Integr. equ. oper. theory 59 (2007), 591–596

0378-620X/040591-6, *published online* October 18, 2007
DOI 10.1007/s00020-007-1535-8

Integral Equations and Operator Theory

Nonautonomous Exponential Dichotomy

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Abstract. In this note we generalize the strongly continuous bisemigroups generated by exponentially dichotomous operators to so-called bievolution families. These families are then related to strongly continuous bisemigroups on certain Banach spaces of continuous and measurable vector-valued functions.

Keywords. Bisemigroup, dichotomous operator, bievolution family.

1. Introduction

In recent years exponentially dichotomous operators $S(X \to X)$ defined on a dense linear subspace of a complex Banach space X have been studied extensively [1, 7, 10, 12]. They can be defined through the Laplace transform relation

$$(\lambda - S)^{-1}x = \int_{-\infty}^{\infty} e^{-\lambda t} E(t;x)\,dt,$$

where, for each $x \in X$, $E(\cdot;x) : \mathbb{R} \to X$ is strongly measurable and satisfies

$$\int_{-\infty}^{\infty} e^{\varepsilon|t|} \|E(t;x)\|_X\,dt \le \text{const.}\|x\|_X, \qquad x \in X,$$

for some constant $\varepsilon > 0$. Then there exists a strongly continuous function $E : \mathbb{R} \to \mathcal{L}(X)$, the so-called bisemigroup, having its values in the complex Banach algebra $\mathcal{L}(X)$ of bounded linear operators on X and having a strong jump discontinuity at $t = 0$ such that $E(t)x = E(t;x)$ for $0 \neq t \in \mathbb{R}$. Also $E(0^+) - E(0^-) = I_X$, the identity operator on X. Further, $\pm E(0^\pm)$ are complementary projections reducing S.

Exponentially bounded evolution families have been defined as strongly continuous operator functions $U : \{(t,s) \in \mathbb{R}^2 : t \ge s\} \to \mathcal{L}(X)$ having the properties (i) $U(t,r)U(r,s) = U(t,s)$ for $t \ge r \ge s$, and (ii) $\|U(t,s)\|_{\mathcal{L}(X)} \le Me^{\varepsilon(t-s)}$ for

The research leading to this note was supported by the Italian Ministry of Education and Research (MIUR) under PRIN grant no. 2006017542-003, and by INdAM-GNCS.

certain constants $M, \varepsilon > 0$. They are the natural generalizations of strongly continuous semigroups when modeling nonautonomous first order initial value problems (cf. [2] and references therein). In the context of [4, 3, 2] exponential dichotomy pertains to the existence of a projection-valued function $P : \mathbb{R} \to \mathcal{L}(X)$ such that (i) $U(t,s)P(s) = P(t)U(t,s)$ for $t \geq s$, (ii) there exists $\varepsilon > 0$ such that $\|U(t,s)P(s)x\|_X \leq \text{const.}e^{\varepsilon(t-s)}\|P(s)x\|_X$ for $t \geq s$, and (iii) the restriction of $U(t,s)$ to the kernel of $P(s)$ is a boundedly invertible operator defined on the kernel of $P(t)$ with norm bounded above by $\text{const.}e^{-\varepsilon(t-s)}$ for some $\varepsilon > 0$. Exponential dichotomy of exponentially bounded evolution families can be proven equivalent to the hyperbolicity of the strongly continuous semigroup $E : \mathbb{R} \to \mathcal{L}(L^p(\mathbb{R}; X))$ defined by $[E(t)f](\tau) = U(\tau, \tau - t)f(\tau - t)$ for $t \geq 0$ and $\tau \in \mathbb{R}$ (cf. [8]).

In this note we generalize the exponentially dichotomous operators as studied in [1, 12] to so-called bievolution families, mimicking the terminology of bisemigroups introduced in [1]. In Theorem 2.2 we prove that U is a bievolution family on X if and only E defined by $[E(t)f](\tau) = U(\tau, \tau - t)f(\tau - t)$ is a strongly continuous bisemigroup on $C_0(\mathbb{R}; X)$. In Proposition 2.1 we also show that E_U is a strongly continuous bisemigroup on $L^p(\mathbb{R}; X)$ $(1 \leq p < \infty)$ if U is a bievolution family.

Exponentially dichotomous operators have among their applications Riccati equations [10], transport equations [5], functional differential equations [9], and noncausal linear systems [6]. Some of these applications have nonautonomous counterparts conductive to treatment as bievolution systems, such as nonautonomous functional differential equations [9] and evolution equations in Banach spaces [11].

Let us introduce some notations. Given a complex Banach space X, we write I_X for the identity operator on X, $\mathcal{L}(X)$ for the Banach algebra of bounded linear operators on X, $C_0(\mathbb{R}; X)$ for the Banach space of strongly continuous functions $f : \mathbb{R} \to X$ such that $\|f(t)\|_X \to 0$ as $t \to \pm\infty$. For $1 \leq p < \infty$ we mean by $L^p(\mathbb{R}; X)$ the Banach space of strongly measurable functions $f : \mathbb{R} \to X$ for which the scalar function $\|f(\cdot)\|_X \in L^p(\mathbb{R})$.

2. Bievolution Families and Main Theorem

Letting $\Delta_\pm = \{(t,s) \in \mathbb{R}^2 : \pm(t-s) \geq 0\}$, the disjoint (set theoretical and topological) union $\Delta = \Delta_+ \cup \Delta_-$ represents the Euclidean plane $\mathbb{R}^2$, where we distinguish between $(t, t^-) \in \Delta_+$ and $(t, t^+) \in \Delta_-$. Letting X be a complex Banach space, by a *bievolution family* on X we mean a strongly continuous operator function $U : \Delta \to \mathcal{L}(X)$ having the following properties:

1. For (t,r) and (r,s) in $\Delta_\pm$ we have the product rule
$$U(t,r)U(r,s) = \pm U(t,s).$$
2. For $(t,\tau) \in \Delta_+$ and $(s,\sigma) \in \Delta_-$ we have
$$U(t,\tau)U(s,\sigma) = U(s,\sigma)U(t,\tau) = 0.$$

3. There exist positive constants M and ε such that
$$\|U(t,s)\|_{\mathcal{L}(X)} \leq Me^{-\varepsilon|t-s|}, \qquad (t,s) \in \Delta_\pm.$$
4. We have
$$U(t^+,t) - U(t^-,t) = I_X, \qquad t \in \mathbb{R}.$$
Then $U(t,t^-)$ and $-U(t,t^+)$ for $(t,t^\mp) \in \Delta_\pm$ are bounded complementary projections on X which are strongly continuous in $t \in \mathbb{R}$.

When $U(t,s)$ only depends on $(t-s) \in \Delta$ and hence we may write $E(t-s) = U(t,s)$ while distinguishing between $E(0^+)$ and $E(0^-)$, we obtain a (*strongly continuous*) *bisemigroup* on X. The separating projection then no longer depends on t and is called the *separating projection* of the bisemigroup. For convenience we write $\dot{\mathbb{R}}$ for the disjoint (set theoretical and topological) union of $\dot{\mathbb{R}}_- = (-\infty,0]$ and $\dot{\mathbb{R}}_+ = [0,\infty)$, so that E can be viewed as a strongly continuous operator function $E : \dot{\mathbb{R}} \to \mathcal{L}(X)$.

Given a bievolution family $U : \Delta \to X$, we define the *evolutionary bisemigroup* $E_U : \dot{\mathbb{R}} \to \mathcal{L}(L^p(\mathbb{R};X))$ $(1 \leq p < \infty)$ or $E_U : \dot{\mathbb{R}} \to \mathcal{L}(C_0(\mathbb{R};X))$ by
$$(E_U(t)f)(\tau) = U(\tau,\tau-t)f(\tau-t), \qquad (\tau,\tau-t) \in \Delta.$$

Proposition 2.1. *Let $1 \leq p < \infty$ and let $U : \Delta \to \mathcal{L}(X)$ be a bievolution family. Then $E_U : \dot{\mathbb{R}} \to L^p(\mathbb{R};X)$ is a strongly continuous bisemigroup.*

Proof. Let $U : \Delta \to \mathcal{L}(X)$ be a bievolution family. For $t \in \dot{\mathbb{R}}$ we have
$$\begin{aligned}\|E_U(t)f\|_{L^p(\mathbb{R};X)} &= \left[\int_{-\infty}^{\infty} \|(E_U(t)f)(\tau)\|^p \, d\tau\right]^{1/p} \\ &= \left[\int_{-\infty}^{\infty} \|U(\tau,\tau-t)f(\tau-t)\|^p \, d\tau\right]^{1/p} \leq Me^{-\varepsilon|t|}\|f\|_{L^p(\mathbb{R};X)},\end{aligned}$$
which implies the boundedness of $E_U(t)$ for $t \in \dot{\mathbb{R}}$ as well as the exponential bound on its norm. Further, for $t,s \in \dot{\mathbb{R}}$ of the same sign we estimate
$$\begin{aligned}&\|E_U(t)f - E_U(s)f\|^p_{L^p(\mathbb{R};X)} \\ &= \int_{-\infty}^{\infty} \|U(\tau,\tau-t)f(\tau-t) - U(\tau,\tau-s)f(\tau-s)\|^p_X \, d\tau \\ &\leq \int_{-\infty}^{\infty} \|[U(\tau+t,\tau) - U(\tau+t,\tau+t-s)]f(\tau)\|^p_X \, d\tau \\ &\quad + Me^{-\varepsilon|s|} \int_{-\infty}^{\infty} \|f(\tau) - f(\tau+t-s)\|^p_X \, d\tau,\end{aligned}$$
which vanishes as $s \to t$ as a result of the strong continuity of U.

For $t,s\in\dot{\mathbb{R}}$ we have

$$\begin{aligned}&(E_U(t)E_U(s)f)(\tau)=U(\tau,\tau-t)(E_U(s)f)(\tau-t)\\&=U(\tau,\tau-t)U(\tau-t,\tau-t-s)f(\tau-t-s)\\&=\begin{cases}U(\tau,\tau-t-s)f(\tau-t-s)=\quad(E_U(t+s)f)(\tau), & t,s\geq 0,\\-U(\tau,\tau-t-s)f(\tau-t-s)=-(E_U(t+s)f)(\tau), & t,s\leq 0,\\0, & ts<0,\end{cases}\end{aligned}$$

which implies the product rule. Next,

$$(E_U(0^+)f)(\tau)-(E_U(0^-)f)(\tau)=U(\tau,\tau^-)f(\tau)-U(\tau,\tau^+)f(\tau)=f(\tau),$$

so that $E_U(0^+)-E_U(0^-)=I_{L^p(\mathbb{R};X)}$. Thus, if $U:\Delta\to\mathcal{L}(X)$ is a bievolution family on X, then E_U is a strongly continuous bisemigroup on $L^p(\mathbb{R};X)$. □

We now derive the main result of this note.

Theorem 2.2. *The operator function $U:\Delta\to\mathcal{L}(X)$ is a bievolution family iff $E_U:\dot{\mathbb{R}}\to\mathcal{L}(X)$ is a strongly continuous bisemigroup on $C_0(\mathbb{R};X)$.*

Proof. For $t\in\dot{\mathbb{R}}$ we have

$$\|E_U(t)f\|_{C_0(\mathbb{R};X)}\leq\sup_{\tau\in\mathbb{R}}\|U(\tau,\tau-t)\|_{\mathcal{L}(X)}\|f\|_{C_0(\mathbb{R};X)}\leq Me^{-\varepsilon|t|}\|f\|_{C_0(\mathbb{R};X)},$$

which yields the boundedness of $E_U(t)$ and the exponential bound on its norm. For $t,s\in\dot{\mathbb{R}}$ of the same sign we estimate

$$\begin{aligned}&\|E_U(t)f-E_U(s)f\|_{C_0(\mathbb{R};X)}\\&=\sup_{\tau\in\mathbb{R}}\|U(\tau,\tau-t)f(\tau-t)-U(\tau,\tau-s)f(\tau-s)\|_X\\&\leq\sup_{\tau\in\mathbb{R}}\|[U(\tau+t,\tau)-U(\tau+t,\tau+t-s)]f(\tau)\|_X\\&\quad+Me^{-\varepsilon|s|}\sup_{\tau\in\mathbb{R}}\|f(\tau)-f(\tau+t-s)\|_X,\end{aligned}$$

proving the strong continuity of $E_U:\dot{\mathbb{R}}\to C_0(\mathbb{R};\mathcal{L}(X))$. Thus, if $U:\Delta\to\mathcal{L}(X)$ is a bievolution family on X, then E_U is a strongly continuous bisemigroup on $C_0(\mathbb{R};X)$.

Conversely, let $E_U:\dot{\mathbb{R}}\to C_0(\mathbb{R};X)$ be a strongly continuous bisemigroup. For $x\in X$ and $\phi\in C_0(\mathbb{R})$ we define $F_{(\phi,x)}\in C_0(\mathbb{R};X)$ by

$$[F_{(\phi,x)}](t)=\phi(t)x,\qquad t\in\mathbb{R}.$$

Then taking a function ϕ without zeros we see from the identity

$$U(\tau,\tau-t)x=\frac{[E(t)F_{(\phi,x)}](\tau)}{\phi(\tau-t)}$$

that $U : \Delta \to \mathcal{L}(X)$ is strongly continuous. For $t, s \in \dot{\mathbb{R}}$ we have on the one hand

$$\begin{aligned}\left[E_U(t)E_U(s)F_{(\phi,x)}\right](\tau) &= U(\tau,\tau-t)\left[E_U(s)F_{(\phi,x)}\right](\tau-t)\\ &= U(\tau,\tau-t)U(\tau-t,\tau-t-s)\left[F_{(\phi,x)}\right](\tau-t-s)\\ &= \phi(\tau-t-s)U(\tau,\tau-t)U(\tau-t,\tau-t-s)x\end{aligned}$$

and on the other hand

$$\begin{aligned}\left[E_U(t+s)F_{(\phi,x)}\right](\tau) &= U(\tau,\tau-t-s)\left[F_{(\phi,x)}\right](\tau-t-s)\\ &= \phi(\tau-t-s)U(\tau,\tau-t-s)x.\end{aligned}$$

Taking $\phi \in C_0(\mathbb{R})$ to be nonzero, we obtain from the product rule for E_U the product properties 1 and 2 for U. Moreover,

$$\phi(\tau)x = \left\{E_U(0^+) - E_U(0^-)\right\}F_{(\phi,x)}(\tau) = \phi(\tau)\left\{U(\tau,\tau^-)x - U(\tau,\tau^+)x\right\}$$

implies that $U(\tau,\tau^-) - U(\tau,\tau^+) = I_X$. Finally,

$$\begin{aligned}\|\phi\|_{C_0(\mathbb{R})}\|U(\tau,\tau-t)x\|_X &= \left\|E_U(t)F_{(\phi,x)}\right\|_{C_0(\mathbb{R};X)}\\ &\le Me^{-\varepsilon|t|}\left\|F_{(\phi,x)}\right\|_{C_0(\mathbb{R};X)} = Me^{-\varepsilon|t|}\|\phi\|_{C_0(\mathbb{R})}\|x\|_X\end{aligned}$$

implies the exponential decay condition 4 on U. Thus, $U : \Delta \to \mathcal{L}(X)$ is a bievolution family. □

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Submitted: April 24, 2007