Exact Solutions of the Heisenberg Ferromagnetic Equation

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Abstract

In this article we construct an explicit (multi)-soliton solution formula for the Heisenberg ferromagnetic chain equation by using the Inverse Scattering Transform and the matrix triplet method.

1 Introduction

In this article we study the Heisenberg Ferromagnetic equation (HF), i.e.,

$$\boldsymbol{m}_t = \boldsymbol{m} \wedge \boldsymbol{m}_{zz}, \tag{1.1}$$

where $\boldsymbol{m}(z,t) \in \mathbb{R}^3$ is a vector function satisfying $\boldsymbol{m}(z,t) \to \boldsymbol{e}_3$ as $z \to \pm\infty$ and $\|\boldsymbol{m}(z,t)\| = \|\boldsymbol{e}_3\| = 1$ with $\boldsymbol{e}_3 = (0,0,1)^T$, and $\wedge$ denotes the vector product. Here z denotes position and t time.

Equation (1.1) describes the dynamics of the magnetization vector $\boldsymbol{m}$ of an isotropic ferromagnetic chain at the nanoscale in the absence of an external magnetic field.

We underline that this article is the preliminary version of a much longer paper currently in preparation, which will account for all the proofs of the outcomes reported herein, and discuss in a more detailed and extensive way the exact solutions arising from our main result, i.e. formula (3.10), along with their interactions and classification.

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It is well known that (1.1) is integrable in the sense that it is possible to find a Lax pair associated with it (see [14]). For later (notational) convenience, let us briefly recall that if V is a 2×2 invertible matrix depending on position $z \in \mathbb{R}$, time $t \in \mathbb{R}$, and a spectral parameter λ, the Lax pair associated to (1.1) is given by:

$$\begin{cases} V_z = AV = [i\lambda(\boldsymbol{m} \cdot \boldsymbol{\sigma})]V \\ V_t = BV = [-2i\lambda^2(\boldsymbol{m} \cdot \boldsymbol{\sigma}) - i\lambda(\boldsymbol{\tau} \cdot \boldsymbol{\sigma})]V \end{cases} \tag{1.2}$$

In (1.2) $\boldsymbol{\tau}$ is defined as $\boldsymbol{\tau} = \boldsymbol{m} \wedge \boldsymbol{m}_z$ and $\boldsymbol{\sigma}$ is the column vector with entries the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Of course, the knowledge of the Lax pairs for (1.1) assures that the Inverse Scattering Tranform (IST) (see [1, 2, 13]) can be applied to solve the initial-value problem for this equation, i.e.,

$$\begin{cases} \boldsymbol{m}_t = \boldsymbol{m} \wedge \boldsymbol{m}_{zz}, \\ \boldsymbol{m}(z, 0) \quad \text{known} \end{cases} \tag{1.3}$$

The first authors who applied the IST to the HF equation were Takhtajan and Zakharov [14, 17].

The aim of this paper is twofold. The first goal is to present a (somehow novel) rigorous theory for the inverse scattering transform for the HF equation. In particular, the direct scattering problem is proved to be well posed for potentials such that:

1. $\boldsymbol{m}(z) \cdot \boldsymbol{\sigma}$ has an almost everywhere existing derivative with respect to z with entries in $L^1(\mathbb{R})$
2. $m_3(z) > -1$ for each $z \in \mathbb{R}$.

Such conditions are less restrictive with respect to the usual Schwartz class hypotheses. For potentials satisfying 1. and 2. we establish the analyticity properties of eigenfunctions and scattering data. In order to obtain these results we derive a convenient set of Jost solutions (see (2.10) below) which allows to study their asymptotic behavior at large λ. The inverse scattering problem is formulated in terms of the Marchenko integral equations. These are obtained by using an appropriate triangular representation of the Jost solutions (see (2.13) in Section 2).

The second objective of this paper is to find an explicit multi-soliton solution formula for (1.1), which allows a classification of all its localized solutions, along with a description of their interaction. In order to achieve this goal, we will apply the matrix triplet method, which has proved successful in solving exactly - in the reflectionless case - several integrable equations [6, 7, 8, 9, 10]. The idea of this method is to represent the Marchenko kernel as $Ce^{-(y+z)A}B$ (where (A, B, C) is a suitable matrix triplet) in such a way that the Marchenko integral equation can be solved explicitly via separation of variables. The solutions obtained in this way will contain nothing more complicated than matrix exponentials and solutions of Lyapunov or Sylvester matrix equations [12, 3], hence can potentially be "unzipped" into lengthy expressions containing elementary functions. In the present preliminary paper we will show how to recover the famous HF one-soliton solution from (3.10).

The paper is organized as follows. In Section 2 we study the analyticity of the Jost solutions and scattering data, and determine their time evolution. Furthermore, we formulate the inverse scattering problem in terms of the Marchenko integral equation. In Section 3, combining the IST and the matrix triplet method, we get an explicit solution formula for (1.1). As a first example, we derive the explicit expression of the one-soliton solution.

2 Direct Scattering Theory

In this section we study the direct and inverse scattering theory associated to the first of equation (1.2). In particular, we analyze the analytic properties and the asymptotic behavior at large λ of the Jost solutions and the scattering data, and formulate the inverse scattering problem in terms of the Marchenko integral equations.

The proofs of the main theorems given in this section, i.e, Propositions 2.3, 2.4 and Theorems 2.5, 2.11, can be found in [16, 11].

2.1 Jost solutions

Let us define the *Jost matrices* $\Psi(z, \lambda)$ and $\Phi(z, \lambda)$ as those solutions of the linear eigenvalue problems $\Psi_z = A\Psi$ and $\Phi_z = A\Phi$ (A is the matrix defined in (1.2)) satisfying the asymptotic conditions:

$$\Psi(z,\lambda) = \begin{pmatrix}\psi(z,\lambda) & \overline{\psi}(z,\lambda)\end{pmatrix} = e^{i\lambda z\sigma_3}[I_2 + o(1)], \qquad z \to +\infty, \tag{2.1a}$$
$$\Phi(z,\lambda) = \begin{pmatrix}\overline{\phi}(z,\lambda) & \phi(z,\lambda)\end{pmatrix} = e^{i\lambda z\sigma_3}[I_2 + o(1)], \qquad z \to -\infty. \tag{2.1b}$$

The columns $\psi(z,\lambda)$, $\overline{\psi}(z,\lambda)$, $\overline{\phi}(z,\lambda)$, and $\phi(z,\lambda)$ are called *Jost functions.* In the sequel, we also use the following notations:

$$\Psi(z,\lambda) = \begin{pmatrix}\psi^{up}(z,\lambda) & \overline{\psi}^{up}(z,\lambda)\\ \psi^{dn}(z,\lambda) & \overline{\psi}^{dn}(z,\lambda)\end{pmatrix}, \quad \Phi(z,\lambda) = \begin{pmatrix}\overline{\phi}^{up}(z,\lambda) & \phi^{up}(z,\lambda)\\ \overline{\phi}^{dn}(z,\lambda) & \phi^{dn}(z,\lambda)\end{pmatrix}.$$

Then the differential equations $\Psi_z = A\Psi$ and $\Phi_z = A\Phi$ (cf. with (1.2)) can be written as

$$\Psi_z = i\lambda(\boldsymbol{m}\cdot\boldsymbol{\sigma})\Psi, \tag{2.2a}$$
$$\Phi_z = i\lambda(\boldsymbol{m}\cdot\boldsymbol{\sigma})\Phi. \tag{2.2b}$$

It is then easily verified[1] that $\Psi(z,\lambda)$ and $\Phi(z,\lambda)$ belong to the group $SU(2)$. As a result,

$$\Psi_{11}(z,\lambda)^* = \Psi_{22}(z,\lambda), \qquad \Psi_{12}(z,\lambda)^* = -\Psi_{21}(z,\lambda), \tag{2.3a}$$
$$\Phi_{11}(z,\lambda)^* = \Phi_{22}(z,\lambda), \qquad \Phi_{12}(z,\lambda)^* = -\Phi_{21}(z,\lambda). \tag{2.3b}$$

Since the two Jost matrices are both solutions to the same first order linear homogeneous differential system, there exists a so-called *transition matrix* $T(\lambda)$, depending on λ and belonging to $SU(2)$, such that

$$\Psi(z,\lambda) = \Phi(z,\lambda)T(\lambda), \qquad \lambda \in \mathbb{R}. \tag{2.4}$$

For $\lambda \in \mathbb{R}$, we have

$$T(\lambda) = \begin{pmatrix}a(\lambda) & -b(\lambda)\\ b(\lambda)^* & a(\lambda)^*\end{pmatrix},$$

where $|a(\lambda)|^2 + |b(\lambda)|^2 = 1$. We assume that $a(\lambda) \neq 0$ for each $\lambda \in \mathbb{R}$, i.e., we assume that no spectral singularities exist.

In order to formulate the Riemann-Hilbert problem we need to establish the properties of analyticity as well as the asymptotic behavior of the Jost

[1] Any square matrix solution $\Xi(z)$ to the differential system $\Xi_z = C(z)\Xi$, where $C(z)$ is skew-hermitian and has zero trace, has $\Xi^\dagger\Xi$ and $\det\Xi$ independent of $z \in \mathbb{R}$. Here the dagger denotes the complex conjugate transpose.

solutions and the coefficients $a(\lambda)$ and $b(\lambda)$ at large λ. To get these results, let us put $\boldsymbol{m}^0 = \boldsymbol{m} - \boldsymbol{e}_3$. We can convert the differential systems (2.2) with corresponding asymptotic conditions (2.1) into the Volterra integral equations

$$\Psi(z,\lambda) = e^{i\lambda z\sigma_3} - i\lambda \int_z^\infty d\hat{z}\, e^{-i\lambda(\hat{z}-z)\sigma_3}(\boldsymbol{m}^0(\hat{z})\cdot\boldsymbol{\sigma})\Psi(\hat{z},\lambda), \tag{2.5a}$$

$$\Phi(z,\lambda) = e^{i\lambda z\sigma_3} + i\lambda \int_{-\infty}^z d\hat{z}\, e^{i\lambda(z-\hat{z})\sigma_3}(\boldsymbol{m}^0(\hat{z})\cdot\boldsymbol{\sigma})\Phi(\hat{z},\lambda). \tag{2.5b}$$

As a result of Gronwall's inequality (see Appendix of [4]) we get for $(z,\lambda) \in \mathbb{R}^2$

$$\|\Psi(z,\lambda)\| \leq \exp\left(|\lambda| \int_z^\infty d\hat{z}\, \|\boldsymbol{m}^0(\hat{z})\|\right), \tag{2.6a}$$

$$\|\Phi(z,\lambda)\| \leq \exp\left(|\lambda| \int_{-\infty}^z d\hat{z}\, \|\boldsymbol{m}^0(\hat{z})\|\right), \tag{2.6b}$$

where we assume that $\boldsymbol{m}^0(z) = \boldsymbol{m}(z) - \boldsymbol{e}_3$ has its entries in $L^1(\mathbb{R})$.

We can easily prove the following

Proposition 2.1 *If $\boldsymbol{m}^0(z) = \boldsymbol{m}(z) - \boldsymbol{e}_3$ has its entries in $L^1(\mathbb{R})$, then the Faddeev functions $e^{-i\lambda z}\psi^{up}(z,\lambda), e^{-i\lambda z}\psi^{dn}(z,\lambda), e^{i\lambda z}\phi^{up}(z,\lambda)$, and $e^{i\lambda z}\phi^{dn}(z,\lambda)$ are analytic in $\lambda \in \mathbb{C}^+$ and continuous in $\lambda \in \mathbb{C}^+ \cup \mathbb{R}$, while the Faddeev functions $e^{i\lambda z}\overline{\psi}^{up}(z,\lambda), e^{i\lambda z}\overline{\psi}^{dn}(z,\lambda), e^{-i\lambda z}\overline{\phi}^{up}(z,\lambda)$ and $e^{-i\lambda z}\overline{\phi}^{dn}(z,\lambda)$ are analytic in $\lambda \in \mathbb{C}^-$ and continuous in $\lambda \in \mathbb{C}^- \cup \mathbb{R}$.*

Proof. We give the proof only for the the Faddeev functions $e^{-i\lambda z}\psi^{\rm up}(z,\lambda)$, $e^{-i\lambda z}\psi^{\rm dn}(z,\lambda), e^{i\lambda z}\overline{\psi}^{\rm up}(z,\lambda)$, $e^{i\lambda z}\overline{\psi}^{\rm dn}(z,\lambda)$ because the proof for the other Faddeev functions is very similar. The Volterra integral equations can be written in the form

$$e^{-i\lambda z}\psi^{\mathrm{up}}(z,\lambda) = 1 - i\lambda \int_z^\infty d\hat{z}\, m^0(\hat{z}) e^{-i\lambda\hat{z}}\psi^{\mathrm{up}}(\hat{z},\lambda) - i\lambda \int_z^\infty d\hat{z}\, m^-(\hat{z}) e^{-i\lambda\hat{z}}\psi^{\mathrm{dn}}(\hat{z},\lambda), \tag{2.7a}$$

$$e^{-i\lambda z}\psi^{\mathrm{dn}}(z,\lambda) = i\lambda \int_z^\infty d\hat{z}\, e^{2i\lambda(\hat{z}-z)} m^0(\hat{z}) e^{-i\lambda\hat{z}}\psi^{\mathrm{dn}}(\hat{z},\lambda) - i\lambda \int_z^\infty d\hat{z}\, e^{2i\lambda(\hat{z}-z)} m^+(\hat{z}) e^{-i\lambda\hat{z}}\psi^{\mathrm{up}}(\hat{z},\lambda), \tag{2.7b}$$

$$e^{i\lambda z}\overline{\psi}^{\mathrm{up}}(z,\lambda) = -i\lambda \int_z^\infty d\hat{z}\, e^{-2i\lambda(\hat{z}-z)} m^0(\hat{z}) e^{i\lambda\hat{z}}\overline{\psi}^{\mathrm{up}}(\hat{z},\lambda) - i\lambda \int_z^\infty d\hat{z}\, e^{-2i\lambda(\hat{z}-z)} m^-(\hat{z}) e^{i\lambda\hat{z}}\overline{\psi}^{\mathrm{dn}}(\hat{z},\lambda), \tag{2.7c}$$

$$e^{i\lambda z}\overline{\psi}^{\mathrm{dn}}(z,\lambda) = 1 + i\lambda \int_z^\infty d\hat{z}\, m^0(\hat{z}) e^{i\lambda\hat{z}}\overline{\psi}^{\mathrm{dn}}(\hat{z},\lambda) - i\lambda \int_z^\infty d\hat{z}\, m^+(\hat{z}) e^{i\lambda\hat{z}}\overline{\psi}^{\mathrm{up}}(\hat{z},\lambda)\,. \tag{2.7d}$$

Using Gronwall's inequality, uniformly in (λ, z) for $\lambda \in \overline{\mathbb{C}}^{\pm}$ and $z \geq z_0 > -\infty$, we obtain continuity in $\lambda \in \overline{\mathbb{C}}^{\pm}$ and analyticity in $\mathbb{C}^{\pm}$. □

Taking the limits of (2.7a) and (2.7d) as $z \to -\infty$ we get

$$a(\lambda) = 1 - i\lambda \int_{-\infty}^\infty d\hat{z}\left[m^0(\hat{z}) e^{-i\lambda\hat{z}}\psi^{\mathrm{up}}(\hat{z},\lambda) + m^-(\hat{z}) e^{-i\lambda\hat{z}}\psi^{\mathrm{dn}}(\hat{z},\lambda)\right], \tag{2.8a}$$

$$a(\lambda^*)^* = 1 + i\lambda \int_{-\infty}^\infty d\hat{z}\left[m^0(\hat{z}) e^{i\lambda\hat{z}}\overline{\psi}^{\mathrm{dn}}(\hat{z},\lambda) - m^+(\hat{z}) e^{i\lambda\hat{z}}\overline{\psi}^{\mathrm{up}}(\hat{z},\lambda)\right], \tag{2.8b}$$

Thus $a(\lambda)$ is continuous in $\lambda \in \overline{\mathbb{C}^+}$, is analytic in $\lambda \in \mathbb{C}^+$, and satisfies $a(0) = 1$. In the same way we get

$$b(\lambda)^* = i\lambda \int_{-\infty}^\infty d\hat{z}\, e^{2i\lambda\hat{z}}\left[m^0(\hat{z}) e^{-i\lambda\hat{z}}\psi^{\mathrm{dn}}(\hat{z},\lambda) - m^+(\hat{z}) e^{-i\lambda\hat{z}}\psi^{\mathrm{up}}(\hat{z},\lambda)\right], \tag{2.9a}$$

$$b(\lambda) = i\lambda \int_{-\infty}^\infty d\hat{z}\, e^{-2i\lambda\hat{z}}\left[m^0(\hat{z}) e^{i\lambda\hat{z}}\overline{\psi}^{\mathrm{up}}(\hat{z},\lambda) + m^-(\hat{z}) e^{i\lambda\hat{z}}\overline{\psi}^{\mathrm{dn}}(\hat{z},\lambda)\right]. \tag{2.9b}$$

Thus $b(0) = 0$ and $b_\lambda(0)$ exists.

From (2.8) and (2.9) it is clear that no information is available on their asymptotics as $\lambda \to \infty$. In order to get such information, let us derive a different set of Volterra integral equations. To do so we need to require that

a. $\boldsymbol{m}(z) \cdot \boldsymbol{\sigma}$ has an almost everywhere existing derivative $\boldsymbol{m}'(z) \cdot \boldsymbol{\sigma}$ with respect to z which has its entries in $L^1(\mathbb{R})$.[2]

b. $m_3(z) > -1$ for each $z \in \mathbb{R}$.

Under the above hypothesis, after straightforward calculations we get

$$D(z)\Psi(z,\lambda) = e^{i\lambda z\sigma_3} - \int_z^\infty d\hat{z}\, e^{-i\lambda(\hat{z}-z)\sigma_3} D'(\hat{z})\Psi(\hat{z},\lambda), \qquad (2.10)$$

where

$$D(z) \stackrel{\text{def}}{=} \tfrac{1}{2}\left[I_2 + \sigma_3(\boldsymbol{m}(z)\cdot\boldsymbol{\sigma})\right] = \tfrac{1}{2}\begin{pmatrix} 1+m_3(z) & m_-(z) \\ -m_+(z) & 1+m_3(z)\end{pmatrix}, \qquad (2.11)$$

which is a matrix of determinant $\frac{1}{2}(1+m_3(z))$. Since we have required that $m_3(z) > -1$ for each $z \in \mathbb{R}$ this matrix $D(z)$ is invertible and its inverse[3] is bounded in $z \in \mathbb{R}$. We may therefore apply Gronwall's inequality to (2.10) and find that

$$\|\Psi(z,\lambda)\| \leq \frac{1}{\sqrt{1+m_3(z)}} \exp\left[\tfrac{1}{2}\int_z^\infty d\hat{z}\, \|(\boldsymbol{m}'(\hat{z})\cdot\boldsymbol{\sigma})\|\right].$$

Remark 2.2 *In the same way and under the assumptions a and b, adapting the procedure above presented to the Jost matrix $\Phi(z,\lambda)$, we get*

$$D(z)\Phi(z,\lambda) = e^{i\lambda z\sigma_3} + \int_{-\infty}^z d\hat{z}\, e^{i\lambda(z-\hat{z})\sigma_3} D'(\hat{z})\Phi(\hat{z},\lambda). \qquad (2.12)$$

We may therefore apply Gronwall's inequality to (2.12) *to obtain*

$$\|\Phi(z,\lambda)\| \leq \frac{1}{\sqrt{1+m_3(z)}} \exp\left[\tfrac{1}{2}\int_{-\infty}^z d\hat{z}\, \|(\boldsymbol{m}'(\hat{z})\cdot\boldsymbol{\sigma})\|\right].$$

[2]Under the first condition, $\boldsymbol{m}(z)$ is absolutely continuous in $z \in \mathbb{R}$. Hence its *pointwise* values make sense and hence it makes mathematical sense to assume that, in addition, $m_3(z) > -1$ for each $z \in \mathbb{R}$.

[3]That is, $D(z)^{-1} = \frac{1}{1+m_3}\begin{pmatrix} 1+m_3 & -m_- \\ m_+ & 1+m_3\end{pmatrix}$, a matrix of norm $(1+m_3)^{-1/2}$. Note that $D(z) \to I_2$ as $z \to \pm\infty$.

Equations (2.10) and (2.12) allow us to prove that the analyticity and continuity properties of the Jost solutions extend to the closed upper and lower half-planes. In other words, the Jost solutions and the coefficient $a(\lambda)$ have a finite limit as $\lambda \to \infty$ from within the closure of its half-plane of analyticity, while $b(\lambda)$ vanishes as $\lambda \to \pm\infty$. In order to prove these results we need to find a "suitable" triangular representation for the Jost solutions. We have the following:

Proposition 2.3 *There exist an auxiliary matrix function $\boldsymbol{K}(x,y)$ such that*

$$\Psi(z,\lambda) = \boldsymbol{H}(z)e^{i\lambda z\sigma_3} + \int_z^\infty d\hat{z}\, \boldsymbol{K}(z,\hat{z})e^{i\lambda\hat{z}\sigma_3}, \tag{2.13}$$

where $\boldsymbol{H}(z)$ is a matrix function satisfying $\boldsymbol{H}(z) = \sigma_2\boldsymbol{H}(z)^*\sigma_2$, $\boldsymbol{H}(z) \to I_2$ as $z \to +\infty$, $\int_z^\infty d\hat{z}\, \|\boldsymbol{K}(z,\hat{z})\|$ converges uniformly in $z \in \mathbb{R}$.

The proof of Proposition 2.3 can be found in [16, 11]. Analogously we have the following

Proposition 2.4 *There exists an auxiliary matrix function $\boldsymbol{N}(z,y)$ such that*

$$\Phi(z,\lambda) = \tilde{\boldsymbol{H}}(z)e^{i\lambda z\sigma_3} + \int_{-\infty}^z d\hat{z}\, \boldsymbol{N}(z,\hat{z})e^{i\lambda\hat{z}\sigma_3}, \tag{2.14}$$

where $\tilde{\boldsymbol{H}}(z)$ is a matrix function satisfying $\tilde{\boldsymbol{H}}(z) = \sigma_2\tilde{\boldsymbol{H}}(z)^*\sigma_2$, $\tilde{\boldsymbol{H}}(z) \to I_2$ as $z \to -\infty$, and $\int_{-\infty}^z d\hat{z}\, \|\boldsymbol{N}(z,\hat{z})\|$ converges uniformly in $z \in \mathbb{R}$.

We remark that the auxiliary matrices $\boldsymbol{K}(z,\lambda)$ and $\boldsymbol{N}(z,\lambda)$ have to satisfy the symmetry relations

$$\boldsymbol{K}(z,y) = \begin{pmatrix} \boldsymbol{K}_1(z,y)^* & \boldsymbol{K}_2(z,y) \\ -\boldsymbol{K}_2(z,y)^* & \boldsymbol{K}_1(z,y) \end{pmatrix}, \quad \boldsymbol{N}(z,y) = \begin{pmatrix} \boldsymbol{N}_1(z,y)^* & \boldsymbol{N}_2(z,y) \\ -\boldsymbol{N}_2(z,y)^* & \boldsymbol{N}_1(z,y) \end{pmatrix}, \tag{2.15}$$

where $\boldsymbol{K}_1(z,y)$, $\boldsymbol{K}_2(z,y)$, $\boldsymbol{N}_1(z,y)$ and $\boldsymbol{N}_2(z,y)$ are scalar functions.

Finally we can prove (see [16, 11] for the proof) the following:

Theorem 2.5 *If* 1) $\boldsymbol{m}^0(z) = \boldsymbol{m}(z) - \boldsymbol{e}_3$ *has its entries in* $L^1(\mathbb{R})$, 2) $\boldsymbol{m}(z)\cdot\boldsymbol{\sigma}$ *has an almost everywhere existing derivative $\boldsymbol{m}'(z)\cdot\boldsymbol{\sigma}$ with respect to z which has its entries in $L^1(\mathbb{R})$, and* 3) *that $m_3(z) > -1$ for each $z \in \mathbb{R}$, then for each $z \in \mathbb{R}$ the functions $e^{-i\lambda z}\psi^{up}(z,\lambda), e^{-i\lambda z}\psi^{dn}(z,\lambda), e^{i\lambda z}\phi^{up}(z,\lambda)$ and $e^{i\lambda z}\phi^{dn}(z,\lambda)$ which are analytic in $\lambda \in \mathbb{C}^+$ and continuous in $\lambda \in$*

$\mathbb{C}^+ \cup \mathbb{R}$ *(see Proposition 2.1) have a finite limit as* $\lambda \to \infty$ *from within the closure of* $\mathbb{C}^+$. *Analogously, for each* $z \in \mathbb{R}$ *the Faddeev functions* $e^{i\lambda z}\overline{\psi}^{up}(z,\lambda), e^{i\lambda z}\overline{\psi}^{dn}(z,\lambda), e^{-i\lambda z}\overline{\phi}^{up}(z,\lambda)$ *and* $e^{-i\lambda z}\overline{\phi}^{dn}(z,\lambda)$ *which are analytic in* $\lambda \in \mathbb{C}^-$ *and continuous in* $\lambda \in \mathbb{C}^- \cup \mathbb{R}$ *(see Proposition 2.1) admit a finite limit as* $\lambda \to \infty$ *from within the closure of* $\mathbb{C}^-$. *Moreover, the coefficient* $a(\lambda)$ *has a finite limit when* $\lambda \to \infty$ *from within* $\overline{\mathbb{C}}^+$ *while* $b(\lambda)$ *may not admit analytical continuation outside the real line and* $b(\lambda) \to 0$ *when* $\lambda \to \pm\infty$.

2.2 Scattering Data

In this subsection, for the sake of completeness, we introduce the scattering matrix and the scattering coefficients.

From now on, we assume that the coefficients $a(\lambda)$ introduced in the preceding section is such that $a(\lambda) \neq 0$ for each $\lambda \in \mathbb{R}$, i.e., there are no spectral singularities. We can write the identity (2.4) as the following Riemann-Hilbert problems:

$$\begin{pmatrix}\overline{\phi}(z,\lambda) & \overline{\psi}(z,\lambda)\end{pmatrix} = \begin{pmatrix}\psi(z,\lambda) & \phi(z,\lambda)\end{pmatrix}\begin{pmatrix}\frac{1}{a(\lambda)} & -\frac{b(\lambda)}{a(\lambda)} \\ -\frac{b(\lambda)^*}{a(\lambda)} & \frac{1}{a(\lambda)}\end{pmatrix}, \tag{2.16a}$$

$$\begin{pmatrix}\psi(z,\lambda) & \phi(z,\lambda)\end{pmatrix} = \begin{pmatrix}\overline{\phi}(z,\lambda) & \overline{\psi}(z,\lambda)\end{pmatrix}\begin{pmatrix}\frac{1}{a(\lambda)^*} & \frac{b(\lambda)}{a(\lambda)^*} \\ \frac{b(\lambda)^*}{a(\lambda)^*} & \frac{1}{a(\lambda)^*}\end{pmatrix}. \tag{2.16b}$$

Putting $\boldsymbol{F}_-(z,\lambda) = \begin{pmatrix}\overline{\phi}(z,\lambda) & \overline{\psi}(z,\lambda)\end{pmatrix}$ and $\boldsymbol{F}_+(z,\lambda) = \begin{pmatrix}\psi(z,\lambda) & \phi(z,\lambda)\end{pmatrix}$, we obtain the *Riemann-Hilbert* problem

$$\boldsymbol{F}_-(z,\lambda) = \boldsymbol{F}_+(z,\lambda)\sigma_3 S(\lambda)\sigma_3, \tag{2.17}$$

where the *scattering matrix* $S(\lambda)$ is defined by

$$S(\lambda) = \begin{pmatrix}T(\lambda) & R(\lambda) \\ L(\lambda) & T(\lambda)\end{pmatrix}.$$

We write

$$T(\lambda) = \frac{1}{a(\lambda)}, \qquad R(\lambda) = \frac{b(\lambda)}{a(\lambda)}, \qquad L(\lambda) = \frac{b(\lambda)^*}{a(\lambda)}, \tag{2.18}$$

where T, R, and L are called the transmission coefficient, the reflection coefficient from the right, and the reflection coefficient from the left, respectively. Equations (2.16) then imply that

$$S(\lambda)^\dagger = \sigma_3 S(\lambda)^{-1}\sigma_3, \qquad \lambda \in \mathbb{R}.$$

Thus $S(\lambda)$ is σ_3-unitary and has determinant $a(\lambda)^*/a(\lambda)$. Also, $S(\lambda) \to e^{-i\alpha} I_2$ as $\lambda \to \pm\infty$ for a suitable complex number $e^{-i\alpha}$ of modulus 1. We easily derive the Fourier representations

$$\boldsymbol{F}_+(z,\lambda)e^{-i\lambda z\sigma_3} = \begin{pmatrix} H_1(z) & \tilde{H}_2(z) \\ H_2(z) & \tilde{H}_1(z) \end{pmatrix} + \int_0^\infty ds\, e^{i\lambda s} \begin{pmatrix} K_1(z,z+s) & N_2(z,z-s) \\ K_2(z,z+s) & N_1(z,z-s) \end{pmatrix}, \tag{2.19a}$$

$$\boldsymbol{F}_-(z,\lambda)e^{-i\lambda z\sigma_3} = \begin{pmatrix} \tilde{H}_1(z)^* & -H_2(z)^* \\ -\tilde{H}_2(z)^* & H_1(z)^* \end{pmatrix} + \int_0^\infty ds\, e^{-i\lambda s} \begin{pmatrix} N_1(z,z-s)^* & -K_2(z,z+s)^* \\ -N_2(z,z-s)^* & K_1(z,z+s)^* \end{pmatrix}, \tag{2.19b}$$

where

$$\int_0^\infty ds\, (|K_1(z,z+s)| + |K_2(z,z+s)| + |N_1(z,z-s)| + |N_2(z,z-s)|)$$

converges uniformly in $z \in \mathbb{R}$.

The **scattering data** associated with the first of equation (1.2) are:

1. one of the reflection coefficients,
2. the poles of the transmission coefficient $T(\lambda)$ (or of $T(\lambda^*)^*$). We call such poles *the discrete eigenvalues* in the upper half-plane $\mathbb{C}^+$ (or in the lower half-plane) and denote them by ia_j (or by $-ia_j^*$) for $j = 1, \ldots, N$.
3. a set of constants N_j $(\overline{N}_j)$ for $j = 1, \ldots, N$ associated to the discrete eigenvalues ia_j $(-ia_j^*)$ $j = 1, \ldots, N$ in the upper half-plane (lower half-plane). These constants are called the *norming constants*.

By using elementary arguments of complex analysis it is possible to prove that if there are no spectral singularities then the number of discrete eigenvalues is finite. We present, for the sake of simplicity, the scattering theory assuming that each pole of the transmission coefficient has multiplicity equal to one (the general case can be treated as shown in [5]). The way to construct the norming constants is standard (see [1, 2, 13]). However, for the sake of completeness, we prefer to insert their construction. Let us assume that there are finitely many poles $ia_1, \ldots, ia_N$ of the transmission coefficient $T(\lambda)$

in the upper half-plane $\mathbb{C}^+$, all of which are assumed to be simple. We let τ_s stand for the residue of $T(\lambda)$ at $\lambda = ia_s$, i.e.,

$$\tau_s = \lim_{\lambda \to ia_s} (\lambda - ia_s) T(\lambda) = \lim_{\lambda \to ia_s} \frac{\lambda - ia_s}{a(\lambda) - a(ia_s)} = \frac{1}{\dot{a}(ia_s)} ,$$

where the dot indicates differentiation with respect to λ. We then introduce the norming constants N_s by

$$\tau_s \phi(z, ia_s) = iN_s \psi(z, ia_s), \qquad s = 1, 2, \ldots, N. \tag{2.20a}$$

By the same token, $T(\lambda^*)^*$ has the simple poles $-ia_1^*, \ldots, -ia_N^*$ in $\mathbb{C}^-$, all of them simple. The corresponding norming constants $\overline{N}_s$ are defined by

$$\tau_s^* \, \overline{\phi}(z, -ia_s^*) = -i\overline{N}_s \overline{\psi}(z, -ia_s^*), \qquad s = 1, 2, \ldots, N. \tag{2.20b}$$

The next proposition shows how the norming constants introduced in the upper half-plane are related to those defined in the lower half-plane.

Proposition 2.6 *The norming constants satisfy the following relations:*

$$\overline{N}_s = -(N_s)^*$$

Proof. By applying the triangular representations to (2.20a) and (2.20b) we get the pair of equalities

$$\begin{aligned}
&\tau_s \left\{ \begin{pmatrix} \tilde{H}_2(z) \\ \tilde{H}_1(z) \end{pmatrix} + \int_0^\infty dw \, e^{-a_s w} \begin{pmatrix} N_2(z, z-w) \\ N_1(z, z-w) \end{pmatrix} \right\} \\
&\quad = iN_s \left\{ \begin{pmatrix} H_1(z) \\ H_2(z) \end{pmatrix} + \int_0^\infty dw \, \begin{pmatrix} K_1(z, z+w) \\ K_2(z, z+w) \end{pmatrix} \right\} , \\
&\tau_s^* \left\{ \begin{pmatrix} \tilde{H}_1(z)^* \\ -\tilde{H}_2(z)^* \end{pmatrix} + \int_0^\infty dw \, e^{-a_s^* w} \begin{pmatrix} N_1(z, z-w)^* \\ -N_2(z, z-w)^* \end{pmatrix} \right\} \\
&\quad = -i\overline{N}_s \left\{ \begin{pmatrix} -H_2(z)^* \\ H_1(z)^* \end{pmatrix} + \int_0^\infty dw \, e^{-a_s^* w} \begin{pmatrix} -K_2(z, z+w)^* \\ K_1(z, z+w)^* \end{pmatrix} \right\} .
\end{aligned}$$

Taking the complex conjugate of the first equation and premultiplying the result by $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \right)$, we obtain the second equation, provided $\overline{N}_s = -(N_s)^*$. □

2.3 Marchenko Equations

In this subsection we formulate the Marchenko integral equations and establish the connection between the solutions of these equations and the solution of the HF equation. We refer the reader to [16, 11, 5] for the details on the derivation of (2.24) below.

In order to derive the Marchenko equations we need the following

Proposition 2.7 *Suppose that hypotheses a and b in Subsection 2.1 hold, and suppose there are no spectral singularities. Then there exist functions ρ and l in $L^1(\mathbb{R})$ such that*

$$R(\lambda) = \int_{-\infty}^{\infty} dw\, e^{-i\lambda w}\rho(w)\,, \quad L(\lambda) = \int_{-\infty}^{\infty} dw\, e^{i\lambda w} l(w)\,. \tag{2.21}$$

We refer to [16, 11] for the proof.

We have the following

Theorem 2.8 *The auxiliary function $\boldsymbol{K}(x,y)$ which appears in (2.13) satisfies the following integral Marchenko equations*

$$\boldsymbol{K}(x,y) + \boldsymbol{H}(z)\boldsymbol{\Omega}(z+y) + \int_{z}^{\infty} du\, \boldsymbol{K}(z,u)\boldsymbol{\Omega}(u+y) = 0_{2\times 2}, \tag{2.22}$$

where

$$\boldsymbol{\Omega}(w) = \begin{pmatrix} 0 & \omega(w) \\ -\omega(w)^* & 0 \end{pmatrix}, \quad \text{with } \omega(w) = \rho(w) + \sum_s e^{-a_s w} N_s\,, \tag{2.23}$$

and $\rho(w)$ is the Fourier transform of the reflection coefficient (see (2.21)).

Recall that $\boldsymbol{H}(z) \in SU(2)$. Writing $\boldsymbol{K}(z,y) = \boldsymbol{H}(z)\boldsymbol{L}(z,y)$ we can convert (2.22) into the traditional Marchenko integral equation. By following the same proof as in the focusing AKNS case [5, 3], we find that the integral equation

$$\boldsymbol{L}(z,y) + \boldsymbol{\Omega}(z+y) + \int_{z}^{\infty} du\, \boldsymbol{L}(z,u)\boldsymbol{\Omega}(u+y) = 0_{2\times 2}, \tag{2.24}$$

is uniquely solvable on the space $L^1(z,+\infty)^{2\times 2}$. We observe that analogous Marchenko equations are satisfied by the auxiliary function $\boldsymbol{N}(x,y)$ which appear in (2.14). More precisely, we have the following

Theorem 2.9 *The auxiliary function $\boldsymbol{N}(x,y)$ which appears in* (2.14) *satisfies the following integral Marchenko equations*

$$\boldsymbol{N}(x,y)+\tilde{\boldsymbol{H}}(z)\tilde{\boldsymbol{\Omega}}(z+y)+\int_{-\infty}^{z}du\,\boldsymbol{N}(z,u)\tilde{\boldsymbol{\Omega}}(u+y)=0_{2\times 2}, \tag{2.25}$$

where

$$\tilde{\boldsymbol{\Omega}}(w)=\begin{pmatrix}0 & \tilde{\omega}(w)\\ -\tilde{\omega}(w)^{*} & 0\end{pmatrix},\quad con\ \tilde{\omega}(w)=l(w)+\sum_{s=1}^{n}e^{-a_s w}\overline{N}_s\,, \tag{2.26}$$

and $l(w)$ is the Fourier transform of the reflection coefficient (see (2.21)*).*

Setting $\boldsymbol{N}(z,y)=\tilde{\boldsymbol{H}}(z)\boldsymbol{P}(z,y)$, we can convert (2.25) in the following integral Marchenko equations

$$\boldsymbol{P}(z,y)+\tilde{\boldsymbol{\Omega}}(z+y)+\int_{-\infty}^{z}du\boldsymbol{P}(z,y)\tilde{\boldsymbol{\Omega}}(u+y)=0_{2\times 2}\,. \tag{2.27}$$

From now on we get our results working with the equations (2.22) and (2.24). We remark that similar results can be established starting with equations (2.25) and (2.27).

For later convenience let us introduce the following notations

$$\tilde{\boldsymbol{K}}(z)=\int_{z}^{\infty}dy\,\boldsymbol{K}(z,y)\,,\quad \tilde{\boldsymbol{L}}(z)=\int_{z}^{\infty}dy\,\boldsymbol{L}(z,y)\,, \tag{2.28}$$

where $\boldsymbol{K}(z,y)$ and $\boldsymbol{L}(z,y)$ satisfy the Marchenko integral equations (2.22) and (2.24), respectively. Using the Volterra equation (2.5a) and the asymptotic relation $\Psi(z,\lambda)e^{-i\lambda z\sigma_3}\to I_2$ as $z\to+\infty$, we get from the triangular representation (2.13)

$$I_2=\Psi(z,0)=\boldsymbol{H}(z)+\tilde{\boldsymbol{K}}(z)=\boldsymbol{H}(z)\,\left[I_2+\tilde{\boldsymbol{L}}(z)\right], \tag{2.29}$$

where $\tilde{\boldsymbol{L}}(z)=\boldsymbol{H}^{-1}(z)\tilde{\boldsymbol{K}}(z)$.

The relationship between the Marchenko integral equation and the solution of equation (1.1) is immediately clarified by the following

Proposition 2.10 *Under the hypoteses a and b of Subsection 2.1, we have:*

$$\boldsymbol{m}(z)\cdot\boldsymbol{\sigma}=\boldsymbol{H}(z)\,\sigma_3\,\boldsymbol{H}(z)^{-1}=\left[I_2+\tilde{\boldsymbol{L}}(z)\right]^{-1}\,\sigma_3\,\left[I_2+\tilde{\boldsymbol{L}}(z)\right]. \tag{2.30}$$

We refer the reader to [16, 11] for the proof.

2.4 Time Evolution of the Scattering Data

In this subsection we derive the time evolution of the scattering data. We shall arrive at the same time evolution as for the NLS equation.

Recall the AKNS pair is given by (1.2). Suppose $V(z,t;\lambda)$ is a nonsingular 2×2 matrix function satisfying

$$V_z = AV, \qquad V_t = BV,$$

where V does not need to be one of the Jost matrices. Then there exist invertible matrices C_Ψ and C_Φ, depending on (t,λ) but not on z, such that $\Psi = VC_\Psi^{-1}$ and $\Phi = VC_\Phi^{-1}$. Thus

$$\begin{aligned}\Psi_t &= V_tC_\Psi^{-1} - VC_\Psi^{-1}[C_\Psi]_tC_\Psi^{-1} = BVC_\Psi^{-1} - VC_\Psi^{-1}[C_\Psi]_tC_\Psi^{-1}\\ &= B\Psi - \Psi[C_\Psi]_tC_\Psi^{-1},\end{aligned}$$

implying

$$[C_\Psi]_tC_\Psi^{-1} = \Psi^{-1}B\Psi - \Psi^{-1}\Psi_t. \tag{2.31}$$

Here the left-hand side does not depend on z, whereas the right-hand side only seemingly depends on z. We may therefore allow z to tend to $+\infty$ without losing the validity of (2.31). Since $B \simeq -2i\lambda^2\sigma_3$ and $\Psi \simeq e^{i\lambda z\sigma_3}$ as $z\to+\infty$, we obtain

$$[C_\Psi]_tC_\Psi^{-1} = -2i\lambda^2\sigma_3. \tag{2.32}$$

The same result can be obtained for the other Jost matrix $\Phi(z,\lambda)$. Using that $\Psi = \Phi T$, we get

$$\begin{aligned}T_t &= (\Phi^{-1}\Psi)_t = \Phi^{-1}\Psi_t - \Phi^{-1}\Phi_t\Phi^{-1}\Psi\\ &= \Phi^{-1}(B\Psi - \Psi[C_\Psi]_tC_\Psi^{-1}) - \Phi^{-1}(B\Phi - \Phi[C_\Phi]_tC_\Phi^{-1})\Phi^{-1}\Psi\\ &= \Phi^{-1}B\Psi - T[C_\Psi]_tC_\Psi^{-1} - \Phi^{-1}B\Phi + [C_\Phi]_tC_\Phi^{-1}T\\ &= 2i\lambda^2\left\{T\sigma_3 - \sigma_3T\right\},\end{aligned}$$

so that

$$T(\lambda,t) = e^{-2i\lambda^2t\sigma_3}T(\lambda,0)e^{2i\lambda^2t\sigma_3}. \tag{2.33}$$

Hence, $a(\lambda)$ and $T(\lambda)$ do not depend on t, whereas

$$R(\lambda,t) = e^{-4i\lambda^2t}R(\lambda,0), \qquad L(\lambda,t) = e^{4i\lambda^2t}L(\lambda,0). \tag{2.34}$$

Differentiating (2.20a) with respect to t, we obtain

$$\tau_s\phi_t(z,ia_s) = iN_s\psi_t(z,ia_s) + i[N_s]_t\psi(z,ia_s).$$

Using (2.31) for Φ and Ψ and using (2.32), we get

$$\tau_s \left\{ B(ia_s)\phi(z, ia_s) - 2ia_s^2\phi(z, ia_s) \right\}$$
$$= iN_s \left\{ B(ia_s)\psi(z, ia_s) + 2ia_s^2\psi(z, ia_s) \right\} + i[N_s]_t\psi(z, ia_s).$$

Using (2.20a) again we obtain

$$[N_s]_t = -4ia_s^2 N_s.$$

Taking into account that $\overline{N}_s = -[N_s]^*$ (see Proposition 2.6), finally we get

$$N_s(t) = e^{-4ia_s^2 t} N_s(0), \qquad \overline{N}_s(t) = e^{4ia_s^{*2} t}\overline{N}(0). \tag{2.35}$$

2.5 Inverse Scattering Transform.

Having presented the *direct scattering problem* (consisting of the construction of the scattering data when $\boldsymbol{m}(z, 0)$ is known), the *inverse scattering problem* (amounting to the construction of $\boldsymbol{m}(z)$ when the scattering data are given), and the *time evolution of the scattering data* associated to the first of equation (1.2), we can discuss how the IST allows us to obtain the solution to the initial value problem for (1.1).

Using the initial condition $\boldsymbol{m}(z, 0)$ as a potential in system (1.3), we develop the direct scattering theory as shown above and build the scattering data. Successively, let the initial scattering data evolve in time in agreement with equation (2.33)-(2.35). The solution of the Heisenberg equation is then obtained by solving the Marchenko equation (2.24) where the kernel $\boldsymbol{\Omega}(w)$ is replaced by $\boldsymbol{\Omega}(w;t)$ (i.e., taking into account (2.33), (2.34), and (2.35)), and then using relation (2.30).

Gauge transformation. In this paragraph we show how the gauge transformation (see [14, 17]) between the solutions of the Heisenberg equation and the solutions of the Nonlinear Schrödinger equation is determined by the matrix $\boldsymbol{H}(z)$ ($\tilde{\boldsymbol{H}}(z)$) introduced in the triangular representation (2.13) ((2.14)).

Let us denote by $\Psi_{ZS}(z, \lambda), \Phi_{ZS}(z, \lambda)$ the Jost matrices of the Zakharov-Shabat system, and by $\Psi(z, \lambda), \Phi(z, \lambda)$ the Jost matrices of the scattering problem (1.2) associated to the HF equation, both corresponding to the same scattering data. We have already established the following relations: $\boldsymbol{H}(z) = \left[I_2 + \int_z^\infty dw \boldsymbol{L}(z, w)\right]^{-1}$ (see (2.28) and (2.29)) and $\tilde{\boldsymbol{H}}(z) = \left[I_2 + \int_z^\infty dw \boldsymbol{N}(z, w)\right]^{-1}$ (see (2.14), (2.15), (2.28) and (2.29)).

Then we have the following:

Theorem 2.11 *The solution of the initial value problem* (1.3) *are expressed in terms of the Jost solutions of the Zakharov-Shabat system as:*

$$\boldsymbol{m}(z) \cdot \boldsymbol{\sigma} = \Psi_{ZS}^{-1}(z,0)\sigma_3\Psi_{ZS}(z,0) = \Phi_{ZS}^{-1}(z,0)\sigma_3\Phi_{ZS}(z,0). \tag{2.36}$$

The proof can be found in [16, 11].

3 Matrix Triplet Method

In this section we construct an explicit soliton solution formula for equation (1.1). We apply the same technique successfully used in [6, 7, 8, 9, 10] to solve the NLS, the cmKdV, the sine-Gordon, the discrete NLS, and the Hirota equations, respectively. Furthermore, we use the triplet method to get explicit expressions for the Jost solutions in the reflectionless case when the corresponding scattering data are specified.

3.1 Explicit soliton solutions for equations (1.1).

We want to restrict ourselves to the case $R(\lambda) = 0$. In this case the expression for $\boldsymbol{\Omega}_l(w;0)$ is given by (2.23), with $\rho = 0$. In particular, we can treat the situation where the discrete eigenvalues are not necessarily simple [5] by introducing the function

$$\omega(w) = \sum_{s=1}^{N} \sum_{s=0}^{n_j-1} N_{js} \frac{w^s}{s!}\, e^{-a_s w}\,, \tag{3.1}$$

where (3.1), $\{i\, a_j\}_{j=1}^N$ are the discrete eigenvalues in $\mathbb{C}^+$, $\left\{ \left\{ \{N_{js}(t)\}_{s=0}^{n_j-1} \right\}_{j=1}^N \right\}$ are the corresponding norming constants, and n_j is the geometric multiplicity of $i\, a_j$.

To recover the solution of (1.3) we follow the three steps indicated below:

a. Assume that the discrete eigenvalues $\{ia_j\}_{j=1}^N$ and the norming constants $\left\{ \left\{ \{N_{js}(t)\}_{s=0}^{n_j-1} \right\}_{j=1}^N \right\}$ are given, with N denoting the number of discrete eigenvalues in $\mathbb{C}^+$, and n_j denoting the geometric multiplicity of $i\,a_j$. By using the scattering data we introduce the matrix function

$$\mathbf{\Omega}(w;t) = \begin{pmatrix} 0 & \sum_{s=1}^{N}\sum_{s=0}^{n_j-1} N_{js}(t)\frac{w^s}{s!}\,e^{-a_s w} \\ -\sum_{s=1}^{N}\sum_{s=0}^{n_j-1} N_{js}^*(t)\frac{w^s}{s!}\,e^{-a_s^* w} & 0 \end{pmatrix}, \tag{3.2}$$

where the norming constants evolve in time according to (2.35).

b. Solve the integral Marchenko equation (2.24):

$$\boldsymbol{L}(z,y;t) + \mathbf{\Omega}(z+y;t) + \int_z^\infty du\, \boldsymbol{L}(z,u;t)\mathbf{\Omega}(u+y;t) = 0_{2\times 2}.$$

where $u > z$, and the kernel $\mathbf{\Omega}(z,y)$ is given by (3.2).

c. Construct the potential $\boldsymbol{m}(z;t)$ by using formula(2.30):

$$\boldsymbol{m}(z;t)\cdot\boldsymbol{\sigma} = \left[I_2 + \tilde{\boldsymbol{L}}(z;t)\right]^{-1}\sigma_3\left[I_2 + \tilde{\boldsymbol{L}}(z;t)\right],$$

where $\tilde{\boldsymbol{L}}(z) = \int_z^\infty dy\,\boldsymbol{L}(z,y)$.

An analogous procedure can be followed by using the Marchenko equation (2.27).

Let us follow the above procedure, momentarily disregarding the time dependence. We will then show how to take into account also the time dependence.

It is well known [12, 3] that it is possible to factorize a matrix function in the form (3.2) by using a suitable triplet of matrices. More precisely, suppose $(\mathcal{A},\mathcal{B},\mathcal{C})$ is a triplet of matrices such that all the eigenvalues of the $q\times q$ matrix $\mathcal{A}$ have positive real parts, $\mathcal{B}$ is $2q\times 2$, and $\mathcal{C}$ is $2\times 2q$. Let $\mathbf{\Omega}(w)$ be defined as

$$\mathbf{\Omega}(w) = \begin{pmatrix} 0 & \sum_{s=1}^{N}\sum_{s=0}^{n_j-1} N_{js}\frac{w^s}{s!}\,e^{-a_s w} \\ -\sum_{s=1}^{N}\sum_{s=0}^{n_j-1} N_{js}^*\frac{w^s}{s!}\,e^{-a_s^* w} & 0 \end{pmatrix} \stackrel{\text{def}}{=} \mathcal{C}e^{-w\mathcal{A}}\mathcal{B}. \tag{3.3}$$

Equation (3.3) can be written as (see (2.23))

$$\mathbf{\Omega}(w) = \begin{pmatrix} 0 & \omega(w) \\ -\omega(w)^* & 0 \end{pmatrix},$$

where the scalar function $\omega(w)$ is defined as $\omega(w) = Ce^{-wA}B$, with A a $q \times q$ matrix having only eigenvalues with positive real part, B a $q \times 1$ matrix, and C a $1 \times q$ matrix. Furthermore, let us assume that the triplet (A, B, C) is a minimal triplet in the sense that the matrix order of A is minimal among all triplets representing the same Marchenko kernel by means of (3.3) [12, 3]. We then define

$$\mathcal{A} = \begin{pmatrix} A & 0_{q\times q} \\ 0_{q\times q} & A^\dagger \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} 0_{q\times 1} & B \\ -C^\dagger & 0_{q\times 1} \end{pmatrix}, \quad \mathcal{C} = \begin{pmatrix} C & 0_{1\times q} \\ 0_{1\times q} & B^\dagger \end{pmatrix}. \tag{3.4}$$

Then

$$\mathcal{P} = \int_0^\infty d\hat{z}\, e^{-\hat{z}\mathcal{A}}\mathcal{B}\mathcal{C}e^{-\hat{z}\mathcal{A}} = \begin{pmatrix} 0_{q\times q} & N \\ -Q & 0_{q\times q} \end{pmatrix},$$

where

$$N = \int_0^\infty dz\, e^{-zA}BB^\dagger e^{-zA^\dagger}, \qquad Q = \int_0^\infty dz\, e^{-zA^\dagger}C^\dagger C e^{-zA}.$$

Then $\mathcal{P}$ is the unique solution of the Sylvester equation $\mathcal{A}\mathcal{P} + \mathcal{P}\mathcal{A} = \mathcal{B}\mathcal{C}$. It follows from the minimality of the triplet (A, B, C) [3, Sec. 4.1] that N and Q are positive Hermitian matrices. Thus $\mathcal{P}$ is invertible and

$$\mathcal{P}^{-1} = \begin{pmatrix} 0_{q\times q} & -Q^{-1} \\ N^{-1} & 0_{q\times q} \end{pmatrix}. \tag{3.5}$$

Substituting the expression for the kernel (3.3) into equation (2.24), we arrive at the following Marchenko equation

$$\boldsymbol{L}(z,y) + \mathcal{C}e^{-(z+y)\mathcal{A}}\mathcal{B} + \int_z^\infty du\, \boldsymbol{L}(z,u)\mathcal{C}e^{-(u+y)\mathcal{A}}\mathcal{B} = 0_{2\times 2}. \tag{3.6}$$

Equation (3.6) can be solved explicitly via separation of variables. In fact, looking for a solution of the following form

$$\boldsymbol{L}(z,y) = -\boldsymbol{F}(z)e^{-y\mathcal{A}}\mathcal{B},$$

after some straightforward calculations, we find

$$\boldsymbol{L}(z,y) = -\mathcal{C}e^{-z\mathcal{A}}[I_{2q} + e^{-z\mathcal{A}}\mathcal{P}e^{-z\mathcal{A}}]^{-1}e^{-y\mathcal{A}}\mathcal{B}, \tag{3.7}$$

provided the inverse matrix exists for each $z \in \mathbb{R}$. Actually, the inverse matrix exists thanks to the unique solvability of (3.6). Finally, in order to reconstruct (1.3) we have to integrate (3.7) with respect to y, obtaining the explicit formula

$$\tilde{\boldsymbol{L}}(z) = -\mathcal{C}e^{-z\mathcal{A}}[I_{2q} + e^{-z\mathcal{A}}\mathcal{P}e^{-z\mathcal{A}}]^{-1}e^{-z\mathcal{A}}A^{-1}\mathcal{B}. \tag{3.8}$$

By means of this latter expression we can write the right-hand side of (2.30) in explicit form, and recover the components $m_j(z)$ of the vector $\boldsymbol{m}(z)$.

In order to introduce the time dependence we have to take into account the time evolution of the scattering data expressed by (2.33)-(2.35). Defining the (reflectionless) Marchenko kernels as follows:

$$\omega(w;t) = Ce^{-wA}e^{-4itA^2}B, \qquad \omega^*(w;t) = B^\dagger e^{-wA^\dagger}e^{4itA^{\dagger 2}t}C^\dagger, \tag{3.9}$$

we may replace the matrix triplet (A, B, C) by (A, B, Ce^{-4itA^2}) in such a way that (2.33), (2.34) and (2.35) are satisfied (A contains the discrete eigenvalues which are time independent, and C the norming constants).

As a consequence, the explicit right-hand side of (2.30) is given by:

$$\tilde{\boldsymbol{L}}(z;t) = -\mathcal{C}(t)e^{-z\mathcal{A}}[I_{2q} + e^{-z\mathcal{A}}\mathcal{P}(t)e^{-z\mathcal{A}}]^{-1}e^{-z\mathcal{A}}A^{-1}\mathcal{B}(t), \tag{3.10}$$

where

$$\mathcal{B}(t) = \begin{pmatrix} 0_{q\times 1} & B \\ -\left(Ce^{-4itA^2}\right)^\dagger & 0_{q\times 1} \end{pmatrix}, \quad \mathcal{C}(t) = \begin{pmatrix} Ce^{-4itA^2} & 0_{1\times q} \\ 0_{1\times q} & B^\dagger \end{pmatrix},$$

$$\mathcal{P}(t) = \begin{pmatrix} 0_{q\times q} & N \\ -Q(t) & 0_{q\times q} \end{pmatrix}, \text{ with } Q(t) = \int_0^\infty dz\, e^{-zA^\dagger}\left(Ce^{-4itA^2}\right)^\dagger Ce^{-4itA^2}e^{-zA}.$$

We remark that formula (3.10) allows us to compute explicitly the functions $m_i(z,t)$ satisfying the HF equation (1.1). The following example illustrates this fact in the most elementary case: the one-soliton solution.

Example 3.1 (One-soliton solution.) The easiest example is the one-soliton solution which is obtained by taking $A = (a)$ with $p = \operatorname{Re} a > 0$, $B = (1)$, and $C = (c)$ with $0 \neq c \in \mathbb{C}$. Then $Q = (|c|^2/2p)$, $N = (1/2p)$. Hence, by substituting (3.10) in the RHS of (2.30), after long but straightforward calculations, we get

$$
\begin{aligned}
m_3(z) &= 1 + \frac{|c|^4}{4p^2|a|^2} \frac{e^{-8pz}}{[1+(|c|^2/4p^2)e^{-4pz}]^2} \\
&\quad - \frac{|c|^2}{|a|^2} \frac{e^{-4pz}}{1+(|c|^2/4p^2)e^{-4pz}} - \frac{|c|^2}{|a|^2} \frac{e^{-4pz}}{[1+(|c|^2/4p^2)e^{-4pz}]^2} \\
&= 1 - 2\frac{|c|^2}{|a|^2} \frac{e^{-4pz}}{[1+(|c|^2/4p^2)e^{-4pz}]^2} = 1 - \frac{2p^2/|a|^2}{\cosh^2[2p(z-z_0)]}, \qquad (3.11) \\
m_+(z) &= -2\frac{c^*}{a^*} \frac{e^{-2a^*z}}{1+(|c|^2/4p^2)e^{-4pz}} \left[1 - \frac{|c|^2}{2pa^*} \frac{e^{-4pz}}{1+(|c|^2/4p^2)e^{-4pz}}\right], \qquad (3.12) \\
m_-(z) &= -2\frac{c}{a} \frac{e^{-2az}}{1+(|c|^2/4p^2)e^{-4pz}} \left[1 - \frac{|c|^2}{2pa} \frac{e^{-4pz}}{1+(|c|^2/4p^2)e^{-4pz}}\right], \qquad (3.13)
\end{aligned}
$$

solution is obtained by replacing the matrix triplet $(a, 1, c)$ by $(a, 1, ce^{-4ita^2})$ in (3.11).

3.2 Reconstruction of the Jost solutions.

By using the same notations introduced in the subsection above, let us compute the Jost solution $\Psi(z,\lambda)$ by substituting the solution of the Marchenko equation (2.22) (see formula (3.7)) into (2.13). We get

$$
\begin{aligned}
\Psi(z,\lambda)e^{-i\lambda z\sigma_3} &= \boldsymbol{H}(z) - \boldsymbol{H}(z)\mathcal{C}e^{-z\mathcal{A}}\left[I_{2q} + e^{-z\mathcal{A}}\mathcal{P}e^{-z\mathcal{A}}\right]^{-1} \int_z^\infty dy\, e^{-y\mathcal{A}}\mathcal{B}e^{i\lambda(y-z)\sigma_3} \\
&= \boldsymbol{H}(z) - \boldsymbol{H}(z)\mathcal{C}\left[\mathcal{P} + e^{2z\mathcal{A}}\right]^{-1} \int_0^\infty ds\, e^{-s\mathcal{A}}\mathcal{B}e^{i\lambda s\sigma_3} \\
&= \boldsymbol{H}(z) - \boldsymbol{H}(z)\mathcal{C}\left[\mathcal{P} + e^{2z\mathcal{A}}\right]^{-1} \mathcal{W}(\lambda),
\end{aligned}
$$

where $-\mathcal{A}\mathcal{W}(\lambda) + i\lambda\mathcal{W}(\lambda)\sigma_3 = \mathcal{B}\mathcal{C}$ has a unique solution.[4] Since σ_3 and $\mathcal{B}$ anticommute, and $\mathcal{A}$ and σ_3 commute [see (3.4)], we have

$$
\begin{aligned}
\mathcal{W}(\lambda) &= \int_0^\infty ds\, e^{-s\mathcal{A}}\mathcal{B}e^{i\lambda s\sigma_3} = \int_0^\infty ds\, e^{-s\mathcal{A}}e^{-i\lambda s\sigma_3}\mathcal{B} = \int_0^\infty ds\, e^{-s\mathcal{A}-i\lambda s\sigma_3}\mathcal{B} \\
&= (i\lambda\sigma_3 + \mathcal{A})^{-1}\mathcal{B} = -i\sigma_3(\lambda I_{2q} - i\sigma_3\mathcal{A})^{-1}\mathcal{B} = i(\lambda I_{2q} - i\sigma_3\mathcal{A})^{-1}\mathcal{B}\sigma_3.
\end{aligned}
$$

[4]Note that $\sigma(\mathcal{A}) \cap \{i\lambda, -i\lambda\} = \emptyset$ for $\lambda \in \mathbb{R}$.

Consequently,

$$\begin{aligned}\Psi(z,\lambda)e^{-i\lambda z\sigma_3} &= \boldsymbol{H}(z)\left[I_{2q} - i\mathcal{C}\left[\mathcal{P}+e^{2z\mathcal{A}}\right]^{-1}(\lambda I_{2q}-i\sigma_3\mathcal{A})^{-1}\mathcal{B}\sigma_3\right] \\ &= \boldsymbol{H}(z)\left[I_{2q} - i\mathcal{C}e^{-z\mathcal{A}}\left[I_{2q}+e^{-z\mathcal{A}}\mathcal{P}e^{-z\mathcal{A}}\right]^{-1}e^{-z\mathcal{A}}(\lambda I_{2q}-i\sigma_3\mathcal{A})^{-1}\mathcal{B}\sigma_3\right].\end{aligned}$$

Taking the limit as $z \to -\infty$, we obtain

$$\begin{pmatrix} a(\lambda) & -\lim_{z\to-\infty} e^{2i\lambda z}b(\lambda) \\ \lim_{z\to-\infty} e^{-2i\lambda z}b(\lambda)^* & a(\lambda)^* \end{pmatrix} = I_2 - i\mathcal{C}\mathcal{P}^{-1}(\lambda I_{2q}-i\sigma_3\mathcal{A})^{-1}\mathcal{B}\sigma_3\,,$$

provided $\rho(w)$ is invertible. Equations (3.4) and (3.5) imply that $b(\lambda) = 0$ [reflectionless case]. Moreover, we find the following:

$$a(\lambda) = 1 - iCQ^{-1}(\lambda I_q + iA^\dagger)^{-1}C^\dagger, \tag{3.14a}$$
$$a(\lambda)^* = 1 + iB^\dagger N^{-1}(\lambda I_q - iA)^{-1}B. \tag{3.14b}$$

Taking the complex conjugate transpose, we get the alternative expressions

$$a(\lambda) = 1 - iB^\dagger(\lambda I_q + iA^\dagger)^{-1}N^{-1}B, \tag{3.14c}$$
$$a(\lambda)^* = 1 + iC(\lambda I_q - iA)^{-1}Q^{-1}C^\dagger. \tag{3.14d}$$

Thus $a(\infty) = 1$. Taking determinants in any of (3.14), we get[5]

$$a(\lambda) = \frac{\det(\lambda I_q - iA)}{\det(\lambda I_q + iA^\dagger)}.$$

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[5] Thus the discrete eigenvalues in $\mathbb{C}^+$ are exactly the eigenvalues of iA.

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