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Explicit solutions of the cubic matrix nonlinear Schrödinger equation

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Abstract
In this paper, we derive a class of explicit solutions, global in $(x, t) \in \mathbb{R}^2$, of the focusing matrix nonlinear Schrödinger equation using straightforward linear algebra. We obtain both the usual and multiple pole multisoliton solutions as well as a new class of solutions exponentially decaying as $x \to \pm\infty$.

1. Introduction

In the focusing case, the cubic matrix nonlinear Schrödinger (mNLS) equation

$$\mathrm{i}u_t + u_{xx} + 2uu^\dagger u = 0, \tag{1.1}$$

where $u(x, t)$ is an $n \times m$ matrix and the dagger denotes the matrix conjugate transpose, has been studied extensively [1–3] as the natural generalization of the nonlinear Schrödinger (NLS) equation. For $n = m = 1$ it describes wave propagation in nonlinear media [33], evolution of surface waves on sufficiently deep waters [32] and signal propagation in optical fibres with anomalous dispersion [16, 17]. For $n = 1$ and $m = 2$ it is known as the Manakov system and arises in a natural way in paraxial light propagation in diffractive media [22], signal transmission in optical fibres with randomly varying birefringence [4, 16, 31] and wave transmission in the half-band-gaps of AlGaAs semiconductors [20]. In this paper, we present a systematic method to derive exact solutions of (1.1) which are global in $(x, t) \in \mathbb{R}^2$ and decay exponentially as $x \to \pm\infty$ for fixed $t \in \mathbb{R}$, i.e., we assume the background field to be zero. In this way we generalize the theory expounded in [6] for obtaining exact NLS solutions and, to a lesser extent, that in [7] to solve the KdV equation on the half-line.

In [6], starting from a $p \times p$ matrix A, a column vector B of length p and a row vector C of length p we derived the exact NLS solutions

$$u(x, t) = -2B^\dagger \mathrm{e}^{-xA^\dagger}[I_p + \boldsymbol{Q}(x; t)\boldsymbol{N}(x)]^{-1} \mathrm{e}^{-xA^\dagger+4\mathrm{i}t(A^\dagger)^2} C^\dagger, \tag{1.2}$$

where

$$\boldsymbol{Q}(x; t) = \mathrm{e}^{-xA^\dagger+4\mathrm{i}t(A^\dagger)^2} Q\, \mathrm{e}^{-xA-4\mathrm{i}tA^2}, \qquad \boldsymbol{N}(x) = \mathrm{e}^{-xA} N \mathrm{e}^{-xA^\dagger},$$

and Q and N are the unique solutions of the Lyapunov equations

$$A^{\dagger}Q + QA = C^{\dagger}C, \qquad AN + NA^{\dagger} = BB^{\dagger}.$$

By choosing a complex $p \times p$ matrix A such that all of its eigenvalues have a positive real part, multisoliton solitons were generated, not only those appearing in [33] but also multiple pole solutions of soliton type. These multiple pole solutions appear when choosing a nondiagonalizable matrix A while satisfying the minimality conditions (2.26), as illustrated by examples 7.3 and 7.4 of [6]. On the other hand, it was suggested in [6] on the basis of the few well-chosen example 7.2 and the corresponding figure 1 that a new class of NLS solutions is generated by choosing the matrix A in such a way that A has eigenvalues in the left and right half-plane and does not have eigenvalues in common with $-A^{\dagger}$.

In the present paper, we generalize the results of [6] to the focusing mNLS equation (1.1) by choosing A, B and C to be $p \times p$, $p \times m$ and $n \times p$ matrices, respectively. We derive the exact solution (1.2) and prove that the trace

$$\mathrm{Tr}[u(x,t)u(x,t)^{\dagger}] = \mathrm{Tr}[u(x,t)^{\dagger}u(x,t)] = \frac{\partial^2}{\partial x^2}\log\det(I_p + \boldsymbol{Q}(x;t)\boldsymbol{N}(x)), \tag{1.3}$$

which generalizes a well-known expression for the squared modulus of the NLS multisoliton solution [6, 33]. Since we do not restrict ourselves to matrices A having all of their eigenvalues in the open right half-plane, we now present a generalization of [6] that is going well beyond a mere matrification of existing equations. One of the major results is that the newly found mNLS solutions are global in $(x,t) \in \mathbb{R}^2$ and decay exponentially as $x \to \pm\infty$. Again multiple pole solutions arise if A is nondiagonalizable and the minimality condition (2.26) is satisfied.

In the literature, there are many different methods to find exact solutions of the NLS equation, such as the Hirota method and other parameter adjustment techniques [5, 9, 15, 18, 24, 25, 28]. In comparison, exact mNLS solutions are far less prevalent in the literature. Though dark soliton solutions of the Manakov system were obtained by Hirota's method [27, 29], it turned out to be difficult to generalize parameter adjustment methods to (1.1) of arbitrary matrix order. We refer to the review paper [21] for further information. In recent years, Park and Shin [23] have applied reduction to a vector eigenvalue problem to systematically obtain dark, bright and 'hybrid' soliton solutions of (1.1). In [26], dark–dark and dark–bright soliton solutions for the defocusing Manakov system were derived using IST techniques, while in [19] the IST method was developed for the square mNLS equation. In [14], matrix realizations were applied to derive exact solutions of (1.1) of arbitrary order, but it was not made clear if these solutions are global in $(x,t) \in \mathbb{R}^2$. Periodic and almost periodic solutions were derived in [13, 30].

In [6], the Marchenko method was used as a tool to derive (1.2), but (1.2) was also shown to satisfy the NLS equation without relying on any Marchenko theory whatsoever. In other words, the Marchenko method was merely used to 'suggest' a concise form of the NLS solution, which could just as well have been derived without it. Moreover, in [6] the power of the method was shown by using Mathematica to produce exact solutions in terms of elementary scalar functions. In this paper, we shall not draw on any Marchenko theory or IST methods to arrive at (1.2) and (1.3), although we could have done so by repeating the arguments of [6, section 4]. Still our method has the advantage of treating the NLS and the mNLS equations in exactly the same way and, most of all, of producing concisely written exact solutions that can be expressed in terms of elementary functions of x and t with the help of Mathematica, REDUCE, Maple, the Symbolic Toolbox of MatLab or other symbolic calculus.

In section 2 we derive the exact solutions (1.2) and (1.3). To present the results more clearly, we have relegated the discussion of Lyapunov equations and the positivity proof of the determinant appearing in (1.3) to subsections 2.1 and 2.2, whereas subsection 2.3 contains the derivation of (1.3) itself. In subsection 2.4, we apply some transformations on the matrices A, B and C to produce mNLS solutions starting from a given mNLS solution. As a result, we manage to derive (1.2) and (1.3) also for matrix triples (A, B, C) of nonminimal matrix size. A few illustrative examples are given in section 3.

2. Deriving explicit mNLS solutions

Let A_1 and A_2 be two square matrices of respective orders p_1 and p_2 and eigenvalues in the open right half-plane such that A_1 and $A_2^\dagger$ (and hence also $A_1^\dagger$ and A_2) do not have eigenvalues in common, and let B_1, B_2, C_1 and C_2 be $p_1 \times n$, $p_2 \times n$, $m \times p_1$ and $m \times p_2$ matrices, respectively. Put

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & -A_2 \end{bmatrix}, \qquad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \qquad C = [C_1 \quad C_2]. \tag{2.1}$$

We assume the (nonessential) minimality conditions

$$\bigcap_{j=0}^{\infty} \mathrm{Ker}(C_s (A_s)^j) = \{0\}, \tag{2.2a}$$

$$\bigcap_{j=0}^{\infty} \mathrm{Ker}\big(B_s^\dagger (A_s^\dagger)^j\big) = \{0\}, \tag{2.2b}$$

where $s = 1, 2$.

2.1. Lyapunov equations and determinant relations

Let

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}, \qquad N = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} \tag{2.3}$$

be the unique solutions of the Lyapunov equations

$$A^\dagger Q + QA = C^\dagger C, \tag{2.4a}$$

$$AN + NA^\dagger = BB^\dagger. \tag{2.4b}$$

Here Q_{11} and N_{11} are $p_1 \times p_1$ matrices, Q_{12} and N_{12} are $p_1 \times p_2$ matrices, Q_{21} and N_{21} are $p_2 \times p_1$ matrices and Q_{22} and N_{22} are $p_2 \times p_2$ matrices. Furthermore, the uniqueness follows from the fact that A and $-A^\dagger$ do not have eigenvalues in common [12, theorem 18.5]. Then the Lyapunov equations

$$A_1^\dagger Q_{11} + Q_{11} A_1 = C_1^\dagger C_1, \qquad A_1 N_{11} + N_{11} A_1^\dagger = B_1 B_1^\dagger,$$
$$A_1^\dagger Q_{12} - Q_{12} A_2 = C_1^\dagger C_2, \qquad A_1 N_{12} - N_{12} A_2^\dagger = B_1 B_2^\dagger$$

and the Sylvester equations

$$A_2^\dagger Q_{21} - Q_{21} A_1 = -C_2^\dagger C_1, \qquad A_2 Q_{21} - Q_{21} A_1^\dagger = -B_2 B_1^\dagger,$$
$$A_2^\dagger Q_{22} + Q_{22} A_2 = -C_2^\dagger C_2, \qquad A_2 N_{22} + N_{22} A_2^\dagger = -B_2 B_2^\dagger$$

are uniquely solvable. Moreover, as is easily verified by taking the adjoints of these matrix equations, their unique solvability implies that $Q_1 \stackrel{\text{def}}{=} Q_{11}, N_1 \stackrel{\text{def}}{=} N_{11}, Q_2 \stackrel{\text{def}}{=} -Q_{22}$ and

$N_2 \stackrel{\text{def}}{=} -N_{22}$ are nonnegative selfadjoint, while $Q_0 \stackrel{\text{def}}{=} Q_{12} = [Q_{21}]^\dagger$ and $N_0 \stackrel{\text{def}}{=} N_{12} = [N_{21}]^\dagger$. In fact [12, exercise 18.7],

$$Q_1 = \int_0^\infty \mathrm{d}z\, \mathrm{e}^{-zA_1^\dagger} C_1^\dagger C_1\, \mathrm{e}^{-zA_1}, \qquad Q_2 = \int_0^\infty \mathrm{d}z\, \mathrm{e}^{-zA_2^\dagger} C_2^\dagger C_2\, \mathrm{e}^{-zA_2},$$
$$N_1 = \int_0^\infty \mathrm{d}z\, \mathrm{e}^{-zA_1} B_1 B_1^\dagger\, \mathrm{e}^{-zA_1^\dagger}, \qquad N_2 = \int_0^\infty \mathrm{d}z\, \mathrm{e}^{-zA_2} B_2 B_2^\dagger\, \mathrm{e}^{-zA_2^\dagger}.$$

Moreover, if γ_1 is a simple positively oriented Jordan contour containing the eigenvalues of $A_1^\dagger$ in its interior region and those of A_2 in its exterior region and γ_2 is a simple positively oriented Jordan contour containing the eigenvalues of A_1 in its interior region and those of $A_2^\dagger$ in its exterior region, we have the contour integral representations [12, exercise 18.8]

$$Q_0 = \frac{1}{2\pi \mathrm{i}} \int_{\gamma_1} \mathrm{d}\lambda \big(\lambda - A_1^\dagger\big)^{-1} C_1^\dagger C_2 (\lambda - A_2)^{-1},$$
$$N_0 = \frac{1}{2\pi \mathrm{i}} \int_{\gamma_2} \mathrm{d}\lambda (\lambda - A_1)^{-1} B_1 B_2^\dagger \big(\lambda - A_2^\dagger\big)^{-1}.$$

As in the proof of [6, theorems 4.2 and 4.4], we easily see that Q_1, Q_2, N_1 and N_2 are (nonsingular) positive selfadjoint matrices if the minimality conditions (2.2) are true. In fact, the identities

$$\langle Q_s x, x\rangle = \int_0^\infty \mathrm{d}z \|C_s\, \mathrm{e}^{-zA_s} x\|^2, \qquad \langle N_s x, x\rangle = \int_0^\infty \mathrm{d}z \big\| B_s^\dagger\, \mathrm{e}^{-zA_s^\dagger} x \big\|^2$$

imply that, for $s = 1, 2$, the positive selfadjointness of Q_s is equivalent to the minimality condition (2.2*a*) and that of N_s to the minimality condition (2.2*b*).

To prove that $\det(I_p + Q(x;t)N(x)) > 0$ for any choice of the triple of matrices (A, B, C) as in (2.1), we begin with some basic linear algebra. The rather elementary proofs of lemmas 2.1 and 2.2 can both be based on the use of Schur complements [12, equations (1.11) and (1.12)]. We give the first proof, since we shall use two equations appearing in it later on.

Lemma 2.1. *Let T_1 and T_2 be two square matrices and let*

$$T = \begin{bmatrix} T_1 & T_0 \\ -T_0^\dagger & T_2 \end{bmatrix}.$$

Then

$$\det T = \begin{cases} (\det T_1)\big(\det T_2^\#\big) & \text{if} \quad \det T_1 > 0, \\ \big(\det T_1^\#\big)(\det T_2) & \text{if} \quad \det T_2 > 0, \end{cases} \tag{2.5}$$

where

$$T_1^\# = T_1 + T_0 (T_2)^{-1} T_0^\dagger, \qquad T_2^\# = T_2 + T_0^\dagger (T_1)^{-1} T_0.$$

In particular, if T_1 and T_2 are positive selfadjoint, then so are $T_1^\#$ and $T_2^\#$, while $\det T > 0$.

Proof. The lemma is immediate from the identities

$$T = \begin{bmatrix} I & 0 \\ -T_0^\dagger T_1^{-1} & I \end{bmatrix} \begin{bmatrix} T_1 & 0 \\ 0 & T_2^\# \end{bmatrix} \begin{bmatrix} I & T_1^{-1} T_0 \\ 0 & I \end{bmatrix}, \tag{2.6a}$$

$$T = \begin{bmatrix} I & T_0 T_2^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} T_1^\# & 0 \\ 0 & T_2 \end{bmatrix} \begin{bmatrix} I & 0 \\ -T_2^{-1} T_0^\dagger & I \end{bmatrix}, \tag{2.6b}$$

because the lateral factors on the right-hand sides of (2.6) have unit determinant. Furthermore, $T_1^\#$ and $T_2^\#$ are positive selfadjoint whenever T_1 and T_2 are, in which case $\det T > 0$. □

Lemma 2.2. *Let T be an $r \times s$ matrix and S an $s \times r$ matrix. Then*

$$\det(I_r + TS) = \det(I_s + ST).$$

We now apply lemmas 2.1 and 2.2 to prove the following:

Theorem 2.3. *Suppose the minimality requirements (2.2) are fulfilled. Then*

$$\det(I_p + QN) > 0,$$

where $p = p_1 + p_2$.

Proof. We first use the positive selfadjointness of Q_1 to write

$$\begin{aligned} I_p + QN &= I_p + \begin{bmatrix} Q_1 & -Q_0 \\ Q_0^\dagger & Q_2 \end{bmatrix} \begin{bmatrix} N_1 & N_0 \\ -N_0^\dagger & N_2 \end{bmatrix} \\ &= I_p + \begin{bmatrix} I_{p_1} & 0 \\ Q_0^\dagger Q_1^{-1} & I_{p_2} \end{bmatrix} \begin{bmatrix} Q_1 & -Q_0 \\ 0 & Q_2^\# \end{bmatrix} \begin{bmatrix} N_1 & N_0 \\ -N_0^\dagger & N_2 \end{bmatrix}, \end{aligned}$$

where $Q_2^\# = Q_2 + Q_0^\dagger Q_1^{-1} Q_0$. Using lemma 2.2 twice we get

$$\begin{aligned} \det(I_p + QN) &= \det\left(I_p + \begin{bmatrix} Q_1 & -Q_0 \\ 0 & Q_2^\# \end{bmatrix} \begin{bmatrix} N_1 & N_0 \\ -N_0^\dagger & N_2 \end{bmatrix} \begin{bmatrix} I_{p_1} & 0 \\ Q_0^\dagger Q_1^{-1} & I_{p_2} \end{bmatrix} \right) \\ &= \det\left(I_p + \begin{bmatrix} Q_1^{1/2} & -Q_1^{-1/2} Q_0 \\ 0 & \left(Q_2^\#\right)^{1/2} \end{bmatrix} \begin{bmatrix} N_1 & N_0 \\ -N_0^\dagger & N_2 \end{bmatrix} \begin{bmatrix} Q_1^{1/2} & 0 \\ Q_0^\dagger Q_1^{-1/2} & \left(Q_2^\#\right)^{1/2} \end{bmatrix} \right) \\ &= \det\left(\begin{bmatrix} Q_1^{1/2} & -Q_1^{-1/2} Q_0 \\ 0 & \left(Q_2^\#\right)^{1/2} \end{bmatrix} \begin{bmatrix} T_1 & T_0 \\ -T_0^\dagger & T_2 \end{bmatrix} \begin{bmatrix} Q_1^{1/2} & 0 \\ Q_0^\dagger Q_1^{-1/2} & \left(Q_2^\#\right)^{1/2} \end{bmatrix} \right) \\ &= \det(Q_1) \det\left(Q_2^\#\right) \det \begin{bmatrix} T_1 & T_0 \\ -T_0^\dagger & T_2 \end{bmatrix}, \end{aligned}$$

where the positive selfadjointness of Q_2 implies that of $Q_2^\#$ and

$$\begin{aligned} T_1 &= N_1 + Q_1^{-1} - Q_1^{-1} Q_0 \left(Q_2^\#\right)^{-1} Q_0^\dagger Q_1^{-1}, \\ T_2 &= N_2 + \left(Q_2^\#\right)^{-1}, \\ T_0 &= N_0 + Q_1^{-1} Q_0 \left(Q_2^\#\right)^{-1}. \end{aligned}$$

Clearly, T_2 is the positive selfadjoint.

To prove the positive selfadjointness of T_1, it suffices to prove the positive selfadjointness of

$$Z \stackrel{\text{def}}{=} Q_1^{-1} - Q_1^{-1} Q_0 \left(Q_2^\#\right)^{-1} Q_0^\dagger Q_1^{-1}.$$

Indeed, we first apply (2.6*b*) and then (2.6*a*) to derive the following three matrix identities:

$$\begin{aligned} \begin{bmatrix} Q_1^{-1} & Q_1^{-1} Q_0 \\ Q_0^\dagger Q_1^{-1} & Q_2^\# \end{bmatrix} &= \begin{bmatrix} I_{p_1} & Q_1^{-1} Q_0 \left(Q_2^\#\right)^{-1} \\ 0 & I_{p_2} \end{bmatrix} \begin{bmatrix} Z & 0 \\ 0 & I_{p_2} \end{bmatrix} \begin{bmatrix} I_{p_1} & 0 \\ \left(Q_2^\#\right)^{-1} Q_0^\dagger Q_1^{-1} & I_{p_2} \end{bmatrix} \\ &= \begin{bmatrix} Q_1^{-1/2} & 0 \\ 0 & I_{p_2} \end{bmatrix} \begin{bmatrix} I_{p_1} & Q_1^{-1/2} Q_0 \\ Q_0^\dagger Q_1^{-1/2} & Q_2^\# \end{bmatrix} \begin{bmatrix} Q_1^{-1/2} & 0 \\ 0 & I_{p_2} \end{bmatrix}, \end{aligned}$$

$$\begin{bmatrix} I_{p_1} & Q_1^{-1/2} Q_0 \\ Q_0^\dagger Q_1^{-1/2} & Q_2^\# \end{bmatrix} = \begin{bmatrix} I_{p_1} & 0 \\ Q_0^\dagger Q_1^{-1/2} & I_{p_2} \end{bmatrix} \begin{bmatrix} I_{p_1} & 0 \\ 0 & Q_2 \end{bmatrix} \begin{bmatrix} I_{p_1} & Q_1^{-1/2} Q_0 \\ 0 & I_{p_2} \end{bmatrix}.$$

Therefore, Z is positive selfadjoint. Furthermore, we easily get the determinant relation

$$\det Z = \frac{\det Q_2}{\det Q_1} > 0.$$

Consequently, $\det Z > 0$, as claimed. □

2.2. *Explicit mNLS solutions*

Given the matrices A, B and C as in (2.1) satisfying the minimality conditions (2.2), we first evaluate the unique solutions Q and N of the Lyapunov equations (2.4). For $(x,t) \in \mathbb{R}^2$ we then define

$$\boldsymbol{Q}(x;t) = \mathrm{e}^{-xA^\dagger + 4\mathrm{i}t(A^\dagger)^2} Q\, \mathrm{e}^{-xA-4\mathrm{i}tA^2}, \tag{2.7a}$$

$$\boldsymbol{N}(x) = \mathrm{e}^{-xA} N\, \mathrm{e}^{-xA^\dagger}. \tag{2.7b}$$

Then the matrices $\boldsymbol{Q}(x;t)$ and $\boldsymbol{N}(x)$ satisfy the Lyapunov equations

$$A^\dagger \boldsymbol{Q}(x;t) + \boldsymbol{Q}(x;t)A = \mathrm{e}^{-xA^\dagger+4\mathrm{i}t(A^\dagger)^2} C^\dagger C\, \mathrm{e}^{-xA-4\mathrm{i}tA^2}, \tag{2.8a}$$

$$A\boldsymbol{N}(x) + \boldsymbol{N}(x)A^\dagger = \mathrm{e}^{-xA} BB^\dagger\, \mathrm{e}^{-xA^\dagger}. \tag{2.8b}$$

Moreover, $\boldsymbol{Q}(x;t)$ and $\boldsymbol{N}(x)$ allow the following partitioning as in (2.3):

$$\boldsymbol{Q}(x;t) = \begin{bmatrix} \boldsymbol{Q}_1(x;t) & \boldsymbol{Q}_0(x;t) \\ \boldsymbol{Q}_0(x;t)^\dagger & -\boldsymbol{Q}_2(x;t) \end{bmatrix}$$

$$= \begin{bmatrix} \mathrm{e}^{-xA_1^\dagger+4\mathrm{i}t(A_1^\dagger)^2} Q_1\, \mathrm{e}^{-xA_1-4\mathrm{i}t(A_1)^2} & \mathrm{e}^{-xA_1^\dagger+4\mathrm{i}t(A_1^\dagger)^2} Q_0\, \mathrm{e}^{xA_2-4\mathrm{i}t(A_2)^2} \\ \mathrm{e}^{xA_2^\dagger+4\mathrm{i}t(A_2^\dagger)^2} Q_0^\dagger\, \mathrm{e}^{-xA_1-4\mathrm{i}t(A_1)^2} & -\mathrm{e}^{xA_2^\dagger+4\mathrm{i}t(A_2^\dagger)^2} Q_2\, \mathrm{e}^{xA_2-4\mathrm{i}t(A_2)^2} \end{bmatrix},$$

$$\boldsymbol{N}(x) = \begin{bmatrix} \boldsymbol{N}_1(x) & \boldsymbol{N}_0(x) \\ \boldsymbol{N}_0(x)^\dagger & -\boldsymbol{N}_2(x) \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{-xA_1} N_1\, \mathrm{e}^{-xA_1^\dagger} & \mathrm{e}^{-xA_1} N_0\, \mathrm{e}^{xA_2^\dagger} \\ \mathrm{e}^{xA_2} N_0^\dagger\, \mathrm{e}^{-xA_1^\dagger} & -\mathrm{e}^{xA_2} N_2\, \mathrm{e}^{xA_2^\dagger} \end{bmatrix}.$$

Replacing C_1 by $C_1\,\mathrm{e}^{-xA_1-4\mathrm{i}t(A_1)^2}$, C_2 by $C_2\,\mathrm{e}^{xA_2-4\mathrm{i}t(A_2)^2}$, B_1 by $\mathrm{e}^{-xA_1}B_1$ and $B_2 = \mathrm{e}^{xA_2}B_2$, we see that the minimality conditions (2.2) remain satisfied. Applying lemma 2.3 to these modified matrices $A \mapsto A$, $B \mapsto \mathrm{e}^{-xA}B$ and $C \mapsto C\,\mathrm{e}^{-xA-4\mathrm{i}tA^2}$, we obtain

Theorem 2.4. *Suppose the minimality requirements (2.2) are fulfilled. Then*

$$\det(I_p + \boldsymbol{Q}(x;t)\boldsymbol{N}(x)) > 0, \qquad (x,t) \in \mathbb{R}^2,$$

where $p = p_1 + p_2$.

We now arrive at the main result of this paper.

Theorem 2.5. *Suppose A, B and C are matrices as in (2.1) satisfying the minimality requirements (2.2) and such that A_1 and $A_2^\dagger$ do not have eigenvalues in common but have all of their eigenvalues in the open right half-plane. Then*

$$u(x,t) = -2B^\dagger\, \mathrm{e}^{-xA^\dagger}[I_p + \boldsymbol{Q}(x;t)\boldsymbol{N}(x)]^{-1}\, \mathrm{e}^{-xA^\dagger+4\mathrm{i}t(A^\dagger)^2} C^\dagger \tag{2.9}$$

is a global in $(x,t) \in \mathbb{R}^2$ solution of the focusing mNLS equation (1.1).

Proof. Putting $\Gamma = I_p + \boldsymbol{Q}\boldsymbol{N}$ (where we have not written the (x,t)-dependence of Γ, $\boldsymbol{Q}$ and $\boldsymbol{N}$), we can mimick the proof given in [6]. Indeed, (2.7) implies that

$$\boldsymbol{Q}_x = -(A^\dagger\boldsymbol{Q} + \boldsymbol{Q}A), \qquad \boldsymbol{N}_x = -(A\boldsymbol{N} + \boldsymbol{N}A^\dagger), \qquad \boldsymbol{Q}_t = 4\mathrm{i}[(A^\dagger)^2\boldsymbol{Q} - \boldsymbol{Q}A^2]. \tag{2.10}$$

We now easily differentiate (2.9) to get

$$
\begin{aligned}
u_x &= 2B^\dagger \,\mathrm{e}^{-xA^\dagger}\Gamma^{-1}[\Gamma A^\dagger + \boldsymbol{Q}_x\boldsymbol{N} + \boldsymbol{Q}\boldsymbol{N}_x + A^\dagger\Gamma]\Gamma^{-1}\,\mathrm{e}^{-xA^\dagger}\,\mathrm{e}^{4\mathrm{i}t(A^\dagger)^2}C^\dagger \\
&= 4B^\dagger \,\mathrm{e}^{-xA^\dagger}\Gamma^{-1}[A^\dagger - \boldsymbol{Q}A\boldsymbol{N}]\Gamma^{-1}\,\mathrm{e}^{-xA^\dagger}\,\mathrm{e}^{4\mathrm{i}t(A^\dagger)^2}C^\dagger,
\end{aligned}
$$

as well as

$$
\begin{aligned}
iu_t &= 8B^\dagger \,\mathrm{e}^{-xA^\dagger}\Gamma^{-1}[-\{(A^\dagger)^2\boldsymbol{Q} - \boldsymbol{Q}A^2\}\boldsymbol{N}\Gamma^{-1} + (A^\dagger)^2]\mathrm{e}^{-xA^\dagger}\,\mathrm{e}^{4\mathrm{i}t(A^\dagger)^2}C^\dagger \\
&= 8B^\dagger \,\mathrm{e}^{-xA^\dagger}\Gamma^{-1}[(A^\dagger)^2 + \boldsymbol{Q}A^2\boldsymbol{N}]\Gamma^{-1}\,\mathrm{e}^{-xA^\dagger}\,\mathrm{e}^{4\mathrm{i}t(A^\dagger)^2}C^\dagger,
\end{aligned}
\tag{2.11a}
$$

$$
\begin{aligned}
u_{xx} &= 8B^\dagger \,\mathrm{e}^{-xA^\dagger}\Gamma^{-1}[(A^\dagger)^2 - 2\boldsymbol{Q}A\boldsymbol{N}\Gamma^{-1}\boldsymbol{Q}A\boldsymbol{N} + 2A^\dagger\Gamma^{-1}\boldsymbol{Q}A\boldsymbol{N} \\
&\quad - 2A^\dagger\Gamma^{-1}A^\dagger + 2\boldsymbol{Q}A\boldsymbol{N}\Gamma^{-1}A^\dagger + \boldsymbol{Q}A^2\boldsymbol{N}]\Gamma^{-1}\,\mathrm{e}^{-xA^\dagger}\,\mathrm{e}^{4\mathrm{i}t(A^\dagger)^2}C^\dagger \\
&= 8B^\dagger \,\mathrm{e}^{-A^\dagger x}\Gamma^{-1}[(A^\dagger)^2 + \boldsymbol{Q}A^2\boldsymbol{N}]\Gamma^{-1}\,\mathrm{e}^{-A^\dagger x + 4\mathrm{i}(A^\dagger)^2 t}C^\dagger \\
&\quad - 16B^\dagger \,\mathrm{e}^{-xA^\dagger}\Gamma^{-1}(A^\dagger - \boldsymbol{Q}A\boldsymbol{N})\Gamma^{-1}(A^\dagger - \boldsymbol{Q}A\boldsymbol{N})\Gamma^{-1}\,\mathrm{e}^{-xA^\dagger}\,\mathrm{e}^{4\mathrm{i}t(A^\dagger)^2}C^\dagger,
\end{aligned}
\tag{2.11b}
$$

$$
\begin{aligned}
2uu^\dagger u &= -16B^\dagger \,\mathrm{e}^{-xA^\dagger}\Gamma^{-1}[(A^\dagger\boldsymbol{Q} + \boldsymbol{Q}A)\hat{\Gamma}^{-1}(A\boldsymbol{N} + \boldsymbol{N}A^\dagger)]\Gamma^{-1}\,\mathrm{e}^{-xA^\dagger}\,\mathrm{e}^{4\mathrm{i}t(A^\dagger)^2}C^\dagger \\
&= -16B^\dagger \,\mathrm{e}^{-xA^\dagger}\Gamma^{-1}[(A^\dagger)^2 + \boldsymbol{Q}A^2\boldsymbol{N} \\
&\quad - (A^\dagger - \boldsymbol{Q}A\boldsymbol{N})\Gamma^{-1}(A^\dagger - \boldsymbol{Q}A\boldsymbol{N})]\Gamma^{-1}\,\mathrm{e}^{-xA^\dagger}\,\mathrm{e}^{4\mathrm{i}t(A^\dagger)^2}C^\dagger,
\end{aligned}
\tag{2.11c}
$$

where $\hat{\Gamma} = I + \boldsymbol{N}\boldsymbol{Q} = \Gamma^\dagger$. Adding (2.11*a*)–(2.11*c*) we obtain (1.1). □

We have derived a straightforward generalization of (4.13) in [6], which contains all of the multisoliton solutions found in [11] (when taking A having all of its eigenvalues in the open right half-plane) as well as a new class of exponentially decaying exact solutions. A solution of the latter type was given in example 7.2 and plotted for various t in figure 1 of [6], while nonscalar examples will be presented in section 3.

It is clear from lemma 2.1 that the Lyapunov solutions

$$
Q = \begin{bmatrix} I_{p_1} & 0 \\ 0 & -I_{p_2} \end{bmatrix}\begin{bmatrix} Q_1 & Q_0 \\ -Q_0^\dagger & Q_2 \end{bmatrix}, \qquad N = \begin{bmatrix} I_{p_1} & 0 \\ 0 & -I_{p_2} \end{bmatrix}\begin{bmatrix} N_1 & N_0 \\ -N_0^\dagger & N_2 \end{bmatrix}
$$

are invertible matrices. Putting $\tilde{Q} = Q^{-1}$ and $\tilde{N} = N^{-1}$ we can write

$$
I_p + \boldsymbol{Q}(x;t)\boldsymbol{N}(x) = \mathrm{e}^{-xA^\dagger + 4\mathrm{i}t(A^\dagger)^2}\,Q\,\mathrm{e}^{-xA - 4\mathrm{i}tA^2}[I_p + \tilde{\boldsymbol{Q}}(x;t)\tilde{\boldsymbol{N}}(x)]\,\mathrm{e}^{-xA}\,N\,\mathrm{e}^{-xA^\dagger},
$$

where

$$
\tilde{\boldsymbol{Q}}(x;t) = \mathrm{e}^{xA + 4\mathrm{i}tA^2}\,\tilde{Q}\,\mathrm{e}^{xA^\dagger - 4\mathrm{i}t(A^\dagger)^2}, \qquad \tilde{\boldsymbol{N}}(x) = \mathrm{e}^{xA^\dagger}\,\tilde{N}\,\mathrm{e}^{xA}.
$$

Rearranging factors in (2.9) we then get

$$
u(x,t) = -2B^\dagger\tilde{N}\,\mathrm{e}^{xA}[I_p + \tilde{\boldsymbol{Q}}(x;t)\tilde{\boldsymbol{N}}(x)]^{-1}\,\mathrm{e}^{xA + 4\mathrm{i}tA^2}\,\tilde{Q}C^\dagger. \tag{2.12}
$$

Applying (2.6*a*) to the (modified) Lyapunov solutions $Q\,\mathrm{diag}(I,-I)$ and $N\mathrm{diag}(I,-I)$ with positive selfadjoint diagonal blocks Q_1, Q_2, N_1 and N_2, we easily compute that

$$
\tilde{Q} = \begin{bmatrix} Q_1^{-1} - Q_1^{-1}Q_0\left(Q_2^\#\right)^{-1}Q_0^\dagger Q_1^{-1} & Q_1^{-1}Q_0\left(Q_2^\#\right)^{-1} \\ \left(Q_2^\#\right)^{-1}Q_0^\dagger Q_1^{-1} & -\left(Q_2^\#\right)^{-1} \end{bmatrix}, \tag{2.13a}
$$

$$
\tilde{N} = \begin{bmatrix} N_1^{-1} - N_1^{-1}N_0\left(N_2^\#\right)^{-1}N_0^\dagger N_1^{-1} & N_1^{-1}N_0\left(N_2^\#\right)^{-1} \\ \left(N_2^\#\right)^{-1}N_0^\dagger N_1^{-1} & -\left(N_2^\#\right)^{-1} \end{bmatrix}, \tag{2.13b}
$$

where $Q_2^\# = Q_2 + Q_0^\dagger(Q_1)^{-1}Q_0$ and $N_2^\# = N_2 + N_0^\dagger(N_1)^{-1}N_0$. Using lemma 2.1, we see that the positivity of the determinants of the matrices $Q_2^\#, N_2^\#$,

$$\tilde{Q}\mathrm{diag}(I,-I) = [\mathrm{diag}(I,-I)Q]^{-1}, \qquad \tilde{N}\mathrm{diag}(I,-I) = [\mathrm{diag}(I,-I)N]^{-1},$$

$$\left[Q_1^{-1} - Q_1^{-1}Q_0\left(Q_2^\#\right)^{-1}Q_0^\dagger Q_1^{-1}\right]^\# = Q_1^{-1},$$

and

$$\left[N_1^{-1} - N_1^{-1}N_0\left(N_2^\#\right)^{-1}N_0^\dagger N_1^{-1}\right]^\# = N_1^{-1},$$

implies the positivity of the determinants of $Q_1^{-1} - Q_1^{-1}Q_0\left(Q_2^\#\right)^{-1}Q_0^\dagger Q_1^{-1}$ and $N_1^{-1} - N_1^{-1}N_0\left(N_2^\#\right)^{-1}N_0^\dagger N_1^{-1}$. Using theorem 5.4 of [6], but also directly, we can prove that the minimality conditions

$$\bigcap_{j=0}^{\infty}\mathrm{Ker}(C\tilde{Q}(A^\dagger)^j) = \{0\} = \bigcap_{j=0}^{\infty}\mathrm{Ker}(B^\dagger\tilde{N}A^j) \tag{2.14}$$

are fulfilled. Thus if replace A, B, C and t by $-A^\dagger, \tilde{N}B, C\tilde{Q}$ and t, we can derive (2.12) directly from theorem 2.5.

To prove the minimality conditions (2.14) from (2.2), we compute that

$$\begin{aligned}[I_p + C(\lambda - A)^{-1}\tilde{Q}C^\dagger]C\tilde{Q}(\lambda + A^\dagger)^{-1} &= C\tilde{Q}(\lambda + A^\dagger)^{-1}\\ &\quad + C(\lambda - A)^{-1}\tilde{Q}[(\lambda + A^\dagger)Q - Q(\lambda - A)]\tilde{Q}(\lambda + A^\dagger)^{-1} = C(\lambda - A)^{-1}\tilde{Q}\end{aligned}$$

and

$$\begin{aligned}[I_p - B^\dagger(\lambda + A^\dagger)^{-1}\tilde{N}B]B^\dagger\tilde{N}(\lambda - A)^{-1} &= B^\dagger\tilde{N}(\lambda - A)^{-1}\\ &\quad - B^\dagger(\lambda + A^\dagger)^{-1}\tilde{N}[N(\lambda + A^\dagger) - (\lambda - A)N]\tilde{N}(\lambda - A)^{-1} = B^\dagger(\lambda + A^\dagger)^{-1}\tilde{N}.\end{aligned}$$

Since the expressions between square brackets are invertible matrices for sufficiently large $|\lambda|$, we see that (2.14) is satisfied if and only if

$$\bigcap_{j=0}^{\infty}\mathrm{Ker}(C\tilde{Q}(A^\dagger)^j) = \bigcap_{j=0}^{\infty}\mathrm{Ker}(CA^j\tilde{Q}) = Q\left[\bigcap_{j=0}^{\infty}\mathrm{Ker}(CA^j)\right],$$

$$\bigcap_{j=0}^{\infty}\mathrm{Ker}(B^\dagger\tilde{N}A^j) = \bigcap_{j=0}^{\infty}\mathrm{Ker}(B^\dagger(A^\dagger)^j\tilde{N}) = N\left[\bigcap_{j=0}^{\infty}\mathrm{Ker}(B^\dagger(A^\dagger)^j)\right].$$

Thus (2.14) follows from the minimality conditions (2.2).

2.3. Hilbert–Schmidt norms of mNLS solutions

To study the asymptotic behaviour of $\|u(x,t)\|$ as $x \to \pm\infty$ for fixed $t \in \mathbb{R}$, we observe that

$$\mathrm{Tr}[u(x,t)u(x,t)^\dagger] = \mathrm{Tr}[u(x,t)^\dagger u(x,t)] = \sum_{r=1}^{n}\sum_{s=1}^{m}|u_{rs}(x,t)|^2, \tag{2.15}$$

which is the squared Hilbert–Schmidt norm of the $n \times m$ matrix $u(x,t)$. Putting $\hat{\Gamma}(x;t) = \Gamma(x;t)^\dagger$, where $\Gamma = I + \boldsymbol{Q}\boldsymbol{N}$ and $\hat{\Gamma} = I + \boldsymbol{N}\boldsymbol{Q}$, we now compute

$$\begin{aligned}\mathrm{Tr}[u(x,t)u(x,t)^\dagger] &= 4\mathrm{Tr}\left[B^\dagger\,\mathrm{e}^{-xA^\dagger}\Gamma(x;t)^{-1}\,\mathrm{e}^{4\mathrm{i}t(A^\dagger)^2}\,\mathrm{e}^{-xA^\dagger}C^\dagger C\,\mathrm{e}^{-xA}\,\mathrm{e}^{-4\mathrm{i}tA^2}\hat{\Gamma}(x;t)^{-1}\,\mathrm{e}^{-xA}B\right]\\ &= 4\mathrm{Tr}\left[\Gamma(x;t)^{-1}\,\mathrm{e}^{4\mathrm{i}t(A^\dagger)^2}\,\mathrm{e}^{-xA^\dagger}C^\dagger C\,\mathrm{e}^{-xA}\,\mathrm{e}^{-4\mathrm{i}tA^2}\hat{\Gamma}(x;t)^{-1}\,\mathrm{e}^{-xA}BB^\dagger\,\mathrm{e}^{-xA^\dagger}\right]\\ &= 4\mathrm{Tr}\,[\Gamma(x;t)^{-1}\boldsymbol{Q}_x(x;t)\hat{\Gamma}(x;t)^{-1}\boldsymbol{N}_x(x)].\end{aligned}$$

Let us prove the following theorem which generalizes a classical result by Zakharov and Shabat [33] (also [6]) for $n = m = 1$.

Theorem 2.6. *Suppose A, B and C are matrices as in (2.1) satisfying the minimality requirements (2.2) and such that A_1 and $A_2^\dagger$ do not have eigenvalues in common but have all of their eigenvalues in the open right half-plane. Then*

$$\mathrm{Tr}[u(x,t)u(x,t)^\dagger] = \mathrm{Tr}[\Gamma(x;t)^{-1}\Gamma_x(x;t)]_x. \tag{2.16}$$

In particular,

$$\mathrm{Tr}[u(x,t)u(x,t)^\dagger] = \frac{\partial^2}{\partial x^2}\log[\det(I_p + \boldsymbol{Q}(x;t)\boldsymbol{N}(x))]. \tag{2.17}$$

Proof. In analogy with [6], we first prove that

$$\mathrm{Tr}[u(x,t)u(x,t)^\dagger] = 2\mathrm{Tr}[\Gamma^{-1}\boldsymbol{Q}\boldsymbol{N}_x]_x, \tag{2.18}$$

where we have not written the x-dependence of $\boldsymbol{Q}$, $\boldsymbol{N}$ and Γ. Indeed, starting from the right-hand side of (2.18) we employ the identities

$$\boldsymbol{N}\Gamma^{-1}\boldsymbol{Q} = I - \hat{\Gamma}^{-1}, \qquad \Gamma^{-1}\boldsymbol{Q}\boldsymbol{N} = I - \Gamma^{-1}, \tag{2.19}$$

and compute

$$\begin{aligned}
-\mathrm{Tr}[\Gamma^{-1}\boldsymbol{Q}\boldsymbol{N}_x]_x
&= \mathrm{Tr}[\Gamma^{-1}(\boldsymbol{Q}_x\boldsymbol{N} + \boldsymbol{Q}\boldsymbol{N}_x)\Gamma^{-1}\boldsymbol{Q}\boldsymbol{N}_x - \Gamma^{-1}\boldsymbol{Q}_x\boldsymbol{N}_x - \Gamma^{-1}\boldsymbol{Q}\boldsymbol{N}_{xx}]\\
&= \mathrm{Tr}[\Gamma^{-1}\boldsymbol{Q}_x(I - \hat{\Gamma}^{-1})\boldsymbol{N}_x + \Gamma^{-1}\boldsymbol{Q}\boldsymbol{N}_x\Gamma^{-1}\boldsymbol{Q}\boldsymbol{N}_x - \Gamma^{-1}\boldsymbol{Q}_x\boldsymbol{N}_x - \Gamma^{-1}\boldsymbol{Q}\boldsymbol{N}_{xx}]\\
&= \mathrm{Tr}[-\Gamma^{-1}\boldsymbol{Q}_x\hat{\Gamma}^{-1}\boldsymbol{N}_x + \Gamma^{-1}\boldsymbol{Q}\boldsymbol{N}_x\Gamma^{-1}\boldsymbol{Q}\boldsymbol{N}_x - \Gamma^{-1}\boldsymbol{Q}\boldsymbol{N}_{xx}]\\
&= \mathrm{Tr}[-\Gamma^{-1}\boldsymbol{Q}_x\hat{\Gamma}^{-1}\boldsymbol{N}_x - \Gamma^{-1}\boldsymbol{Q}(A\boldsymbol{N} + \boldsymbol{N}A^\dagger)\Gamma^{-1}\boldsymbol{Q}\boldsymbol{N}_x\\
&\quad + \Gamma^{-1}\boldsymbol{Q}(A\boldsymbol{N}_x + \boldsymbol{N}_xA^\dagger)]\\
&= \mathrm{Tr}[-\Gamma^{-1}\boldsymbol{Q}_x\hat{\Gamma}^{-1}\boldsymbol{N}_x - \Gamma^{-1}\boldsymbol{Q}A(I - \hat{\Gamma}^{-1})\boldsymbol{N}_x\\
&\quad - (I - \Gamma^{-1})A^\dagger\Gamma^{-1}\boldsymbol{Q}\boldsymbol{N}_x + \Gamma^{-1}\boldsymbol{Q}(A\boldsymbol{N}_x + \boldsymbol{N}_xA^\dagger)]\\
&= \mathrm{Tr}[-\Gamma^{-1}\boldsymbol{Q}_x\hat{\Gamma}^{-1}\boldsymbol{N}_x + \Gamma^{-1}\boldsymbol{Q}A\hat{\Gamma}^{-1}\boldsymbol{N}_x + \Gamma^{-1}A^\dagger\Gamma^{-1}\boldsymbol{Q}\boldsymbol{N}_x]\\
&= \mathrm{Tr}[-\Gamma^{-1}\boldsymbol{Q}_x\hat{\Gamma}^{-1}\boldsymbol{N}_x + \Gamma^{-1}(\boldsymbol{Q}A + A^\dagger\boldsymbol{Q})\hat{\Gamma}^{-1}\boldsymbol{N}_x]\\
&= -2\mathrm{Tr}[\Gamma^{-1}\boldsymbol{Q}_x\hat{\Gamma}^{-1}\boldsymbol{N}_x] = -\tfrac{1}{2}\mathrm{Tr}[u(x,t)u(x,t)^\dagger],
\end{aligned}$$

which proves (2.18).

The remainder of the proof proceeds as in [6]. Using (2.19) repeatedly, it is easily verified that

$$\begin{aligned}
\mathrm{Tr}[\Gamma^{-1}\boldsymbol{Q}\boldsymbol{N}_x] &= -\mathrm{Tr}[\Gamma^{-1}\boldsymbol{Q}(A\boldsymbol{N} + \boldsymbol{N}A^\dagger)]\\
&= -\mathrm{Tr}[\boldsymbol{N}\Gamma^{-1}\boldsymbol{Q}A + \Gamma^{-1}\boldsymbol{Q}\boldsymbol{N}A^\dagger]\\
&= -\mathrm{Tr}[(I - \hat{\Gamma}^{-1})A + (I - \Gamma^{-1})A^\dagger]\\
&= -\mathrm{Tr}[(I - \Gamma^{-1})A^\dagger + (I - \hat{\Gamma}^{-1})A]\\
&= -\mathrm{Tr}[\boldsymbol{Q}\boldsymbol{N}\Gamma^{-1}A^\dagger + \boldsymbol{N}\Gamma^{-1}\boldsymbol{Q}A]\\
&= -\mathrm{Tr}(\Gamma^{-1}A^\dagger\boldsymbol{Q}\boldsymbol{N} + \Gamma^{-1}\boldsymbol{Q}A\boldsymbol{N})\\
&= -\mathrm{Tr}(\Gamma^{-1}[A^\dagger\boldsymbol{Q} + \boldsymbol{Q}A]\boldsymbol{N})\\
&= \mathrm{Tr}[\Gamma^{-1}\boldsymbol{Q}_x\boldsymbol{N}].
\end{aligned}$$

In combination with (2.18) we finally derive that

$$\mathrm{Tr}[u(x,t)u^\dagger(x,t)] = \mathrm{Tr}[\Gamma^{-1}(\boldsymbol{Q}_x\boldsymbol{N} + \boldsymbol{Q}\boldsymbol{N}_x)]_x = \mathrm{Tr}[\Gamma^{-1}\Gamma_x]_x = \mathrm{Tr}[\Gamma_x\Gamma^{-1}]_x.$$

Equation (2.17) is now immediate from [10, theorem I 7.3] by taking $\Phi = \Gamma$ and $A = \Gamma_x\Gamma^{-1}$ and differentiating the result with respect to x. □

We now establish the $x \to \pm\infty$ behaviour of the mNLS solutions (2.9).

Theorem 2.7. *Suppose A, B and C are matrices as in (2.1) satisfying the minimality requirements (2.2) and such that A_1 and $A_2^\dagger$ do not have eigenvalues in common but have all of their eigenvalues in the open right half-plane. Then the mNLS solution (2.9) is exponentially decaying as $x \to \pm\infty$ for fixed $t \in \mathbb{R}$.*

Proof. With no loss of generality we give the proof for $t = 0$. By PolTrig we denote the complex vector space of all finite linear combinations of functions $x^n, x^n\cos(\gamma x)$ and $x^n\sin(\gamma x)$, where $n = 0, 1, 2, \ldots$ and $\gamma > 0$. Then the entries of the matrix exponentials e^{-xA} and $\mathrm{e}^{-xA^\dagger}$ are finite linear combination of functions of the type $q(x)\,\mathrm{e}^{rx}$ for $q \in$ PolTrig and $r \in \mathbb{R}$, as is easily seen by using the similarity of A to a matrix in Jordan normal form. Thus there exists a finite set $\mathcal{F}$ of distinct real numbers such that

$$F(x) \stackrel{\text{def}}{=} \det\Gamma(x; 0) = \sum_{f\in\mathcal{F}} q_f(x)\,\mathrm{e}^{fx}$$

for certain $q_f \in \mathcal{F}$. Putting

$$f_+ = \max\mathcal{F}, \qquad f_- = \min\mathcal{F}, \qquad \varepsilon_\pm = \operatorname{dist}(f_\pm, \mathcal{F}\setminus\{f_\pm\}),$$

we have the asymptotic expression

$$\det\Gamma(x; 0) = \begin{cases} q_{f_+}(x)\,\mathrm{e}^{f_+x}[1 + O(\mathrm{e}^{-\varepsilon_1 x})], & x \to +\infty, \\ q_{f_-}(x)\,\mathrm{e}^{f_-x}[1 + O(\mathrm{e}^{\varepsilon_2 x})], & x \to -\infty, \end{cases}$$

for any $\varepsilon_1 \in (0, \varepsilon_+)$ and $\varepsilon_2 \in (0, \varepsilon_-)$, where $q_{f_\pm}$ are elements of PolTrig that are positive as $x \to \pm\infty$. Now

$$\operatorname{Tr}[u(x, 0)u(x, 0)^\dagger] = \frac{F(x)F''(x) - F'(x)^2}{F(x)^2}.$$

Then it is clear that the numerator has the form

$$F(x)F''(x) - F'(x)^2 = \sum_{f\in\mathcal{F}} r_f(x)\,\mathrm{e}^{fx},$$

where

$$r_f(x) = q_f(x)q_f''(x) - [q_f'(x)]^2$$

belongs to PolTrig. It is now easily verified that

$$\operatorname{Tr}[u(x, 0)u(x, 0)^\dagger] = \begin{cases} \dfrac{r_{f_+}(x)\,\mathrm{e}^{f_+x}}{[q_{f_+}(x)\,\mathrm{e}^{f_+x}]^2}[1 + O(\mathrm{e}^{-\varepsilon_1 x})], & x \to +\infty, \\ \dfrac{r_{f_-}(x)\,\mathrm{e}^{f_-x}}{[q_{f_-}(x)\,\mathrm{e}^{f_-x}]^2}[1 + O(\mathrm{e}^{\varepsilon_2 x})], & x \to -\infty, \end{cases}$$

for any $\varepsilon_1 \in (0, \varepsilon_+)$ and $\varepsilon_2 \in (0, \varepsilon_-)$. Hence,

$$\operatorname{Tr}[u(x, 0)u(x, 0)^\dagger] = \begin{cases} \dfrac{r_{f_+}(x)}{q_{f_+}(x)^2}\,\mathrm{e}^{-f_+x}[1 + O(\mathrm{e}^{-\varepsilon_1 x})], & x \to +\infty, \\ \dfrac{r_{f_-}(x)}{q_{f_-}(x)^2}\,\mathrm{e}^{-f_-x}[1 + O(\mathrm{e}^{\varepsilon_2 x})], & x \to -\infty, \end{cases}$$

for any $\varepsilon_1 \in (0, \varepsilon_+)$ and $\varepsilon_2 \in (0, \varepsilon_-)$. Since $f_+ > 0$ (unless A_2 is the zero matrix) and $f_- < 0$ (unless A_1 is the zero matrix), we get exponential decay, as claimed. □

2.4. Transforming the triple (A, B, C)

In this subsection, we discuss the effect on the mNLS solution $u(x,t)$ of certain transformations on the triple (A, B, C), where A is a square matrix of order p such that A and $-A^\dagger$ do not have eigenvalues in common, C is an $m \times p$ matrix, and B is a $p \times n$ matrix.

1. Multiplying B and C by unitary matrices. By replacing B by BV and C by UC for unitary matrices U and V, we leave $C^\dagger C$ and $BB^\dagger$ (and hence Q, N and $\Gamma(x;t)$) unaltered but replace $u(x,t)$ by $V^\dagger u(x,t)U^\dagger$ which is again a solution of the mNLS equation.

2. Similarity. By replacing (A, B, C) by (SAS^{-1}, SB, CS^{-1}) for some nonsingular $p \times p$ matrix S, we replace the matrix $C^\dagger C$ by $(S^\dagger)^{-1}C^\dagger CS^{-1}$, the matrix Q by $(S^\dagger)^{-1}QS^{-1}$, $BB^\dagger$ by $SBB^\dagger S^\dagger$, N by $SNS^\dagger$ and $\Gamma(x;t)$ by $(S^\dagger)^{-1}\Gamma(x;t)S^\dagger$. Thus $\det\Gamma(x;t)$ and $u(x,t)$ remain unchanged. Analogous similarity results can be found in [6, 11].

3. Parity. Let us replace (A, B, C) by $(-A, B, C)$. Then Q and N are replaced by $-Q$ and $-N$ and $\Gamma(x;t)$ by $\Gamma(-x;t)$. As a result, $u(x,t)$ is converted to $u(-x,t)$, which solves the mNLS equation.

4. Conjugate transposition. Let us replace (A, B, C) by $(A^\dagger, C^\dagger, B^\dagger)$, which switches the roles of n and m. Then Q and N switch places and $\Gamma(x;t)$ is replaced by $\mathrm{e}^{4\mathrm{i}tA^2}\Gamma(x;-t)^\dagger\,\mathrm{e}^{-4\mathrm{i}tA^2}$. Thus $u(x,t)$ is converted to the mNLS solution

$$u(x,-t)^\dagger = -2C\,\mathrm{e}^{-xA}\,\mathrm{e}^{4\mathrm{i}tA^2}[I_p + \boldsymbol{N}(x)\boldsymbol{Q}(x;-t)]^{-1}\,\mathrm{e}^{-xA}B.$$

5. Self-similarity. Let us replace (A, B, C) by $(A + \mathrm{i}(c/2)I, B, C)$, where $c \in \mathbb{R}$. Then N and Q do not change, while $\Gamma(x;t)$ is replaced by

$$\mathrm{e}^{2ctA^\dagger}[I_p + \boldsymbol{Q}(x-2ct,t)\boldsymbol{N}(x-ct)]\mathrm{e}^{-2ctA^\dagger} = \mathrm{e}^{2ctA^\dagger}\Gamma(x-2ct;t)\,\mathrm{e}^{-2ctA^\dagger}.$$

As a result, $u(x,t)$ is replaced by

$$-2B^\dagger\,\mathrm{e}^{-xA^\dagger}\,\mathrm{e}^{\mathrm{i}(c/2)x}\,\mathrm{e}^{2ctA^\dagger}\Gamma(x-2ct;t)^{-1}\mathrm{e}^{-2ctA^\dagger}\,\mathrm{e}^{-xA^\dagger}\,\mathrm{e}^{\mathrm{i}(c/2)x}\,\mathrm{e}^{4\mathrm{i}t(A^\dagger)^2}\,\mathrm{e}^{-\mathrm{i}c^2t}\,\mathrm{e}^{4ctA^\dagger}C^\dagger,$$

which coincides with

$$\mathrm{e}^{\mathrm{i}c(x-ct)}u(x-2ct,t),$$

in compliance with a well-known self-similarity relation for mNLS solutions.

6. Extension. Let us replace (A, B, C) by $(\check{A}, \check{B}, \check{C})$, where

$$\check{A} = \begin{bmatrix} *_1 & *_2 & *_5 \\ 0 & A & *_4 \\ 0 & 0 & *_3 \end{bmatrix}, \qquad \check{B} = \begin{bmatrix} *_6 \\ B \\ 0 \end{bmatrix}, \qquad \check{C} = [0 \quad C \quad *_7], \tag{2.20}$$

while $\check{A}$ and $-\check{A}^\dagger$ do not have eigenvalues in common and, apart from that, the asterisks are arbitrary matrices of compatible sizes. Then it is easily verified that

$$\check{C}\check{A}^j\check{B} = CA^jB, \qquad j = 0, 1, 2, \ldots. \tag{2.21}$$

By expansion into a power series in x we get from (2.21)

$$\check{C}\,\mathrm{e}^{-x\check{A}}\check{B} = C\,\mathrm{e}^{-xA}B, \qquad x \in \mathbb{R}. \tag{2.22}$$

It is now easily seen that

$$\mathrm{e}^{-x\check{A}-4\mathrm{i}t\check{A}^2} = \begin{bmatrix} * & * & * \\ 0 & \mathrm{e}^{-xA-4\mathrm{i}tA^2} & * \\ 0 & 0 & * \end{bmatrix}, \qquad \mathrm{e}^{-x\check{A}^\dagger+4\mathrm{i}t(\check{A})^2} = \begin{bmatrix} * & 0 & 0 \\ * & \mathrm{e}^{-xA^\dagger+4\mathrm{i}t(A^\dagger)^2} & 0 \\ * & * & * \end{bmatrix}.$$

As a result,

$$\check{B}^\dagger \mathrm{e}^{-x\check{A}^\dagger} = [* \quad B^\dagger \mathrm{e}^{-xA^\dagger} \quad 0], \qquad \mathrm{e}^{-x\check{A}^\dagger + 4\mathrm{i}t(\check{A}^\dagger)^2}\check{C}^\dagger = \begin{bmatrix} 0 \\ \mathrm{e}^{-xA^\dagger + 4\mathrm{i}t(A^\dagger)^2} C^\dagger \\ * \end{bmatrix}.$$

Proposition 2.8. *The solutions* $\check{Q}$ *and* $\check{N}$ *of the Lyapunov equations*

$$\check{A}^\dagger \check{Q} + \check{Q}\check{A} = \check{C}^\dagger \check{C}, \tag{2.23a}$$

$$\check{A}\check{N} + \check{N}\check{A}^\dagger = \check{B}\check{B}^\dagger, \tag{2.23b}$$

have the form

$$\check{Q} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & Q & Q_{[23]} \\ 0 & Q_{[23]}^\dagger & Q_{[33]} \end{bmatrix}, \qquad \check{N} = \begin{bmatrix} N_{[11]} & N_{[12]} & 0 \\ N_{[12]}^\dagger & N & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{2.24}$$

where $Q_{[33]}$ *and* $N_{[11]}$ *are selfadjoint.*

Proof. Put

$$\check{N} = \begin{bmatrix} N_{[11]} & N_{[12]} & N_{[13]} \\ N_{[21]} & N_{[22]} & N_{[23]} \\ N_{[31]} & N_{[32]} & N_{[33]} \end{bmatrix}.$$

Then $\check{N}$ is selfadjoint and hence $N_{[rs]} = N_{[sr]}^\dagger$ $(r, s = 1, 2, 3)$, because $\check{A}$ and $-\check{A}^\dagger$ are assumed not to have any eigenvalues in common. From (2.20) and (2.23*b*) we get

$$*_3 N_{[33]} + N_{[33]} *_3^\dagger = 0,$$

$${}_3 N_{[32]} + N_{[32]} A^\dagger + N_{[33]} *_4^\dagger = 0,$$

$${}_3 N_{[31]} + N_{[31]} *_1^\dagger + N_{[32]} *_2^\dagger + N_{[33]} *_3^\dagger = 0.$$

Since the eigenvalues $*_1$, A and $*_3$ are eigenvalues of $\check{A}$ and $\check{A}$ and $-\check{A}^\dagger$ do not have eigenvalues in common, we successively get $N_{[33]} = 0$, $N_{[32]} = 0$ and $N_{[31]} = 0$ [12, theorem 18.5]. Furthermore,

$$AN_{[22]} + N_{[22]}A^\dagger = BB^\dagger,$$

which implies that $N_{[22]} = N$ [cf (2.4*b*)]. Hence, $\check{N}$ has the form (2.24). The proof for $\check{Q}$ is analogous. □

Proposition 2.8 implies that

$$\check{\Gamma}(x;t) \stackrel{\text{def}}{=} I + \check{\boldsymbol{Q}}(x;t)\check{\boldsymbol{N}}(x) = I + \mathrm{e}^{-x\check{A}^\dagger + 4\mathrm{i}t(\check{A})^2}\check{Q}\,\mathrm{e}^{-2x\check{A} - 4\mathrm{i}t\check{A}^2}\check{N}\,\mathrm{e}^{-x\check{A}^\dagger}$$
$$= \begin{bmatrix} I & 0 & 0 \\ ? & I_p + \boldsymbol{Q}(x;t)\boldsymbol{N}(x) & 0 \\ ? & ? & I \end{bmatrix}, \tag{2.25}$$

where the question marks stand for unspecified matrices. Hence,

$$\det(I + \check{\boldsymbol{Q}}(x;t)\check{\boldsymbol{N}}(x)) = \det(I_p + \boldsymbol{Q}(x;t)\boldsymbol{N}(x)) > 0, \qquad (x,t) \in \mathbb{R}^2.$$

Using (2.25) we get

$$\check{\Gamma}(x;t)^{-1} = \begin{bmatrix} I & 0 & 0 \\ ? & \Gamma(x;t)^{-1} & 0 \\ ? & ? & I \end{bmatrix},$$

where the question marks stand for unspecified matrices. Therefore,

$$\begin{aligned}\check{u}(x,t) &\overset{\text{def}}{=} -2\check{B}^\dagger\,\mathrm{e}^{-x\check{A}^\dagger}\check{\Gamma}(x;t)^{-1}\,\mathrm{e}^{-x\check{A}^\dagger}\,\mathrm{e}^{4\mathrm{i}t(\check{A}^\dagger)^2}\check{C}^\dagger\\ &= -2B^\dagger\,\mathrm{e}^{-xA^\dagger}\Gamma(x;t)^{-1}\,\mathrm{e}^{-xA^\dagger}\,\mathrm{e}^{4\mathrm{i}t(A^\dagger)^2}C^\dagger.\end{aligned}$$

Consequently,

$$u(x,t) = -2B^\dagger\,\mathrm{e}^{-xA^\dagger}\Gamma(x;t)^{-1}\,\mathrm{e}^{-xA^\dagger}\,\mathrm{e}^{4\mathrm{i}t(A^\dagger)^2}C^\dagger$$

is a global in $(x,t)\in\mathbb{R}^2$ solution of the focusing mNLS equation (1.1). In other words, $u(x,t)$ does not change upon extension of the matrix triple (A,B,C).

2.5. Removing the minimality conditions

Throughout section 2 we have assumed that the triple of matrices (A,B,C) satisfies the minimality conditions

$$\bigcap_{j=0}^{\infty}\mathrm{Ker}(CA^j)=\{0\},\qquad \bigcap_{j=0}^{\infty}\mathrm{Ker}(B^\dagger(A^\dagger)^j)=\{0\},\tag{2.26}$$

while A and $-A^\dagger$ do not have eigenvalues in common. When partitioning A, B and C as in (2.1), we have assumed (2.2), which is, in fact, equivalent to (2.26). In this subsection we shall prove theorems 2.4–2.6 without assuming (2.1). Since theorem 2.3 cannot easily be generalized to noninvertible Q_1 and Q_2, we shall study the effect of extending 'minimal' triples (A,B,C) on theorems 2.4–2.6 instead.

Let (A,B,C) be a minimal matrix triple in the sense of (2.26). Then any matrix triple $(\check{A},\check{B},\check{C})$ satisfying (2.22) is given by [8, theorem 3.2]

$$\check{A}=S\begin{bmatrix}* & * & *\\ 0 & A & *\\ 0 & 0 & *\end{bmatrix}S^{-1},\qquad \check{B}=S\begin{bmatrix}*\\ B\\ 0\end{bmatrix},\qquad \check{C}=[0\quad C\quad *]S^{-1},\tag{2.27}$$

where S is a nonsingular matrix, $\check{A}$ and $-\check{A}^\dagger$ do not have eigenvalues in common and, apart from that, the asterisks are arbitrary matrices of compatible sizes. In other words, $(\check{A},\check{B},\check{C})$ is obtained from (A,B,C) by extension and similarity, which are operations that do not affect the mNLS solution $u(x,t)$. Thus we have proved the following generalization of theorems 2.4–2.6.

Theorem 2.9. *Suppose A, B and C are matrices as in (2.1) such that A_1 and $A_2^\dagger$ do not have eigenvalues in common but have all of their eigenvalues in the open right half-plane. Then*

$$u(x,t) = -2B^\dagger\,\mathrm{e}^{-xA^\dagger}\Gamma(x;t)^{-1}\,\mathrm{e}^{-xA^\dagger}\,\mathrm{e}^{4\mathrm{i}t(A^\dagger)^2}C^\dagger$$

is a global in $(x,t)\in\mathbb{R}^2$ solution of the focusing mNLS equation (1.1). Moreover,

$$\mathrm{Tr}[u(x,t)u(x,t)^\dagger]=\frac{\partial^2}{\partial x^2}\log[\det(I_p+\boldsymbol{Q}(x;t)\boldsymbol{N}(x))]>0.$$

3. Some illustrative examples

In this section we present a few illustrative examples. For $n=m=1$, other examples (and in particular multiple pole examples) can be found in [6], complete with plots for various values of t.

Example 3.1. Let $A = a$ be a scalar off the imaginary axis, $B = b$ a complex nonzero row vector of length m and $C = c$ a complex nonzero column vector of length n. Then

$$Q = \frac{\|c\|^2}{2p}, \qquad N = \frac{\|b\|^2}{2p}, \qquad \Gamma(x;t) = 1 + \frac{\|c\|^2\|b\|^2}{4p^2}\,\mathrm{e}^{-4px}\,\mathrm{e}^{16pqt},$$

where $p = \operatorname{Re} a \neq 0$ and $q = \operatorname{Im} a$. Then we obtain the one-soliton solution

$$u(x,t) = \frac{-2\mathrm{e}^{-2p(x-4qt)}\,\mathrm{e}^{2\mathrm{i}(qx+2(p^2-q^2)t)}}{1 + \frac{\|c\|^2\|b\|^2}{4p^2}\,\mathrm{e}^{-4px}\,\mathrm{e}^{16pqt}}\rho = -\frac{\mathrm{e}^{-2px_0}\,\mathrm{e}^{2\mathrm{i}(qx+2(p^2-q^2)t)}}{\cosh[2p(x-x_0-4qt)]}\rho,$$

where $\rho = b^\dagger c^\dagger$ is an $n \times m$ matrix of rank 1, $x_0 = (1/2p)\log(\|\rho\|/2|p|)$, $u(x,t) = \phi(x,t)\rho$ and $\phi(x,t)$ is a scalar function satisfying the rescaled focusing NLS equation

$$\mathrm{i}\phi_t + \phi_{xx} + 2\|\rho\|^2|\phi|^2\phi = 0.$$

Example 3.2. Consider distinct $\alpha, \beta > 0$, nonzero complex row vectors b_1, b_2 of length m and nonzero complex column vectors c_1, c_2 of length n, and put

$$A = \begin{bmatrix} \alpha & 0 \\ 0 & -\beta \end{bmatrix}, \qquad B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \qquad C = [c_1 \quad c_2]. \tag{3.1}$$

Then we easily compute that

$$Q = \begin{bmatrix} \dfrac{c_1^\dagger c_1}{2\alpha} & \dfrac{c_1^\dagger c_2}{\alpha-\beta} \\ \dfrac{c_2^\dagger c_1}{\alpha-\beta} & -\dfrac{c_2^\dagger c_2}{2\beta} \end{bmatrix}, \qquad N = \begin{bmatrix} \dfrac{b_1 b_1^\dagger}{2\alpha} & \dfrac{b_1 b_2^\dagger}{\alpha-\beta} \\ \dfrac{b_2 b_1^\dagger}{\alpha-\beta} & -\dfrac{b_2 b_2^\dagger}{2\beta} \end{bmatrix}.$$

Consequently,

$$\begin{aligned}
\det(I_2 + \boldsymbol{Q}(x;t)\boldsymbol{N}(x)) &= 1 + \frac{\|c_1\|^2\|b_1\|^2}{4\alpha^2}\,\mathrm{e}^{-4\alpha x} + \frac{\|c_2\|^2\|b_2\|^2}{4\beta^2}\,\mathrm{e}^{4\beta x} \\
&\quad + \frac{\mathrm{e}^{4\mathrm{i}(\alpha^2-\beta^2)t}c_1^\dagger c_2 b_2 b_1^\dagger + \mathrm{e}^{-4\mathrm{i}(\alpha^2-\beta^2)t}c_2^\dagger c_1 b_1 b_2^\dagger}{(\alpha-\beta)^2}\,\mathrm{e}^{-2(\alpha-\beta)x} \\
&\quad + \left(\frac{\|c_1\|^2\|c_2\|^2}{4\alpha\beta} + \frac{\left|c_1^\dagger c_2\right|^2}{(\alpha-\beta)^2}\right)\left(\frac{\|b_1\|^2\|b_2\|^2}{4\alpha\beta} + \frac{\left|b_1 b_2^\dagger\right|^2}{(\alpha-\beta)^2}\right)\mathrm{e}^{-4(\alpha-\beta)x} \\
&= \left|1 + \frac{\mathrm{e}^{4\mathrm{i}(\alpha^2-\beta^2)t}c_1^\dagger c_2 b_2 b_1^\dagger}{(\alpha-\beta)^2}\,\mathrm{e}^{-2(\alpha-\beta)x}\right|^2 + \frac{\|c_1\|^2\|b_1\|^2}{4\alpha^2}\,\mathrm{e}^{-4\alpha x} + \frac{\|c_2\|^2\|b_2\|^2}{4\beta^2}\,\mathrm{e}^{4\beta x} \\
&\quad + \left(\frac{\|c_1\|^2\|b_1\|^2\|c_2\|^2\|b_2\|^2}{16\alpha^2\beta^2} + \frac{\left|c_1^\dagger c_2\right|^2\|b_1\|^2\|b_2\|^2 + \|c_1\|^2\|c_2\|^2\left|b_1 b_2^\dagger\right|^2}{(\alpha-\beta)^2}\right)\mathrm{e}^{-4(\alpha-\beta)}.
\end{aligned}$$

Hence, this determinant is positive irrespective of the choice of $(x,t) \in \mathbb{R}^2$. Asymptotically we have

$$\det(I_2 + \boldsymbol{Q}(x;t)\boldsymbol{N}(x)) = \begin{cases} \dfrac{\|c_2\|^2\|b_2\|^2}{4\beta^2}(1 + O(\mathrm{e}^{-4\min(\alpha,\beta)x}))\,\mathrm{e}^{4\beta x}, & x \to +\infty, \\ \dfrac{\|c_1\|^2\|b_1\|^2}{4\alpha^2}(1 + O(\mathrm{e}^{4\min(\alpha,\beta)x}))\,\mathrm{e}^{-4\alpha x}, & x \to -\infty. \end{cases}$$

Using (2.9) we obtain

$$u(x,t) = -2\frac{\mathrm{e}^{4\mathrm{i}\alpha^2 t}\mathrm{Num}_1(x) + \mathrm{e}^{4\mathrm{i}\beta^2 t}\mathrm{Num}_2(x)}{\det(I_2 + \boldsymbol{Q}(x;t)\boldsymbol{N}(x))},$$

where

$$\mathrm{Num}_1(x) = b_1^\dagger c_1^\dagger \, \mathrm{e}^{-2\alpha x} + \left(\frac{\|b_2\|^2}{2\beta} b_1^\dagger + \frac{b_2 b_1^\dagger}{\alpha - \beta} b_2^\dagger \right) \left(\frac{\|c_2\|^2}{2\beta} c_1^\dagger + \frac{c_1^\dagger c_2}{\alpha - \beta} c_2^\dagger \right) \mathrm{e}^{-(2\alpha - 4\beta)x},$$

$$\mathrm{Num}_2(x) = b_2^\dagger c_2^\dagger \, \mathrm{e}^{2\beta x} + \left(\frac{\|b_1\|^2}{2\alpha} b_2^\dagger - \frac{b_1 b_2^\dagger}{\alpha - \beta} b_1^\dagger \right) \left(\frac{\|c_1\|^2}{2\alpha} c_2^\dagger - \frac{c_2^\dagger c_1}{\alpha - \beta} c_1^\dagger \right) \mathrm{e}^{-(4\alpha - 2\beta)x}.$$

Thus there exist bounded functions $F_1(t)$ and $F_2(t)$ of $t \in \mathbb{R}$ such that

$$u(x,t) = \begin{cases} F_1(t)\, \mathrm{e}^{-2\alpha x}(1 + O(\mathrm{e}^{-4\min(\alpha,\beta)x})), & x \to +\infty, \\ F_2(t)\, \mathrm{e}^{2\beta x}(1 + O(\mathrm{e}^{4\min(\alpha,\beta)x})), & x \to -\infty. \end{cases}$$

Example 3.3. Let $A = \mathrm{diag}(a_1, \ldots, a_p)$ be a diagonal matrix such that $a_j \neq -\overline{a_k}$ for $j, k = 1, \ldots, p$, and let B and C be given by

$$B = \begin{bmatrix} b_1 \\ \vdots \\ b_p \end{bmatrix}, \qquad C = [c_1 \ldots c_p],$$

where $b_1, \ldots, b_p$ are complex nonzero row vectors of length m and $c_1, \ldots, c_p$ are complex nonzero column vectors of length n. Then

$$Q = \left[\frac{c_j^\dagger c_k}{\overline{a_j} + a_k} \right]_{j,k=1}^p, \qquad N = \left[\frac{b_j b_k^\dagger}{a_j + \overline{a_k}} \right]_{j,k=1}^p.$$

As a result, for $j, k = 1, \ldots, p$ we have

$$\Gamma(x;t)_{j,k} = \delta_{jk} + \sum_{l=1}^p \mathrm{e}^{-x(\overline{a_j + a_k} + 2a_l)}\, \mathrm{e}^{4\mathrm{i}t(\overline{a_j}^2 - a_l^2)} \frac{c_j^\dagger c_l}{\overline{a_j} + a_l} \frac{b_l b_k^\dagger}{a_l + \overline{b_k}},$$

and therefore

$$u(x,t) = -2\left[\mathrm{e}^{-x\overline{a_1}} b_1^\dagger \ldots \mathrm{e}^{-x\overline{a_p}} b_p^\dagger\right] \Gamma(x;t)^{-1} \begin{bmatrix} \mathrm{e}^{-x\overline{a_1}}\, \mathrm{e}^{4\mathrm{i}t\overline{a_1}^2} c_1^\dagger \\ \vdots \\ \mathrm{e}^{-x\overline{a_p}}\, \mathrm{e}^{4\mathrm{i}t\overline{a_p}^2} c_p^\dagger \end{bmatrix}.$$

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