iJ\frac{dX}{dx}(x,\lambda) - V(x)X(x,\lambda) = \lambda X(x,\lambda), \qquad x \in \mathbb{R}, \tag{1}$$

where

$$J = \begin{pmatrix} I_n & 0_{n\times m} \\ 0_{m\times n} & -I_m \end{pmatrix}, \qquad V(x) = \begin{pmatrix} 0_{n\times n} & iq(x) \\ \mp iq(x)^\dagger & 0_{m\times m} \end{pmatrix}, \tag{2}$$

I_p is the identity matrix of order p, the dagger stands for the conjugate transpose, and the entries of $q(x)$ belong to $L^1(\mathbb{R})$. The plus sign in (2) occurs in the focusing case and the minus sign in the defocusing case. Equation (1) has been studied extensively. We mention the original articles by Zakharov and Shabat [25] ($n = m = 1$) and Manakov [15] ($n = 1$ and $m = 2$) and in particular [1, 2, 3], where also some of the applications are discussed. For the applications to fiber optics we

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refer to [13, 21]. Equation (1) can also be viewed as a so-called canonical system (cf. [20, 5] and references therein).

As in [4, 22, 8], for $\lambda \in \mathbb{R}$ we define the *Jost solution from the left*,[1] $F_l(x, \lambda)$, and the *Jost solution from the right*, $F_r(x, \lambda)$, as the $(n+m) \times (n+m)$ matrix solutions of (1) that satisfy the boundary conditions

$$F_l(x, \lambda) = e^{i\lambda Jx}[I_{n+m} + o(1)], \qquad x \to +\infty, \tag{3a}$$

$$F_r(x, \lambda) = e^{i\lambda Jx}[I_{n+m} + o(1)], \qquad x \to -\infty. \tag{3b}$$

Actually, the Jost solutions follow from the Volterra integral equations

$$F_l(x, \lambda) = e^{i\lambda Jx} - iJ \int_x^\infty dy\, e^{i\lambda J(x-y)} V(y) F_l(y, \lambda), \tag{4a}$$

$$F_r(x, \lambda) = e^{i\lambda Jx} + iJ \int_{-\infty}^x dy\, e^{i\lambda J(x-y)} V(y) F_r(y, \lambda). \tag{4b}$$

Then these two Jost solutions satisfy

$$F_l(x, \lambda) = e^{i\lambda Jx}[a_l(\lambda) + o(1)], \qquad x \to -\infty, \tag{5a}$$

$$F_r(x, \lambda) = e^{i\lambda Jx}[a_r(\lambda) + o(1)], \qquad x \to +\infty, \tag{5b}$$

where $a_l(\lambda)$ and $a_r(\lambda)$ are called *transition matrices*. It is easily seen that $a_l(\lambda)$ and $a_r(\lambda)$ are each other's inverses, while

$$a_l(\lambda)^{-1} = J a_l(\lambda)^\dagger J, \quad a_r(\lambda)^{-1} = J a_r(\lambda)^\dagger J, \qquad \text{defocusing case,}$$

$$a_l(\lambda)^{-1} = a_l(\lambda)^\dagger, \qquad a_r(\lambda)^{-1} = a_r(\lambda)^\dagger, \qquad \text{focusing case.}$$

Moreover, since (1) is a first order system, we have

$$F_l(x, \lambda) = F_r(x, \lambda) a_l(\lambda), \qquad F_r(x, \lambda) = F_l(x, \lambda) a_r(\lambda). \tag{6}$$

Introducing the *Faddeev functions*

$$M_l(x, \lambda) = F_l(x, \lambda) e^{-i\lambda Jx}, \qquad M_r(x, \lambda) = F_r(x, \lambda) e^{-i\lambda Jx}, \tag{7}$$

we obtain from (4) the Volterra integral equations

$$M_l(x, \lambda) = I_{n+m} - iJ \int_x^\infty dy\, e^{i\lambda J(x-y)} V(y) M_l(y, \lambda) e^{i\lambda J(y-x)}, \tag{8a}$$

$$M_r(x, \lambda) = I_{n+m} + iJ \int_{-\infty}^x dy\, e^{i\lambda J(x-y)} V(y) M_r(y, \lambda) e^{i\lambda J(y-x)}. \tag{8b}$$

Defining the *modified Faddeev functions*

$$m_+(x, \lambda) = M_l(x, \lambda) \left(I_n \dot{+} 0_{m\times m}\right) + M_r(x, \lambda) \left(0_{n\times n} \dot{+} I_m\right), \tag{9a}$$

$$m_-(x, \lambda) = M_r(x, \lambda) \left(I_n \dot{+} 0_{m\times m}\right) + M_l(x, \lambda) \left(0_{n\times n} \dot{+} I_m\right), \tag{9b}$$

where $A \dot{+} B$ denotes the direct sum of the square matrices A and B, we see that $m_+(x, \lambda)$ is analytic in $\lambda \in \mathbb{C}^+$ and continuous in $\lambda \in \overline{\mathbb{C}^+}$ and $m_-(x, \lambda)$ is analytic

[1] In [3] the term "Jost function" is used for the $(n+m)\times n$ and $(n+m)\times m$ submatrices composed of the first n and last m columns, respectively.

in $\lambda \in \mathbb{C}^-$ and continuous in $\lambda \in \overline{\mathbb{C}^-}$. The corresponding *modified Jost solutions* are then defined by

$$f_+(x,\lambda) = m_+(x,\lambda)e^{i\lambda Jx}, \qquad f_-(x,\lambda) = m_-(x,\lambda)e^{i\lambda Jx}. \tag{10}$$

The modified Faddeev functions are related by the *scattering matrix* $\boldsymbol{S}(\lambda)$ by means of the Riemann-Hilbert problem

$$m_-(x,\lambda) = m_+(x,\lambda)J\boldsymbol{S}(\lambda)J = m_+(x,\lambda)\begin{pmatrix} T_l(\lambda) & -R(\lambda) \\ -L(\lambda) & T_r(\lambda) \end{pmatrix}, \tag{11}$$

where $\boldsymbol{S}(\lambda)$ is unitary in the defocusing case and J-unitary in the focusing case. The $n \times n$ matrix $T_l(\lambda)$ and the $m \times m$ matrix $T_r(\lambda)$ are called *transmission coefficients*, while the $n \times m$ matrix $R(\lambda)$ and the $m \times n$ matrix $L(\lambda)$ are called *reflection coefficients*.

In [4, 3, 22] the scattering coefficients were introduced ad hoc to create an operational inverse scattering theory without relating them to time dependent scattering theory [14, 19, 24, 23]. In this article we introduce the scattering operator as in time dependent scattering theory by

$$S = \Omega_+(\Omega_-)^\dagger,$$

where $\Omega_\pm$ are the Moeller wave operators. In the defocusing case, where the free Hamiltonian $H_0 = -iJ(d/dx)$ and the full Hamiltonian $H = H_0 - V$ are both selfadjoint on the direct sum of $n+m$ copies of $L^2(\mathbb{R})$, we then prove the existence and asymptotic completeness of $\Omega_\pm$ and hence the unitarity of S for potentials with entries in $L^1(\mathbb{R})$. Applying the unitary equivalence by the (modified) Fourier transform $\mathbb{F}$, we then go on to prove that $\mathbb{F}S\mathbb{F}^{-1}$ is the multiplication by a unitary matrix function $\mathfrak{S}(\lambda)$ of order $n+m$. We then proceed to identify $\mathfrak{S}(\lambda)$ with the scattering matrix $\boldsymbol{S}(\lambda)$ given by (11) and defined in [4] in terms of the transition matrices.

Finally, we briefly touch on the Marchenko integral equation method for recovering the potential from one of the reflection coefficients (See [4, 3, 8] in the defocusing case, [22, 3] in the focusing case without bound states, and [8, 3] in the focusing case) and characterize the scattering data that lead to a defocusing L^1-potential. We note that a full characterization of the scattering data for the Schrödinger equation on the line is known [16, 18] as is a characterization of the scattering data for the matrix Zakharov-Shabat system on the half-line [17].

Let us discuss the contents of the various sections. In Sec. 2 we derive expressions for the resolvent and the spectral decomposition of the full Hamiltonian. Section 3 contains the results on the wave operators, the scattering operator, and the scattering matrix. We also prove the absolute continuity of the full Hamiltonian. In Sec. 4 we review the necessary Marchenko theory, prove boundedness, compactness, and continuous dependence of the Marchenko integral operator on x, and solve the characterization problem. In Appendix A we discuss the precise definition of the full Hamiltonian $-iJ(d/dx) - V$.

We now introduce some notations. By $\mathbb{C}^+$ and $\mathbb{C}^-$ we denote the open upper and lower complex half-planes. We put $\overline{\mathbb{C}^\pm} = \mathbb{C}^\pm \cup \mathbb{R}$. The orthogonal direct sum of p copies of $L^2(\mathbb{R})$ is written as $\mathcal{H}_p$. Let $\mathcal{H}_p^s$ denote the orthogonal direct sum of p copies of the Sobolev space $H^s(\mathbb{R})$ of measurable functions ϕ on $\mathbb{R}$ whose Fourier transforms $\hat{\phi}$ satisfy

$$\|\phi\|_{2,s} = \left[\int_{-\infty}^{\infty} d\xi\,(1+\xi^2)^{s/2}|\hat{\phi}(\xi)|^2\right]^{1/2}.$$

The ℓ^1-direct sum of p copies of $L^1(\mathbb{R})$ is written as $\mathcal{L}_p$. By $\mathcal{L}_p^1$ we denote the ℓ^1-direct sum of p copies of the weighted space $\mathcal{L}^1 \stackrel{\text{def}}{=} L^1(\mathbb{R};(1+|x|)dx)$. The corresponding spaces of $p \times q$ matrices of vectors in the same space are denoted by $\mathcal{H}_{p\times q}$, $\mathcal{H}_{p\times q}^s$, $\mathcal{L}_{p\times q}$, and $\mathcal{L}_{p\times q}^1$, respectively. Throughout this article we partition square matrices $\mathbb{H}$ of order $n+m$ as follows:

$$\mathbb{H} = \begin{pmatrix} \mathbb{H}_1 & \mathbb{H}_2 \\ \mathbb{H}_3 & \mathbb{H}_4 \end{pmatrix},$$

where $\mathbb{H}_1$ is $n \times n$, $\mathbb{H}_2$ is $n \times m$, $\mathbb{H}_3$ is $m \times n$, and $\mathbb{H}_4$ is $m \times m$. This partitioning will be applied in particular to Jost solutions, Faddeev matrices, and transition matrices.

2. Resolvent of the Full Hamiltonian

In this section we derive an expression for the resolvent operator $(\lambda - H)^{-1}$ for $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and apply it to find the spectral decomposition of H.

Let $\phi \in \mathcal{H}_{n+m}$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}$. In order to find the resolvent $(\lambda - H)^{-1}$, we need to determine $\Psi(\cdot, \lambda) \in \mathcal{H}_{n+m}$ such that

$$(H-\lambda)\Psi = -iJ\frac{d}{dx}\Psi(x,\lambda) - V(x)\Psi(x,\lambda) - \lambda\Psi(x,\lambda) = -\phi(x).$$

Letting $\lambda \in \mathbb{C}^+$ and writing $\Psi(x,\lambda) = f_+(x,\lambda)\Phi(x,\lambda)$, we get

$$-iJf_+(x,\lambda)\frac{d}{dx}\Phi(x,\lambda) + \underbrace{\left[-iJ\frac{d}{dx}f_+(x,\lambda) - V(x)f_+(x,\lambda) - \lambda f_+(x,\lambda)\right]}_{=0_{(n+m)\times(n+m)}}\Phi(x,\lambda)$$
$$= -\phi(x),$$

and hence

$$\frac{d}{dx}\Phi(x,\lambda) = -if_+(x,\lambda)^{-1}J\phi(x) = -iJg_+(x,\lambda)\phi(x),$$

where $g_+(x,\lambda) = Jf_+(x,\lambda)^{-1}J$. Thus, writing $g_+(x,\lambda) = e^{-i\lambda Jx}n_+(x,\lambda)$ we get

$$\Phi^{\rm up}(x,\lambda) = -i\int_{-\infty}^{x} dy\, e^{-i\lambda y}[n_+(y,\lambda)\phi(y)]^{\rm up} = -i\int_{-\infty}^{x} dy\,[g_+(y,\lambda)\phi(y)]^{\rm up},$$
$$\Phi^{\rm dn}(x,\lambda) = -i\int_{x}^{\infty} dy\, e^{i\lambda y}[n_+(y,\lambda)\phi(y)]^{\rm dn} = -i\int_{x}^{\infty} dy\,[g_+(y,\lambda)\phi(y)]^{\rm dn},$$

where $\Phi^{\rm up}(x,\lambda) = \begin{pmatrix} I_n & 0_{n\times m}\end{pmatrix}\Phi(x,\lambda)$ and $\Phi^{\rm dn}(x,\lambda) = \begin{pmatrix} 0_{m\times n} & I_m\end{pmatrix}\Phi(x,\lambda)$. Analogously, letting $\lambda \in \mathbb{C}^-$ and writing $\Psi(x,\lambda) = f_-(x,\lambda)\Phi(x,\lambda)$, we get

$$-iJf_-(x,\lambda)\frac{d}{dx}\Phi(x,\lambda) + \underbrace{\left[-iJ\frac{d}{dx}f_-(x,\lambda) - V(x)f_-(x,\lambda) - \lambda f_-(x,\lambda)\right]}_{=0_{(n+m)\times(n+m)}}\Phi(x,\lambda)$$
$$= -\phi(x),$$

and hence

$$\frac{d}{dx}\Phi(x,\lambda) = -if_-(x,\lambda)^{-1}J\phi(x) = -iJg_-(x,\lambda)\phi(x),$$

where $g_-(x,\lambda) = Jf_-(x,\lambda)^{-1}J$. Thus, writing $g_-(x,\lambda) = e^{-i\lambda Jx}n_-(x,\lambda)$ we get

$$\Phi^{\rm up}(x,\lambda) = +i\int_{x}^{\infty} dy\, e^{-i\lambda y}[n_-(y,\lambda)\phi(y)]^{\rm up} = +i\int_{x}^{\infty} dy\,[g_-(y,\lambda)\phi(y)]^{\rm up},$$
$$\Phi^{\rm dn}(x,\lambda) = +i\int_{-\infty}^{x} dy\, e^{i\lambda y}[n_-(y,\lambda)\phi(y)]^{\rm dn} = +i\int_{-\infty}^{x} dy\,[g_-(y,\lambda)\phi(y)]^{\rm dn}.$$

Consequently,

$$[(\lambda - H)^{-1}\phi](x) = \int_{-\infty}^{\infty} dy\,\mathcal{G}(x,y;\lambda)\phi(y), \tag{12}$$

where the Green's function $\mathcal{G}(x,y;\lambda)$ is given by

$$\mathcal{G}(x,y;\lambda) = \begin{cases} -i\,f_+(x,\lambda)\frac{1}{2}(I+J)g_+(y,\lambda), & \operatorname{Im}\lambda > 0,\ x > y,\\ -i\,f_+(x,\lambda)\frac{1}{2}(I-J)g_+(y,\lambda), & \operatorname{Im}\lambda > 0,\ x < y,\\ +i\,f_-(x,\lambda)\frac{1}{2}(I+J)g_-(y,\lambda), & \operatorname{Im}\lambda < 0,\ x < y,\\ +i\,f_-(x,\lambda)\frac{1}{2}(I-J)g_-(y,\lambda), & \operatorname{Im}\lambda < 0,\ x > y. \end{cases} \tag{13}$$

Now recall that for $\lambda \in \mathbb{R}$

$$f_+(x,\lambda) = F_l(x,\lambda)\frac{1}{2}(I+J) + F_r(x,\lambda)\frac{1}{2}(I-J)$$
$$= F_l(x,\lambda)\left[\frac{1}{2}(I+J) + a_r(\lambda)\frac{1}{2}(I-J)\right] = F_l(x,\lambda)\begin{pmatrix} I_n & a_{r2}(\lambda)\\ 0_{m\times n} & a_{r4}(\lambda)\end{pmatrix}.$$

We now easily verify that for $\lambda \in \mathbb{R}$

$$g_+(x,\lambda) = Jf_+(x,\lambda)^{-1}J = \begin{pmatrix} I_n & a_{r2}(\lambda)a_{r4}(\lambda)^{-1}\\ 0_{m\times n} & a_{r4}(\lambda)^{-1}\end{pmatrix} JF_l(x,\lambda)^{-1}J.$$

In the same way we get for $\lambda \in \mathbb{R}$

$$\begin{aligned} f_-(x,\lambda) &= F_r(x,\lambda)\frac{1}{2}(I+J) + F_l(x,\lambda)\frac{1}{2}(I-J) \\ &= F_r(x,\lambda)\left[\frac{1}{2}(I+J) + a_l(\lambda)\frac{1}{2}(I-J)\right] = F_r(x,\lambda)\begin{pmatrix} I_n & a_{l2}(\lambda) \\ 0_{m\times n} & a_{l4}(\lambda)\end{pmatrix} \\ &= F_l(x,\lambda)a_r(\lambda)\begin{pmatrix} I_n & a_{l2}(\lambda) \\ 0_{m\times n} & a_{l4}(\lambda)\end{pmatrix} = F_l(x,\lambda)\begin{pmatrix} a_{r1}(\lambda) & 0_{n\times m} \\ a_{r3}(\lambda) & I_m\end{pmatrix}, \end{aligned}$$

where we have used that $a_r(\lambda)a_l(\lambda) = I_{n+m}$. Thus for $\lambda \in \mathbb{R}$

$$\begin{aligned} g_-(x,\lambda) = Jf_-(x,\lambda)^{-1}J &= \begin{pmatrix} I_n & a_{l2}(\lambda)a_{l4}(\lambda)^{-1} \\ 0_{m\times n} & a_{l4}(\lambda)^{-1}\end{pmatrix} JF_r(x,\lambda)^{-1}J \\ &= \begin{pmatrix} a_{r1}(\lambda)^{-1} & 0_{n\times m} \\ a_{r3}(\lambda)a_{r1}(\lambda)^{-1} & I_m\end{pmatrix} JF_l(x,\lambda)^{-1}J. \end{aligned}$$

By taking the adjoints we get for $\lambda \in \mathbb{R}$

$$\begin{aligned} f_+(x,\lambda)^\dagger &= \begin{pmatrix} I_n & 0_{n\times m} \\ a_{r2}(\lambda)^\dagger & a_{r4}(\lambda)^\dagger\end{pmatrix} F_l(x,\lambda)^\dagger = \begin{pmatrix} I_n & 0_{n\times m} \\ -a_{l3}(\lambda) & a_{l4}(\lambda)\end{pmatrix} JF_l(x,\lambda)^{-1}J, \\ f_-(x,\lambda)^\dagger &= \begin{pmatrix} a_{r1}(\lambda)^\dagger & a_{r3}(\lambda)^\dagger \\ 0_{m\times n} & I_m\end{pmatrix} F_l(x,\lambda)^\dagger = \begin{pmatrix} a_{l1}(\lambda) & -a_{l2}(\lambda) \\ 0_{m\times n} & I_m\end{pmatrix} JF_l(x,\lambda)^{-1}J. \end{aligned}$$

Thus for $\lambda \in \mathbb{R}$ we have

$$\begin{aligned} g_+(x,\lambda) &= \begin{pmatrix} I_n & a_{r2}(\lambda)a_{r4}(\lambda)^{-1} \\ 0_{m\times n} & a_{r4}(\lambda)^{-1}\end{pmatrix}\begin{pmatrix} a_{l1}(\lambda)^{-1} & a_{l1}(\lambda)^{-1}a_{l2}(\lambda) \\ 0_{m\times n} & I_m\end{pmatrix} f_-(x,\lambda)^\dagger \\ &= \begin{pmatrix} a_{l1}(\lambda)^{-1} & 0_{n\times m} \\ 0_{m\times n} & a_{r4}(\lambda)^{-1}\end{pmatrix} f_-(x,\lambda)^\dagger, \qquad (14a) \\ g_-(x,\lambda) &= \begin{pmatrix} a_{r1}(\lambda)^{-1} & 0_{n\times m} \\ a_{l3}(\lambda)a_{r1}(\lambda)^{-1} & I_m\end{pmatrix}\begin{pmatrix} I_n & 0_{n\times m} \\ a_{l4}(\lambda)^{-1}a_{l3}(\lambda) & a_{l4}(\lambda)^{-1}\end{pmatrix} f_+(x,\lambda)^\dagger \\ &= \begin{pmatrix} a_{r1}(\lambda)^{-1} & 0_{n\times m} \\ 0_{m\times n} & a_{l4}(\lambda)^{-1}\end{pmatrix} f_+(x,\lambda)^\dagger. \qquad (14b) \end{aligned}$$

Using estimates as derived in [7] for the Schrödinger equation on the line, we prove the following

Proposition 1. *Suppose the entries of $V(x)$ belong to $\mathcal{L}^1$. Then*

$$\|M_l(x,\lambda) - I_{n+m}\| \le const.\frac{1+\max(0,-x)}{1+|x|}, \qquad (15a)$$

$$\|M_r(x,\lambda) - I_{n+m}\| \le const.\frac{1+\max(0,x)}{1+|x|}, \qquad (15b)$$

uniformly in $\lambda \in \mathbb{R}$.

Proof. From (8a) we easily derive by iteration

$$\|M_l(x,\lambda)\| \le 1 + \int_x^\infty dy\,\|V(y)\|\|M_l(y,\lambda)\| \le \exp\left(\int_x^\infty dy\,\|V(y)\|\right) \le e^{\|V\|_1}.$$

Next, we apply (8a) again and get

$$\begin{aligned}\|M_l(x,\lambda)-I_{n+m}\| &\le e^{\|V\|_1}\int_x^\infty dy\,\|V(y)\| \\ &\le e^{\|V\|_1}\frac{1+\max(0,-x)}{1+|x|}\int_x^\infty dy\,(1+|y|)\|V(y)\| \\ &\le \|V\|_{\mathcal{L}^1}\,e^{\|V\|_1}\frac{1+\max(0,-x)}{1+|x|},\end{aligned}$$

which proves (15a). The estimate (15b) follows from (8b) in an analogous way. □

3. Wave Operators

Consider the free Hamiltonian H_0 and the full Hamiltonian H defined by

$$H_0 = -iJ\frac{d}{dx}, \qquad H = H_0 - V = -iJ\frac{d}{dx} - V,$$

defined on dense domains in $\mathcal{H}_{n+m}$. Here we assume V to have its entries in $L^1(\mathbb{R})$. Then H_0 is an absolutely continuous selfadjoint operator with domain $\mathcal{H}^1_{n+m}$ and spectrum $\mathbb{R}$. In the defocusing case H is a selfadjoint operator with essential spectrum $\mathbb{R}$. Note that iH_0 and iH generate the strongly continuous unitary groups $\{e^{itH_0}\}_{t\in\mathbb{R}}$ and $\{e^{itH}\}_{t\in\mathbb{R}}$ on $\mathcal{H}_{n+m}$ (cf. [14]).

We first prove the following elementary result [8]. It has a well-known analog in the case of the Schrödinger equation [14, Sec. 5.3].

Proposition 2. *Let $\lambda\in\mathbb{C}\setminus\mathbb{R}$ and let W have its entries in $L^2(\mathbb{R})$. Then $W(\lambda-H_0)^{-1}$ is a Hilbert-Schmidt operator on $\mathcal{H}_{n+m}$. Moreover, if W_1 and W_2 have their entries in $L^2(\mathbb{R})$, then $W_1(\lambda-H_0)^{-1}W_2$ is a Hilbert-Schmidt operator on $\mathcal{H}_{n+m}$.*

Proof. It suffices to prove that, for $T=-i(d/dx)$ defined on $L^2(\mathbb{R})$ with domain $H^1(\mathbb{R})$ and for $W\in L^2(\mathbb{R})$, the operator $W(\lambda-T)^{-1}$ is Hilbert-Schmidt. Indeed, letting $\mathcal{F}$ stand for the Fourier transform, we have

$$(\mathcal{F}W(\lambda-T)^{-1}\mathcal{F}^{-1}\hat{\phi})(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty d\eta\,\frac{\hat{W}(\xi-\eta)}{\lambda+\xi}\,\hat{\phi}(\eta).$$

Hence, $\mathcal{F}W(\lambda-T)^{-1}\mathcal{F}^{-1}$ is an integral operator on $L^2(\mathbb{R})$ with square integrable kernel and hence of Hilbert-Schmidt type.

The second part is immediate from

$$[W_1(\lambda-H_0)^{-1}W_2\phi](x) = \begin{cases} -iW_1(x)\displaystyle\int_{-\infty}^x dy\,e^{i\lambda(x-y)}W_2(y)\phi(y), & \operatorname{Im}\lambda>0,\\ +iW_1(x)\displaystyle\int_x^\infty dy\,e^{-i\lambda(y-x)}W_2(y)\phi(y), & \operatorname{Im}\lambda<0,\end{cases}$$

which shows $W_1(\lambda-H_0)^{-1}W_2$ to be a matrix of integral operators on $L^2(\mathbb{R})$ with square integrable kernels. □

Corollary 3. *Let the potential V have its entries in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Then the domains of the free Hamiltonian H_0 and the full Hamiltonian H coincide.*

In the defocusing case we now define the *wave operators* $\tilde{\Omega}_\pm$ by

$$\tilde{\Omega}_\pm \phi = \lim_{\tau \to \pm\infty} e^{i\tau H} e^{-i\tau H_0} \phi, \tag{16}$$

where the limit is taken in the norm of $\mathcal{H}_{n+m}$. If the limits in (16) exist, then $\tilde{\Omega}_\pm$ maps the domain of H_0 into the domain of H and

$$\tilde{\Omega}_\pm H_0 = H \tilde{\Omega}_\pm \quad \text{on} \quad \mathbb{D}(H_0). \tag{17}$$

By the same token, in the defocusing case we define the *wave operators*

$$\Omega_\pm \phi = \lim_{\tau \to \pm\infty} e^{i\tau H_0} e^{-i\tau H} P_{ac}(H) \phi, \tag{18}$$

where $P_{ac}(H)$ is the orthogonal projection onto the absolutely continuous subspace of H. If the limits in (18) exist, then $\Omega_\pm$ maps the domain of H into the domain of H_0 and

$$\Omega_\pm H = H_0 \Omega_\pm \quad \text{on} \quad \mathbb{D}(H), \tag{19}$$

while

$$\tilde{\Omega}_\pm = (\Omega_\pm)^\dagger.$$

For the basic theory of wave operators we refer to [14, 19, 23, 24].

We summarize the above results in the following

Theorem 4. *Let the entries of $V(x)$ belong to $L^1(\mathbb{R})$. Then the wave operators $\Omega_\pm$ defined by* (16) *exist and have the absolutely continuous subspace of H as their range. Moreover, the scattering operator*

$$S = \Omega_+ \tilde{\Omega}_- \tag{20}$$

is unitary.

Proof. Since V has its entries in $L^1(\mathbb{R})$, we can write $V = V_1 V_2$, where V_1 and V_2 both have their entries in $L^2(\mathbb{R})$. Put

$$W(\lambda) = I + V_2(\lambda - H_0)^{-1} V_1, \qquad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

Then, according to the second part of Proposition 2, for nonreal λ the operator $W(\lambda)$ is a Hilbert-Schmidt perturbation of the identity. Then for $\lambda \in \mathbb{C} \setminus \mathbb{R}$ we apply Proposition 2 and derive that

$$\begin{aligned}(\lambda - H)^{-1} - (\lambda - H_0)^{-1} &= -(\lambda - H_0)^{-1} V_1 W(\lambda)^{-1} V_2 (\lambda - H_0)^{-1} \\ &= -\Bigg[\underbrace{\overline{V_1}(\overline{\lambda} - H_0)^{-1}}_{\text{Hilbert-Schmidt}}\Bigg]^\dagger \underbrace{W(\lambda)^{-1}}_{\text{bounded}} \underbrace{V_2(\lambda - H_0)^{-1}}_{\text{Hilbert-Schmidt}}\end{aligned}$$

is a trace class operator. We refer to Appendix A for details on the precise definition of H. According to [23, Theorem 22.19], the wave operators $\Omega_\pm$ and $\tilde{\Omega}_\pm$ defined by (17) and (18) exist. Then $\Omega_\pm$ and $\tilde{\Omega}_\pm$ are partial isometries such that $\tilde{\Omega}_\pm = (\Omega_\pm)^\dagger$, where $\Omega_\pm$ has full range and the absolutely continuous subspace of H as its cokernel (see [23, Theorem 21.3], [14, Lemma X 4.11]). Moreover, S as defined in (20) is a unitary operator on $\mathcal{H}_{n+m}$. $\square$

Equation (16) implies that the wave operators $\Omega_\pm$ satisfy the intertwining relations (17) and (19). As a result, the scattering operator S leaves invariant the domain of H_0 and commutes with H_0.

For $\phi \in \mathcal{H}_{n+m}$, put

$$\breve{\phi}(\lambda) = (\mathbb{F}\phi)(\lambda) = \int_{-\infty}^{\infty} dx\, e^{-i\lambda Jx}\phi(x). \tag{21a}$$

Then

$$\phi(x) = (\mathbb{F}^{-1}\breve{\phi})(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda\, e^{i\lambda Jx}\breve{\phi}(\lambda). \tag{21b}$$

For $\phi \in \mathcal{H}^1_{n+m}$ we get

$$\breve{\psi}(\lambda) = -iJ\breve{\phi}'(\lambda) = \lambda\breve{\phi}(\lambda).$$

where $\psi = H_0\phi$. Thus

$$(\mathbb{F}H_0\phi)(\lambda) = \lambda(\mathbb{F}\phi)(\lambda), \qquad \phi \in \mathbb{D}(H_0) = \mathcal{H}^1_{n+m}. \tag{22}$$

Since this operator commutes with $\mathbb{F}S\mathbb{F}^{-1}$, the unitary operator $\mathbb{F}S\mathbb{F}^{-1}$ coincides with the operator of multiplication by an (almost everywhere existing) unitary scattering matrix [23, Theorem 21.15], $\mathfrak{S}(\lambda)$ say. In other words,

$$(\mathbb{F}S\mathbb{F}^{-1}\hat{\phi})(\lambda) = \mathfrak{S}(\lambda)\hat{\phi}(\lambda), \qquad \hat{\phi} \in \mathcal{H}_{n+m}. \tag{23}$$

Let us now introduce the transformation $\mathbb{G}$ that diagonalizes the full Hamiltonian H.

Lemma 5. *Suppose the entries of $V(x)$ belong to $\mathcal{L}^1 \cap L^2(\mathbb{R})$. Then the linear operators $\mathbb{G}_l$ and $\mathbb{G}_r$ defined by*

$$(\mathbb{G}_l\phi)(\lambda) = \int_{-\infty}^{\infty} dx\, [a_{l1}(\lambda)^{-1}\dot{+}a_{l4}(\lambda)^{-1}]F_r(x,\lambda)^\dagger\phi(x), \tag{24a}$$

$$(\mathbb{G}_r\phi)(\lambda) = \int_{-\infty}^{\infty} dx\, [a_{r1}(\lambda)^{-1}\dot{+}a_{r4}(\lambda)^{-1}]F_l(x,\lambda)^\dagger\phi(x), \tag{24b}$$

are bounded on $\mathcal{H}_{n+m}$. Moreover, $\mathbb{G}_l$ and $\mathbb{G}_r$ are boundedly invertible on $\mathcal{H}_{n+m}$ and their inverses are given by

$$(\mathbb{G}_l^{-1}\hat{\phi})(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda\, F_l(x,\lambda)\hat{\phi}(\lambda), \tag{24c}$$

$$(\mathbb{G}_r^{-1}\hat{\phi})(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda\, F_r(x,\lambda)\hat{\phi}(\lambda). \tag{24d}$$

Further, $\mathbb{G}_l$ and $\mathbb{G}_r$ diagonalize the Hamiltonian H in the sense that for each $\phi \in \mathbb{D}(H)$

$$(\mathbb{G}_l H\phi)(\lambda) = \lambda(\mathbb{G}_l\phi)(\lambda), \tag{25a}$$

$$(\mathbb{G}_r H\phi)(\lambda) = \lambda(\mathbb{G}_r\phi)(\lambda). \tag{25b}$$

Proof. Proposition 1 implies that

$$F_l(x,\lambda) = e^{i\lambda Jx} + o((1+|x|)^{-1}), \qquad x\to+\infty,$$
$$= e^{i\lambda Jx}a_l(\lambda) + o((1+|x|)^{-1}), \qquad x\to-\infty,$$

where the entries of $V(x)$ are assumed to be in $\mathcal{L}^1$ for the right-hand sides to be true. Therefore,

$$\mathbb{G}_l^{-1} = 2\pi\mathbb{F}^{-1}\left(\Pi_+ + [a_l(\lambda)]\Pi_-\right) + K,$$

where $\Pi_\pm$ are the restrictions of vectors in $\mathcal{H}_{n+m}$ to $\mathbb{R}^\pm$ and K is a bounded linear operator on $\mathcal{H}_{n+m}$. The boundedness of K follows from the estimate

$$\|K\phi\| \le \text{const.}\int_{-\infty}^{\infty} dx\,\frac{\|\phi(x)\|}{\sqrt{1+x^2}} \le \pi\,\text{const.}\|\phi\|.$$

Consequently, $\mathbb{G}_l^{-1}$ itself is bounded on $\mathcal{H}_{n+m}$. The boundedness proofs for $\mathbb{G}_r^{-1}$, $\mathbb{G}_l$, and $\mathbb{G}_r$ are similar, but depart from (4b) and the adjoints of (4). It is now immediate from the two versions of (28) that the bounded linear operators defined by (24a) and (24b) are the inverses of those defined by (24c) and (24d).

Since

$$G(x,\lambda) \overset{\text{def}}{=} [a_{l1}(\lambda)^{-1}\dot{+}a_{l4}(\lambda)^{-1}]F_r(x,\lambda)^\dagger = [a_{l1}(\lambda)^{-1}\dot{+}a_{l4}(\lambda)^{-1}]JF_r(x,\lambda)^{-1}J \tag{26}$$

satisfies the adjoint matrix Zakharov-Shabat system

$$i\frac{\partial G}{\partial x}(x,\lambda)J - G(x,\lambda)V(x) = \lambda G(x,\lambda), \tag{27}$$

we easily verify that for $\phi\in\mathbb{D}(H)=\mathbb{D}(H_0)=\mathcal{H}^1_{n+m}$

$$\begin{aligned}\lambda(\mathbb{G}_l\phi)(\lambda) &= [iG(x,\lambda)J\phi(x)]_{x=-\infty}^{\infty}\\ &\quad+\int_{-\infty}^{\infty}dx\,G(x,\lambda)\left(-iJ\phi'(x)-V(x)\phi(x)\right)\\ &= (\mathbb{G}_lH\phi)(\lambda),\end{aligned}$$

which implies (25a). In the same way we prove (25b). □

We now prove that the full Hamiltonian H, like the free Hamiltonian H_0, is absolutely continuous.

Theorem 6. *Suppose the entries of $V(x)$ belong to $\mathcal{L}^1\cap L^2(\mathbb{R})$. Then the Hamiltonian operator H is absolutely continuous.*

Proof. We divide the proof into three parts.

1. Let σ denote the resolution of the identity of the selfadjoint operator H, where $\sigma(E)$ is an orthogonal projection on $\mathcal{H}_{n+m}$ for each real Borel set E. Then for each pair of real numbers a,b with $a<b$ and each $\phi\in\mathcal{H}_{n+m}$ we have

$$\frac{\sigma((a,b))+\sigma([a,b])}{2}\phi = \lim_{\varepsilon\to0^+}\frac{1}{2\pi i}\int_a^b d\tau\left((\lambda-i\varepsilon-H)^{-1}-(\lambda+i\varepsilon-H)^{-1}\right)\phi,$$

where the limit is taken in the strong sense [14]. Thus if a and/or b is an eigenvalue of H, which results in a nonzero eigenprojection $\sigma(\{a\})$ and/or $\sigma(\{b\})$, then these eigenprojections are only taken account of with weight $\frac{1}{2}$. Let us now apply this identity. Using (12) and (13) we take the limit as $\varepsilon \to 0^+$ and get

$$\left[\frac{\sigma((a,b)) + \sigma([a,b])}{2}\phi\right](x) = \int_a^b d\zeta \int_{-\infty}^{\infty} dy\, \mathcal{J}(x,y;\zeta)\phi(y),$$

where

$$\mathcal{J}(x,y;\zeta) = \\ = \begin{cases} \frac{1}{2\pi}\left[f_+(x,\zeta)\frac{1}{2}(I+J)g_+(y,\zeta) + f_-(x,\zeta)\frac{1}{2}(I-J)g_-(y,\zeta)\right], & x > y, \\ \frac{1}{2\pi}\left[f_+(x,\zeta)\frac{1}{2}(I-J)g_+(y,\zeta) + f_-(x,\zeta)\frac{1}{2}(I+J)g_-(y,\zeta)\right], & x < y. \end{cases}$$

2. Let us now write $\mathcal{J}(x,y;\zeta)$ in a different way. Using (9) and (10) we write (14) in the form

$$\mathcal{J}(x,y;\zeta) = \begin{cases} \frac{1}{2\pi}F_l(x,\zeta)[a_{l1}(\zeta)^{-1}\dot{+}a_{l4}(\zeta)^{-1}]F_r(y,\zeta)^\dagger, & x > y, \\ \frac{1}{2\pi}F_r(x,\zeta)[a_{r1}(\zeta)^{-1}\dot{+}a_{r4}(\zeta)^{-1}]F_l(y,\zeta)^\dagger, & x < y. \end{cases}$$

Using (6) and the J-unitarity of the transition matrices, we compute for $x > y$

$$\begin{aligned} 2\pi\mathcal{J}(x,y;\zeta) &= F_l(x,\zeta)[a_{l1}(\zeta)^{-1}\dot{+}a_{l4}(\zeta)^{-1}]F_r(y,\zeta)^\dagger \\ &= F_r(x,\zeta)a_l(\zeta)[a_{l1}(\zeta)^{-1}\dot{+}a_{l4}(\zeta)^{-1}]a_r(\lambda)^\dagger F_l(x,\lambda)^\dagger \\ &= F_r(x,\zeta)a_l(\zeta)[a_{l1}(\zeta)^{-1}\dot{+}a_{l4}(\zeta)^{-1}]Ja_r(\zeta)^{-1}JF_l(x,\zeta)^\dagger \\ &= F_r(x,\zeta)a_l(\zeta)[a_{l1}(\zeta)^{-1}\dot{+}a_{l4}(\zeta)^{-1}]Ja_l(\zeta)JF_l(x,\zeta)^\dagger \\ &= F_r(x,\zeta)[(a_{l1}(\zeta) - a_{l2}(\zeta)a_{l4}(\zeta)^{-1}a_{l3}(\zeta)) \\ &\qquad \dot{+}(a_{l4}(\zeta) - a_{l3}(\zeta)a_{l1}(\zeta)^{-1}a_{l2}(\zeta))]F_l(x,\zeta)^\dagger \\ &= F_r(x,\zeta)[a_{r1}(\zeta)^{-1}\dot{+}a_{r4}(\zeta)^{-1}]F_l(x,\zeta)^\dagger, \end{aligned}$$

where we have used $a_l(\zeta)a_r(\zeta) = I_{n+m}$ at the last step. Consequently,

$$\begin{aligned} \mathcal{J}(x,y;\zeta) &= \frac{1}{2\pi}F_l(x,\zeta)[a_{l1}(\zeta)^{-1}\dot{+}a_{l4}(\zeta)^{-1}]F_r(y,\zeta)^\dagger \\ &= \frac{1}{2\pi}F_r(x,\zeta)[a_{r1}(\zeta)^{-1}\dot{+}a_{r4}(\zeta)^{-1}]F_l(y,\zeta)^\dagger, \end{aligned} \tag{28}$$

irrespective of whether $x > y$ or $x < y$.

3. Equations (24) imply that

$$\int_{-\infty}^{\infty} dy\, \mathcal{J}(x,y;\zeta)\phi(y) = \frac{1}{2\pi}F_l(x,\zeta)(\mathbb{G}_l\phi)(\zeta) = \frac{1}{2\pi}F_r(x,\zeta)(\mathbb{G}_r\phi)(\zeta),$$

where $\mathbb{G}_l$ and $\mathbb{G}_r$ are bounded operators. Hence, for $\phi, \psi \in \mathcal{H}_{n+m}$ we have

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \, \langle \mathcal{C}(x,y;\zeta)\phi(y), \psi(x)\rangle$$

$$= \begin{cases} \frac{1}{2\pi}\langle(\mathbb{G}_l\phi)(\zeta), [a_{r1}(\zeta)\dot{+}a_{r4}(\zeta)](\mathbb{G}_r\psi)(\zeta)\rangle, \\ \frac{1}{2\pi}\langle(\mathbb{G}_r\phi)(\zeta), [a_{l1}(\zeta)\dot{+}a_{l4}(\zeta)](\mathbb{G}_l\psi)(\zeta)\rangle, \end{cases}$$

which implies the boundedness of the integral operator $\mathcal{J}(\zeta)$ with integral kernel $\mathcal{J}(x,y;\zeta)$ for each $\zeta \in \mathbb{R}$. Consequently, H is absolutely continuous with $\sigma'(\zeta) = \mathcal{J}(\zeta)$, as claimed. □

We now relate the scattering operator S to the scattering matrix

$$\boldsymbol{S}(\lambda) = \begin{pmatrix} T_l(\lambda) & R(\lambda) \\ L(\lambda) & T_r(\lambda) \end{pmatrix},$$

where

$$T_l(\lambda) = a_{l1}(\lambda)^{-1}, \qquad R(\lambda) = a_{r2}(\lambda)a_{r4}(\lambda)^{-1}$$
$$L(\lambda) = a_{l3}(\lambda)a_{l1}(\lambda)^{-1}, \qquad T_r(\lambda) = a_{r4}(\lambda)^{-1}.$$

Here we have defined the scattering coefficients as in [4, 22, 8]. We also have the alternative expressions

$$R(\lambda) = -a_{l1}(\lambda)^{-1}a_{l2}(\lambda), \qquad L(\lambda) = -a_{r4}(\lambda)^{-1}a_{r3}(\lambda).$$

Recall that in the defocusing case the scattering matrix $\boldsymbol{S}(\lambda)$ is unitary. Using the scattering matrix we can write (28) in the form

$$\mathcal{J}(x,y;\lambda) = F_l(x,\lambda)\begin{pmatrix} I_n & R(\lambda) \\ R(\lambda)^\dagger & I_m \end{pmatrix} F_l(x,\lambda)^\dagger$$
$$= F_r(x,\lambda)\begin{pmatrix} I_n & L(\lambda)^\dagger \\ L(\lambda) & I_m \end{pmatrix} F_r(x,\lambda)^\dagger.$$

To prove that $\mathcal{S}(\lambda) = \boldsymbol{S}(\lambda)$ for each $\lambda \in \mathbb{R}$, we follow the path taken in [23] fot the Schrödinger equation on the line.

Theorem 7. *Let the entries of $V(x)$ belong to $\mathcal{L}^1 \cap L^2(\mathbb{R})$. Then the Fourier transformed scattering operator $\mathbb{F}S\mathbb{F}^{-1}$ coincides with the operator of premultiplication by the unitary $(n+m)\times(n+m)$ matrix function $\boldsymbol{S}(\lambda)$.*

Proof. Equations (17), (22), and (25) imply that on $\mathbb{F}[\mathbb{D}(H_0)]$

$$\mathbb{G}_l\tilde{\Omega}_\pm\mathbb{F}^{-1}[\lambda] \overset{(22)}{=} \mathbb{G}_l\tilde{\Omega}_\pm H_0\mathbb{F}^{-1} \overset{(17)}{=} \mathbb{G}_l H\tilde{\Omega}_\pm\mathbb{F}^{-1} \overset{(25a)}{=} [\lambda]\mathbb{G}_l\tilde{\Omega}_\pm\mathbb{F}^{-1},$$

where $[\lambda]$ is the operator of multiplication by the independent variable. Analogously, (19), (22), and (25a) imply that on $\mathbb{G}[\mathbb{D}(H)]$

$$\mathbb{F}\Omega_\pm\mathbb{G}_l^{-1}[\lambda] \overset{(25a)}{=} \mathbb{F}\Omega_\pm H\mathbb{G}_l^{-1} \overset{(19)}{=} \mathbb{F}H_0\Omega_\pm\mathbb{G}_l^{-1} \overset{(22)}{=} [\lambda]\mathbb{F}\Omega_\pm\mathbb{G}_l^{-1}.$$

Consequently, both $\mathbb{G}_l\tilde{\Omega}_\pm\mathbb{F}^{-1}$ and $\mathbb{F}\Omega_\pm\mathbb{G}_l^{-1}$ are operators of premultiplication by an $(n+m)\times(n+m)$ matrix function [23, Theorem 21.15]. Hence there exist matrix functions $\tilde{\boldsymbol{w}}_\pm(\lambda)$ and $\boldsymbol{w}_\pm(\lambda)$ such that

$$(\mathbb{G}_l\tilde{\Omega}_\pm\mathbb{F}^{-1}\hat{\phi})(\lambda) = \tilde{\boldsymbol{w}}_\pm(\lambda)\hat{\phi}(\lambda), \qquad (\mathbb{F}\Omega_\pm\mathbb{G}_l^{-1}\hat{\phi})(\lambda) = \boldsymbol{w}_\pm(\lambda)\hat{\phi}(\lambda), \tag{29}$$

where $\hat{\phi} \in \mathcal{H}_{n+m}$. Then using (20) and (23) it follows that

$$\mathfrak{S}(\lambda) = \boldsymbol{w}_+(\lambda)\tilde{\boldsymbol{w}}_-(\lambda), \qquad \lambda \in \mathbb{R}. \tag{30}$$

In the above derivation we could have employed $\mathbb{G}_r$ instead of $\mathbb{G}_l$.

Let us now compute $\tilde{\boldsymbol{w}}_\pm(\lambda)$. Writing $G(x,\lambda)$ defined by (26) in the form

$$G(x,\lambda) = e^{-i\lambda Jx}N(x,\lambda), \tag{31}$$

we obtain from (27) the Volterra integral equations

$$N(x,\lambda) = e^{i\lambda Jx}N_{+\infty}(\lambda)e^{-i\lambda Jx} + i\int_x^\infty dy\, e^{i\lambda J(x-y)}N(y,\lambda)V(y)e^{i\lambda J(y-x)}J, \tag{32a}$$

$$N(x,\lambda) = e^{i\lambda Jx}N_{-\infty}(\lambda)e^{-i\lambda Jx} - i\int_{-\infty}^x dy\, e^{i\lambda J(x-y)}N(y,\lambda)V(y)e^{i\lambda J(y-x)}J, \tag{32b}$$

where

$$\begin{aligned} N_{+\infty}(\lambda) &= [a_{l1}(\lambda)^{-1}\dot{+}a_{l4}(\lambda)^{-1}]a_r(\lambda)^\dagger = [a_{l1}(\lambda)^{-1}\dot{+}a_{l4}(\lambda)^{-1}]Ja_r(\lambda)^{-1}J \\ &= \begin{pmatrix} I_n & -a_{l1}(\lambda)^{-1}a_{l2}(\lambda) \\ -a_{l4}(\lambda)^{-1}a_{l3}(\lambda) & I_m \end{pmatrix}, \end{aligned} \tag{33a}$$

$$N_{-\infty}(\lambda) = a_{l1}(\lambda)^{-1}\dot{+}a_{l4}(\lambda)^{-1}. \tag{33b}$$

We then compute

$$\begin{aligned} \tilde{\boldsymbol{w}}_\pm(\lambda)\hat{\phi}(\lambda) &\overset{(29)}{=} (\mathbb{G}_l\tilde{\Omega}_\pm\mathbb{F}^{-1}\hat{\phi})(\lambda) \\ &\overset{(16)}{=} \lim_{\tau\to\pm\infty} (\mathbb{G}_l e^{i\tau H}e^{-i\tau H_0}\mathbb{F}^{-1}\hat{\phi})(\lambda) \\ &\overset{(22)}{=} \lim_{\tau\to\pm\infty} e^{i\lambda\tau}(\mathbb{G}_l\mathbb{F}^{-1}e^{-i[\xi]\tau}\hat{\phi})(\lambda) \\ &\overset{(24)}{=} \lim_{\tau\to\pm\infty} e^{i\lambda\tau}\int_{-\infty}^\infty dx\, G(x,\lambda)(\mathbb{F}^{-1}e^{-i[\xi]\tau}\hat{\phi})(x) \\ &\overset{(21b)}{=} \lim_{\tau\to\pm\infty} e^{i\lambda\tau}\int_{-\infty}^\infty dx\, G(x,\lambda)\frac{1}{2\pi}\int_{-\infty}^\infty d\xi\, e^{i\xi Jx}e^{-i\xi\tau}\hat{\phi}(\xi) \\ &= \lim_{\tau\to\pm\infty}\int_{-\infty}^\infty d\xi\, e^{i(\lambda-\xi)\tau}\left(\frac{1}{2\pi}\int_{-\infty}^\infty dx\, G(x,\lambda)e^{i\xi Jx}\right)\hat{\phi}(\xi) \\ &\overset{(31)}{=} \lim_{\tau\to\pm\infty}\int_{-\infty}^\infty d\xi\, e^{i(\lambda-\xi)\tau}\left(\frac{1}{2\pi}\int_{-\infty}^\infty dx\, e^{-i\lambda Jx}N(x,\lambda)e^{i\xi Jx}\right)\hat{\phi}(\xi), \end{aligned}$$

where we have changed the order of integration at the penultimate transition. Substituting (32) we get

$$\tilde{\boldsymbol{w}}_\pm(\lambda)\hat{\phi}(\lambda) = \lim_{\tau\to\pm\infty} \frac{1}{2\pi}\int_{-\infty}^{\infty} d\xi\, e^{i(\lambda-\xi)\tau}\left[\int_0^\infty dx\, N_{+\infty}(\lambda)e^{i(\xi-\lambda)Jx} + \int_{-\infty}^0 dx\, N_{-\infty}(\lambda)e^{i(\xi-\lambda)Jx} - \int_{-\infty}^\infty dx\, \Phi(x,\lambda)Je^{i(\xi-\lambda)Jx}\right]\hat{\phi}(\xi),$$

where

$$\Phi(x,\lambda) = \begin{cases} +i\displaystyle\int_{-\infty}^x dy\, e^{-i\lambda Jy}N(y,\lambda)V(y)e^{i\lambda Jy}, & x<0,\\ -i\displaystyle\int_x^\infty dy\, e^{-i\lambda Jy}N(y,\lambda)V(y)e^{i\lambda Jy}, & x>0,\end{cases}$$

is continuous in $0\neq x\in\mathbb{R}$, vanishes as $x\to\pm\infty$, and satisfies [cf. (32)]

$$\Phi(0^+,\lambda)-\Phi(0^-,\lambda) = N_{+\infty}(\lambda)-N_{-\infty}(\lambda).$$

Moreover, for potentials V whose entries belong to $\mathcal{L}^1$, the entries of $\Phi(\cdot,\lambda)$ belong to $L^1(\mathbb{R})$ for each $\lambda\in\mathbb{R}$. When taking the limit as $\tau\to\pm\infty$, the terms involving $N_{\pm\infty}(\lambda)$ either lead to a delta function integration or vanish and the term involving $\Phi(x,\lambda)$ vanishes. As a result,

$$\tilde{\boldsymbol{w}}_\pm(\lambda)\hat{\phi}(\lambda) = \left\{N_{\pm\infty}(\lambda)\frac{1}{2}(I+J)+N_{\mp\infty}(\lambda)\frac{1}{2}(I-J)\right\}\hat{\phi}(\lambda). \tag{34}$$

Next, we compute

$$\begin{aligned}\boldsymbol{w}_\pm(\lambda)\hat{\phi}(\lambda) &\overset{(29)}{=} (\mathbb{F}\Omega_\pm\mathbb{G}_l^{-1}\hat{\phi})(\lambda)\\ &\overset{(18)}{=} \lim_{\tau\to\pm\infty}(\mathbb{F}e^{i\tau H_0}e^{-i\tau H}P_{ac}(H)\mathbb{G}_l^{-1}\hat{\phi})(\lambda)\\ &= \lim_{\tau\to\pm\infty}(\mathbb{F}e^{i\tau H_0}e^{-i\tau H}\mathbb{G}_l^{-1}\hat{\phi})(\lambda)\\ &\overset{(22)}{=} \lim_{\tau\to\pm\infty} e^{i\lambda\tau}(\mathbb{F}\mathbb{G}_l^{-1}e^{-i[\xi]\tau}\hat{\phi})(\lambda)\\ &\overset{(21a)}{=} \lim_{\tau\to\pm\infty} e^{i\lambda\tau}\int_{-\infty}^\infty dx\, e^{-i\lambda Jx}(\mathbb{G}_l^{-1}e^{-i[\xi]\tau}\hat{\phi})(x)\\ &\overset{(25)}{=} \lim_{\tau\to\pm\infty} e^{i\lambda\tau}\int_{-\infty}^\infty dx\, e^{-i\lambda Jx}\frac{1}{2\pi}\int_{-\infty}^\infty d\xi\, F_l(x,\xi)e^{-i\xi\tau}\hat{\phi}(\xi)\\ &= \lim_{\tau\to\pm\infty}\int_{-\infty}^\infty d\xi\, e^{i(\lambda-\xi)\tau}\left(\frac{1}{2\pi}\int_{-\infty}^\infty dx\, e^{-i\lambda Jx}F_l(x,\xi)\right)\hat{\phi}(\xi)\\ &\overset{(13)}{=} \lim_{\tau\to\pm\infty}\int_{-\infty}^\infty d\xi\, e^{i(\lambda-\xi)\tau}\left(\frac{1}{2\pi}\int_{-\infty}^\infty dx\, e^{-i\lambda Jx}M_l(x,\xi)e^{i\xi Jx}\right)\hat{\phi}(\xi),\end{aligned}$$

where we have used the absolute continuity [Lemma 6] at the third transition and changed the order of integration at the penultimate transition. Substituting (8) we get

$$\boldsymbol{w}_\pm(\lambda)\hat{\phi}(\lambda) = \lim_{\tau\to\pm\infty} \frac{1}{2\pi}\int_{-\infty}^{\infty} d\xi\, e^{i(\lambda-\xi)\tau}\left[\int_0^\infty dx\, e^{i(\xi-\lambda)Jx}M_{+\infty}(\xi)\right.$$
$$\left. + \int_{-\infty}^{0} dx\, e^{i(\xi-\lambda)Jx}M_{-\infty}(\xi) - \int_{-\infty}^{\infty} dx\, J\tilde{\Phi}(x,\xi)\right]\hat{\phi}(\xi),$$

where $M_l(x,\lambda)\to M_{\pm\infty}(\lambda)$ as $x\to\pm\infty$ and

$$\tilde{\Phi}(x,\lambda) = \begin{cases} -i\displaystyle\int_{-\infty}^{x} dy\, e^{-i\lambda Jy}V(y)M(y,\lambda)e^{i\lambda Jy}, & x<0,\\ +i\displaystyle\int_{x}^{\infty} dy\, e^{-i\lambda Jy}V(y)M(y,\lambda)e^{i\lambda Jy}, & x>0. \end{cases}$$

As above, we obtain for potentials V with entries in $\mathcal{L}^1$

$$\boldsymbol{w}_\pm(\lambda)\hat{\phi}(\lambda) = \left\{\frac{1}{2}(I+J)M_{\pm\infty}(\lambda) + \frac{1}{2}(I-J)M_{\mp\infty}(\lambda)\right\}\hat{\phi}(\lambda), \tag{35}$$

where $M_{+\infty}(\lambda) = I_{n+m}$ and $M_{-\infty}(\lambda) = a_l(\lambda)$.

Let us compute $\mathfrak{S}(\lambda)$:

$$\begin{aligned}
\mathfrak{S}(\lambda) &\overset{(30)}{=} \boldsymbol{w}_+(\lambda)\tilde{\boldsymbol{w}}_-(\lambda)\\
&\overset{(34)}{=} \begin{pmatrix} I_n & 0_{n\times m}\\ a_{l3}(\lambda) & a_{l4}(\lambda)\end{pmatrix}\tilde{\boldsymbol{w}}_-(\lambda)\\
&\overset{(33)}{=} \begin{pmatrix} I_n & 0_{n\times m}\\ a_{l3}(\lambda) & a_{l4}(\lambda)\end{pmatrix}\begin{pmatrix} a_{l1}(\lambda)^{-1} & -a_{l1}(\lambda)^{-1}a_{l2}(\lambda)\\ 0_{m\times n} & I_m\end{pmatrix}\\
&= \begin{pmatrix} a_{l1}(\lambda)^{-1} & -a_{l1}(\lambda)^{-1}a_{l2}(\lambda)\\ a_{l3}(\lambda)a_{l4}(\lambda)^{-1} & a_{l4}(\lambda) - a_{l3}(\lambda)a_{l1}(\lambda)^{-1}a_{l2}(\lambda)\end{pmatrix}\\
&= \begin{pmatrix} a_{l1}(\lambda)^{-1} & a_{r2}(\lambda)a_{r4}(\lambda)^{-1}\\ a_{l3}(\lambda)a_{l4}(\lambda)^{-1} & a_{r4}(\lambda)^{-1}\end{pmatrix}\\
&= \begin{pmatrix} a_{l1}(\lambda)^{-1} & a_{r2}(\lambda)a_{r4}(\lambda)^{-1}\\ a_{l3}(\lambda)a_{l1}(\lambda)^{-1} & a_{r4}(\lambda)^{-1}\end{pmatrix} = \begin{pmatrix} T_l(\lambda) & R(\lambda)\\ L(\lambda) & T_r(\lambda)\end{pmatrix} = \boldsymbol{S}(\lambda),
\end{aligned}$$

which concludes the proof. □

4. Marchenko Theory

In this section we prove the unique solvability of the Marchenko integral equations that lead to the solution of the inverse scattering problem of determining the potential in the defocusing matrix Zakharov-Shabat system from one of the reflection coefficients. The information obtained will then be used to solve the characterization problem of describing the scattering data leading to a unique determination of a potential having its entries in $L^1(\mathbb{R})$.

4.1. From Reflection Coefficient to Scattering Matrix

We first summarize some of the properties of the scattering coefficients, referring to [4, Theorem 3.1] and [8, Proposition 3.13 and the two lines above its statement] for the proof.

Proposition 8. *In the defocusing case the scattering matrix* $\boldsymbol{S}(\lambda)$ *is unitary, i.e.,*

$$\boldsymbol{S}(\lambda)^{-1} = \boldsymbol{S}(\lambda)^{\dagger}, \quad \lambda \in \mathbb{R}.$$

Further, the reflection and transmission matrices are continuous in $\lambda \in \mathbb{R}$*, while as* $\lambda \to \pm\infty$ *the reflection coefficients vanish and the transmission coefficients tend to the identity. Moreover, the transmission coefficients* $T_l(\lambda)$ *and* $T_r(\lambda)$ *are continuous in* $\lambda \in \overline{\mathbb{C}^+}$ *and analytic in* $\lambda \in \mathbb{C}^+$*, while*

$$\sup_{\lambda\in\overline{\mathbb{C}^+}} \|T_l(\lambda)\| > 0, \qquad \sup_{\lambda\in\overline{\mathbb{C}^+}} \|T_r(\lambda)\| > 0. \tag{36a}$$

The reflection coefficients $R(\lambda)$ *and* $L(\lambda)$ *satisfy the inequalities*

$$\sup_{\lambda\in\mathbb{R}} \|R(\lambda)\| < 1, \qquad \sup_{\lambda\in\mathbb{R}} \|L(\lambda)\| < 1. \tag{36b}$$

In order to construct the scattering matrix from one of the reflection coefficients, we define the so-called Wiener algebra $\mathcal{W}^q$ denote of all $q \times q$ matrix functions of the form [9, 10, 11]

$$Z(\lambda) = Z_\infty + \int_{-\infty}^{\infty} d\alpha\, z(\alpha) e^{i\lambda\alpha}, \tag{37}$$

where $z(\alpha)$ is a $q \times q$ matrix function whose entries belong to $L^1(\mathbb{R})$ and $Z_\infty = Z(\pm\infty)$. Then $\mathcal{W}^q$ is a Banach algebra with unit element endowed with the norm

$$\|Z(\lambda)\|_{\mathcal{W}^q} = \|Z_\infty\| + \int_{-\infty}^{\infty} d\alpha\, \|z(\alpha)\|.$$

The analogous Banach algebra of $q \times r$ matrix functions is denoted as $\mathcal{W}^{q\times r}$, so that $\mathcal{W}^{q\times q} = \mathcal{W}^q$. By $\mathcal{W}_+^{q\times r}$ we denote the closed subalgebra of $\mathcal{W}^{q\times r}$ consisting of those $Z(\lambda)$ of the type (37) for which $z(\alpha)$ is supported on the positive real line. We write $\mathcal{W}_+^q$ instead of $\mathcal{W}_+^{q\times q}$.

It is important to recall the following result [4, 8].

Theorem 9. *The coefficients* $a_l(\lambda)$ *and* $a_r(\lambda)$ *are elements of* $\mathcal{W}^{n+m}$.

Now it is simple to show that in the defocusing case the scattering data consist of just one reflection coefficient, either $R(\lambda)$ or $L(\lambda)$, while the other reflection coefficient and the transmission coefficients can computed in the process. Indeed, using the unitarity of $\boldsymbol{S}(\lambda)$ we first determine the unique matrix functions $T_l(\lambda)$ and $T_r(\lambda)$ such that T_l is an invertible element of $\mathcal{W}_+^{n\times n}$ with $T_l(\pm\infty) = I_n$, T_r

is an invertible element of $\mathcal{W}_+^{m\times m}$ with $T_r(\pm\infty) = I_m$, and the following two equations are true:

$$T_l(\lambda)\,T_l(\lambda)^\dagger = I_n - R(\lambda)\,R(\lambda)^\dagger, \qquad \lambda \in \mathbb{R},$$
$$T_r(\lambda)^\dagger\,T_r(\lambda) = I_m - R(\lambda)^\dagger\,R(\lambda), \qquad \lambda \in \mathbb{R}.$$

These factorization problems have a unique solution, as a result of the following [12, 10, 6]

Theorem 10. *Let $F \in L^1(\mathbb{R}; \mathbb{C}^{p\times p})$ be such that*

$$\hat{W}(\lambda) = I_p + \int_{-\infty}^{\infty} dt\, e^{i\lambda t} F(t)$$

is positive and selfadjoint for $\lambda \in \mathbb{R}$. Then there exist unique functions $F_+ \in L^1(\mathbb{R}^+; \mathbb{C}^{p\times p})$ and $G_+ \in L^1(\mathbb{R}^+; \mathbb{C}^{p\times p})$ such that

$$\hat{W}(\lambda) = \left[I_p + \int_0^\infty dt\, e^{i\lambda t} F_+(t)\right] \left[I_p + \int_0^\infty dt\, e^{-i\lambda t} F_+(t)\right]^\dagger,$$
$$\hat{W}(\lambda) = \left[I_p + \int_0^\infty dt\, e^{-i\lambda t} G_+(t)\right]^\dagger \left[I_p + \int_0^\infty dt\, e^{i\lambda t} G_+(t)\right],$$

while

$$\det\left(\left[I_p + \int_0^\infty dt\, e^{i\lambda t} F_+(t)\right]\right) \neq 0, \qquad \lambda \in \overline{\mathbb{C}^+},$$
$$\det\left(\left[I_p + \int_0^\infty dt\, e^{i\lambda t} G_+(t)\right]\right) \neq 0, \qquad \lambda \in \overline{\mathbb{C}^+}.$$

Finally we define $L(\lambda)$ by

$$L(\lambda) = -T_r(\lambda)\,R(\lambda)^\dagger\,[T_l(\lambda)^\dagger]^{-1}, \qquad \lambda \in \mathbb{R}.$$

On the other hand, given $L \in \mathcal{W}^{m\times n}$ satisfying the second of (36b), we first determine the unique matrix functions $T_l(\lambda)$ and $T_r(\lambda)$ such that T_l is an invertible element of $\mathcal{W}_+^{n\times n}$ with $T_l(\pm\infty) = I_n$, T_r is an invertible element of $\mathcal{W}_+^{m\times m}$ with $T_r(\pm\infty) = I_m$, and the following two equations are true:

$$T_l^\dagger(\lambda)\,T_l(\lambda) = I_n - L^\dagger(\lambda)\,L(\lambda), \qquad \lambda \in \mathbb{R},$$
$$T_r(\lambda)\,T_r(\lambda)^\dagger = I_m - L(\lambda)\,L^\dagger(\lambda), \qquad \lambda \in \mathbb{R}.$$

By Theorem 10, these factorization problems again have a unique solution. We then define

$$R(\lambda) = -T_l(\lambda)L(\lambda)^\dagger[T_r(\lambda)^\dagger]^{-1}, \qquad \lambda \in \mathbb{R}.$$

4.2. Coupled and uncoupled Marchenko equation

It is well-known [4, 8] that the Marchenko integral equation can be written in two different and equivalent forms. In the defocusing case the *coupled Marchenko equations*[2] are given by

$$K_1(x,\alpha) = -\int_0^\infty d\beta\, K_2(x,\beta)\,\Omega(\alpha+\beta+2x)^\dagger, \tag{38a}$$

$$K_2(x,\alpha) = -\Omega(\alpha+2x) - \int_0^\infty d\beta\, K_1(x,\beta)\,\Omega(\alpha+\beta+2x), \tag{38b}$$

where

$$\Omega(\alpha) = \frac{1}{2\pi}\int_{-\infty}^\infty d\lambda\, R(\lambda)e^{-i\alpha\lambda}, \qquad R(\lambda) = \int_{-\infty}^\infty d\alpha\, \Omega(\alpha)e^{i\alpha\lambda}. \tag{39}$$

Substituting (38a) into (38b), we obtain the *uncoupled Marchenko equation*

$$\begin{aligned} &K_2(x,\alpha) + \Omega(\alpha+2x) \\ &- \int_0^\infty d\beta\, K_2(x,\beta)\int_0^\infty d\gamma\, \Omega(\beta+\gamma+2x)^\dagger\Omega(\alpha+\gamma+2x) = 0_{n\times m}. \end{aligned} \tag{40}$$

Formally we can write the adjoint of the uncoupled Marchenko equation (40) as

$$\left(I - \Omega^\dagger\Omega\right) K_2^\dagger = -\Omega^\dagger.$$

The potential $q(x)$ is obtained from the solution of the Marchenko equation (40) as follows:

$$q(x) = 2K_2(x,0^+). \tag{41}$$

In the same way we get[3]

$$K_3(x,\alpha) = -\int_0^\infty d\beta\, K_4(x,\beta)\,\tilde{\Omega}(\alpha+\beta-2x) \tag{42a}$$

$$K_4(x,\alpha) = -\tilde{\Omega}(\alpha-2x)^\dagger - \int_0^\infty d\beta\, K_3(x,\beta)\,\tilde{\Omega}(\alpha+\beta-2x)^\dagger, \tag{42b}$$

where

$$\tilde{\Omega}(\alpha) = \frac{1}{2\pi}\int_{-\infty}^\infty d\lambda\, L(\lambda)e^{-i\alpha\lambda}, \qquad L(\lambda) = \int_{-\infty}^\infty d\alpha\, \tilde{\Omega}(\alpha)e^{i\alpha\lambda}.$$

Substituting (42a) into (42b), we obtain the *uncoupled Marchenko equation*

$$\begin{aligned} &K_4(x,\alpha) + \tilde{\Omega}(\alpha-2x)^\dagger \\ &- \int_0^\infty d\beta\, K_4(x,\beta)\int_0^\infty d\gamma\, \tilde{\Omega}(\beta+\gamma-2x)\tilde{\Omega}(\alpha+\gamma-2x)^\dagger = 0_{n\times m}. \end{aligned} \tag{43}$$

The potential $q(x)$ is obtained from the solution of the Marchenko equation (43) as follows:

$$q(x) = -2K_4(x,0^+).$$

[2] In [4, 8] the notations $K_1 = B_{l1}$ and $K_2 = B_{l2}$ were used.
[3] In [4, 8] the notations $K_3 = B_{r1}$ and $K_4 = B_{r2}$ were used.

To prove the unique solvability of (40) we need the following two elementary results. A proof of Lemma 12 can be found in [11].

Lemma 11. *Let Ω belong to $L^1(\mathbb{R}^+)$. Then the operator K_Ω defined by*

$$(K_\Omega b)(\alpha) = \int_0^\infty d\beta\, \Omega(\alpha+\beta) b(\beta) \tag{44}$$

is bounded on $L^2(\mathbb{R}^+)$ and satisfies the norm estimate

$$\|K_\Omega\| \le \|\hat{\Omega}\|_\infty \le \|\Omega\|_1,$$

where $\hat{\Omega}$ denotes the Fourier transform of Ω.

Proof. The estimate $\|K_\Omega\| \le \|\Omega\|_1$ is immediate, while the estimate $\|K_\Omega\| \le \|\hat{\Omega}\|_\infty$ follows using the commutative diagram

$$\begin{array}{ccccc}
L^2(\mathbb{R}^+) & \xrightarrow[\text{imbedding plus inversion}]{} & L^2(\mathbb{R}) & \xrightarrow[\mathcal{F}]{} & L^2(\mathbb{R}) \\
{\scriptstyle K_\Omega}\downarrow & & & & \downarrow{\scriptstyle [\hat{\Omega}]} \\
L^2(\mathbb{R}^+) & \xleftarrow[\text{orthogonal projection}]{} & L^2(\mathbb{R}) & \xleftarrow[\mathcal{F}^{-1}]{} & L^2(\mathbb{R})
\end{array}$$

where the first step involves the imbedding-plus-inversion $f(\beta) \mapsto f(-\beta)$ and the third step multiplication by $\hat{\Omega}$. □

Lemma 12. *Let Ω belong to $L^1(\mathbb{R}^+)$. Then for $1 \le p < +\infty$ the operator K_Ω defined by* (44) *is compact on $L^p(\mathbb{R}^+)$ and has the norm estimate*

$$\|K_\Omega\| \le \|\Omega\|_1.$$

Using Lemma 11, Lemma 12, and Proposition 8, it is easy to prove the following

Theorem 13. *For each $x \in \mathbb{R}$ and $1 \le p < +\infty$ the Marchenko equations* (38) *are uniquely solvable for $K_1(x,\cdot)$ and $K_2(x,\cdot)$ with entries in $L^p(\mathbb{R}^+)$.*

Proof. In the defocusing case $\hat{\Omega}(\lambda) = R(\lambda)$ and (36b) is true. Then the integral operators appearing in (38) have an operator norm bounded above by $\|R\|_\infty$, which is strictly less than 1. The unique solvability of (38) now follows also in the other L^p-spaces as a result of the compactness of the integral operators involved. Indeed, putting $T_\Omega = I - K_\Omega$ and $X = L^2(\mathbb{R}^+) \cap L^p(\mathbb{R}^+)$ endowed with the sum of the L^2 and L^p norms, we first prove the compactness of K_Ω on X. Next, since T_Ω is Fredholm of index zero on $L^p(\mathbb{R}^+)$ and X and invertible on $L^2(\Omega)$ and X is continuously and densely imbedded in both $L^2(\Omega)$ and $L^p(\Omega)$, the invertibility of T_Ω is also true on X and $L^p(\mathbb{R}^+)$. □

4.3. Characterization problem

The characterization problem can be described as follows:

> Give necessary and sufficient conditions for an $n \times m$ matrix function $R(\lambda)$ to be the right reflection coefficient of a defocusing matrix Zakharov-Shabat system (1) whose potential $q(x)$ has its entries in $L^1(\mathbb{R})$.

In this subsection we shall solve this characterization problem. A similar characterization problem can be formulated and solved in terms of the left reflection coefficient $L(\lambda)$. In fact, the solutions of the direct and inverse scattering problems for (1) provide a 1,1-corresponding between potentials $q(x)$ with entries in $L^1(\mathbb{R})$ and a suitable class of $n \times m$ matrix functions $R(\lambda)$ or $m \times n$ matrix functions $L(\lambda)$ on the line, as depicted in the following diagram:

$$\boxed{q(x) \text{ with entries in } L^1(\mathbb{R})} \quad \underset{\text{inverse scattering problem}}{\overset{\text{direct scattering problem}}{\rightleftarrows}} \quad \boxed{R(\lambda) \text{ or } L(\lambda)}$$

The solution of the characterization problem for the Schrödinger equation on the line is well-known [16, 18]. As far as we know, no solution of the characterization problem for the matrix Zakharov-Shabat system has been published. In the defocusing case on the half-line Melik-Adamjan [17] has given a complete characterization of the Jost solution as scattering data to retrieve an L^1-potential.

Theorem 14. *In the defocusing case it is possible to determine a unique potential $q(x)$ with entries in $L^1(\mathbb{R})$ from the right reflection coefficient $R(\lambda)$ if and only if the following conditions are satisfied:*

1) $\sup_{\lambda\in\mathbb{R}} \|R(\lambda)\| < 1$, *and*
2) *the $n \times m$ matrix function $\Omega(\alpha)$ given by* (39) *has its entries in $L^1(\mathbb{R})$.*

A similar characterization result holds in the case of a left reflection coefficient $L(\lambda)$ as scattering data.

Proof. It is well-known [4, 8] that a defocusing matrix Zakharov-Shabat system (1) with L^1-potential has a right reflection coefficient $R(\lambda)$ satisfying conditions 1)-2) of Theorem 14.

To prove the converse, we assume to have an $n \times m$ matrix function $R(\lambda)$ with properties 1)-2). We then prove uniquely solvable the Marchenko equations (38) for $K_1(x,\cdot)$ and $K_2(x,\cdot)$ with entries in $L^1(\mathbb{R}^+)$. We then consider the integral operator $\mathcal{K}_\Omega^{(x)}$ as a function of $x \in \mathbb{R}$. Then for $b(\alpha)$ with entries in $L^p(\mathbb{R}^+)$ we have

$$\|[\mathcal{K}_\Omega^{(x_1)} - \mathcal{K}_\Omega^{(x_2)}]b\|_p \leq \|b\|_p \int_0^\infty d\beta \, \|\Omega(\beta + 2x_1) - \Omega(\beta + 2x_2)\|,$$

so that $K_\Omega^{(x)}$ depends continuously on $x \in \mathbb{R}$ in the operator norm. Moreover, its operator norm

$$\|K_\Omega^{(x)}\| \leq \int_0^\infty d\beta \, \|\Omega(\beta + 2x)\| = \int_{2x}^\infty d\gamma \, \|\Omega(\gamma)\| \to 0, \qquad x \to +\infty.$$

Since (38) are uniquely solvable for $K_1(x, \cdot)$ and $K_2(x, \cdot)$ with entries in $L^1(\mathbb{R}^+)$, we see that for any $x_0 \in \mathbb{R}$ we have

$$C(x_0) \stackrel{\text{def}}{=} \sup_{x \geq x_0} \int_0^\infty d\beta \, \|K_1(x, \beta)\| < +\infty.$$

Defining $q(x)$ by (41), i.e., by

$$q(x) = -2\Omega(2x) - 2 \int_0^\infty d\beta \, K_1(x, \beta)\Omega(\beta + 2x), \tag{45}$$

by integrating (45) we get for each $x_0 \in \mathbb{R}$

$$\int_{x_0}^\infty dx \, \|q(x)\| \leq (1 + C(x_0))\|\Omega\|_1 < +\infty,$$

which proves that any right tail of the potential obtained has L^1 entries.

Next, we apply the same argument to the identity

$$q(x) = 2\tilde{\Omega}(-2x)^\dagger + 2 \int_0^\infty d\beta \, K_1(x, \beta)\tilde{\Omega}(\beta - 2x)^\dagger, \tag{46}$$

which follows directly from (42). Arguing that

$$D(x_0) \stackrel{\text{def}}{=} \sup_{x \leq x_0} \int_0^\infty d\beta \, \|K_3(x, \beta)\| < +\infty,$$

we now get

$$\int_{-\infty}^{x_0} dx \, \|q(x)\| \leq (1 + D(x_0))\|\Omega\|_1 < +\infty,$$

which proves that any left tail of the potential obtained has L^1 entries.

As a result, the potential as a whole has L^1 entries. □

Appendix A. Definition of the Full Hamiltonian

Write $V = V_1 V_2$ as the product of the two matrix functions V_1 and V_2 with L^2 entries. Then for $\lambda \in \mathbb{C} \setminus \mathbb{R}$

$$W(\lambda) = I + V_2(\lambda - H_0)^{-1}V_1 \tag{47}$$

is a Hilbert-Schmidt perturbation of the identity, so that $W(\lambda)$ is invertible except possibly on a discrete subset of $\mathbb{C} \setminus \mathbb{R}$. Now put

$$R(\lambda) \stackrel{\text{def}}{=} (\lambda - H_0)^{-1} - (\lambda - H_0)^{-1}V_1 W(\lambda)^{-1} V_2 (\lambda - H_0)^{-1}, \tag{48}$$

which implies

$$R(\overline{\lambda})^\dagger = (\lambda - H_0)^{-1} - (\lambda - H_0)^{-1} V_2^\dagger [W(\overline{\lambda})^\dagger]^{-1} V_1^\dagger (\lambda - H_0)^{-1}. \tag{49}$$

Then for nonreal λ the operators $R(\lambda)$ and $R(\overline{\lambda})^\dagger$ both have a zero kernel, because $(\lambda - H_0)^{-1}$ does. Hence, $R(\lambda)$ has a zero kernel and a dense range.

Next, we show that $R(\lambda)$ satisfies the resolvent identity. Indeed,

$$\begin{aligned}
&R(\lambda)R(\zeta) = (\lambda - H_0)^{-1}(\zeta - H_0)^{-1}\\
&- (\lambda - H_0)^{-1}(\zeta - H_0)^{-1}V_1W(\zeta)^{-1}V_2(\zeta - H_0)^{-1}\\
&- (\lambda - H_0)^{-1}V_1W(\lambda)^{-1}V_2(\lambda - H_0)^{-1}(\zeta - H_0)^{-1}\\
&+ (\lambda - H_0)^{-1}V_1W(\lambda)^{-1}V_2(\lambda - H_0)^{-1}(\zeta - H_0)^{-1}V_1W(\zeta)^{-1}V_2(\zeta - H_0)^{-1}.
\end{aligned}$$

Multiplying each term (called I, II, III, and IV) by $(\zeta - \lambda)$ and using theresolvent identity

$$(\zeta - \lambda)(\lambda - H_0)^{-1}(\zeta - H_0)^{-1} = (\lambda - H_0)^{-1} - (\zeta - H_0)^{-1},$$

we obtain

$$\begin{aligned}
(\zeta - \lambda)R(\lambda)R(\zeta) = {} & (\lambda - H_0)^{-1} - (\zeta - H_0)^{-1}\\
& - \{(\lambda - H_0)^{-1} - (\zeta - H_0)^{-1}\}V_1W(\zeta)^{-1}V_2(\zeta - H_0)^{-1}\\
& - (\lambda - H_0)^{-1}V_1W(\lambda)^{-1}V_2\{(\lambda - H_0)^{-1} - (\zeta - H_0)^{-1}\}\\
& - (\lambda - H_0)^{-1}V_1W(\lambda)^{-1}V_2(\zeta - H_0)^{-1}\\
& + (\lambda - H_0)^{-1}V_1W(\zeta)^{-1}V_2(\zeta - H_0)^{-1},
\end{aligned}$$

where the last two terms occur by writing $W(\lambda)^{-1}\{V_2(\lambda - H_0)^{-1}V_1 - V_2(\zeta - H_0)^{-1}V_1\}W(\zeta)^{-1}$ as the difference of two terms. Letting the terms in the right-hand side be called Ia, Ib, IIa, IIb, IIIa, IIIb, IVa, and IVb, we see that IVaand IIIb cancel out and IVb and IIa cancel out. At the end, we get

$$(\zeta - \lambda)R(\lambda)R(\zeta) = R(\lambda) - R(\zeta),$$

which is the resolvent identity. Thus there exists a closed and densely defined linear operator $\tilde{H}$ such that

$$R(\lambda) = (\lambda - \tilde{H})^{-1}, \qquad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

Clearly, $(\lambda - \tilde{H})^{-1} - (\lambda - H_0)^{-1}$ is a trace class operator.

To derive "mixed" resolvent identities, we first derive from (48) and (49) with the help of (47)

$$\begin{aligned}
V_2R(\lambda) &= V_2(\lambda - H_0)^{-1} - [W(\lambda) - I]W(\lambda)^{-1}V_2(\lambda - H_0)^{-1}\\
&= W(\lambda)^{-1}V_2(\lambda - H_0)^{-1},\\
V_1^\dagger R(\overline{\lambda})^\dagger &= V_1^\dagger(\lambda - H_0)^{-1} - [W(\overline{\lambda})^\dagger - I][W(\overline{\lambda})^\dagger]^{-1}V_1^\dagger(\lambda - H_0)^{-1}\\
&= [W(\overline{\lambda})^\dagger]^{-1}V_1^\dagger(\lambda - H_0)^{-1},
\end{aligned}$$

which are Hilbert-Schmidt operators [cf. Proposition 2] except possibly on a discrete subset of $\mathbb{C} \setminus \mathbb{R}$. These identities in turn imply

$$(\lambda - H_0)^{-1} - (\lambda - H_0)^{-1}V_1V_2R(\lambda) = (\lambda - H_0)^{-1} \\ - (\lambda - H_0)^{-1}V_1W(\lambda)^{-1}V_2(\lambda - H_0)^{-1} = R(\lambda),$$

$$(\lambda - H_0)^{-1} - (\lambda - H_0)^{-1}V_2^\dagger V_1^\dagger R(\overline{\lambda})^\dagger = (\lambda - H_0)^{-1} \\ - (\lambda - H_0)^{-1}V_2^\dagger[W(\overline{\lambda})^\dagger]^{-1}V_1^\dagger(\lambda - H_0)^{-1} = R(\overline{\lambda})^\dagger,$$

which again imply

$$(\lambda - H_0)^{-1} - [(\lambda - H_0)^{-1}V_1][V_2R(\lambda)] = R(\lambda), \tag{50a}$$

$$(\lambda - H_0)^{-1} - [R(\lambda)V_1][V_2(\lambda - H_0)^{-1}] = R(\lambda). \tag{50b}$$

Equations (50) imply that $R(\overline{\lambda})^\dagger = R(\lambda)$ and hence that $\tilde{H}$ is a selfadjoint operator on $\mathcal{H}_{n+m}$. Moreover, $\tilde{H}$ is an extension of $H_0 - V$ (which can in principle be defined on the dense domain $\mathcal{H}^2_{n+m}$, but not as a closed operator). We therefore define the full Hamiltonian H as follows:

$$H = \tilde{H}.$$

If the entries of V belong to $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then $H = \tilde{H}$ has the same domain as H_0, namely $\mathcal{H}^1_{n+m}$.

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