Integr. equ. oper. theory 58 (2007), 65–86

0378-620X/010065-22, *published online* April 16, 2007
DOI 10.1007/s00020-007-1491-3

Integral Equations and Operator Theory

Factorization in Weighted Wiener Matrix Algebras on Linearly Ordered Abelian Groups

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Abstract. Factorizations of Wiener-Hopf type of elements of weighted Wiener algebras of continuous matrix-valued functions on a compact abelian group are studied. The factorizations are with respect to a fixed linear order in the character group (considered with the discrete topology). Among other results, it is proved that if a matrix function has a canonical factorization in one such matrix Wiener algebra then it belongs to the connected component of the identity of the group of invertible elements in the algebra, and moreover, the factors of the canonical factorization depend continuously on the matrix function. In the scalar case, complete characterizations of canonical and noncanonical factorability are given in terms of abstract winding numbers. Wiener-Hopf equivalence of matrix functions with elements in weighted Wiener algebras is also discussed.

Mathematics Subject Classification (2000). Primary 46J10; Secondary 43A20.
Keywords. Wiener algebra, Wiener-Hopf factorization, compact abelian group.

1. Introduction

Let G be a compact multiplicative abelian group and Γ its additive character group equipped with the discrete topology. We denote by $\langle j, g\rangle$ the action of the character $j \in \Gamma$ on the group element $g \in G$ or, by Pontryagin duality, of the character $g \in G$ on the group element $j \in \Gamma$.

It is well-known [29] that Γ can be made into a linearly ordered group if and only if G is connected; the latter hypothesis will be maintained throughout the

The second author is supported by COFIN grant 2004015437 and by INdAM; the third and the fourth authors are partially supported by NSF grant DMS-0456625; the third author is also partially supported by the Faculty Research Assignment from the College of William and Mary.

paper. We fix a linear order $\preceq$ on Γ such that Γ is an ordered group with respect to $\preceq$, i.e., $x+z \preceq y+z$ if $x, y, z \in \Gamma$ and $x \preceq y$. The notations $\prec$, $\succeq$, $\succ$, max, min (with obvious meaning) will also be used. We introduce the additive semigroups $\Gamma_+ = \{x \in \Gamma : x \succeq 0\}$ and $\Gamma_- = \{x \in \Gamma : x \preceq 0\}$. In applications, often Γ is an additive subgroup of $\mathbb{R}^k$ so that G is its Bohr compactification, or $\Gamma = \mathbb{Z}^d$ so that $G = \mathbb{T}^d$ is the d-torus.

In this paper we study factorization of Wiener-Hopf type of matrix valued functions, as shown in formula (1.1) below, where we use diag $(x_1, \ldots, x_n)$ to denote the $n \times n$ diagonal matrix with $x_1, \ldots, x_n$ on the main diagonal, in that order:

$$\widehat{A}(g) = \widehat{A}_+(g)\,(\mathrm{diag}\,(\langle j_1, g\rangle, \ldots, \langle j_n, g\rangle))\,\widehat{A}_-(g), \quad g \in G. \tag{1.1}$$

Here $\widehat{A}_+$ and $(\widehat{A}_+)^{-1}$ belong to the $n \times n$ matrix function algebra of abstract Fourier transforms of a weighted ℓ^1 space indexed by Γ_+, $\widehat{A}_-$ and $(\widehat{A}_-)^{-1}$ belong to the $n \times n$ matrix function algebra of abstract Fourier transforms of a weighted ℓ^1 space indexed by Γ_-, and $j_1, \ldots, j_n \in \Gamma$.

Factorization of type (1.1), for ℓ^1 spaces without weights, is classical when G is the unit circle; it goes back to [14], and see also [10, 6], among many books on this subject. Factorization of type (1.1) (without weights) in the case when Γ is a subgroup of the additive group $\mathbb{R}^k$ (endowed with the discrete topology) and its numerous applications have been extensively studied in the literature. A very partial list of relevant references here include [30, 31, 17, 18, 19, 3, 26, 25, 27], and see also the recent book [4]. In the abstract setting, but still for ℓ^1 spaces without weights, the factorization (1.1) was studied in [22, 21, 24]. On the other hand, in the paper [8] the weighted case was studied for scalar valued functions when G is the d-dimensional torus.

In the present paper we continue this line of investigation, and focus on the abstract compact multiplicative abelian connected group G and its additive ordered character group Γ. The factorization (1.1) will be considered in the matrix function algebras of abstract Fourier transforms of weighted ℓ^1 spaces, with arbitrary weights subject only to natural admissibility assumptions (see Section 3 for details).

Let us discuss the contents of this article. Sections 2 and 3 are devoted to Wiener algebras of scalar functions, without and with weights, and contain a full characterization of Wiener-Hopf factorizations. In Section 4 the matrix analog is discussed, in particular the uniqueness of factorization indices, hereditary properties of factors with respect to subalgebras, and connectedness. In Section 5 we relate canonical factorability to the unique solvability of certain Toeplitz equations. In Section 6 we conclude the paper with a discussion of Wiener-Hopf equivalence.

The following notation is used throughout the paper: $\mathbb{N}$ the set of positive integers, $\mathbb{Z}$ the set of integers, $\mathbb{R}$ the set of reals, $\mathbb{T}$ the unit circle, $\mathbb{C}$ the set of complex numbers.

2. Unweighted Wiener Algebras

Let G be a compact connected multiplicative abelian group and Γ its additive character group equipped with the discrete topology and the linear order $\preceq$.

For any nonempty set M, let $\ell^1(M)$ stand for the complex Banach space of all complex-valued M-indexed sequences $x = \{x_j\}_{j\in M}$ having at most countably many nonzero terms that are finite with respect to the norm

$$\|x\|_1 = \sum_{j\in M} |x_j|.$$

Then it is clear that

$$\begin{aligned}\ell^1(\Gamma) &= \ell^1(\Gamma_+)\dot{+}\ell^1(\Gamma_- \setminus \{0\}) = \ell^1(\Gamma_+ \setminus \{0\})\dot{+}\ell^1(\Gamma_-)\\ &= \ell^1(\Gamma_+ \setminus \{0\})\dot{+}\mathbb{C}\dot{+}\ell^1(\Gamma_- \setminus \{0\}),\end{aligned} \tag{2.1}$$

where the projections involved all have unit norm. Moreover, $\ell^1(\Gamma)$ is a commutative Banach algebra with unit element with respect to the convolution product

$$(x * y)_j = \sum_{k\in\Gamma} x_k\, y_{j-k}.$$

Further, $\ell^1(\Gamma_+)$ and $\ell^1(\Gamma_-)$ are closed subalgebras of $\ell^1(\Gamma)$ containing the unit element.

For every Banach algebra $\mathcal{A}$ with identity element we denote its group of invertible elements by $\mathcal{G}(\mathcal{A})$ and the connected component of $\mathcal{G}(\mathcal{A})$ containing the identity by $\mathcal{G}_0(\mathcal{A})$. It is well-known that

$$\mathcal{G}_0(\mathcal{A}) = \{\exp(b_1)\cdots\exp(b_n) : b_1, \ldots, b_n \in \mathcal{A},\ n \in \mathbb{N}\}$$

for arbitrary Banach algebras with identity element and

$$\mathcal{G}_0(\mathcal{A}) = \{\exp(b) : b \in \mathcal{A}\}$$

for those that are commutative, see [7], for example.

Given $a = \{a_j\}_{j\in\Gamma} \in \ell^1(\Gamma)$, by the *symbol* of a we mean the complex-valued continuous function $\widehat{a}$ on G defined by

$$\widehat{a}(g) = \sum_{j\in\Gamma} a_j\langle j, g\rangle, \qquad g \in G. \tag{2.2}$$

The set $\{j \in \Gamma : a_j \neq 0\}$ will be called the *Fourier spectrum* of $\widehat{a}$ given by (2.2). Since Γ is written additively and G multiplicatively, we have

$$\langle j + k, g\rangle = \langle j, g\rangle \cdot \langle k, g\rangle, \qquad j, k \in \Gamma, \quad g \in G,$$

$$\langle j, gh\rangle = \langle j, g\rangle \cdot \langle j, h\rangle, \qquad j \in \Gamma, \quad g, h \in G.$$

We will also use the shorthand notation e_j for the function

$$e_j(g) = \langle j, g\rangle, \qquad g \in G. \tag{2.3}$$

Thus, $e_{j+k} = e_j e_k$, $j, k \in \Gamma$.

The set of all symbols of elements $a \in \ell^1(\Gamma)$ forms an algebra $W(G)$ of continuous functions on G (with pointwise addition, scalar multiplication and multiplication). The algebra $W(G)$ is made into a Banach algebra isomorphic to $\ell^1(\Gamma)$ by letting $\Lambda : a \mapsto \widehat{a}$ be an isometry. (This is possible since Λ is injective, which follows from [29, Sec. 1.3.6].) Standard Gelfand theory implies that the algebra $W(G)$ is inverse closed in the algebra of all continuous functions on G (indeed, it is well-known that the maximal ideal space of $\ell^1(\Gamma)$ can be identified with G, see, for example, [9, Section 21]).

Given $a = \{a_j\}_{j\in\Gamma} \in \ell^1(\Gamma)$, by a *canonical factorization* of $\widehat{a}$ we mean a factorization of the symbol $\widehat{a}$ of the form

$$\widehat{a}(g) = \widehat{a}_+(g)\widehat{a}_-(g), \qquad g \in G, \tag{2.4}$$

where $a_+ \in \mathcal{G}(\ell^1(\Gamma_+))$ and $a_- \in \mathcal{G}(\ell^1(\Gamma_-))$. In that case we obviously have $a \in \mathcal{G}(\ell^1(\Gamma))$ and $a = a_+ * a_- = a_- * a_+$. Moreover, for any two canonical factorizations of $\widehat{a}$, say (2.4) and $\widehat{a}(g) = \widehat{b}_+(g)\widehat{b}_-(g)$ for $g \in G$, there exists a nonzero complex number c such that $b_+ = c\, a_+$ and $a_- = c\, b_-$.

The following result has been established in [15] for the torus $G = \mathbb{T}^2$. It has been generalized to compact connected groups G with finitely generated character group Γ and to weighted generalizations of the Banach algebra $\ell^1(\Gamma)$ in [8]. Here we achieve full generality in the unweighted case.

Theorem 2.1. *Let G be a compact multiplicative abelian group with ordered character group $(\Gamma, \preceq)$, and let $a \in \ell^1(\Gamma)$. Then the following statements are equivalent:*

(a) *$\widehat{a}$ has a canonical factorization.*
(b) *$a \in \mathcal{G}_0(\ell^1(\Gamma))$.*

Proof. (b) $\Rightarrow$ (a) Suppose $a \in \mathcal{G}_0(\ell^1(\Gamma))$. Then there exists $b \in \ell^1(\Gamma)$ such that $a = \exp(b)$. We can now find $b_+ \in \ell^1(\Gamma_+)$ and $b_- \in \ell^1(\Gamma_-)$ such that $b = b_+ + b_-$. Then, by commutativity, we have the canonical factorization (2.4), where $a_+ = \exp(b_+)$ and $a_- = \exp(b_-)$.

(a) $\Rightarrow$ (b) For $a \in \ell^1(\Gamma)$, let $\widehat{a}$ have a canonical factorization. Then $a \in \mathcal{G}(\ell^1(\Gamma))$. Let $b \in \ell^1(\Gamma)$ be such that $\|b - a\| < \|a^{-1}\|^{-1}$ and $\{j \in \Gamma : b_j \neq 0\}$ is a finite subset of $\{j \in \Gamma : a_j \neq 0\}$. Then the symbols $(1-t)\widehat{a} + t\widehat{b}$ $(0 \le t \le 1)$ all have a canonical factorization in $\ell^1(\Gamma)$, because

$$\|[(1-t)a + tb] - a\| = t\|b - a\| < \|a^{-1}\|^{-1}, \qquad 0 \le t \le 1.$$

(This follows from a general result on factorization in decomposable Banach algebras, see for example [10, Lemma I.5.1] or [11, Theorem XXIX.9.1].) On the other hand, the canonical factorization of $\widehat{b}$ with respect to $\ell^1(\Gamma)$ is actually a canonical factorization of $\widehat{b}$ with respect to $\ell^1(\Gamma_0)$, where Γ_0 is the additive group generated by the finite set $\{j \in \Gamma : b_j \neq 0\}$, see [24, Theorem 1]. According to [8], $b \in \mathcal{G}_0(\ell^1(\Gamma_0))$. Then a is continuously connected to b within $\mathcal{G}(\ell^1(\Gamma))$ and $b \in \mathcal{G}_0(\ell^1(\Gamma))$, and hence $a \in \mathcal{G}_0(\ell^1(\Gamma))$. ∎

For any commutative Banach algebra $\mathcal{A}$ with unit element we have the group isomorphisms

$$\frac{\mathcal{G}(\mathcal{A})}{\mathcal{G}_0(\mathcal{A})} \simeq \frac{\mathcal{G}(C(\mathcal{M}))}{\mathcal{G}_0(C(\mathcal{M}))} \simeq \pi^1(\mathcal{M}), \tag{2.5}$$

where $\mathcal{M}$ stands for the maximal ideal space of $\mathcal{A}$, $C(\mathcal{M})$ is the Banach algebra of continuous functions on $\mathcal{M}$, and $\pi^1(\mathcal{M})$ denotes the first cohomotopy group of $\mathcal{M}$ (see, for example, [16]). The first isomorphism follows from the Arens-Royden theorem [33] (also [1, 28]). The second isomorphism follows from [7, Theorem 2.18]. It is most instructive to spell out the equivalences in (2.5). For every $m \in \mathcal{M}$ we let ϕ_m stand for the unique multiplicative linear functional on $\mathcal{A}$ with kernel m. Then $a \in \mathcal{G}(\mathcal{A})$ is mapped onto the homotopy class $[F_a]$ of the continuous function $F_a : \mathcal{M} \to \mathbb{T}$ defined by

$$F_a(m) = \frac{\phi_m(a)}{|\phi_m(a)|}, \qquad a \in \mathcal{A}.$$

In the situation we are interested in, where $\mathcal{A} = \ell^1(\Gamma)$, we have $\mathcal{M} = G$ (see [9, Section 21]) and F_a becomes

$$F_a(g) = \frac{\sum_{j\in\Gamma} \langle j, g\rangle\, a_j}{\left|\sum_{j\in\Gamma} \langle j, g\rangle\, a_j\right|}, \qquad a = \{a_j\}_{j\in\Gamma} \in \ell^1(\Gamma). \tag{2.6}$$

Moreover, the groups in (2.5) can be identified explicitly. Since G is connected, we have the group isomorphism

$$\frac{\mathcal{G}(C(G))}{\mathcal{G}_0(C(G))} \cong \Gamma \tag{2.7}$$

(see [32, Proposition in Subsection 8.3.2]). More specifically, each continuous function $\widehat{a} \in \mathcal{G}(C(G))$ can be written as $\widehat{a}(g) = e_j(g)\widehat{b}(g)$ with uniquely determined $j \in \Gamma$ and $\widehat{b} \in \mathcal{G}_0(C(G))$. As a result we obtain the group isomorphism

$$\frac{\mathcal{G}(\ell^1(\Gamma))}{\mathcal{G}_0(\ell^1(\Gamma))} \simeq \Gamma \tag{2.8}$$

between the so-called *abstract index group* of $\ell^1(\Gamma)$ on the left-hand side (cf. [7]) and the given discrete group Γ. Given $a \in \mathcal{G}(\ell^1(\Gamma))$, the element $j \in \Gamma$ uniquely determined by the above isomorphism will be called the *abstract winding number* of $\widehat{a}$.

If $\Gamma = \mathbb{Z}$, $G = \mathbb{T}$, and $a \in \mathcal{G}(\ell^1(\mathbb{Z}))$, the abstract winding number of $\widehat{a}$ coincides with the usual winding number of the function $\widehat{a} \in \mathcal{G}(C(\mathbb{T}))$. Clearly, $\pi^1(\mathbb{T}) \cong \mathbb{Z}$.

If $\Gamma = \mathbb{R}$ with discrete topology, G is the Bohr compactification of $\mathbb{R}$, and $a \in \mathcal{G}(\ell^1(\Gamma))$, the abstract winding number of $\widehat{a} \in \mathcal{G}(C(G))$ is known as the *mean motion* of $\widehat{a}$ (more precisely, of the function $\widehat{a}$ restricted to $\mathbb{R}$). In this case, $\pi^1(G) \cong \mathbb{R}$.

Corollary 2.2. *If $a \in \ell^1(\Gamma)$ and $\widehat{a}(g) \neq 0$ for every $g \in G$, then $\widehat{a}$ admits a factorization*

$$\widehat{a}(g) = \widehat{a}_+(g) e_j(g) \widehat{a}_-(g), \qquad g \in G, \tag{2.9}$$

where $a_+ \in \mathcal{G}(\ell^1(\Gamma_+))$, $a_- \in \mathcal{G}(\ell^1(\Gamma_-))$, and $j \in \Gamma$ is the abstract winding number of $\widehat{a}$.

Proof. Clearly, $a \in \mathcal{G}(\ell^1(\Gamma))$. The specific form of the isomorphisms (2.5) and (2.7) implies that $\widehat{a}(g) = e_j(g)\widehat{b}(g)$ with $b \in \mathcal{G}_0(\ell^1(\Gamma))$ and $j \in \Gamma$. It now remains to apply Theorem 2.1. ☐

For the particular case $\Gamma = \mathbb{R}^k$, Corollary 2.2 was proved in [23] by elementary means. See also [26], where the case when Γ is a subgroup of $\mathbb{R}^k$ is treated. For $\Gamma = \mathbb{Z}^d$ and $G = \mathbb{T}^d$ Corollary 2.2 was proved in [8].

Note that either condition (a) or (b) of Theorem 2.1 is equivalent to $\widehat{a}(g) \neq 0$ for every $g \in G$ and $\widehat{a}(g)$ having winding number zero.

We now prove that the abstract index groups of $\ell^1(\Gamma_+)$ and $\ell^1(\Gamma_-)$ are trivial, as is immediate in the case $\Gamma = \mathbb{Z}$.

Proposition 2.3. *The groups $\mathcal{G}(\ell^1(\Gamma_+))$ and $\mathcal{G}(\ell^1(\Gamma_-))$ are connected.*

Proof. Let $a \in \mathcal{G}(\ell^1(\Gamma_+))$. Trivially, $\widehat{a}$ admits a canonical factorization and hence, by Theorem 2.1, $a \in \mathcal{G}_0(\ell^1(\Gamma))$. This means that $a = \exp(b)$ for some $b \in \ell^1(\Gamma)$. Writing $b = b_+ + b_-$ with $b_+ \in \ell^1(\Gamma_+)$ and $b_- \in \ell^1(\Gamma_- \setminus \{0\})$, we have $a * \exp(-b_+) = \exp(b_-)$. Since the left-hand side belongs to $\ell^1(\Gamma_+)$ and the right-hand side to $e + \ell^1(\Gamma_- \setminus \{0\})$, either side equals e and hence $a = \exp(b_+)$. Consequently, $a \in \mathcal{G}_0(\ell^1(\Gamma_+))$. In a similar way we prove that $\mathcal{G}(\ell^1(\Gamma_-))$ is connected. ☐

3. Weighted Wiener Algebras

Let us now introduce weighted Wiener algebras on an arbitrary ordered discrete abelian group $(\Gamma, \preceq)$. An *admissible weight* $\boldsymbol{\beta} = \{\beta_j\}_{j \in \Gamma}$ is defined as a Γ-indexed sequence of positive numbers β_j such that

$$1 \leq \beta_{i+j} \leq \beta_i \, \beta_j, \qquad i, j \in \Gamma. \tag{3.1}$$

For a subset M of Γ, let $\ell^1_{\boldsymbol{\beta}}(M)$ stand for the complex Banach space of all complex-valued M-indexed sequences $x = \{x_j\}_{j \in M}$ having at most countably many nonzero terms for which

$$\|x\|_{1,\boldsymbol{\beta}} := \sum_{j \in M} \beta_j \, |x_j| < \infty.$$

A decomposition analogous to (2.1) holds, again with the projections having all unit norm. Further, $\ell^1_{\boldsymbol{\beta}}(\Gamma)$, $\ell^1_{\boldsymbol{\beta}}(\Gamma_+)$ and $\ell^1_{\boldsymbol{\beta}}(\Gamma_-)$ are commutative Banach algebras with unit element. As sets, we clearly have $\ell^1_{\boldsymbol{\beta}}(M) \subseteq \ell^1(M)$ for every set $M \subseteq \Gamma$. We now introduce the weighted Wiener algebra $W(G)_{\boldsymbol{\beta}} = \{\widehat{a} : a \in \ell^1_{\boldsymbol{\beta}}(\Gamma)\}$ which inherits its norm from $\ell^1_{\boldsymbol{\beta}}(\Gamma)$ by natural isometry, and its subalgebras

$$(W(G)_{\boldsymbol{\beta}})_\pm = \{\widehat{a} : a \in \ell^1_{\boldsymbol{\beta}}(\Gamma_\pm)\}.$$

Clearly, $W(G)_{\boldsymbol{\beta}}$ coincides with $W(G)$ if $\beta_j \equiv 1$.

Moreover, due to Gelfand's theorem, $\widehat{a} \in \mathcal{G}(W(G)_{\boldsymbol{\beta}})$ if and only if $\widehat{a} \in W(G)_{\boldsymbol{\beta}}$ and $0 \notin \{\widehat{a}(g) : g \in \mathcal{M}_{\boldsymbol{\beta}}\}$, where $\mathcal{M}_{\boldsymbol{\beta}}$ stands for the maximal ideal space of $W(G)_{\boldsymbol{\beta}}$ (or of $\ell^1_{\boldsymbol{\beta}}(\Gamma)$). Here we observe that every multiplicative linear functional ϕ on $\ell^1_{\boldsymbol{\beta}}(\Gamma)$ corresponds to a Γ-indexed sequence of nonzero complex numbers ϕ_j such that $\phi_{i+j} = \phi_i \phi_j$ for all $i, j \in \Gamma$ and $\sup_{j\in\Gamma} (|\phi_j|/\beta_j) < +\infty$, while

$$\phi(x) = \sum_{j\in\Gamma} \phi_j \, x_j, \qquad x = \{x_j\}_{j\in\Gamma} \in \ell^1_{\boldsymbol{\beta}}(\Gamma). \tag{3.2}$$

Thus $\mathcal{M}_{\boldsymbol{\beta}}$ contains the maximal ideal space G of $W(G)$.

Before generalizing Theorem 2.1 to $W(G)_{\boldsymbol{\beta}}$, we derive two propositions on the unique extension of functions $\widehat{a} \in W(G)_{\boldsymbol{\beta}}$ to the (generally larger) maximal ideal space $\mathcal{M}_{\boldsymbol{\beta}}$ of a weighted Wiener algebra. This justifies our usage of the notation $\widehat{a}$ also for the Gelfand transform of $a \in \ell^1_{\boldsymbol{\beta}}(\Gamma)$.

Proposition 3.1. *Let G be a compact abelian group with ordered character group $(\Gamma, \preceq)$, $\boldsymbol{\beta} = \{\beta_j\}_{j\in\Gamma}$ an admissible weight. Then two functions $\widehat{a}, \widehat{b} \in W(G)_{\boldsymbol{\beta}}$ coincide on $\mathcal{M}_{\boldsymbol{\beta}}$ if and only if they coincide on G.*

Proof. Certainly $\widehat{a}, \widehat{b} \in W(G)$. If they coincide on G, then they are Fourier transforms of sequences in $\ell^1(\Gamma)$ which must coincide. Since these two sequences also belong to $\ell^1_{\boldsymbol{\beta}}(\Gamma)$, we may apply any multiplicative functional on $\ell^1_{\boldsymbol{\beta}}(\Gamma)$ and show that $\widehat{a}, \widehat{b}$ coincide on $\mathcal{M}_{\boldsymbol{\beta}}$, as claimed. □

This easy result generalizes the analytic continuation property for symbols on annuli in the case of $\Gamma = \mathbb{Z}$. Proposition 3.1 is also true for the corresponding algebra of matrix functions.

A similar argument, where the maximal ideal space $\mathcal{M}^{\pm}_{\boldsymbol{\beta}}$ of $\ell^1_{\boldsymbol{\beta}}(\Gamma_\pm)$ extends that of $\ell^1(\Gamma)$, can be used to prove the following.

Proposition 3.2. *Let G be a compact abelian group with ordered character group $(\Gamma, \preceq)$, $\boldsymbol{\beta} = \{\beta_j\}_{j\in\Gamma}$ an admissible weight. Then two functions $\widehat{a}, \widehat{b} \in (W(G)_{\boldsymbol{\beta}})_\pm$ coincide on $\mathcal{M}^{\pm}_{\boldsymbol{\beta}}$ if and only if they coincide on G.*

The next theorem generalizes the main result of [8] from finitely generated discrete abelian groups Γ to arbitrary discrete ordered abelian groups. It also generalizes Theorem 2.1 to the setting of weighted Wiener algebras.

Theorem 3.3. *Let G be a compact abelian group with ordered character group $(\Gamma, \preceq)$, $\boldsymbol{\beta} = \{\beta_j\}_{j\in\Gamma}$ an admissible weight, and $a \in \ell^1_{\boldsymbol{\beta}}(\Gamma)$. Then the following statements are equivalent:*

(a) *$\widehat{a}$ has a canonical factorization of the form (2.4), where $a_+ \in \mathcal{G}(\ell^1_{\boldsymbol{\beta}}(\Gamma_+))$ and $a_- \in \mathcal{G}(\ell^1_{\boldsymbol{\beta}}(\Gamma_-))$.*

(b) *$a \in \mathcal{G}_0(\ell^1_{\boldsymbol{\beta}}(\Gamma))$.*

(c) *$\widehat{a}(\phi) \neq 0$ for every $\phi \in \mathcal{M}_{\boldsymbol{\beta}}$ and the abstract winding number of $\widehat{a}$ is zero.*

Proof. The equivalence of (b) and (a) is proved as in Theorem 2.1.

To show the equivalence of (b) and (c) we have to show that the natural group homomorphism

$$\frac{\mathcal{G}(\ell^1_{\boldsymbol{\beta}}(\Gamma))}{\mathcal{G}_0(\ell^1_{\boldsymbol{\beta}}(\Gamma))} \to \frac{\mathcal{G}(\ell^1(\Gamma))}{\mathcal{G}_0(\ell^1(\Gamma))}$$

is an isomorphism. In view of the concrete form of the isomorphism in (2.5) this amounts to proving that the natural group homomorphism

$$\mathrm{Inj} : [f] \in \pi^1(\mathcal{M}_{\boldsymbol{\beta}}) \to [f|_G] \in \pi^1(G) \tag{3.3}$$

is an isomorphism. This natural group-isomorphism assigns to the homotopy class $[f]$ of a continuous function $f : \mathcal{M}_{\boldsymbol{\beta}} \to \mathbb{T}$ the homotopy class of the restriction of f onto G. (Notice that $G \subseteq \mathcal{M}_{\boldsymbol{\beta}}$, and that $f_1|_G$ and $f_2|_G$ are homotopic whenever f_1 and f_2 are homotopic.)

For the map Inj to be an isomorphism it is sufficient to show that G is a strong deformation retract of $\mathcal{M}_{\boldsymbol{\beta}}$ [16]. The latter means that there exists a continuous function

$$F : \mathcal{M}_{\boldsymbol{\beta}} \times [0,1] \to \mathcal{M}_{\boldsymbol{\beta}} \tag{3.4}$$

such that $F(\phi, 0) = \phi$ for all $\phi \in \mathcal{M}_{\boldsymbol{\beta}}$, $F(g,t) = g$ for all $g \in G$, $t \in [0,1]$, and $F(\phi, 1) \in G$ for all $\phi \in \mathcal{M}_{\boldsymbol{\beta}}$. Indeed, in order to see the sufficiency consider the map

$$\mathrm{Proj} : [h] \in \pi^1(G) \mapsto [\widetilde{h}] \in \pi^1(\mathcal{M}_{\boldsymbol{\beta}}), \quad \widetilde{h}(\phi) = h(F(\phi,1)).$$

(Again, notice that if h_1 and h_2 are homotopic, then $\widetilde{h}_1$ and $\widetilde{h}_2$ are homotopic, too.) Since for $h : G \to \mathbb{T}$ we have $\widetilde{h}|_G(g) = h(F(g,1)) = h(g)$, $g \in G$, it follows that $\mathrm{Inj} \circ \mathrm{Proj} = \mathrm{id}$. On the other hand, each continuous function $f : \mathcal{M}_{\boldsymbol{\beta}} \to \mathbb{T}$, which equals $f(F(\phi,0))$, is homotopic to $f(F(\phi,1))$ with the connecting function $f(F(\phi,t))$, $t \in [0,1]$, $\phi \in \mathcal{M}_{\boldsymbol{\beta}}$. This implies that $\mathrm{Proj} \circ \mathrm{Inj} = \mathrm{id}$. Hence Inj is indeed an isomorphism once we have shown that a retracting deformation (3.4) exists.

Recall that every multiplicative linear functional ϕ on $\ell^1_{\boldsymbol{\beta}}(\Gamma)$ corresponds to a Γ-indexed sequence of nonzero complex numbers ϕ_j such that $\phi_{i+j} = \phi_i \phi_j$ for all $i, j \in \Gamma$ and

$$\|\phi\| := \sup_{j \in \Gamma} \frac{|\phi_j|}{\beta_j} = 1, \tag{3.5}$$

by means of (3.2). (Every multiplicative linear functional has norm one.) The topology on the set of all such sequences $\phi = \{\phi_j\}_{j \in \Gamma}$ which is compatible with that on $\mathcal{M}_{\boldsymbol{\beta}}$ can be defined as the weakest topology that makes each function $\phi \mapsto \phi_j \in \mathbb{C}$ continuous, $j \in \Gamma$. Indeed, if $\phi^{(\alpha)} = \{\phi_j^{(\alpha)}\}_{j \in \Gamma}$ is a net converging to $\phi = \{\phi_j\}_{j \in \Gamma}$ such that the continuity condition just mentioned holds, then

$$\lim_{\alpha} \sum_{j \in \Gamma} \phi_j^{(\alpha)} x_j = \sum_{j \in \Gamma} \phi_j x_j \quad \text{ for each } x = \{x_j\}_{j \in \Gamma} \in \ell^1_{\boldsymbol{\beta}}(\Gamma)$$

by (3.5) and by the dominated convergence theorem.

Clearly, when $\beta_i \equiv 1$, then the above characterization applies to G in place of $\mathcal{M}_{\boldsymbol{\beta}}$ and expresses the fact that G, the group of characters on Γ, can be identified with the maximal ideal space of $\ell^1(\Gamma)$. Notice that condition (3.5) then forces ϕ_i to have modulus one.

We now define the mapping F as follows:

$$F(\phi, t) = \psi^t \quad \text{with} \quad \psi_i^t = \frac{\phi_i}{|\phi_i|^t} \tag{3.6}$$

It is easy to see that ψ^t is a Γ-indexed sequence enjoying the multiplicativity property and (3.5). Moreover, $\psi^0 = \phi$, $\psi^1 \in G$ (since $|\psi_i^1| = 1$), and $\psi^t \in G$ whenever $\phi \in G$ (i.e., $|\phi_i| = 1$). The continuity of F can be seen as follows. Let $\phi^{(\alpha)}$ be a net of such sequences converging ϕ, and consider a net t^α of numbers in $[0, 1]$ converging to t. Then $\phi_i^{(\alpha)} \to \phi_i$ for each i. Hence

$$\frac{\phi_i^{(\alpha)}}{|\phi_i^{(\alpha)}|^{t^{(\alpha)}}} \to \frac{\phi_i}{|\phi_i|^t}$$

for each $i \in \Gamma$, which means nothing but $\lim_\alpha F(\phi^{(\alpha)}, t^{(\alpha)}) = F(\phi, t)$. □

Elaborating on the maximal ideal space $\mathcal{M}_{\boldsymbol{\beta}}$ (= space of all multiplicative Γ-index sequences $\phi = \{\phi_j\}_{j\in\Gamma}$ satisfying (3.5)) it is easy to show that

$$\mathcal{M}_{\boldsymbol{\beta}} \cong \mathcal{M}_{\boldsymbol{\beta},\mathrm{pos}} \times G, \tag{3.7}$$

both topologically and algebraically, where $\mathcal{M}_{\boldsymbol{\beta},\mathrm{pos}}$ stands for the subset of all real positive valued sequences satisfying (3.5) and G can be identified with the set of all unimodular valued sequences. The multiplication of two such sequences is defined pointwise, i.e., $(\phi\psi)_j = \phi_j\psi_j$, $j \in \Gamma$.

The set $\mathcal{M}_{\boldsymbol{\beta},\mathrm{pos}}$ can be shown to be a convex subset of the set $\mathcal{M}$ of all multiplicative nonzero Γ-indexed sequences ϕ not necessarily satisfying (3.5). The set $\mathcal{M}$ becomes a topological vector space when introducing the algebraic operations as $(\phi + \psi)_i := \phi_i\psi_i$, $(\lambda\phi)_i = (\phi_i)^\lambda$ and the topology as the weakest topology that makes each mapping $\phi \in \mathcal{M} \mapsto \phi_j \in \mathbb{C} \setminus \{0\}$, $j \in \Gamma$, continuous. In fact, $\mathcal{M}_{\boldsymbol{\beta},\mathrm{pos}}$ is contractible to the neutral element in $\mathcal{M}$, namely $\phi \equiv 1$. The contracting deformation is given by $F(\phi, \lambda) = \lambda\phi$ (with the scalar multiplication as defined above). This provides another argument for the fact that G is the strong deformation retract of $\mathcal{M}_{\boldsymbol{\beta}}$.

Moreover, this fact generalizes to some extent the explicit characterization of the maximal ideal space $\mathcal{M}_{\boldsymbol{\beta}}$ of $\ell^1_{\boldsymbol{\beta}}(\Gamma)$ given in [8] for the case $\Gamma = \mathbb{Z}^d$, $G = \mathbb{T}^d$ (see also [11] if $d = 1$). Using the map

$$z = (z_1, \ldots, z_d) \in \mathbb{C}^d \mapsto \Big(x \in \ell^1(\Gamma) \mapsto \sum_{i\in\mathbb{Z}^d} z^i x_i\Big) \in \mathcal{M}_{\boldsymbol{\beta}},$$

(where $z^i = z_1^{i_1} \cdots z_d^{i_d}$), $\mathcal{M}_{\boldsymbol{\beta}}$ corresponds exactly to the set

$$\begin{aligned}\Omega_{\boldsymbol{\beta}} &= \left\{ z \in \mathbb{C}^d : \sup_{i \in \mathbb{Z}^d} \frac{|z^d|}{\beta_i} < \infty \right\} \\ &= \left\{ (t_1 e^{\xi_1}, \ldots, t_d e^{\xi_d}) : (t_1, \ldots, t_d) \in \mathbb{T}^d, (\xi_1, \ldots, \xi_d) \in K_{\boldsymbol{\beta}} \right\}, \qquad (3.8)\end{aligned}$$

where $K_{\boldsymbol{\beta}}$ is the compact convex subset of $\mathbb{R}^d$ given by

$$K_{\boldsymbol{\beta}} = \left\{ y \in \mathbb{R}^d : \sup_{i \in \mathbb{Z}^d} \left(\langle i, y \rangle - \log(\beta_i) \right) < \infty \right\}.$$

In this case (3.7) has the form $\mathcal{M}_{\boldsymbol{\beta}} \simeq \exp(K_{\boldsymbol{\beta}}) \times \mathbb{T}^d$, both topologically and algebraically.

Finally, let us mention the analogue of Corollary 2.2 in the weighted case.

Corollary 3.4. *Let G be a compact abelian group with ordered character group $(\Gamma, \preceq)$, $\boldsymbol{\beta} = \{\beta_j\}_{j \in \Gamma}$ an admissible weight, and $a \in \ell^1_{\boldsymbol{\beta}}(\Gamma)$. If $\widehat{a}(\phi) \neq 0$ for every $\phi \in \mathcal{M}_{\boldsymbol{\beta}}$, then $\widehat{a}$ admits a factorization*

$$\widehat{a}(g) = \widehat{a}_+(g) e_j(g) \widehat{a}_-(g), \qquad g \in G, \qquad (3.9)$$

where $a_+ \in \mathcal{G}(\ell^1_{\boldsymbol{\beta}}(\Gamma_+))$, $a_- \in \mathcal{G}(\ell^1_{\boldsymbol{\beta}}(\Gamma_-))$, and $j \in \Gamma$ is the abstract winding number of $\widehat{a}$.

Proof. As in the proof of Corollary 2.2 we can write $\widehat{a}(g) = e_j(g)\widehat{b}(g)$ with $b \in \mathcal{G}_0(\ell^1(\Gamma))$ and $j \in \Gamma$ being the abstract winding number. Clearly, $b \in \mathcal{G}(\ell^1_{\boldsymbol{\beta}}(\Gamma))$ and the abstract winding number of $\widehat{b}$ is zero. It remains to apply Theorem 3.3. □

4. Weighted Wiener Algebras of Matrix Valued Functions

If $\mathcal{A}$ is a commutative Banach algebra, we denote by $\mathcal{A}^{n \times n}$ the Banach algebra of $n \times n$ matrices with entries in $\mathcal{A}$. Invertibility in the weighted algebra $(W(G)_{\boldsymbol{\beta}})^{n \times n}$ is characterized in terms of pointwise invertibility, as immediately follows from the natural matrix generalization of Gelfand's theorem:

Proposition 4.1. *Let G be a compact abelian group with character group Γ, and let $(W(G)_{\boldsymbol{\beta}})^{n \times n}$ be the corresponding Wiener algebra of $n \times n$ matrix functions, where the weight $\boldsymbol{\beta}$ satisfies* (3.1)*. Then $\widehat{A} \in \mathcal{G}((W(G)_{\boldsymbol{\beta}})^{n \times n})$ if and only if $\widehat{A}(g) \in \mathcal{G}(\mathbb{C}^{n \times n})$ for every $g \in \mathcal{M}_{\boldsymbol{\beta}}$, where $\mathcal{M}_{\boldsymbol{\beta}}$ is the maximal ideal space of $W(G)_{\boldsymbol{\beta}}$.*

The concept of factorization as in Proposition 2.2 extends to $n \times n$ matrix functions in $(W(G)_{\boldsymbol{\beta}})^{n \times n}$. A (*left*) *factorization* of $\widehat{A} \in (W(G)_{\boldsymbol{\beta}})^{n \times n}$ is a representation of the form

$$\widehat{A}(g) = \widehat{A}_+(g) \left(\operatorname{diag}\left(e_{j_1}(g), \ldots, e_{j_n}(g) \right) \right) \widehat{A}_-(g), \quad g \in G, \qquad (4.1)$$

where $\widehat{A}_+ \in \mathcal{G}((W(G)_+)^{n \times n}_{\boldsymbol{\beta}})$, $\widehat{A}_- \in \mathcal{G}((W(G)_-)^{n \times n}_{\boldsymbol{\beta}})$, and $j_1, \ldots, j_n \in \Gamma$.

We remark that if $\widehat{A} \in (W(G)_{\boldsymbol{\beta}})^{n \times n}$ has the left factorization (4.1), where $g \in G$, then (4.1) holds automatically for all $g \in \mathcal{M}_{\boldsymbol{\beta}}$, the maximal ideal space of $W(G)_{\boldsymbol{\beta}}$. Indeed, since (4.1) obviously is a left factorization in $W(G)^{n\times n}$, each factor is the Fourier transform of a Γ-indexed sequence of $n\times n$ matrices belonging to $(\ell^1(\Gamma))^{n\times n}$. Since each sequence in fact belongs to $(\ell^1_{\boldsymbol{\beta}}(\Gamma))^{n\times n}$, we may apply any multiplicative functional to either side of the equation obtained from (4.1) by restricting it to the (i,j)-element and conclude, using Proposition 3.1, that (4.1) holds for each $g\in\mathcal{M}_{\boldsymbol{\beta}}$.

Proposition 4.2. *If $\widehat{A}(g)\in (W(G)_{\boldsymbol{\beta}})^{n\times n}$ admits a factorization, then the elements j_k are uniquely defined (if ordered $j_1\preceq j_2\preceq\ldots\preceq j_n$).*

The elements $j_1,\ldots,j_n$ in (4.1) are called the (*left*) *factorization indices* of A. For $\Gamma=\mathbb{Z}$ and $\beta_j\equiv 1$ Proposition 4.2 is a classical result (see [10, Theorem VIII.1.1]). The same method can be used to prove Proposition 4.2, and this was done in [26] in the context of almost periodic matrix functions of several variables. We omit further details.

Analogously, by a *right factorization* of $\widehat{A}\in (W(G)_{\boldsymbol{\beta}})^{n\times n}$ we mean a representation of the form

$$\widehat{A}(g)=\widehat{A}_-(g)\,(\operatorname{diag}(e_{j_1}(g),\ldots,e_{j_n}(g)))\,\widehat{A}_+(g),\quad g\in G, \tag{4.2}$$

where $\widehat{A}_+\in\mathcal{G}((W(G)_+)^{n\times n}_{\boldsymbol{\beta}})$, $\widehat{A}_-\in\mathcal{G}((W(G)_-)^{n\times n}_{\boldsymbol{\beta}})$, and $j_1,\ldots,j_n\in\Gamma$. We can prove as above that (4.2) in fact holds for $g\in\mathcal{M}_{\boldsymbol{\beta}}$, the maximal ideal space of $W(G)_{\boldsymbol{\beta}}$. Unless stated otherwise, all notions involving factorization will pertain to left factorization.

If all factorization indices are zero, the factorization is called *canonical*. If a factorization of $\widehat{A}$ exists, the function $\widehat{A}$ is called *factorable*. For $\Gamma=\mathbb{Z}$, G the unit circle, and $\beta_j\equiv 1$, the definitions and results are classical [14, 10, 6]. Many of these results have been generalized to unweighted Wiener algebras for the cases when $\Gamma=\mathbb{R}^k$ (see [4] and references there) and Γ a subgroup of $\mathbb{R}^k$ (see [25, 26]). The factorability and nonfactorability of certain block triangular matrix functions has been studied in [22, 21], generalizing respective results for $\Gamma=\mathbb{R}$ from [17, 18, 3], see also [4].

Wiener-Hopf factorization is hereditary (in the terminology of [26]) with respect to subgroups of Γ, with the induced order:

Theorem 4.3. *Let Γ' be a subgroup of Γ, and let $\widehat{A}\in (W(G)_{\boldsymbol{\beta}})^{n\times n}$ be such that the Fourier spectrum of $\widehat{A}$ is contained in Γ'. If $\widehat{A}$ admits a Wiener-Hopf factorization* (4.1)*, then the factorization indices belong to Γ', and there exists a Wiener-Hopf factorization* (4.1) *in which the factors $\widehat{A}_\pm\in\mathcal{G}((W(G)_\pm)^{n\times n}_{\boldsymbol{\beta}})$ and their inverses have their Fourier spectrum also contained in Γ'.*

In particular, if $\widehat{A}$ admits a canonical Wiener-Hopf factorization, then the Fourier spectra of its factors and of the inverses of the factors belong to Γ'.

Proof. Since a Wiener-Hopf factorization in the weighted Wiener algebra is automatically a Wiener-Hopf factorization in the corresponding unweighted Wiener algebra, it follows from [24, Theorem 1] that the factorization indices of (4.1) belong to Γ'. Now repeat the arguments from the proof of [24, Theorem 3], working with the weighted Wiener algebra rather than with the unweighted one as in [24]. The statement about the canonical factorization follows from the uniqueness of the canonical factorization up to a constant invertible multiplier. □

The following result relates canonical factorizations and the connected component of the identity in the group $\mathcal{G}((\ell^1_{\boldsymbol\beta}(\Gamma))^{n\times n})$.

Theorem 4.4. *Let G be a compact abelian group with ordered character group $(\Gamma, \preceq)$, $\boldsymbol\beta = \{\beta_j\}_{j\in\Gamma}$ an admissible weight on Γ, $\mathcal{M}_{\boldsymbol\beta}$ the maximal ideal space of $\ell^1_{\boldsymbol\beta}(\Gamma)$, and $a \in (\ell^1_{\boldsymbol\beta}(\Gamma))^{n\times n}$.*

(a) *If $a \in \mathcal{G}_0((\ell^1_{\boldsymbol\beta}(\Gamma))^{n\times n})$ then $\widehat{a}(g)$ is invertible for every $g \in \mathcal{M}_{\boldsymbol\beta}$ and the winding number of $\det\widehat{a}$ is zero.*

(b) *If $\widehat{a}$ has a canonical factorization with factors and their inverses belonging to $(W(G)_{\boldsymbol\beta})^{n\times n}$, then $a \in \mathcal{G}_0((\ell^1_{\boldsymbol\beta}(\Gamma))^{n\times n})$.*

Before proving Theorem 4.4 we recall some topology [16]. For any two topological spaces X and Y we denote by $[X, Y]$ the set of homotopy classes of continuous maps $f : X \to Y$. We say that $f_0, f_1 : X \to Y$ are homotopic if there exists a continuous map $F : [0, 1] \times X \to Y$ such that $F(0, z) = f_0(z)$ and $F(1, z) = f_1(z)$. We deviate here from the usual definition, which requires fixing base points $x_0 \in X$, $y_0 \in Y_0$ and imposing the additional assumption that $f(x_0) = y_0$, at the expense that $[X, Y]$ is not necessarily a group (cf. [16], where setups with and without fixed base points are presented).

Proof. For part (a) observe that the invertibility of $\widehat{a}(g)$ is obvious, and the statement concerning the winding number follows from Theorem 3.3.

To prove part (b), let us assume that $\widehat{a}$ has a canonical factorization in $(W(G)_{\boldsymbol\beta})^{n\times n}$. Then, using the argument for proving the implication $(a) \Longrightarrow (b)$ in the proof of Theorem 2.1, we connect $\widehat{a}$ to another element $\widehat{a}_1$ within $(W(G)_{\boldsymbol\beta})^{n\times n}$ which has finite Fourier spectrum. Then, replacing Γ by a finitely generated group containing the Fourier spectrum of $\widehat{a}_1$, in view of Theorem 4.3 we may assume that $G \simeq \mathbb{T}^d$ and $\Gamma \simeq \mathbb{Z}^d$.

Letting $\Gamma \simeq \mathbb{Z}^d$ and $G \simeq \mathbb{T}^d$, we denote by $\mathcal{M}^+_{\boldsymbol\beta}$ the maximal ideal space of $\ell^1_{\boldsymbol\beta}(\Gamma_+)$, which can be identified with a compact subset $\Omega^+_{\boldsymbol\beta}$ of $\mathbb{C}^d$ [8, Theorem 5.2]. Recall the generalized Arens' theorem [33, 1] according to which for each commutative Banach unital algebra $\mathcal{A}$ the quotient group $\mathcal{G}(\mathcal{A}^{n\times n})/\mathcal{G}_0(\mathcal{A}^{n\times n})$ depends (up to an isomorphism) only on the maximal ideal space of $\mathcal{A}$. Using the obvious result that $\mathcal{M}^+_{\boldsymbol\beta}$ is the maximal ideal space of $C(\mathcal{M}^+_{\boldsymbol\beta})$, we have the group isomorphism

$$\frac{\mathcal{G}(\ell^1(\Gamma_+)^{n\times n}_{\boldsymbol\beta})}{\mathcal{G}_0(\ell^1(\Gamma_+)^{n\times n}_{\boldsymbol\beta})} \simeq \frac{\mathcal{G}(C(\mathcal{M}^+_{\boldsymbol\beta})^{n\times n})}{\mathcal{G}_0(C(\mathcal{M}^+_{\boldsymbol\beta})^{n\times n})}.$$

Moreover, we have

$$\frac{\mathcal{G}(C(\mathcal{M}_{\boldsymbol{\beta}}^+)^{n\times n})}{\mathcal{G}_0(C(\mathcal{M}_{\boldsymbol{\beta}}^+)^{n\times n})} \simeq [\mathcal{M}_{\boldsymbol{\beta}}^+, GL(\mathbb{C}, n)].$$

This follows from the fact (see [7, Theorem 2.18], generalized to matrix valued functions) that $f_1, f_2 \in \mathcal{G}(C(\mathcal{M}_{\boldsymbol{\beta}}^+)^{n\times n})$ are equivalent modulo $\mathcal{G}_0(C(\mathcal{M}_{\boldsymbol{\beta}}^+)^{n\times n})$ if and only if $f_1 f_2^{-1}$ is path-connected to the identity element e of the group $\mathcal{G}(C(\mathcal{M}_{\boldsymbol{\beta}}^+)^{n\times n})$, which means that $f_1 f_2^{-1}$ and e are homotopic. Clearly, this is equivalent to f_1 and f_2 being homotopic (as defined in the paragraph after Theorem 4.4; see [7, Def. 2.17 and Th. 2.18]).

In [8, Corollary 5.3] it is proved that $\mathcal{M}_{\boldsymbol{\beta}}^+$ is contractible to the trivial multiplicative functional ϕ_0 on $\ell^1(\Gamma_+)$ that sends $a = \{a_j\}_{j\in\Gamma_+} \in \ell^1(\Gamma_+)$ to a_0. This means that there exists a continuous function $F : [0, 1] \times \mathcal{M}_{\boldsymbol{\beta}}^+ \to \mathcal{M}_{\boldsymbol{\beta}}^+$ such that $F(0, z) = z$ and $F(1, z) = \phi_0$. Now given a representative $f : \mathcal{M}_{\boldsymbol{\beta}}^+ \to GL(\mathbb{C}, n)$ of some homotopy class belonging to $[\mathcal{M}_{\boldsymbol{\beta}}^+, GL(\mathbb{C}, n)]$ we define $f_r(z) = f(F(r, z))$, $0 \le r \le 1$, which implies that f is homotopic to the constant map $f_1(z) = f(\phi_0)$. Because $GL(\mathbb{C}, n)$ is arcwise connected, all constant maps are homotopic to each other. This proves that $[\mathcal{M}_{\boldsymbol{\beta}}^+, GL(\mathbb{C}, n)]$ is trivial.

In the same way we prove that

$$\frac{\mathcal{G}(\ell_{\boldsymbol{\beta}}^1(\Gamma_-)^{n\times n})}{\mathcal{G}_0(\ell_{\boldsymbol{\beta}}^1(\Gamma_-)^{n\times n})} \simeq [\mathcal{M}_{\boldsymbol{\beta}}^-, GL(\mathbb{C}, n)],$$

where $\mathcal{M}_{\boldsymbol{\beta}}^-$ is the space of multiplicative functionals on $\ell_{\boldsymbol{\beta}}^1(\Gamma_-)$, and that this group is trivial.

We have proved that the groups $\mathcal{G}(\ell_{\boldsymbol{\beta}}^1(\Gamma_\pm)^{n\times n})$ are connected. Now argue as in the proof of Proposition 2.3 to complete the proof. □

5. Canonical Factorization and Toeplitz Operators

In the classical case ($G = \mathbb{T}$, $\beta_j \equiv 1$) Wiener-Hopf factorization and Toeplitz operators are closely related to each other, see, for example, [10]. This relationship can be generalized to a more abstract context, e.g., Toeplitz operators acting on $(\ell^1(\Gamma_+))^n$, the space of column vectors with entries in $\ell_{\boldsymbol{\beta}}^1(\Gamma_+)$. As before the setting is that of a compact abelian group G with ordered character group $(\Gamma, \preceq)$ and an admissible weight $\boldsymbol{\beta}$. The symbols of such Toeplitz operators are supposed to belong to $(W(G)_{\boldsymbol{\beta}})^{n\times n}$ (or, equivalently, to $(\ell_{\boldsymbol{\beta}}^1(\Gamma))^{n\times n}$). Given $A \in (\ell_{\boldsymbol{\beta}}^1(\Gamma))^{n\times n}$ we define T_A by

$$T_A x = y, \quad \sum_{j\in\Gamma_+} A_{i-j} x_j = y_i, \quad i \in \Gamma_+, \tag{5.1}$$

where $x = \{x_i\}_{i\in\Gamma}$ and $y = \{y_i\}_{i\in\Gamma}$. This operator is well-defined and bounded on $(\ell^1_{\boldsymbol\beta}(\Gamma_+))^n$ since $\boldsymbol\beta$ is an admissible weight and the norm can be estimated by

$$\|T_A\| \le \|\widehat{A}\|_{(W(G)_{\boldsymbol\beta})^{n\times n}}. \tag{5.2}$$

This can be seen from the inequality

$$\sum_{i\in\Gamma_+} \beta_i\|y_i\| \le \sum_{i,j\in\Gamma_+} \beta_{i-j}\beta_j\|A_{i-j}\|\cdot\|x\|_j \le \Big(\sum_{i\in\Gamma}\beta_i\|A_i\|\Big)\Big(\sum_{j\in\Gamma_+}\beta_j\|x_j\|\Big).$$

If $A, B \in (\ell^1_{\boldsymbol\beta}(\Gamma))^{n\times n}$, then $y = T_{AB}x - T_AT_Bx$ evaluates to

$$y_i = \sum_{j\prec 0}\sum_{k\succeq 0} A_{i-j}B_{j-k}x_k, \quad i \in \Gamma_+,$$

which vanishes if the corresponding entries of either A or B vanish. More precisely, we have the formula

$$T_{AB} = T_AT_B \tag{5.3}$$

if $A \in (\ell^1_{\boldsymbol\beta}(\Gamma_-))^{n\times n}$ or $B \in (\ell^1_{\boldsymbol\beta}(\Gamma_+))^{n\times n}$. From this it follows easily that if $A \in \mathcal{G}(\ell^1_{\boldsymbol\beta}(\Gamma_-))^{n\times n}$ or $A \in \mathcal{G}(\ell^1_{\boldsymbol\beta}(\Gamma_+))^{n\times n}$, then the Toeplitz operator T_A is invertible and the inverse is given by

$$(T_A)^{-1} = T_{A^{-1}}. \tag{5.4}$$

Now assume that $A \in (\ell^1_{\boldsymbol\beta}(\Gamma))^{n\times n}$ and $\widehat{A}$ possesses a right canonical factorization $\widehat{A}(g) = \widehat{A}_-(g)\widehat{A}_+(g)$. Then (5.3) implies the factorization $T_A = T_{A_-}T_{A_+}$ from which we can conclude that T_A is invertible and its inverse is given by

$$(T_A)^{-1} = T_{A_+^{-1}}T_{A_-^{-1}}. \tag{5.5}$$

The right canonical factorization of a symbol $\widehat{A} \in (W(G)_{\boldsymbol\beta})^{n\times n}$ implies the invertibility of yet another Toeplitz operator. Given $A \in (\ell^1_{\boldsymbol\beta}(\Gamma))^{n\times n}$, we define

$$A^* = \{A^T_{-j}\}_{j\in\Gamma}, \quad \text{where } A = \{A_j\}_{j\in\Gamma} \tag{5.6}$$

and A^T_{-j} refers to the matrix transpose of the matrix A_{-j}. Clearly, A^* belongs to $(\ell^1_{\boldsymbol\beta^*}(\Gamma))^{n\times n}$ with the underlying weight being defined by $\boldsymbol\beta^* = \{\beta_{-j}\}_{j\in\Gamma}$ where $\boldsymbol\beta = \{\beta_j\}_{j\in\Gamma}$.

Indeed, if $A \in (\ell^1_{\boldsymbol\beta}(\Gamma))^{n\times n}$ and $\widehat{A}(g) = \widehat{A}_-(g)\widehat{A}_+(g)$ is a right canonical factorization, then

$$\widehat{A^*}(g) = \widehat{A^*_+}(g)\widehat{A^*_-}(g)$$

is also a right canonical factorization. Hence the Toeplitz operator T_{A^*} acting on $(\ell^1_{\boldsymbol\beta^*}(\Gamma_+))^n$ is invertible and its inverse is given by

$$(T_{A^*})^{-1} = T_{(A^*_-)^{-1}}T_{(A^*_+)^{-1}}. \tag{5.7}$$

Let us remark that the defining equation for the Toeplitz operator T_{A^*},

$$T_{A^*}x = y, \quad \sum_{j\in\Gamma_+} A^T_{j-i}x_j = y_i, \quad i \in \Gamma_+, \tag{5.8}$$

can be equivalently written as

$$\sum_{j\in\Gamma_+} u_j A_{j-i} = v_i, , \quad i \in \Gamma_+ \tag{5.9}$$

when passing to the transpose. Therein, $u^T = \{u_j^T\}_{j\in\Gamma_+}$ and $v^T = \{v_j^T\}_{j\in\Gamma_+}$ belong to $(\ell^1_{\boldsymbol{\beta}^*}(\Gamma_+))^n$.

The following result is well-known for $G = \mathbb{T}$ (see [10]) if $\beta_j \equiv 1$. It represents to some extent the converse of the observations just made that the canonical Wiener-Hopf factorization of a symbol $\widehat{A}$ implies the (unique) solvability of the equations (5.1) and (5.9). In fact, the solutions to these equations for particular right hand sides allow the construction of the right canonical Wiener-Hopf factorization.

Let $\delta_{i,j}$ stand for the Kronecker symbol and I_n for the $n \times n$ identity matrix.

Theorem 5.1. *Let G be a compact abelian group with ordered character group $(\Gamma, \preceq)$, let $\boldsymbol{\beta} = \{\beta_j\}_{j\in\Gamma}$ be a Γ-indexed sequence of positive numbers satisfying (3.1), and let $A \in (\ell^1_{\boldsymbol{\beta}}(\Gamma))^{n\times n}$. If the convolution equations*

$$\sum_{j\in\Gamma_+} A_{i-j} X_j = \delta_{i,0}\, I_n, \qquad i \in \Gamma_+, \tag{5.10}$$

and

$$\sum_{j\in\Gamma_+} U_j A_{j-i} = \delta_{i,0}\, I_n, \qquad i \in \Gamma_+, \tag{5.11}$$

each have a solution such that $\sum_{i\in\Gamma_+}(\beta_i\|X_i\| + \beta_{-i}\|U_i\|) < \infty$, and either of the two conditions

(a) $\det(X_0) \neq 0$, *equivalently,* $\det(U_0) \neq 0$, *or*
(b) $A \in \mathcal{G}(\ell^1_{\boldsymbol{\beta}}(\Gamma))^{n\times n}$

is fulfilled, then $\widehat{A} \in (W(G)_{\boldsymbol{\beta}})^{n\times n}$ has a right canonical factorization with factors and their inverses belonging to $(W(G)_{\boldsymbol{\beta}})^{n\times n}$.

Before we give the proof, let us remark that if (5.10) and (5.11) hold, then necessarily $X_0 = U_0$ since

$$X_0 = \sum_{i,j\succeq 0} U_i A_{i-j} X_j = U_0.$$

Proof. Given the solutions of the equations (5.10) and (5.11) we can define $Y \in (\ell^1_{\boldsymbol{\beta}}(\Gamma_-))^{n\times n}$ and $V \in (\ell^1_{\boldsymbol{\beta}^*}(\Gamma_-))^{n\times n})$ by

$$Y_i = \sum_{j\in\Gamma_+} A_{i-j} X_j, \qquad i \in \Gamma, \tag{5.12}$$

$$V_i = \sum_{j\in\Gamma_+} U_j A_{j-i}, \qquad i \in \Gamma. \tag{5.13}$$

Clearly, $Y_i = V_i = 0$ for $i \succ 0$, and (5.12) represents a convolution of two sequences belonging to $(\ell^1_{\boldsymbol{\beta}}(\Gamma))^{n\times n}$ while (5.13) represents the convolution of two sequences belonging to $(\ell^1_{\boldsymbol{\beta}^*}(\Gamma)))^{n\times n}$, namely $\{U_j\}_{j\in\Gamma_+}$ and $\{A_{-j}\}_{j\in\Gamma}$.

The first equation can be rewritten in terms of the corresponding symbols as

$$\Big(\sum_{j\preceq 0}\langle j,g\rangle Y_j\Big) = \widehat{A}(g)\Big(\sum_{j\succeq 0}\langle j,g\rangle X_j\Big), \qquad g\in\mathcal{M}_{\boldsymbol{\beta}},$$

while the second one turns into

$$\Big(\sum_{j\succeq 0}\langle j,g\rangle V_{-j}\Big) = \Big(\sum_{j\preceq 0}\langle j,g\rangle U_{-j}\Big)\widehat{A}(g), \qquad g\in\mathcal{M}_{\boldsymbol{\beta}}. \tag{5.14}$$

All of the symbols encountered here belong to $(W(G)_{\boldsymbol{\beta}})^{n\times n}$, while the previous two equations, which obviously hold for $g\in G$, are easily seen to be true for $g\in\mathcal{M}_{\boldsymbol{\beta}}$ as well (cf. Proposition 3.1). Using obvious abbreviations we rewrite (5.14) as

$$\widehat{Y}_-(g) = \widehat{A}(g)\widehat{X}_+(g), \qquad \widehat{V}_+(g) = \widehat{U}_-(g)\widehat{A}(g) \tag{5.15}$$

and conclude that

$$\widehat{V}_+(g)\widehat{X}_+(g) = \widehat{U}_-(g)\widehat{A}(g)\widehat{X}_+(g) = \widehat{U}_-(g)\widehat{Y}_-(g).$$

Inspecting the Fourier spectra of the products on either side it follows that they must be constant, and since $V_0 = Y_0 = I_n$ we obtain that they are equal to $X_0 = U_0$. Hence

$$X_0 = \widehat{V}_+(g)\widehat{X}_+(g) = \widehat{U}_-(g)\widehat{A}(g)\widehat{X}_+(g) = \widehat{U}_-(g)\widehat{Y}_-(g) = U_0. \tag{5.16}$$

(a) If we assume that $X_0 = U_0$ is non-singular, then $\widehat{X}_+ \in \mathcal{G}(W(G)_{\boldsymbol{\beta}})^{n\times n}_+$ and $\widehat{U}_- \in \mathcal{G}(W(G)_{\boldsymbol{\beta}})^{n\times n}_-$ with

$$\widehat{X}_+^{-1}(g) = X_0^{-1}\widehat{V}_+(g), \qquad \widehat{U}_-^{-1}(g) = \widehat{Y}_-(g)U_0^{-1},$$

and the right canonical factorization is given by

$$\widehat{A}(g) = \hat{U}_-^{-1}(g)X_0\widehat{X}_+^{-1}(g).$$

(b) If we assume that $\widehat{A}(g)$ is invertible on all of $\mathcal{M}_{\boldsymbol{\beta}}$, then clearly $\det\widehat{A}\in \mathcal{G}W(G)_{\boldsymbol{\beta}}\subseteq\mathcal{G}W(G)$ and we can define the winding number $\gamma\in\Gamma$ of $\det\widehat{A}$ as in Section 2. By Theorem 3.3 we have a factorization

$$\det\widehat{A}(g) = \widehat{a}_-(g)e_\gamma(g)\widehat{a}_+(g), \qquad g\in\mathcal{M}_{\boldsymbol{\beta}}, \tag{5.17}$$

with $\widehat{a}_\pm\in\mathcal{G}(W(G)_{\boldsymbol{\beta}})_\pm$. Taking determinants in (5.15) and using this factorization we obtain

$$\widehat{a}_-(g)^{-1}\det\widehat{Y}_-(g) = e_\gamma(g)\widehat{a}_+(g)\det\widehat{X}_+(g), \tag{5.18}$$

$$\det\widehat{V}_+(g)\widehat{a}_+(g)^{-1} = \det\widehat{U}_-(g)\widehat{a}_-(g)e_\gamma(g). \tag{5.19}$$

This implies $\gamma = 0$ because otherwise in one of these equations the left and right hand sides must vanish identically, which contradicts the fact that $[\det\widehat{V}_+]_0 = \det V_0 = 1$ and $[\det\widehat{Y}_-]_0 = \det Y_0 = 1$. Here $[\ldots]_0$ stands for the zero-th Fourier coefficient of the underlying function. We use the fact that the map $\widehat{c} \mapsto [\widehat{c}]_0$ is a multiplicative linear functional on both of $(W(G)_{\boldsymbol{\beta}})_\pm$. Thus, (5.17) is in fact a canonical factorization.

With $\gamma = 0$ it follows that both sides of equations (5.18) and (5.19) must be constants, and for the same reasons as just pointed out these constants must be nonzero. Hence $\det \widehat{X}_+(g)$ and $\det \widehat{U}_-(g)$ are nonzero on all of $\mathcal{M}_{\boldsymbol{\beta}}$, which by (5.16) implies that $\det X_0 = \det U_0 \neq 0$. Hence this case is reduced to case (a). □

Suppose that $\widehat{A} \in (W(G)_{\boldsymbol{\beta}})^{n\times n}$ possesses a right canonical factorization $\widehat{A}(g) = \widehat{A}_-(g)\widehat{A}_+(g)$. Then the zero-th Fourier coefficients of $\widehat{A}_-$ and $\widehat{A}_+$ must be non-singular matrices, which can be pulled out from those factors. Thus one arrives at what might be called a *normalized* right canonical factorization,

$$\widehat{A}(g) = \widehat{A}_-(g)C\widehat{A}_+(g),$$

where $\widehat{A}_\pm \in \mathcal{G}(W(G)_{\boldsymbol{\beta}})^{n\times n}$ with $[A_\pm]_0 = I_n$ and $C \in \mathbb{C}^{n\times n}$ is invertible. The significance of this normalized representation is that both left and right factors as well as C are uniquely determined. Moreover, the solutions of the equations (5.10) and (5.11) are given by

$$\widehat{X}_+(g) = \widehat{A}_+^{-1}(g)C^{-1}, \qquad \widehat{U}_-(g) = C^{-1}\widehat{A}_-^{-1}(g),$$

as follows easily from (5.15). In particular, $X_0 = C^{-1} = U_0$, and hence condition (a) in Theorem 5.1 is also necessary for the existence of a right canonical factorization.

In the case $n = 1$, the constant C can be interpreted as the geometric mean of $\widehat{A}$, a well-known notion in the almost periodic case (see [4, Chapter 3]). In fact, if $\widehat{a} \in \mathcal{G}_0(W(G)_{\boldsymbol{\beta}})$, then

$$C = \exp([\log a]_0).$$

The results of this section allow us to easily obtain the continuity property of canonical factorizations. A proof of this property is known in many particular situations (see, for example, [27] for the case of almost periodic functions of several variables), and is presented here for completeness. It will be convenient to work with normalized right canonical factorizations, and with analogously defined normalized left canonical factorizations, in the next theorem.

Theorem 5.2. *The set of all matrix functions $\widehat{A} \in (W(G)_{\boldsymbol{\beta}})^{n\times n}$ which have a left* (*resp., a right*) *canonical factorization with factors and their inverses belonging to $(W(G)_{\boldsymbol{\beta}})^{n\times n}$, is open in $(W(G)_{\boldsymbol{\beta}})^{n\times n}$. Further, the factors in a normalized left* (*resp., right*) *canonical factorization depend continuously on $\widehat{A}$ in $(W(G)_{\boldsymbol{\beta}})^{n\times n}$.*

Proof. Using the remarks made at the beginning of this section, namely, that existence of a right canonical factorization implies invertibility of the corresponding Toeplitz operator, and using Theorem 5.1, the result of Theorem 5.2 follows from the well-known continuity of inversion of invertible bounded operators in a Banach space. □

A different proof of Theorem 5.2 may be given using the properties of decomposing algebras (see [5, 10]); note that $(W(G)_{\boldsymbol{\beta}})^{n\times n}$ is a decomposing algebra.

6. Wiener-Hopf Equivalence

Two matrix functions $\widehat{A},\widehat{B}\in(W(G)_{\boldsymbol{\beta}})^{n\times n}$ are called *left Wiener-Hopf equivalent* if there exist $\widehat{C}_+\in\mathcal{G}((W(G)_+)^{n\times n}_{\boldsymbol{\beta}})$ and $\widehat{C}_-\in\mathcal{G}((W(G)_-)^{n\times n}_{\boldsymbol{\beta}})$ such that

$$\widehat{A}(g)=\widehat{C}_+(g)\widehat{B}(g)\widehat{C}_-(g),\qquad g\in G. \tag{6.1}$$

It is easily seen that left Wiener-Hopf equivalence is indeed an equivalence relation on $\mathcal{G}((W(G)_{\boldsymbol{\beta}})^{n\times n})$. Similarly, we define right Wiener-Hopf equivalence. Clearly, either notion depends essentially on the weight. For $n=1$ the notions of left and right Wiener-Hopf equivalence obviously coincide, but this is not the case for $n\geq 2$.

The concept of Wiener-Hopf equivalence has been introduced and studied in [2, 13, 12] in the context of operator polynomials and analytic operator valued functions. For $\Gamma=\mathbb{R}$ the notion of Wiener-Hopf equivalence was implicitly discussed in [20, Section 2.3]. Observe that the Portuguese transformation, introduced in the setting of $\Gamma=\mathbb{R}$ in [3], christened in [4] and then further used in the setting of ordered abelian groups in [22], is in fact a convenient tool for establishing Wiener-Hopf equivalence of some block triangular matrix functions.

For $\Gamma=\mathbb{Z}$, two matrix functions $\widehat{A},\widehat{B}\in\mathcal{G}((W(\mathbb{T}))^{n\times n})$ are left Wiener-Hopf equivalent if and only if up to rearrangement they have the same left partial indices, as it easily follows from the classical results of [10, 14]. For $n=1$ it is clear from Theorem 2.2 that two nowhere zero scalar functions $\widehat{a},\widehat{b}\in W(G)_{\boldsymbol{\beta}}$ are (left and right) Wiener-Hopf equivalent if and only if they have the same (abstract) winding number.

The following result serves primarily to illustrate the scope of the Wiener-Hopf equivalence problem. We only state and prove it in the left Wiener-Hopf equivalence case. The right Wiener-Hopf equivalence class follows by reversing the order of the character group.

Proposition 6.1. *Let G be a compact abelian group with ordered character group $(\Gamma,\preceq)$ and $\boldsymbol{\beta}=\{\beta_j\}_{j\in\Gamma}$ an admissible weight. Then the left (or, right, resp.) Wiener-Hopf equivalence classes containing at least one diagonal $n\times n$ matrix function in $\mathcal{G}((W(G))^{n\times n}_{\boldsymbol{\beta}})$ are completely specified by the elements of the set*

$$\{(\gamma_1,\ldots,\gamma_d)\in\Gamma^n:\gamma_1\preceq\ldots\preceq\gamma_n\}\,. \tag{6.2}$$

Above we have used the term "factorable" for those matrix functions which are left Wiener-Hopf equivalent to a diagonal matrix with entries e_j. This proposition characterizes the left Wiener-Hopf equivalence classes of the factorable matrix functions.

Proof. Given a left Wiener-Hopf equivalence class, let

$$\operatorname{diag}(\widehat{a}_1, \dots, \widehat{a}_n), \text{ with } \widehat{a}_1, \dots, \widehat{a}_n \in W(G)_{\boldsymbol{\beta}},$$

be one of its elements. Let us denote the (abstract) winding number of $\widehat{a}_s$ by $\widetilde{\gamma}_s$ $(s = 1, \dots, n)$. Then the above element is obviously left Wiener-Hopf equivalent to $\operatorname{diag}(e_{\gamma_1}, \dots, e_{\gamma_n})$, where $(\gamma_1, \dots, \gamma_n)$ is the rearrangement of the n-tuple $(\widetilde{\gamma}_1, \dots, \widetilde{\gamma}_n)$ that satisfies $\gamma_1 \preceq \dots \preceq \gamma_n$. Since, by Proposition 4.2, the diagonal factor in (4.1) is uniquely determined by the left Wiener-Hopf equivalence class up to rearrangement of diagonal entries, there exists a one-to-one correspondence between the left Wiener-Hopf equivalence classes containing at least one diagonal matrix function in $\mathcal{G}((W(G))_{\boldsymbol{\beta}}^{n\times n})$ and the elements of the set given in (6.2). □

It is possible to construct matrix functions $\widehat{A} \in \mathcal{G}((W(G)_{\boldsymbol{\beta}})^{n\times n})$ which are *not* left Wiener-Hopf equivalent to a diagonal matrix function. Such examples have been constructed in the following cases: (1) $\Gamma = \mathbb{R}$, $n \geq 2$, and $\beta_j \equiv 1$ (see [18] and the book [4]), (2) Γ not isomorphic to a subgroup of the additive group of the rational numbers, $n \geq 2$, and $\beta_j \equiv 1$ (see [22]). We mention here the case when

$$\widehat{A} = \begin{bmatrix} e_\lambda & 0 \\ c_{-1}e_{-\nu} + c_0 + c_1 e_\alpha & e_{-\lambda} \end{bmatrix}, \tag{6.3}$$

where $\alpha, \nu \in \Gamma = \mathbb{R}$ are positive with an irrational ratio, $\lambda = \alpha + \nu$, and the coefficients $c_j \in \mathbb{C}$ are such that

$$|c_{-1}|^\alpha \, |c_1|^\nu = |c_0|^\lambda \neq 0.$$

According to [19], the Wiener-Hopf equivalence class of (6.3) does not contain any diagonal matrix functions at all. On the other hand, the proof of this result given in [19] (see also [4]) implies that this equivalence class contains a sequence of triangular matrix functions $\widehat{A}_j$ of the same type (6.3) with the diagonal exponents λ_j going to zero.

One reduction of the problem is the following. A left Wiener-Hopf equivalence class of functions in $(W(G)_{\boldsymbol{\beta}})^{n\times n}$ is called *reducible* if there exist $\widehat{A}_s \in (W(G)_{\boldsymbol{\beta}})^{n_s\times n_s}$ $(s = 1, 2)$ with $n_1, n_2 \in \mathbb{N}$ and $n_1 + n_2 = n$ such that the direct sum $\widehat{A}_1 \dot{+} \widehat{A}_2$ is contained in the class. It is then sufficient to characterize the irreducible classes varying the matrix order n. For $\Gamma = \mathbb{Z}$ and $n \geq 2$ all left Wiener-Hopf equivalence classes of everywhere invertible matrix functions are reducible. Obviously, the example of [18] belongs to an irreducible class. A sufficient reducibility condition, in terms of the Toeplitz operators associated with $e_\alpha \widehat{A}$, is given in [20, Theorem 2.6].

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Submitted: May 15, 2006
Revised: December 20, 2006