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TRANSPORT EQUATION ON A FINITE DOMAIN
II. REDUCTION TO X- AND Y-FUNCTIONS

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In this article the solution of the time-independent linear transport equation in a finite homogeneous and non-multiplying medium is expressed in Chandrasekhar's X- and Y-functions through the solution of two linear systems of equations of finite order. The existence of the X- and Y-functions is proved in general.

INTRODUCTION

Being a continuation of the first part [15] this article contains a rigorous study of the integro-differential equation

$$\mu \frac{\partial\psi}{\partial x}(x,\mu) + \psi(x,\mu) =$$

$$(0.1) \qquad = \int_{-1}^{+1} \left[\frac{1}{2\pi} \int_{0}^{2\pi} \hat{g}(\mu\mu' + \sqrt{1-\mu^2}\,\sqrt{1-\mu'^2}\,\cos\alpha)\,d\alpha\right]\psi(x,\mu')\,d\mu'$$

$$(-1 \le \mu \le +1,\ 0 < x < \tau < +\infty)$$

with boundary conditions

$$(0.2) \qquad \psi(0,\mu) = \varphi(\mu)\ (0 \le \mu \le 1)\ ,\ \psi(\tau,\mu) = \varphi(\mu)\ (-1 \le \mu < 0).$$

This so-called "finite-slab problem" plays an important role in radiative transfer of unpolarized light (cf. [5,22,11]) and in neutron transport with uniform speed (cf. [6]). Given the nonnegative "phase function" $\hat{g} \in L_1[-1,+1]$ and the boundary value function $\varphi \in L_p[-1,+1]$ $(1 \le p < +\infty)$, the problem is to compute the solution ψ of the boundary value problem (0.1)-(0.2). More precisely, introducing the vector $\psi(x)$ in $L_p[-1,+1]$, the operators T and B and the projections P_+ and P_- on $L_p[-1,+1]$ by

(0.2a) $\psi(x)(\mu) = \psi(x,\mu)$, $(Th)(\mu) = \mu h(\mu)$; $(-1\le\mu\le+1, 0<x<\tau)$

(0.3b) $$(Bh)(\mu) = \int_{-1}^{+1}\left[\frac{1}{2\pi}\int_0^{2\pi}\hat{g}(\mu\mu'+\sqrt{1-\mu^2}\sqrt{1-\mu'^2}\cos\alpha)d\alpha\right]h(\mu')d\mu';$$

(0.3c) $$(P_+h)(\mu) = \begin{cases} h(\mu), \mu\ge 0; \\ 0 \quad , \mu<0; \end{cases} \quad (P_-h)(\mu) = \begin{cases} 0 \quad , \mu\ge 0; \\ h(\mu), \mu<0, \end{cases}$$

the problem is to find a vector-valued function $\psi:(0,\tau)\to L_p[-1,+1]$ such that $T\psi$ is strongly differentiable and ψ satisfies the equations

(0.4) $(T\psi)'(x) = -(I-B)\psi(x) \quad (0<x<\tau)$;

(0.5) $\lim_{x\downarrow 0}||P_+\psi(x)-P_+\varphi||_p = 0$, $\lim_{x\uparrow\tau}||P_-\psi(x)-P_-\varphi||_p = 0$.

Instead of (0.5) for $\chi\in L_p[-1,+1]$ one might also consider the more general boundary conditions

(0.6) $\lim_{x\downarrow 0}||TP_+\psi(x)-P_+\chi||_p = 0$, $\lim_{x\uparrow\tau}||TP_-\psi(x)-P_-\chi||_p = 0$.

For p=2 the finite-slab problem was stated in the form (0.4)-(0.5) by Hangelbroek [8]. Assuming that $\hat{g}\in L_r[-1,+1]$ for some $r>1$, is nonnegative and fulfills $c = \int_{-1}^{+1}\hat{g}(t)dt\le 1$, on $L_p[-1,+1]$ $(1\le p<+\infty)$ the boundary value problems (0.4)-(0.5) and (0.4)-(0.6) were proved to have a unique solution (see [14]; for p=2 the problem (0.4)-(0.5) was shown to be well-posed in [12]).

In most practical situations one cuts off the Legendre series expansion of the phase function $\hat{g}$ and confines the description to polynomial phase functions of the form

(0.7) $$\hat{g}(t) = \sum_{n=0}^{N} a_n(n+\tfrac{1}{2})P_n(t) \qquad (-1\le t\le+1),$$

where $P_n(t) = (2^n.n!)^{-1}\left(\frac{d}{dt}\right)^n(t^2-1)^n$ is the usual Legendre polynomial. The constraints on $\hat{g}$ imply that $0\le a_0\le 1$ and $-a_0\le a_n\le a_0$ $(n=1,2,...,N)$. The cases $0\le a_0<1$ and $a_0=1$ are usually called the non-conservative and the conservative case. Astrophysicists are accustomed to write the solution of (0.1)-(0.2) (with $\varphi(\mu)=0$ for $-1\leq\mu<0$) in terms of the reflection and transmission functions S and T (resp. ρ and σ) of Chandrasekhar [5] (resp. Sobolev [22]). Recently symmetries of this problem induced Hovenier [9] to use the so-called exit function instead.

The article [15] and its present continuation aim at a synthesis of the rigorous theory in mathematics ([8,12,13,14], for instance) and the analytic expressions partly derived and partly stipulated by astrophysicists ([5,18,21,10], for instance). In [15] reflection and transmission operators were introduced; in terms of the unique solution of the boundary value problem (0.4)-(0.5) they were defined as follows:

$$\psi(0) = R_{+\tau}P_{+}\varphi+T_{-\tau}P_{-}\varphi \quad , \quad \psi(\tau) = R_{-\tau}P_{-}\varphi+T_{+\tau}P_{+}\varphi.$$

The connection with Sobolev's reflection and transmission functions is given by

$$\begin{aligned} (R_{+\tau}\varphi)(-\mu) &= 2\int_0^1 \nu\, \rho(\nu,\mu)\varphi(\nu)d\nu; \\ (T_{+\tau}\varphi)(\mu)-e^{-\tau/\mu}\varphi(\mu) &= 2\int_0^1 \nu\, \sigma(\nu,\mu)\varphi(\nu)d\nu. \end{aligned} \qquad (0\leq\mu\leq 1)$$

In [15] these operators were expressed in the 2N+2 auxiliary functions $R^{*}_{+\tau}P_n$ and $T^{*}_{+\tau}P_n$ (n=0,1,...,N) and these functions were related to functions studied in [5,18,21].

In this second part we shall reduce the operators $R_{+\tau}$ and $T_{+\tau}$ further by expressing $R^{*}_{+\tau}P_n$ and $T^{*}_{+\tau}P_n$ in X- and Y-functions through a pair of polynomials. For the isotropic case (N=0) the X- and Y-functions were introduced by Ambartsumian [1] and generalized to the anisotropic case by Chandrasekhar [5]. For nonnegative characteristic functions $\psi(\mu)$ their existence was established by Busbridge [2] and constraints on the equations they satisfy were derived by Mullikin [16,17] (also [3]). Inspired by partial results of Chandrasekhar [5] (for $N\leq 2$) and Mullikin [18] Sobolev [21] accomplished a complete reduction of the reflection and transmission functions to X- and Y-functions. Hovenier [10] exploited the exit function to get formulas more expedient than the ones of Sobolev [21]. In the non-conservative case the polynomials appearing in the reduction formulas ([21,10]) are commonly believed to be uniquely specified by the equations given for them.

In this article we construct the physically relevant solutions X and Y of Chandrasekhar's X- and Y- equations by setting

$$X(\mu) \pm Y(\mu) = [(R^*_{+\tau} \pm T^*_{+\tau}J)p^{\pm}](\mu),$$

where $p^{\pm}$ is some polynomial of degree $\leq N$ and $(Jp)(\mu) = p(-\mu)$, and derive reduction formulas of the type

(0.8a) $$(R^*_{+\tau}P_n)(\mu) = q_n(\mu)X(\mu)+(-1)^n s_n(-\mu)Y(\mu);$$

$(0\leq\mu\leq 1, n=0,1,\ldots,N)$

(0.8b) $$(T^*_{+\tau}P_n)(\mu) = s_n(\mu)X(\mu)+(-1)^n q_n(-\mu)Y(\mu),$$

where q_n and s_n are polynomials of degree $\leq N$. Up to notation these formulas were stipulated by Sobolev [21]. We exploit the Hölder continuity of the functions $R^*_{+\tau}P_n$ and $T^*_{+\tau}P_n$ on [0,1] (established in [15]) to construct their analytic continuations and these continuations in turn enable us to prove the existence of unique polynomials q_n and s_n such that (0.8) holds true. Further, we derive linear equations for the linear combinations $q_n+(-1)^n s_n$ and $q_n-(-1)s_n$; these equations were found by Hovenier [10] by decoupling related equations due to Sobolev [21]. Here we study the invertibility properties of these linear equations in detail and in the conservative case $a_0=1$ this analysis will produce additional constraints on the polynomials $q_n\pm(-1)^n s_n$.

This article draws back on [15], but it is of a less operator-theoretical nature. The first section is devoted to the analytic continuation of $R^*_{+\tau}P_n$ and $T^*_{+\tau}P_n$ and some of its consequences. The existence of the X- and Y-functions and their connection to solutions of a convolution equation make up the contents of Section 2. In Section 3 the representations (0.8) are deduced. A detailed study of the polynomial $t_n^{\pm}=q_n\pm(-1)^n s_n$ follows in Section 4.

We conclude the introduction with notational remarks. By J we denote the "inversion symmetry" $(Jh)(\mu) = h(-\mu)$, by $\mathbb{C}_\infty$ the Riemann sphere $\mathbb{C}\cup\{\infty\}$ and by P_n the usual Legendre polynomial (so that $P_n(1)=1$). The degree of a polynomial p is written as deg p; deg 0 = -1. All Hilbert and Banach spaces will be complex and <.,.> is the usual inner product on $L_2[-1,+1]$. The algebra of bounded linear operators on the Banach space H is written as $L(H)$ and its unit element as I_H (or I). The spectrum, null space and range of an operator T are denoted by $\sigma(T)$, Ker T and ImT, respectively.

1. ANALYTIC CONTINUATION

In this section for phase functions of the form (0.7) we prove the following analytic continuation result and some corollaries.

THEOREM 1.1. *Let* $0\le a_0\le 1$ *and* $-a_n\le a_0\le a_n$ $(n=1,2,\ldots,N)$. *Then for every polynomial* p *the functions* $R^*_{+\tau}p$ *and* $T^*_{+\tau}Jp$ *on* [0,1] *can be extended to functions analytic on* $\mathbb{C}\setminus\{0\}$, *uniformly Hölder continuous on bounded parts of the closed right half-plane and satisfying the following identities*:

$$(1.1a)\qquad \lim_{\mu\downarrow 0}(R^*_{+\tau}p)(\mu) = p(0)\quad ,\quad \lim_{\mu\downarrow 0}(T^*_{+\tau}Jp)(\mu) = 0;$$

$$(1.1b)\qquad (R^*_{+\tau}p)(-\mu) = e^{+\tau/\mu}(T^*_{+\tau}Jp)(\mu)\qquad (0\neq\mu\in\mathbb{C}).$$

Proof. We recall the definitions of the polynomials $H_0, H_1, H_2,\ldots$, the characteristic binomial $\psi(\nu,\mu)$, the dispersion function $\Lambda(\lambda)$ and the function $\lambda(\nu)$ (cf.[15],(4.1)-(4.3)), various symmetry relations ([15],(4.4)), the limit relationship (4.6) of [15], the absence of common zeros of $\lambda(\nu)$ and $\psi(\nu)$ on (-1,+1) and the non-vanishing of the limit of $\Lambda(\lambda)$ as $\lambda\to\pm 1$ ([15], Proposition 4.1). These results will be used in the proof.

According to Theorem 5.1 of [15] there exists a right invertible operator $F^+ : L_2[-1,+1]\to L_2(N)_\sigma$, with $N = [-1,+1]\cup\{\nu\notin[-1,+1]: \Lambda(\nu) = 0\}$ and σ a finite Borel measure on N, such that

$$(1.2) \qquad (F^+P_n)(\nu) = H_n(\nu) \quad (\nu\in N,\ n=0,1,2,\dots).$$

In Section 1 of [15] a spectral decomposition of AT^{-1} was presented, where $A = I - B$; in terms of related concepts we have the diagonalization properties

$$(1.3a) \qquad (F^+e^{-\tau AT^{-1}}P_p^*h)(\nu) = \begin{cases} e^{-\tau/\nu}(F^+h)(\nu), & \nu\in N\cup(0,+\infty); \\ 0 & ,\nu\in N\cup(-\infty,0); \end{cases}$$

$$(1.3b) \qquad (F^+e^{+\tau AT^{-1}}P_m^*h)(\nu) = \begin{cases} 0 & ,\nu\in N\cup(0,+\infty); \\ e^{+\tau/\nu}(F^+h)(\nu), & \nu\in N\cup(-\infty,0); \end{cases}$$

$$(1.3c) \qquad (F^+P_0^*h)(\nu)\equiv 0 \ , \quad (F^+(I-\tau AT^{-1})P_0^*h)(\nu)\equiv 0 \ ; \ \nu\in N.$$

These identities are immediate from the diagonalization

$$(1.3d) \qquad (F^+h)(\nu)\equiv 0 \ (h\in \mathrm{Im}P_0^*),(F^+S^+h)(\nu)=\nu(F^+h)(\nu) \ (h\in \mathrm{Ker}P_0^*),$$

where S^+ is the unique bounded operator on $\mathrm{Ker}P_0^*$ such that $TP_0 + S^+A(I-P_0) = T$ ([15]; Th.5.1 and Eq.(5.4), also the definition of S^+ in Section 1). In terms of the inversion symmetry $(Jh)(\mu) = h(-\mu)$ we have

$$(1.3e) \qquad (F^+Jh)(\nu) = (F^+h)(-\nu) \qquad (\nu\in N).$$

Let us recall how the reflection and transmission operators are defined ([15],(2.1),(2.2),(2.6)). For every $p\in L_2[-1,+1]$ formulas (1.3a)-(1.3c) imply that

$$(F^+R_{+\tau}^*p)(\nu) = (F^+(I-R_{-\tau}^+)p)(\nu) = (F^+p)(\nu) - $$
$$-e^{-\tau/\nu}(F^+T_{-\tau}^+p)(\nu) = (F^+p)(\nu)-e^{-\tau/\nu}(F^+T_{-\tau}^*p)(\nu);$$

$$(F^+T^*_{+\tau}Jp)(\nu) = (F^+T^+_{+\tau}Jp)(\nu) = e^{-\tau/\nu}(F^+R^+_{+\tau}Jp)(\nu) =$$
$$= e^{-\tau/\nu}(F^+(I-R^*_{-\tau})Jp)(\nu) = e^{-\tau/\nu}(F^+Jp)(\nu)-e^{-\tau/\nu}(F^+R^*_{-\tau}Jp)(\nu).$$

Applying Eq.(3.2b) of [15] and (1.3e) we get

$$(F^+R^*_{+\tau}p)(\nu) = (F^+p)(\nu) \qquad -e^{-\tau/\nu}(F^+T^*_{+\tau}Jp)(-\nu);$$
$$(F^+T^*_{+\tau}Jp)(\nu)= e^{-\tau/\nu}(F^+p)(-\nu)-e^{-\tau/\nu}(F^+R^*_{+\tau}p)(-\nu).$$

Adding and subtracting these equations and abbreviating $\Gamma^\pm := R^*_{+\tau}\pm T^*_{+\tau}J$ we obtain

$$(1.4) \qquad (F^+\Gamma^\pm p)(\nu)\pm e^{-\tau/\nu}(F^+\Gamma^\pm p)(-\nu) = (F^+p)(\nu)\pm e^{-\tau/\nu}(F^+p)(-\nu).$$

Observe that $\Gamma^\pm p = R^*_{+\tau}p\pm T^*_{+\tau}Jp\in H_+ := L_2[0,1]$ (i.e., $(\Gamma^\pm p)(\nu) = 0$ for $\nu\in[-1,0)$). So for $\nu\in N\cup(0,+\infty)$ the substitution of an expression for F^+ (i.e., Eq.(5.1) of [15]) into (1.4) yields

$$(1.5) \qquad \begin{aligned} &\lambda(\nu)(\Gamma^\pm p)(\nu) - \oint_0^1 \nu(\mu-\nu)^{-1}\psi(\nu,\mu)(\Gamma^\pm p)(\mu)d\mu\pm \\ &\pm e^{-\tau/\nu}\int_0^1 \nu(\nu+\mu)^{-1}\psi(-\nu,\mu)(\Gamma^\pm p)(\mu)d\mu=(F^+p)(\nu)\pm e^{-\tau/\nu}(F^+p)(-\nu), \end{aligned}$$

where $0<\nu\leq 1$ or $\nu>1$ with $\Lambda(\nu)=0$. If $p=P_n$ is a Legendre polynomial, then (1.2) yields that $(F^+p)(\nu)\pm e^{-\tau/\nu}(F^+p)(-\nu) = [1\pm(-1)^n e^{-\tau/\nu}]H_n(\nu)$. Formula (1.5) will be crucial to the remaining part of this article.

Let us introduce the function $\Delta^\pm p$ implicitly by

$$(1.6) \qquad \begin{aligned} &\Lambda(\lambda)(\Delta^\pm p)(\lambda)-\int_0^1 \lambda(\mu-\lambda)^{-1}\psi(\lambda,\mu)(\Gamma^\pm p)(\mu)d\mu\pm \\ &\pm e^{-\tau/\lambda}\int_0^1 \lambda(\lambda+\mu)^{-1}\psi(-\lambda,\mu)(\Gamma^\pm p)(\mu)d\mu=(F^+p)(\lambda)\pm e^{-\tau/\lambda}(F^+p)(-\lambda), \end{aligned}$$

where p (and thus F^+p) is a polynomial. This equation defines $(\Delta^\pm p)(\lambda)$ uniquely for $\lambda\notin[-1,+1]$ as a meromorphic function whose poles could only be zeros of $\Lambda(\lambda)$. Because of Corollary 5.3 of [15] the function $\Gamma^\pm p=R^*_{+\tau}p\pm T^*_{+\tau}Jp$ is Hölder continuous on [0,1] of exponent $0<\alpha<1$ (i.e., $|\mu-\nu|^{-\alpha}|(\Gamma^\pm p)(\mu)-(\Gamma^\pm p)(\nu)|$ has a finite supremum for $0\leq\mu\neq\nu\leq 1$). The Hölder continuity will be exploited to prove that $\Delta^{\pm}p$ is the analytic continuation of $\Gamma^{\pm}p$ to $\mathbb{C}\setminus\{0\}$.

Clearly, $\Delta^{\pm}p$ has its poles within the set of zeros of $\Lambda(\lambda)$. But from (1.5) (applied for $1<\nu<+\infty$ with $\Lambda(\nu)=\lambda(\nu)=0$) it follows that $\Delta(\lambda)(\Delta^{\pm}p)(\lambda)\to 0$ as $\lambda\to\nu$. As $\Lambda(\lambda)$ has simple zeros only (see Section 4 of [15] and the references given there), it follows that $\Delta^{\pm}p$ has an analytic continuation outside the set $[-1,+1]\cup\{\lambda\in(-\infty,-1):\Lambda(\lambda)=0\}$.

Recall that $\Gamma^{\pm}p$ is uniformly Hölder continuous on $[0,1]$ (cf. Corollary 5.3 of [15]). It is well known (Proposition 4.1 of [15] and the references given there) that

$$\lim_{\epsilon\downarrow 0}\Lambda(t\pm i\epsilon) = \lambda(t)\pm i\pi t\psi(t) \neq 0 \quad (-1<t<+1).$$

From (1.6) it is clear that the limits $\lim_{\epsilon\downarrow 0}\Gamma^{+}(t\pm i\epsilon)$ and $\lim_{\epsilon\downarrow 0}\Gamma^{-}(t\pm i\epsilon)$ exist $(-1<t<+1, t\neq 0)$. Further, since obviously $\Delta^{\pm}(\lambda)$ and $\Delta^{\pm}(\bar{\lambda})$ are complex conjugates, the Cauchy-Schwarz reflection principle implies the existence of functions $\alpha^{\pm},\beta^{\pm}:(-1,0)\cup(0,1)\to\mathbb{R}$ such that

$$\text{(1.7a)} \qquad \lim_{\epsilon\downarrow 0}\Gamma^{+}(t\pm i\epsilon) = \alpha^{+}(t)\pm i\beta^{+}(t);$$

$$\text{(1.7b)} \qquad \lim_{\epsilon\downarrow 0}\Gamma^{-}(t\pm i\epsilon) = \alpha^{-}(t)\pm i\beta^{-}(t).$$

To prove that $\alpha^{\pm}(t) = \Gamma^{\pm}(t)$ and $\beta^{\pm}(t) = 0$ $(0<t<1)$, we substitute $\lambda=t+i\epsilon$ and $\lambda=t-i\epsilon$ into (1.6), compute the limits as $\epsilon\downarrow 0$ and add and subtract the resulting equations. Here we make use of the uniform Hölder continuity of $\Gamma^{\pm}p$ in an essential way. We obtain the following linear system of equations:

$$\begin{bmatrix}\lambda(t) & -\pi t\psi(t)\\ \pi t\psi(t) & \lambda(t)\end{bmatrix}\begin{bmatrix}\alpha^{\pm}(t)\\ \beta^{\pm}(t)\end{bmatrix} = \begin{bmatrix} c^{\pm}(t)\\ \pi t\psi(t)(\Gamma^{\pm}p)(t)\end{bmatrix}, 0<t<1,$$

where

$$c^{\pm}(t)=(F^{+}p)(t)\pm e^{-\tau/t}(F^{+}p)(-t)+\fint_0^1 t(\mu-t)^{-1}\psi(t,\mu)(\Gamma^{\pm}p)(\mu)d\mu\mp$$

$$\mp e^{-\tau/t}\int_0^1 t(t+\mu)^{-1}\psi(-t,\mu)(\Gamma^{\pm}p)(\mu)d\mu = \lambda(t)(\Gamma^{\pm}p)(t)$$

(cf.(1.5) with $\nu=t$). As the determinant $\lambda^2(t)+\pi^2t^2\psi(t)^2\neq 0$ (see Proposition 4.1 of [15] and the references given there), the linear system has a unique solution, namely $\alpha^{\pm}(t)=(\Gamma^{\pm}p)(t)$ and $\beta^{\pm}(t)=0$. Hence, $\Delta^{\pm}p$ is the analytic continuation of $\Gamma^{\pm}p$ to the set $\mathbb{C}\setminus\{[-1,0]\cup\{\nu\in(-\infty,-1):\Lambda(\nu)=0\}\}$.

To continue $\Gamma^{\pm}p$ to $\mathbb{C}\setminus\{-1,0,1\}$ analytically, we define $\alpha^{\pm}$ and $\beta^{\pm}$ on $(-1,0)$ (as in (1.7)) and derive in an analogous way the following linear system of equations:

$$\begin{bmatrix}\lambda(t) & -\pi t\psi(t)\\ \pi t\psi(t) & \lambda(t)\end{bmatrix}\begin{bmatrix}\alpha^{\pm}(t)\\ \beta^{\pm}(t)\end{bmatrix}=\begin{bmatrix}d^{\pm}(t)\\ \pm e^{-\tau/t}\pi t\psi(t)(\Gamma^{\pm}p)(-t)\end{bmatrix},0<t<1,$$

where

$$d^{\pm}(t)=(F^{+}p)(t)\pm e^{-\tau/t}(F^{+}p)(-t)-\int_0^1 t(\mu-t)^{-1}\psi(t,\mu)(\Gamma^{\pm}p)(\mu)d\mu\mp$$

$$\mp e^{-\tau/t}\fint_0^1 t(t+\mu)^{-1}\psi(-t,\mu)(\Gamma^{\pm}p)(\mu)d\mu = \pm e^{-\tau/t}\lambda(t)(\Gamma^{\pm}p)(-t).$$

Solving the system we get $\alpha^{\pm}(t) = \pm e^{-\tau/t}(\Gamma^{\pm}p)(-t)$ and $\beta^{\pm}(t)=0$, $-1<t<0$. Hence, the analytic continuation $\Delta^{\pm}p$ of $\Gamma^{\pm}p$ has the property

$$(1.8)\qquad (\Delta^{\pm}p)(\lambda) = \pm e^{-\tau/\lambda}(\Delta^{\pm}p)(-\lambda).$$

So $\Delta^{\pm}p$ does not have poles in the left half-plane and is analytic on $\mathbb{C}\setminus\{-1,0,1\}$.

To show that the singularities of $\Delta^{\pm}p$ at +1 and -1 are removable, one has to distinguish between two cases. In case $\psi(1) = \sum_{n=0}^{N} a_n(n+\frac{1}{2})H_n(1)P_n(1) = 0$, $\Lambda(\lambda)$ has a finite and non-zero limit as $\lambda\to 1$ and $\lambda\notin[0,1]$ (see Proposition 4.1 of [15] and the references given there). Now the right-hand side of (1.6) has a finite limit as $\lambda\to 1$ and $\lambda\notin[0,1]$ (see Eq.(29.4) of [19]), and thus $\Delta^{\pm}p$ tends to a finite limit as $\lambda\to 1$ and $\lambda\notin[0,1]$. Next assume $\psi(1)\neq 0$. If $\Delta^{\pm}p$ would not be analytic at $\lambda=1$, it would have an essential singularity there (note that $(\Delta^{\pm}p)(\lambda)\to(\Gamma^{\pm}p)$ as $\lambda\uparrow 1$). According to the Casorati-Weierstrass theorem, for every $c\in\mathbb{C}$ there would be a path Γ_c in $\mathbb{C}\setminus[0,1]$ such that $|(\Delta^{\pm}p)(\lambda)-c|\to 0$ as $\lambda\to 1$ along Γ_c. From (1.6) it is clear that for some function γ bounded on Γ_c Eq.(1.6) may be written as

$$c\psi(1)\log(\lambda-1)=\psi(1)(\Gamma^{\pm}p)(1)\log(\lambda-1)+\gamma(\lambda) \quad ; \quad \lambda\in\Gamma_c$$

([19], Eq.(29.4)). Here the branch cut of $\log(\lambda-1)$ is chosen to be the half-line $(-\infty,1)$. For $c\neq(\Gamma^{\pm}p)(1)$ a contradiction arises. So in this case too the function $\Delta^{\pm}p$ is analytic at $\lambda=1$. By (1.8) it is analytic at $\lambda=-1$ too.

We now know that for any polynomial p the function $\Gamma^{\pm}p$ has an analytic continuation to $\mathbb{C}\setminus\{0\}$. But $\Gamma^{\pm}p=R^*_{+\tau}p\pm T^*_{+\tau}Jp$. So $R^*_{+\tau}p$ and $T^*_{+\tau}Jp$ have analytic continuations to $\mathbb{C}\setminus\{0\}$ too. Further, (1.8) implies (1.1b).

Finally, if $E\subset\{\lambda:\mathrm{Re}\lambda\geq 0\}$ is bounded, $[0,1]\subset E$ and $\bar{E}\cup\{\nu\in(1,+\infty):\Lambda(\nu)=0\}=\emptyset$, then $\Lambda(\lambda)$ is Hölder continuous and bounded away from zero on $E\setminus[0,1]$. Using this we easily prove that $\Gamma^{\pm}p$ (and thus $R^*_{+\tau}p$ and $T^*_{+\tau}Jp$) are uniformly Hölder continuous on E (cf.(1.6)). This completes the proof.□

COROLLARY 1.2. *Let* $0<a_0\leq 1$ *and* $-a_0\leq a_n\leq a_0$, *and put* $m=\max\{n:a_{n-1}=1\}$ *for* $a_0=1$ *and* $m=0$ *for* $0<a_0<1$. *Let* $s=m$ *for even* m *and* $s=m+1$ *for odd* m. *Then the following identities are equivalent:*

(i) $R^*_{+\tau}p+T^*_{+\tau}q=0$;

(ii) $R_{+\tau}p+T_{+\tau}q$ *has an analytic continuation to a neighbourhood of* $\lambda=0$;

(iii) *there exists* $h_0 \in \mathrm{span}\{P_0,P_1,\ldots P_{s-1}\}$ *such that* $p=Th_0$ *and* $q=-(\tau A+T)h$, *where* $A=I-B$ *and* T *and* B *are given by* (0.3).

Here p *and* q *are polynomials. In particular, if* $0<a<1$, *there is a one-to-one correspondence between pairs of polynomials* p, q *and functions* $R^*_{+\tau}+T^*_{+\tau}q$.

PROOF. (i)⇒(ii) Trivial.

(iii)⇒(i) Let $p=Th_0$ and $q=-(\tau A+T)h_0$ for some $h_0 \in \mathrm{span}\,\{P_0,P_1,\ldots,P_{s-1}\}$. From Proposition 4.2 of [15] it appears that $\mathrm{span}\,\{P_0,P_1,\ldots P_{s-1}\}$ is the "singular subspace" H_0 connected to the spectrum of $T^{-1}A$ at $\lambda=0$. Using the definitions of $R_{+\tau}$ and $T_{+\tau}$ (i.e.,(2.2a)-(2.2b) in [15]) and the orthogonality properties (1.6a)-(1.6b) in [15] we obtain

$$R^*_{+\tau}p+T^*_{+\tau}q=P_+\left(V^*_\tau\right)^{-1}\left\{\left(U^\tau_p\right)^*Th_0-\left(U^\tau_m\right)^*\left(\tau A+T\right)h_0\right\}=$$

$$=P_+\left(V^*_\tau\right)^{-1}\left\{Th_0-\left(I-\tau AT^{-1}\right)\left(\tau A+T\right)h_0\right\}=0,$$

where we have used Proposition III 3.2 of [12].

(ii)⇒(iii) If $R^*_{+\tau}+T^*_{+\tau}q$ has an analytic continuation at $\lambda=0$, it is an entire function (see Theorem 1.1). Since $\psi(\nu,\mu)$ is a binomial in ν and μ, F^+p is a polynomial whenever p is a polynomial, and $\Lambda(\lambda)$ has a zero at infinity of order s (see Section 4 of [15]), formula (1.6) implies that

$$(1.9)\qquad \left(\Delta^{\pm}p\right)\left(\lambda\right) = O\left(\lambda^{\max(N,\deg p)}\right)\ (\lambda\to\infty).$$

Hence, $R^*_{+\tau}p+T^*_{+\tau}q = \frac{1}{2}\Gamma^+(p+Jq)+\frac{1}{2}\Gamma^-(p-Jq)$ is a polynomial of degree at most max (N,deg p, deg q).

As derived at the beginning of the proof of Theorem 1.1,

$$(F^+R^*_{+\tau}p)(\nu) = (F^+p)(\nu)-e^{-\tau/\nu}(F^+T^*_{-\tau}p)(\nu);$$
$$(\nu\in[-1,+1]\cup\{\mu\notin[-1,+1]:\Lambda(\mu)=0\})$$
$$(F^+T^*_{+\tau}q)(\nu) = e^{-\tau/\nu}\{(F^+q)(\nu)-(F^+R^*_{-\tau}q)(\nu)\}$$

So $(F^+(q-R^*_{-\tau}q-T^*_{-\tau}p))(\tau) = e^{+\tau/\nu}r(\nu)$ for some polynomial r. But by Corollary 5.3 of [15] the left-hand side is Hölder continuous on [-1,+1] except for a jump at $\nu=0$, and therefore it is bounded in a neighbourhood of $\nu=0$. So $r(\nu)\equiv 0$, and thus

$$(I-R^*_{-\tau})q-T^*_{-\tau}p\in \operatorname{Ker}F^+ = \operatorname{span}\{TP_0,TP_1,\ldots,TP_{s-1}\}$$

(cf.[15], Theorem 5.1). Therefore, there exists a unique $k_0\in H_0 = \operatorname{span}\{P_0,P_1,\ldots,P_{s-1}\}$ such that $(I-R^*_{-\tau})q-T^*_{-\tau}p=Tk_0$. Lemma 2.1 of [15] implies that

$$R^+_{+\tau}q-T^+_{-\tau}p = Tk_0.$$

Substitute Eq.(2.1) of [15], premultiply by P^*_p and P^*_m and conclude that

$$(V^+_\tau)^{-1}(P_+q-P_-p) = Tk_0.$$

But $TV_\tau=V^+_\tau T$ ([15],(1.10)). As p and q are polynomials, one has $p=Tp_0$ and $q=Tq_0$ for certain $p_0,q_0 \in \operatorname{span}\{P_0,P_1,\ldots,P_{s-1}\}$. Note that $P_+q_0-P_-p_0=V_\tau k_0=P_+k_0 + P_-(I-\tau T^{-1}A)\, k_0$ ([15],(1.8b)). So $P_+q_0=P_+k_0$ and $-P_-p_0=P_-(I-\tau T^{-1}A)k_0$. As these equations concern polynomials, we conclude that $q_0=k_0$ and $-p_0=(I-\tau T^{-1}A)k_0$. Put $h_0=-(I-\tau T^{-1}A)k_0$. Then $p=Th_0$ and $-(\tau A+T)h_0=(\tau A+T)(I-\tau T^{-1}A)k_0$

$=\tau Ak_0+Tk_0-\tau Ak_0-\tau^2AT^{-1}Ak_0=Tk_0=\dot{q}$, because $(T^{-1}A)^2k_0=0$ ([12], Proposition III3.2).□

From the corollary it follows that $R^*_{+\tau}p$ and $T^*_{+\tau}Jp$ have an essential singularity at $\lambda=0$ whenever $p\neq 0$. It is more complicated to find all polynomials p such that $\Gamma^{\pm}p=R^*_{+\tau}p\pm T^*_{+\tau}Jp=0$. Such a polynomial p has the form $p=Th_0$ with deg $h_0\leq s-1$ and $\pm Jp=-(\tau A+T)h_0$ (thus $(I\pm J)p=-\tau Ah_0$). If $m=\max\{n:a_{n-1}=1\}$ is even, then $s=m$ and $T^{-1}Ah_0=0$ ([15], Proposition 4.2), and thus $(I\pm J)p=0$ (i.e., p is an odd resp. even polynomial; thus $h_0=0$ is an even resp. odd polynomial). If $m=\max\{n:a_{n-1}=1\}$ is odd, then $T^{-1}Ah_0\in\operatorname{span}\{T^{-1}P_m\}$ which is a set of even polynomials (cf.[15], Proposition 4.2). Then $\Gamma^+p=0$ and $p=Th_0$ imply that the even polynomial $(I+J)p\in\operatorname{span}\{P_m\}$, and thus $(I+J)p=0$ (i.e., p is odd and therefore h_0 is even). On the contrary, for m odd $\Gamma^-p=0$ and $p=Th_0$ imply that the odd polynomial $(I-J)p\in\operatorname{span}\{P_m\}$. As deg $h_0\leq s-1=m$, we get $h_0=\frac{1}{2}(I-J)h_0+\frac{1}{2}(I+J)h_0\in\operatorname{span}\{P_1,P_3,\ldots,P_m\}\oplus\operatorname{span}\{T^{-1}P_m\}$ and thus for $h_0=\xi_1P_1+\xi_3P_3+\ldots+\xi_mP_m+\eta T^{-1}P_m$ the identity $(I-J)Th_0=-\tau Ah_0$ (and thus $(I+J)h_0=-\tau T^{-1}Ah_0$) yields $2\eta T^{-1}P_m=-\tau\xi_m(1-a_m)T^{-1}P_m$. So $\eta=-\frac{1}{2}\tau(1-a_m)\xi_m$. Summarizing these results we get

$$\{p:\Gamma^+p=0\}=\operatorname{span}\{TP_0,TP_2,\ldots,TP_{s-2}\};$$

(1.10)

$$\{p:\Gamma^-p=0\}=\begin{cases}\operatorname{span}\{TP_1,TP_3,\ldots,TP_{m-1}\} & \text{for even } m;\\ \operatorname{span}\{TP_1,TP_3,\ldots,TP_{m-2}\}\oplus \\ \oplus\operatorname{span}\{TP_m-\frac{1}{2}\tau(1-a_m)P_m\} & \text{for odd } m.\end{cases}$$

Observe that $\dim\{p:\Gamma^{\pm}p=0\}=\frac{1}{2}s$ in all cases.

We conclude this section with historical references. Eq. (1.5) (or (1.6)) is a linear singular integral equation for $\Gamma^{\pm}p=0$ and the problem is to find a solution that admits an analytic continuation to $\mathbb{C}\setminus\{0\}$. Such linear singular equations appeared in [18,21,10]. Adding and subtracting Eq.(1.5) (for Γ^+p) and Eq.(1.5) (for Γ^-p) and using that $R^*_{+\tau}p=\frac{1}{2}(\Gamma^+p+\Gamma^-p)$

and $T^*_{+\tau}Jp=\frac{1}{2}(\Gamma^+p-\Gamma^-p)$, one obtains a coupled system of linear singular integral equations for $R^*_{+\tau}p$ and $T^*_{+\tau}Jp$. For $p=P_n$ the identities $JP_n=(-1)^nP_n$, $F^+P_n=H_n$ and $JH_n=(-1)^nH_n$ can be applied to obtain a coupled system of linear singular integral equations for $R^*_{+\tau}P_n$ and $T^*_{+\tau}P_n$. They were found by Mullikin ([18],(3.39)-(3.40)) and Sobolev ([21],(35)-(36)). For $p=P_n$ the same transformations applied to (1.5) lead to separate linear singular integral equations for Γ^+P_n and Γ^-P_n ([10],(12)-(13)).

2. THE X- AND Y-FUNCTIONS

In astrophysics the X- and Y-functions are very important, at least from a historical point of view. First introduced for the isotropic case by Ambartsumian [1], they were studied further and generalized for polynomial phase functions by Chandrasekhar [5]. In a first mathematical study Busbridge [2] found them in the form

$$(2.1)\quad X(\mu)=1+\int_0^\tau e^{-x/\mu}\xi(x)dx\ ,\ Y(\mu)=e^{-\tau/\mu}+\int_0^\tau e^{-(\tau-x)/\mu}\xi(x)dx,$$

where $\xi:(0,\tau)\to L_1(0,\tau)$ is the unique solution of the convolution equation

$$(2.2)\qquad \xi(x)-\int_0^\tau \kappa(x-y)\xi(y)dy=\kappa(x)\qquad (0<x<\tau)$$

and $\kappa:(-\tau,+\tau)\to\mathbb{R}$ is what we call the <u>dispersion kernel</u>

$$(2.3)\qquad \kappa(x)=\int_0^1 z^{-1}\psi(z)e^{-|x|/z}dz\qquad (0\neq x\in\mathbb{R}).$$

The dispersion kernel and the dispersion function are related as follows:

$$(2.4)\qquad \Lambda(\lambda)=1-\int_{-\infty}^{+\infty} e^{x/\lambda}\kappa(x)dx\quad ,\quad \mathrm{Re}\ \lambda=0.$$

In [13] (Theorem 5.1) it was proved that a solution ξ of Eq.(2.2) in $L_1(0,\tau)$ is unique and for this solution the functions X and Y in (2.1) satisfy two systems of singular integral equations:

(1) the (nonlinear) X- and Y-equations

$$X(\mu)=1+\mu\int_0^1 \frac{X(\mu)X(\nu)-Y(\mu)Y(\nu)}{\nu+\mu}\,\psi(\nu)d\nu; \tag{2.5a}$$

$$Y(\mu)=e^{-\tau/\mu}+\mu\int_0^1 \frac{X(\mu)Y(\nu)-Y(\mu)X(\nu)}{\nu-\mu}\,\psi(\nu)d\nu; \tag{2.5b}$$

(2) the linear X- and Y-equations

$$\Lambda(\mu)X(\mu)=1+\mu\int_0^1 \frac{\psi(\nu)X(\nu)}{\nu-\mu}\,d\nu-e^{-\tau/\mu}\mu\int_0^1 \frac{\psi(\nu)Y(\nu)}{\nu+\mu}\,d\nu; \tag{2.6a}$$

$$\Lambda(\mu)Y(\mu)=e^{-\tau/\mu}+\mu\int_0^1 \frac{\psi(\nu)Y(\nu)}{\nu-\mu}\,d\nu-e^{-\tau/\mu}\mu\int_0^1\frac{\psi(\nu)X(\nu)}{\nu+\mu}\,d\nu. \tag{2.6b}$$

The linear equations (2.6) were first derived from Chandrasekhar's X- and Y-equations (2.5) by Busbridge ([2], Section 40). For nonnegative $\psi(\mu)$ Busbridge [2] proved Eq.(2.2) to have a solution ξ, provided $\int_0^1 \psi(\mu)d\mu\leq\frac{1}{2}$.

PROPOSITION 2.1. Let $0\leq a_0\leq 1$ and $-a_0\leq a_n\leq a_0$ $(n=1,2,..,N)$. If either $\kappa(x)$ is nonnegative on $(0,\tau)$ or $\int_0^1|\psi(z)|dz\leq\frac{1}{2}$, then Eq.(2.2) has a unique solution ξ in $L_1(0,\tau)$.

PROOF. If $0\leq a_0\leq 1$ and $-a_0\leq a_n\leq a_0$ $(n=1,2,...N)$, one has $1-2\int_0^1 \psi(z)dz=\Lambda(\infty)\geq 0$ (cf.[15],Section 4), and therefore $\int_0^1 \psi(z)dz\leq\frac{1}{2}$. So if $\kappa(x)\geq 0$ on $(0,\tau)$ (and thus on $(-\tau,+\tau)$), then

$$\int_{-\tau}^{+\tau} |\kappa(x)|dx=2\int_0^\tau \kappa(x)dx=2\int_0^1 \psi(z)(1-e^{-\tau/z})dz<1;$$

if $\int_0^1 |\psi(z)|dz\leq\frac{1}{2}$, then

$$\int_{-\tau}^{+\tau}|\kappa(x)|dx=2\int_0^\tau |\kappa(x)|dx\leq 2\int_0^1|\psi(z)|(1-e^{-\tau/z})dz<1.$$

As the norm of the operator $(K\zeta)(x) = \int_0^1\kappa(x-y)\zeta(y)dy$ does not exceed $\int_{-\tau}^{+\tau}|\kappa(x)|dx$, the norm of K is strictly less than +1, which completes the proof. □

The generalization of this proposition is not straightforward. To prove the existence of a solution of Eq.(2.2) in general, we establish the following lemma first.

LEMMA 2.2. *Let* $0 \le a_0 \le 1$ *and* $-a_0 \le a_n \le a_0$ $(n=1,2,\dots,N)$. *Then there exists a unique pair of functions* X *and* Y *that are analytic on* $\mathbb{C}_\infty \setminus \{0\}$ *and satisfy the linear* X- *and* Y-*equations* (2.6). *For this pair of functions one can find polynomials* $p^\pm$ *such that*

$$X(\mu) \pm Y(\mu) = (\Gamma^\pm p^\pm)(\mu), \quad 0 \le \mu \le 1.$$

PROOF. Let us rewrite (1.6) using Eqs (4.2)-(4.3a) of [15] and obtain

$$\begin{aligned}(2.7)\qquad &\Lambda(\lambda)(\Delta^\pm p)(\lambda) - \lambda\int_0^1 (\mu-\lambda)^{-1}\psi(\mu)(\Gamma^\pm p)(\mu)d\mu \pm \\ &\pm\lambda e^{-\tau/\lambda}\int_0^1 (\mu+\lambda)^{-1}\psi(\mu)(\Gamma^\pm p)(\mu)d\mu = r^\pm(\lambda) \pm e^{-\tau/\lambda} r^\pm(-\lambda),\end{aligned}$$

where $r^\pm(\lambda)$ is the following polynomial of degree $\le$max (N-s, deg p-s):

$$(2.8)\qquad r^\pm(\lambda) = (F^+p)(\lambda) - \lambda\sum_{n=0}^{N} a_n(n+\tfrac{1}{2})\int_0^1 \frac{H_n(\lambda)-H_n(\mu)}{\lambda-\mu} P_n(\mu)(\Gamma^\pm p)(\mu)d\mu.$$

Next write $\Lambda(\lambda) = 1 + \lambda\int_0^1(\mu-\lambda)^{-1}\psi(\mu)d\mu - \lambda\int_0^1(\mu+\lambda)^{-1}\psi(\mu)d\mu$ (cf.[15], (4.3a)), substitute (1.8) and rewrite (2.7) as follows:

$$\begin{aligned}(2.9)\qquad (\Delta^\pm p)(\lambda) = &\left\{r^\pm(\lambda) \pm e^{-\tau/\lambda} r^\pm(-\lambda)\right\} + \lambda\int_0^1 \frac{(\Delta^\pm p)(\mu) - (\Delta^\pm p)(\lambda)}{\mu-\lambda}\psi(\mu)d\mu \mp \\ &\mp\lambda e^{-\tau/\lambda}\int_0^1 \frac{(\Delta^\pm p)(\mu) - (\Delta^\pm p)(-\lambda)}{\mu+\lambda}\psi(\mu)d\mu.\end{aligned}$$

Suppose p is a polynomial for which $r^\pm = 0$ (see (2.8)). Then p satisfies Eq.(2.9) with $r^\pm(\lambda) \pm e^{-\tau/\lambda} r^\pm(-\lambda) \equiv 0$. If $\Delta^\pm p$ would have an essential singularity at $\lambda=0$, then, because of the identity $\lim(\Delta^{\pm}p)(\lambda)=(\Gamma^{\pm}p)(0)$ (as $\lambda\to 0$, Re $\lambda\geq 0$), for every $c\neq\Gamma^{\pm}(0)$ there would exist a path Γ_c in the open left half-plane such that $|(\Delta^{\pm}p)(\lambda)-c|\to 0$ as $\lambda\to 0$ along Γ_c. Then Eq.(2.9) (for $\lambda\to 0$ along Γ_c) would imply $c=0$, contradicting the free choice of c. So $\Delta^{\pm}p$ would be analytic at $\lambda=0$ and therefore $\Gamma^{\pm}p$ would vanish. Conversely, if $\Gamma^{\pm}p=0$, then $F^{+}p=0$ (cf.(1.10)) and thus $r^{\pm}=0$ (cf.(2.8)).

If the non-conservative case $s=0$ we have $r^{\pm}=0$ if and only if $p=0$, and so a simple dimension argument involving the vector space of polynomials of degree $\leq N$ yields the existence of a unique polynomial $p^{\pm}$ such that

$$r^{\pm}(\lambda)=(F^{+}p^{\pm})(\lambda)-\lambda\sum_{n=0}^{N}a_n(n+\tfrac{1}{2})\int_0^1\frac{H_n(\lambda)-H_n(\mu)}{\lambda-\mu}P_n(\mu)(\Gamma^{\pm}p^{\pm})(\mu)d\mu\equiv 1.$$

Then $X=\frac{1}{2}(\Delta^{+}p^{+}+\Delta^{-}p^{-})$ and $Y=\frac{1}{2}(\Delta^{+}p^{+}-\Delta^{-}p^{-})$ are analytic functions on $\mathbb{C}_{\infty}\setminus\{0\}$ that satisfy the linear equations (2.6). (To see this, add and subtract Eq.(2.7) for $\Delta^{+}p^{+}$ and Eq.(2.7) for $\Delta^{-}p^{-}$, and use that $r^{+}p=r^{-}p^{-}\equiv 1$). For general s we remark that $R^{\pm}p=r^{\pm}$ maps the space of polynomials of degree $\leq N$ into the space of polynomials of degree $\leq N-s$, while $\{p:R^{\pm}p=0\}$ is a space of dimension $\leq s$. Hence, $R^{\pm}$ is surjective, and so there exists a polynomial $p^{\pm}$ of degree $\leq N$ such that $r^{\pm}(\lambda)\equiv 1$. In the same way as for $s=0$ we prove the existence of analytic functions X and Y on $\mathbb{C}_{\infty}\setminus\{0\}$ that satisfy Eqs(2.6).

It remains to prove the uniquness of solutions X and Y that are analytic on $\mathbb{C}_{\infty}\setminus\{0\}$. But this is clear from the uniqueness of a solution $\Delta^{\pm}p$ of Eq.(2.7) that is analytic on $\mathbb{C}_{\infty}\setminus\{0\}$ and continuous on the closed right half-plane. The latter can be shown with the help of the argument of the second paragraph of this proof.□

THEOREM 2.4. *Let $0\leq a_0\leq 1$ and $-a_0\leq a_n\leq a_0$ (n=1,2,...,N). Then there exists a unique solution ξ of the convolution equation (2.2) in $L_1(0,\tau)$. The functions X and Y defined in terms of ξ by (2.1) satisfy Eqs(2.5) and (2.6).*

PROOF. According to Lemma 2.2 there exist polynomials

$p^{\pm}$ such that

$$X(\mu)=\tfrac{1}{2}\left[R^{*}_{+\tau}(p^{+}+p^{-})\right](\mu)+\tfrac{1}{2}\left[T^{*}_{+\tau}J(p^{+}-p^{-})\right](\mu);$$

$$Y(\mu)=\tfrac{1}{2}\left[R^{*}_{+\tau}(p^{+}-p^{-})\right](\mu)+\tfrac{1}{2}\left[T^{*}_{+\tau}J(p^{+}+p^{-})\right](\mu).$$

As the function X is uniformly Hölder continuous on any bounded subset E of the closed right half-plane (see Theorem 1.1), any Hölder exponent $0<\alpha<1$ may be taken (which appears from the proof) and $X(0)=1$, it follows that there exists $\frac{1}{2}<\alpha<1$ such that $|(X(\mu)-1)/\mu|=O(|\mu|^{\alpha-1})(\mu\to 0\ ,\mathrm{Re}\ \mu=0)$. Therefore, $\int_{-i\infty}^{+i\infty}|X(\mu^{-1})-1|^{2}d|\mu|<+\infty$. So there exists $\xi\in L_2(-\infty,+\infty)$ such that $X(\mu)-1=\int_{-\infty}^{+\infty}e^{-x/\mu}\xi(x)dx$, Re $\mu=0$. However, X has an essential singularity at $\mu=0$ of order $\leq\tau$ (see Corollary 1.2) and is analytic on the open right half-plane and continuous up to the imaginary line. The Paley-Wiener theorem implies that $\xi(x)=0$ for $x\notin(0,\tau)$, and therefore $\xi\in L_2(0,\tau)\subset L_1(0,\tau)$.
This proves the first part of (2.1). The second part follows with the help of the symmetry $Y(\mu)=e^{-\tau/\mu}X(-\mu)$.

Using the first part of (2.1) one easily reduces $\Lambda(\mu)X(\mu)=\Lambda(-\mu)X(\mu)$ to

$$\Lambda(\mu)X(\mu)=1+\int_{-\infty}^{+\infty}e^{-x/\mu}\{\xi(x)-\int_{0}^{\tau}\kappa(x-y)\xi(y)dy-\kappa(x)\}dx,$$

where $\xi(x)=0$ for $x\notin(0,\tau)$. We have to show that

$$\text{(2.10)}\quad \Lambda(\mu)X(\mu)=1+\int_{-\infty}^{0}e^{-x/\mu}\ell(x)dx+e^{-\tau/\mu}\int_{\tau}^{+\infty}e^{(\tau-x)/\mu}\ell(x)dx,\mathrm{Re}\mu\geq 0,$$

where $\ell\in L_1(-\infty,+\infty)$. (We have put $\ell(x)=0$ for $0\leq x\leq\tau$).

Consider the Wiener algebra **A** of functions h on the extended imaginary line of the form $h(\mu)=c+\int_{-\infty}^{+\infty}e^{-x/\mu}z(x)dx$ with $c\in\mathbb{C}$ and $z\in L_1(-\infty,+\infty)$ (see [7] for this algebra; however, in [7] the Fourier transform is used). Then $\Lambda(\mu)X(\mu)$ belongs to **A** for $c=1$ and $z=\ell$. According to Eq.(2.6a) one can write

$$\Lambda(\mu)X(\mu)=1+g_{-}(\mu)+e^{-\tau/\mu}g_{+}(\mu)\quad(\mathrm{Re}\ \mu=0),$$

where $g_-(\mu) = \mu\int_0^1 (\nu-\mu)^{-1}\psi(\nu)X(\nu)d\nu$ is analytic on the open left half-plane and continuous up to the boundary, whereas $g_+(\mu) = -\mu\int_0^1 (\nu+\mu)^{-1}\psi(\nu)Y(\nu)d\nu$ is analytic on the open right half-plane and continuous up to the boundary. Hence, $\Lambda(\mu)X(\mu)$ admits the representation (2.10) and Eq.(2.2) is clear.

The derivation of the non-linear equations (2.5) from (2.1) and (2.2) is a standard argument that can be found in Section 5 of [13], for instance.□

In many cases the functions X and Y in (2.1) do not provide the only solutions of Eqs (2.5) and (2.6). If the dispersion function Λ has zeros on $\mathbb{C}_\infty \setminus [-1,+1]$, these equations have infinitely many solutions. However, imposing suitable constraints one may specify X and Y by Eqs (2.5) (or (2.6)) completely (cf. [16,17]; also the **erratum** in Astrophys. J. 147, 858, 1967).

3. REDUCTION TO X- AND Y-functions

In Section 3 of [15] the search for analytic expressions for the reflection and transmission operators was reduced to the computation of the 2N+2 functions $R^*_{+\tau}P_n$ and $T^*_{+\tau}P_n$ $(n=0,1,...,N)$. In the present section a further reduction is accomplished, namely to X- and Y-functions.

THEOREM 3.1. *Let* $0\le a_0\le 1$ *and* $-a_0\le a_n\le a_0$ $(n=1,2,...,N)$. *For* $n=0,1,...,N$ *there exist unique polynomials* q_n *and* s_n *such that*

$$(R^*_{+\tau}P_n)(\mu)=q_n(\mu)X(\mu)+(-1)^n s_n(-\mu)Y(\mu); \tag{3.1a}$$

$$(T^*_{+\tau}P_n)(\mu)=s_n(\mu)X(\mu)+(-1)^n q_n(-\mu)Y(\mu). \tag{3.1b}$$

Here the degrees of q_n *and* s_n *do not exceed* $\max(N,n)$. *The polynomials* $t_n^\pm=q_n\pm(-1)^n s_n$ *satisfy the linear equation*

$$t_n^\pm(\mu)=\mu\int_0^1 \frac{\psi(\mu,\nu)t_n^\pm(\nu)-\psi(\gamma,\nu)t_n^\pm(\mu)}{\nu-\mu} X(\nu)d\nu \pm$$

(3.2)
$$\pm\mu\int_0^1 \frac{\psi(\mu,\nu)t_n^{\pm}(-\nu)-\psi(\gamma,\nu)t_n^{\pm}(-\mu)}{\nu-\mu} Y(\nu)d\nu+H_n(\mu).$$

In the next section we shall investigate the properties of Eq.(3.2) further. Here we mainly aim at proving the representation (3.1).

PROOF OF THEOREM 3.1. Let P_n be the (n+1)-dimensional vector space of polynomials of degree n, and let $P=\bigcup_{n=0}^{+\infty}P_n$. First we show that for $p,q\in P$ the function $pX+qY = 0$ if and only if $p=q=0$. This will imply the uniqueness of the polynomials q_n and s_n in (3.1) once the representation (3.1) has been established. If q would be non-zero, then $r=Y/X$ would be a rational function satisfying $r(\lambda)\ r(-\lambda)\equiv 1$ and $\lim_{\lambda\downarrow 0} r(\lambda)=0$ (this follows from the identities $X(-\mu)=e^{-\tau/\mu}Y(\mu)$, $X(0)=1$ and $Y(0)=0$, which in turn follow from (2.1)). Contradiction. So $p=q=0$. Hence, for n=0,1, 2,... the set

$$Z_n^{\pm} = \{tX\pm(Jt)Y:t\in P_n\}$$

is a complex vector space of dimension n+1.

Recall that $X\pm Y = R^*_{+\tau}p^{\pm}\pm T^*_{+\tau}Jp^{\pm}$ for some $p^{\pm}\in P_N$. So using the commutator relations (2.17a)-(2.17b) repeatedly, one proves the existence of polynomials $q_1^{\pm}$ and $q_2^{\pm}$ such that

(3.3a)
$$tX\pm(Jt)Y = R^*_{+\tau}q_1^{\pm}\pm T^*_{+\tau}Jq_2^{\pm}.$$

However, $f=tX\pm(Jt)Y$ satisfies the symmetry $f(\mu)=\pm e^{-\tau/\mu}f(-\mu)$. So using (1.1b) we get

(3.3b)
$$tX\pm(Jt)Y = R^*_{+\tau}q_2^{\pm}\pm T^*_{+\tau}Jq_1^{\pm}.$$

Subtracting (3.3a) and (3.3b) and applying (1.10) yields that $q_1^{\pm}-q_2^{\pm}\in\{p\in P:\Gamma^{\mp}p = 0\}$. Hence, for $n\geq N$ we have

$$Z_n^{\pm} \subset \{\Gamma^{\pm}p : p \in \mathcal{P}_n\} + \{T^*_{+\tau}Jq : \Gamma^{\mp}q = 0\}.$$

But the left-hand side is a space of dimension n+1, whereas the right-hand has dimension $\leq (n+1-\frac{1}{2}s)+\frac{1}{2}s = n+1$. So equality holds and therefore there exist polynomials q_n and s_n such that $R^*_{+\tau}P_n = \frac{1}{2}\Gamma^+P_n + \frac{1}{2}\Gamma^-P_n = q_nX + s_nY$. With the help of the symmetries (1.1b), $Y(\mu) = e^{-\tau/\mu}X(-\mu)$ and $JP_n = (-1)^nP_n$ we derive the other one of the representations (3.1). Furthermore, for $n \geq N$ we necessarily have deg $q_n \leq n$ and deg $s_n \leq n$.

Note that, for $t_n^{\pm} = q_n \pm (-1)^n s_n$,

$$\Gamma^{\pm}P_n = t_n^{\pm}X \pm (Jt_n^{\pm})Y \in Z_{\max(n,N)}. \tag{3.4}$$

Substituting this into (1.6) (with $p = P_n, F^+p = H_n$; see Section 4 of [15]) and employing the linear X- and Y-equation (2.6) one obtains

$$Q_n^{\pm}(\lambda) \pm e^{-\tau/\lambda}Q_n^{\pm}(-\lambda) = H_n(\lambda) \pm e^{-\tau/\lambda}H_n(-\lambda), \tag{3.5}$$

where $Q_n^{\pm}$ is the following polynomial:

$$\begin{aligned} Q_n^{\pm}(\lambda) = {} & t_n^{\pm}(\lambda) - \lambda\int_0^1 \frac{\psi(\lambda,\mu)t_n^{\pm}(\mu) - \psi(\mu,\mu)t_n^{\pm}(\lambda)}{\mu-\lambda} X(\mu)d\mu \mp \\ & \mp\lambda\int_0^1 \frac{\psi(\lambda,\mu)t_n^{\pm}(-\mu) - \psi(\mu,\mu)t_n^{\pm}(-\lambda)}{\mu-\lambda} Y(\mu)d\mu. \end{aligned} \tag{3.6}$$

As $Q_n^{\pm}$ and H_n are polynomials, (3.5) implies $Q_n^{\pm} = H_n$. □

The representations (3.1) for $\varphi_n = R^*_{+\tau}P_n$ and $\psi_n = T^*_{+\tau}P_n$ were first stipulated by Sobolev ([21],(47)-(48)), but no information was given on the conditions under which (3.1) would be valid nor on the uniqueness of the polynomials q_n and s_n. The equations (3.2) were first obtained by Hovenier ([10],(35)-(36)) for the non-conservative case.

4. COEFFICIENT POLYNOMIALS AND THEIR CONSTRAINTS

In this section we study the "coefficient polynomials" $t_n^{\pm}$ in detail.

PROPOSITION 4.1. *Let* $0\le a_0\le 1$ *and* $-a_0\le a_n\le a_0$ $(n=1,2,\dots,N)$. *Then for certain* $(\xi_n)_{n=0}^{k}$ *the polynomial* $\sum_{n=0}^{k}\xi_n t_n^{\pm}=0$ *if and only if* $\sum_{n=0}^{k}\xi_n P_n$ *belongs to the kernel of* $\Gamma^{\pm}$. *In particular, if* $0\le a_0<1$, *then the polynomials* $t_n^{\pm}$ $(n=0,1,2,\dots)$ *are linearly independent.*

PROOF. If $\sum_{n=0}^{k}\xi_n t_n^{\pm}=0$, then (3.4) yields $\Gamma^{\pm}\left(\sum_{n=0}^{k}\xi_n P_n\right)=0$. Conversely, if $\Gamma^{\pm}\left(\sum_{n=0}^{k}\xi_n P_n\right)=0$, then for $t=\sum_{n=0}^{k}\xi_n t_n^{\pm}$ we have $tX\pm(Jt)Y=0$ (see(3.4)). Reasoning as in the beginning of the proof of Theorem 3.1 one gets $t=0$.□

Let $V_m^{\pm}$ be the linear operator on P_m with property $V_m^{\pm}t_n^{\pm}=H_n$ $(n=0,1,2,\dots,m)$. For $m\ge N$ the linear span of $H_0,H_1,\dots,H_m$ has dimension $m-s$ and the span of $t_0^{\pm},t_1^{\pm},\dots,t_m^{\pm}$ dimension $m-\frac{1}{2}s$. So $\dim \operatorname{Ker} V_m^{\pm}=\frac{1}{2}s$ for $m\ge N$. Hence, for $0\le a_0<1$ Eq.(3.2) has $t_n^{\pm}$ as a unique polynomial solution. For $a_0=1$ Eq.(3.2) does not specify $t_n^{\pm}$ completely. To deduce additional constraints we first derive the following

LEMMA 4.2. *Let* $a_0=1$ *and* $f\in H_0$ *with* $Jf=\pm f$. *If* χ *is a polynomial, then*

$$\langle\Gamma^{\pm}\chi,f\rangle+\tfrac{1}{2}\tau\langle\Gamma^{\pm}\chi,T^{-1}Af\rangle=\langle\chi,f\rangle+\tfrac{1}{2}\tau\langle\chi,T^{-1}Af\rangle. \tag{4.1}$$

PROOF. According to Theorem 1.1 of [15] the solution of the boundary value problem (0.4)-(0.5) on $L_2[-1,+1]$ has the form

$$\psi(x)=\left[e^{-xT^{-1}A}P_p+e^{(\tau-x)T^{-1}A}P_m+(I-xT^{-1}A)P_0\right]V_\tau^{-1}\varphi,\ 0<x<\tau.$$

With the help of formulas (1.7) of [15] one gets for $f\in H_0$:

$$<T\psi(x),f> = <T(I-xT^{-1}A)P_0V_\tau^{-1}\varphi,f> = <TP_0V_\tau^{-1}\varphi,f> -$$

$$-x<TP_0V_\tau^{-1}\varphi,T^{-1}Af>.$$

For $\varphi\in L_2[-1,0]$ this implies

$$-<TR_{-\tau}\varphi,f>+<TT_{-\tau}\varphi,f> = <T\psi(0),f>-<T\psi(\tau),f> =$$

$$= \tau<TP_0V_\tau^{-1}\varphi, \quad T^{-1}Af> = \tau<TT_{-\tau}\varphi,T^{-1}Af>.$$

Note that $<TR_{-\tau}\varphi,f> = <TT_{-\tau}\varphi,f>$ whenever $Af = 0$. Using formulas (2.4) of [15], putting $\chi = T\varphi$ and extending continuously to all $\chi\in L_2[-1,+1]$ one gets

$$-<R^+_{-\tau}\chi,f>+<T^+_{-\tau}\chi,f> = \tau<T^+_{-\tau}\chi,T^{-1}Af> =$$

$$= \tau<R^+_{-\tau}\chi,T^{-1}Af>.$$

Inserting Lemma 2.1 and formulas (3.2b) of [15] and specializing to $f\in H_0$ with $Jf = \pm f$ (and thus $JT^{-1}Af = \mp T^{-1}Af$) one obtains

$$<\left[R^*_{+\tau}\pm T^*_{+\tau}J\right]\chi,f>+\tfrac{1}{2}\tau<\left[R^*_{+\tau}\pm T^*_{+\tau}J\right]\chi,T^{-1}Af> =$$

$$= <\chi,f>+\tfrac{1}{2}\tau<\chi,T^{-1}Af>.$$

From this identity formula (4.1) is clear.□

With the help of Lemma 4.2 two theorems are deduced concerning constraints on t_n^+ and t_n^- under which Eq.(3.2) has a unique solution.

THEOREM 4.3. _Let_ $a_0=a_1=\ldots=a_{m-1}=1$, $-1\leq a_n<+1$ $(n=m,m+1,\ldots,N)$, $s=m+1$ _for even_ m _and_ $s=m$ _for odd_ m. _Then_

(4.2) $$t_1^+ = t_3^+ = t_5^+ = \ldots = t_{s-1}^+ = 0$$

<u>and</u> t_n^+ <u>is the unique polynomial solution of</u> Eq.(3.2) <u>under the</u> $\frac{1}{2}s$ <u>constraints</u>

$$(4.3)\qquad \int_0^1 \left\{t_n^+(\mu)X(\mu)+t_n^+(-\mu)Y(\mu)\right\}P_{2k}(\mu)d\mu = \delta_{n,2k}(2k+\tfrac{1}{2})^{-1}.$$

$$(k=0,1,\ldots,\tfrac{1}{2}s-1)$$

PROOF. The functions $P_0,P_2,\ldots,P_{s-2}$ belong to Ker A and satisfy $Jf = f$. According to (4.1) we have $\langle\Gamma^+\chi,P_{2k}\rangle=\langle\chi,P_{2k}\rangle$. For $\chi=P_n$ we insert the inner product $\langle.,.\rangle$ in $L_2[-1,+1]$ employ (3.4) and obtain (4.3). Formula (4.2) is clear from Proposition 4.1 and the form $\{p\in P:\Gamma^+p=0\}$ has (cf. 1.10)).

To prove the uniqueness of the solution t_n^+ of Eq.(3.2) under the constraints (4.3), it suffices to prove that a polynomial $t=0$ whenever $V_m^+\ t=0$ and $\int_0^1\{t(\mu)X(\mu)+t(-\mu)Y(\mu)\}P_{2k}(\mu)d\mu$ 0 $(k=0,1,\ldots,\frac{1}{2}s-1)$. As an application of Proposition 4.1 we see that $\eta_0t_0^++\eta_2t_2^++\ldots+\eta_st_s^+=0$ if and only if $\Gamma^+(\eta_0P_0+\eta_2P_2+\ldots+\eta_sP_s)=$ 0. But $\{p\in P:\Gamma^+p=0\}$ consists of odd functions only (cf.(1.10)) and so $\eta_0=\eta_2=\ldots=\eta_s=0$. Thus $t_0^+,t_2^+,\ldots,t_s^+$ are linearly independen In view of the constancy of $H_0,H_2,\ldots,H_s$ (in fact, $H_{2k}(\mu)\equiv P_{2k}(0)$ for $k=0,1,\ldots\frac{1}{2}s$) one derives that $\{t_{2k}^+-P_{2k}(0)t_0^+\}_{k=1}^{\frac{1}{2}s}$ is a basis of Ker V_m. So for certain coefficients $\zeta_1,\zeta_2,\ldots,\zeta_{\frac{1}{2}s}$ we have

$$(4.4)\qquad t = \sum_{k=1}^{\frac{1}{2}s} \zeta_k(t_{2k}^+-P_{2k}(0)t_0^+).$$

The constraints (4.3), Eq.(4.4) and $\int^1\{t(\mu)X(\mu)+t(-\mu)Y(\mu)\}P_{2k}(\mu)d$ 0 $(k=0,1,\ldots,\frac{1}{2}s-1)$ together yield $t^0= 0$.□

THEOREM 4.4. <u>Let</u> $a_0=a_1=\ldots=a_{m-1}=1$, $-1\leq a_n<+1$ $(n=m,m+1,\ldots,N)$. <u>If</u> m <u>is even, then</u>

$$(4.5)\qquad t_2^- = P_2(0)t_0^-\ ,\ t_4^- = P_4(0)t_0^-,\ldots,t_m^- = P_m(0)t_0^-\ ,$$

<u>while</u> t_n^- <u>is the unique polynomial solution of</u> Eq.(3.2) <u>under the</u> $\frac{1}{2}m$ <u>constraints</u>

$$(4.6)\qquad \int_0^1 \{t_n^-(\mu)X(\mu)-t_n^-(-\mu)Y(\mu)\}P_{2k+1}(\mu)d\mu=\delta_{n,2k+1}\cdot(2k+\tfrac{3}{2})^{-1}.$$

$$(k = 0,1,\ldots,\tfrac{1}{2}m-1)$$

If m *is odd, then*

$$(4.7a)\qquad (k+1)t_{k+1}^- + k\, t_{k-1}^- = 0 \ (k = 1,3,5,\ldots,m-2);$$

$$(4.7b)\qquad (m+1)t_{m+1}^- + m\, t_{m-1}^- = \tau(m+\tfrac{1}{2})(1-a_m)t_m^-,$$

while t_n^- *is the unique polynomial solution of* Eq.(3.2) *under the* $\frac{1}{2}(m+1)$ *constraints*

$$(4.8a)\qquad \int_0^1 \{t_n^-(\mu)X(\mu)-t_n^-(-\mu)Y(\mu)\}P_{2k+1}(\mu)d\mu=\delta_{n,2k+1}\cdot(2k+\tfrac{3}{2})^{-1}.$$

$$(k = 0,1,\ldots,\tfrac{1}{2}m - \tfrac{3}{2})$$

$$(4.8b)\qquad \int_0^1 \{t_n^-(\mu)X(\mu)-t_n^-(-\mu)Y(\mu)\}(1 - \tfrac{1}{2}\tau(1-a_m)/\mu)P_m(\mu)d\mu = $$

$$= \int_{-1}^{+1} P_n(\mu)(1 - \tfrac{1}{2}\tau(1-a_m)/\mu)P_m(\mu)d\mu.$$

The proof is analogous to the one of Theorem 4.3 and will be omitted. Let us work out the example m=1 (i.e., $a_0=1$ and $a_1\neq 1$). Then s=2, and thus one constraint has to be considered only. From Theorem 4.3 one finds

$$(4.9)\qquad t_1^+ = 0\ ,\ \int_0^1 \{t_n^+(\mu)X(\mu) + t_n^+(-\mu)Y(\mu)\}d\mu = 2\delta_{no}.$$

However, from Theorem 4.4 one derives that

$$(4.10a)\qquad 2t_2^- + t_0^- = \tfrac{3}{2}\,\tau(1-a_1)t_1^-\ ;$$

$$(4.10b)\qquad \int_0^1 \{t_n^-(\mu)X(\mu)-t_n^-(-\mu)Y(\mu)\}(\mu-\tfrac{1}{2}\tau(1-a_1)) = \tfrac{2}{3}\,\delta_{n1}-\tau(1-a_1)\delta_{no}.$$

For the half-space problem with $a_0 = 1$ (and $a_1 \neq 1$) one may derive the solution in terms of the H-function and a set of polynomials; the latter ones are the unique solutions of Eq.(3.2) (with X replaced by H and Y by 0) under a constraint of the form (4.9). In physical literature such observation was made by Pahor [20] and by Busbridge and Orchard [4]. Clearly the conditions (4.9) may be viewed as generalizations. To see that the conditions (4.10) are generalizations also, one has to divide Eqs(4.10) by τ before taking the limit as $\tau \to +\infty$.

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