STATIONARY KINETIC EQUATIONS WITH COLLISION TERMS RELATIVELY BOUNDED WITH RESPECT TO THE COLLISION FREQUENCY*

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In this article the existence and uniqueness theory of stationary kinetic equations in L^1-spaces is developed for collision terms dominated in the norm by the collision frequency.

1. Introduction

In this article we study boundary value problems of the type

$$v \cdot \frac{\partial u}{\partial x} + a(x,v) \cdot \frac{\partial u}{\partial v} + h(x,v)u(x,v) = (Ju)(x,v) + f(x,v), \quad (x,v) \in \Sigma; \tag{1}$$

$$u_-(x,v) = (Ku_+)(x,v) + g_-(x,v), \qquad (x,v) \in \Sigma_-; \tag{2}$$

where the position $x \in \Omega$ (Ω an open subset of $\mathbb{R}^n$), the velocity $v \in \mathcal{V}$ ($\mathcal{V}$ a subset of $\mathbb{R}^n$ equipped with a positive Borel measure μ_0 such that all bounded Borel sets in $\mathbb{R}^n$ have finite μ_0-measure), and $\Sigma = \Omega \times \mathcal{V}$ equipped with the product measure $d\mu(x,v) = dx d\mu_0(v)$. We assume that $a(x,v)$ is real and continuous in (x,v) and Lipschitz continuous in v on the closure of Σ, introduce the vector field

$$X = v \cdot \frac{\partial}{\partial x} + a(x,v) \cdot \frac{\partial}{\partial v},$$

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and suppose that for any C^1-function ϕ of compact support in Σ

$$\int_\Sigma X\phi \, d\mu_T = 0,$$

meaning that X is divergence free. Then through every point of Σ there passes exactly one integral curve of X. The left endpoints form the incoming boundary Σ_- and the right endpoints the outgoing boundary Σ_+. We assume in addition that no maximal integral curve of X can have a left or right endpoint in $\partial\Sigma$ where $v = a(x, v) = 0$.

Next, we assume that (1) the function $h(x, v)$ is nonnegative and locally μ-integrable, (2) the operator J is real and satisfies

$$\|Ju\|_1 \leq \delta \|hu\|_1 \tag{3}$$

for some $\delta \in (0, 1)$, and (3) the operator K has norm strictly less than 1. If J and K are positive operators (in lattice sense), then we shall allow K to have unit norm and δ to equal 1.

A comprehensive theory of the existence and uniqueness of the time dependent counterpart of Eqs. (1)-(2) has been developed by Beals and Protopopescu [3] (also Chapter XI of Greenberg et al.[9]) to cover situations where the operator J is bounded. It has recently been extended by Van der Mee [15] to deal with situations where J is the sum of a bounded operator and one satisfying (3). We mention that important earlier work on the time dependent problem was done by Voigt [21] for the case where $a \equiv 0$ and $J = 0$ and Ukai [20] for $J = 0$. In addition to these papers, the literature is littered with treatments of particular examples, but discussing them is beyond the scope of this article.

Let us outline the basic method of Beals and Protopopescu,[3] Greenberg et al.,[9] and Van der Mee.[15] Assuming a phase space Σ equipped with a Borel measure μ and a vector field independent of t and writing

$$X = v \cdot \frac{\partial}{\partial x} + a(x, v) \cdot \frac{\partial}{\partial v},$$

the fact that the vector field is divergence free may be expressed through the Green's identity

$$\int_\Sigma X\phi \, d\mu = \int_{\Sigma_+} \phi \, d\mu_+ - \int_{\Sigma_-} \phi \, d\mu_-$$

for ϕ in a suitable test function space, where $\mu_\pm$ are suitable measures on $\Sigma_\pm$. After constructing the boundary measures $\mu^\pm$ and the test function space pertaining to the vector field $Y = \frac{\partial}{\partial t} + X$, Eqs. (1)-(2) with $J = 0$ and $K = 0$ reduce to ordinary first order differential equations along the integral curves of Y which can be solved trivially. Two perturbation arguments then allow one to incorporate a bounded J and K with $\|K\| < 1$ into the theory. If J and K are positive operators, a monotonicity argument allows one to extend the existence and uniqueness result to operators K of unit norm.

When developing existence and uniqueness theory for the stationary kinetic boundary value problem (1)-(2), there are essentially two approaches. One approach, favored by the French school in kinetic theory, is to prove that the corresponding time evolution semigroup has a negative spectral bound and hence has $\lambda = 0$ in its resolvent set. This immediately implies that Eq. (1) with the homogeneous boundary condition (2) (i.e., with $g_- = 0$) is uniquely solvable in the functional setting to which the spectral result pertains. The case $g_- \not\equiv 0$ can be dealt with by subtracting a solution of Eqs. (1)-(2) for exactly that g_- but possibly different J and f and solving a boundary value problem with homogeneous boundary condition.

The second approach is restricted to plane-parallel homogeneous spatial domains without external forces. This approach in fact boils down to the study of vector-valued differential equations of the type

$$(T\psi)'(x) = -A\psi(x) + f(x), \qquad x \in (0,\tau), \tag{4}$$

on a finite interval or on the half-line under boundary conditions involving projected boundary data. Starting from the operator-theoretic formulation of the one-speed neutron transport equation with isotropic scattering by Hangelbroek and Lekkerkerker,[10] one can in fact distinguish two major subapproaches. In the subapproach launched by Beals [1] the solutions $\psi(x)$ are sought in an extended Hilbert space for which two natural scalar products are proven to be equivalent and to yield existence and uniqueness as a corollary. This subapproach, originally developed for positive selfadjoint operators A, has been made to apply also to indefinite Sturm-Liouville boundary value problems [2] and bounded and accretive A.[16] In the second subapproach, initiated by Van der Mee,[13] by assuming compactness of $B = I - A$ one is able to (1) seek solutions within the given Hilbert space, and (2) convert Eq. (4) with boundary conditions into a vector-valued convolution equation of the form

$$\psi(x) - \int_0^\tau \mathcal{H}(x-y)B\psi(y)\,dy = \omega(x), \quad x \in (0,\tau), \tag{5}$$

where Fredholm techniques can be applied to either the given boundary value problem or the convolution equation (5). An up-to-date account of the two subapproaches can be found in Chapters II-IX of Greenberg et al.[9]

In problems where the natural functional space is $L^1(\Sigma; d\mu)$ and some equilibrium condition demands that

$$\int_\Sigma \{hu - Ju\}\, d\mu = 0, \qquad u \in L^1(\Sigma; d\mu) \cap L^1(\Sigma; hd\mu). \tag{6}$$

a theory in an L^1-setting for J satisfying (3) comes to mind in a natural way. To mention a few applications with unbounded $h \geq 0$ and positive J satisfying $\|Ju\|_1 \leq \delta\|hu\|_1$ for some $\delta \in [0,1]$, just consider (1) neutron transport where the collision frequency dominates the collision kernel integrated over outgoing velocities if the medium is nonmultiplying,[4] (2) radiative transfer where the phase function integrated over postscattering directions is dominated by the extinction coefficient,[6 19] (3) cell growth modeling,[18 14] (4) electron transport in weakly ionized gases,[8] (5) rarefied gas dynamics,[5] (6) electron-phonon interaction in semiconductors,[11 12] and (7) the linearized Boltzmann equation with infinite range forces.[17 7] In many (if not all) of these applications, the integrated (nonnegative) collision kernel is exactly equal to the collision frequency. In fact, Eq. (6) is the linear counterpart of the balance condition involved in the nonlinear Boltzmann equation.

In the time dependent counterpart of Eqs. (1)-(2) the vector field to consider on the spatial-velocity-time phase space $\Lambda_T = \Sigma \times (0,T)$, namely $Y = (\partial/\partial t) + X$, has only integral curves on which the travel time does not exceed T and which have both a left and right endpoint.[a] This allows us to parametrize the points of $\Sigma \times (0,T)$ as (z,s), where z is a left endpoint of a maximal integral curve of Y and $s \in (0, \ell(z))$ is the travel time parameter, $\ell(z)$ standing for the total travel time along this curve. We may then write the initial-boundary value problem as an elementary initial value problem by combining the initial data g_0 and the boundary data g_- into one initial data $g^- = (g_0, g_-)$ on the incoming boundary $(\Lambda_T)^-$ and solve the resulting initial value problem in $L^1(\Lambda_T, d\mu_T)$, where $d\mu_T = d\mu dt$ is the product measure. This can be done explicitly if $K = 0$ and $J = 0$ and by contraction mapping and monotonicity arguments for more general K and J.

The more extensive variety of integral curve parametrizations complicates the study of the stationary Eqs. (1)-(2) in comparison to their time dependent counterpart. Integral curves may or may not have a left and/or a right endpoint, may be closed loops and may allow an infinite travel time. Thus when parametrizing them using the travel time parameter s, the do-

[a] One also assumes that the integral curves of Y do not run off to infinity in finite time.

main of parametrization is either a finite interval, a left half-line, a right half-line, the full real line, or a circle. In this article *we shall limit ourselves to the case in which all of the integral curves have a left endpoint.* We can then parametrize Σ as

$$\Sigma = \{(z, s) : z \in \Sigma_-, \ s \in (0, \ell(z))\}$$

and identify the measure μ with the product measure $d\mu_- ds$.

We now briefly describe the organization of the paper. In Sec. 2 we discuss the Green's identity for the vector field X in the case in which all integral curves of X have a left endpoint. We also derive solutions in $L^1(\Sigma; hd\mu)$ if $J = 0$ and $K = 0$. In Sec. 3 we obtain solutions in $L^1(\Sigma; hd\mu)$ for general J and K with $\delta, \kappa \in [0, 1)$ and explore extension to the case $\delta < 1$ and $\kappa = 1$. In Sec. 4 we explore the stationary problems for which X has only closed loop integral curves.

2. The Green's Identity

Let us define $L^{1,loc}(\Sigma; d\mu)$ as the linear space of all μ-measurable functions u on Σ which are μ-integrable on every bounded μ-measurable subset of Σ on which $\ell(z, s) \equiv \ell(z)$ is bounded away from zero. Further, let Φ_T be the test function space of all Borel functions u on Σ such that (i) u is continuously differentiable on each integral curve of X, (ii) u and Xu are bounded, and (iii) the support of u is bounded and the travel time along the integral curves meeting the support of u is bounded away from zero. Then if $u, Xu \in L^1(\Sigma; d\mu)$, we define a *trace* for u as a pair of functions $u_\pm \in L^{1,loc}(\Sigma_\pm; d\mu_\pm)$ such that for each $\phi \in \Phi$

$$< Xu, \phi > + < u, X\phi > = \int_{\Sigma_+} u_+ \phi \, d\mu_+ - \int_{\Sigma_-} u_- \phi \, d\mu_- .$$

Then if $\{u, (X + h)u\} \subset L^1(\Sigma, d\mu)$, u has a unique trace $u_\pm$. Moreover, if $u_- \in L^1(\Sigma_-; d\mu_-)$, then $u_+ \in L^1(\Sigma_+; d\mu_+)$, hu and Xu are μ-integrable and

$$\int_{\Sigma_+} |u_+| \, d\mu_+ + \int_\Sigma h|u| \, d\mu = \int_{\Sigma_-} |u_-| \, d\mu_- + \int_\Sigma \operatorname{sgn}(u)(X + h)u \, d\mu. \quad (7)$$

Observing that

$$u(z, s) = \exp\left[-\int_0^s h(z, \hat{\sigma}) \, d\hat{\sigma}\right] u_-(z) + \int_0^s \exp\left[-\int_\sigma^s h(z, \hat{\sigma}) \, d\hat{\sigma}\right] f(z, \sigma) \, d\sigma, \quad (8)$$

we now immediately have

Proposition 2.1. *Given* $f \in L^1(\Sigma, d\mu)$ *and* $g_- \in L^1(\Sigma_-, d\mu_-)$*, the unique solution* $u = S(f, g_-)$ *of the boundary value problem*

$$v \cdot \frac{\partial u}{\partial x} + a(x,v) \cdot \frac{\partial u}{\partial v} + h(x,v)u(x,v) = f(x,v), \quad (x,v) \in \Sigma; \tag{9}$$

$$u_-(x,v) = g_-(x,v), \qquad (x,v) \in \Sigma_-; \tag{10}$$

satisfies

$$\|hu\|_1 + \|u_+\|_1 \leq \|f\|_1 + \|g_-\|_1, \tag{11}$$

where the equality sign holds if $f \geq 0$ *and* $g_- \geq 0$.

We remark that the above solution u of Eqs. (9)-(10) belongs to $L^1(\Sigma; d\mu)$ whenever h is essentially bounded away from zero (i.e., if $h^{-1} \in L^\infty(\Sigma; d\mu)$).

3. Using the Method of Characteristics

In this section we shall prove the unique solvability of Eqs. (1)-(2).

Theorem 3.1. *Given* $f \in L^1(\Sigma, d\mu)$ *and* $g_- \in L^1(\Sigma_-, d\mu_-)$*, the boundary value problem* (1)-(2) *has a unique solution* $u \in L^1(\Sigma; hd\mu)$ *having trace* $u_\pm \in L^1(\Sigma_\pm; d\mu_\pm)$*, provided there exist* $\delta, \kappa \in [0,1)$ *such that*

$$\|Ju\|_1 \leq \delta \|hu\|_1, \qquad \|Ku_+\|_1 \leq \kappa \|u_+\|_1.$$

Further, u *is nonnegative if* J*,* K*,* f *and* g_- *are nonnegative. Finally,* $u \in L^1(\Sigma; d\mu)$ *whenever* h *is essentially bounded away from zero.*

Proof. Suppose $J = 0$ and $\kappa \in [0,1)$. Then any solution of Eqs. (1)-(2) satisfies $u = S(f, Ku_+ + g_-)$, where

$$u_+ = S(0, Ku_+)_+ + S(f, g_-)_+.$$

Since $\|S(0, Ku_+)_+\|_1 \leq \|Ku_+\|_1 \leq \kappa \|u_+\|_1$, a contraction mapping argument yields $u_+ \in L^1(\Sigma_+; d\mu_+)$ uniquely. Denoting the so-obtained solution by $u = Z(f, g_-)$, we have

$$\begin{aligned} \|hZ(f,g_-)\|_1 + \|Z(f,g_-)_+\|_1 &\leq \|f\|_1 + \|KZ(f,g_-)_+ + g_-\|_1 \\ &\leq \|f\|_1 + \|g_-\|_1 + \kappa \|Z(f,g_-)_+\|_1 \leq \frac{1}{1-\kappa} \left(\|f\|_1 + \|g_-\|_1 \right). \end{aligned}$$

Let us now consider Eqs. (1)-(2) for J and K with $\delta + \kappa < 1$. Then any solution u satisfies

$$u = Z(Ju + f, g_-) = Z(Ju, 0) + Z(f, g_-).$$

Moreover, since

$$\|hZ(Ju,0)\|_1 \leq \frac{1}{1-\kappa}\|Ju\|_1 \leq \frac{\delta}{1-\kappa}\|hu\|_1,$$

a contraction mapping argument yields the existence of u if $\delta + \kappa < 1$. Denoting the so-obtained solution by $u = W(f, g_-)$, we get

$$\begin{aligned}\|hW(f,g_-)\|_1 + \|W(f,g_-)_+\|_1 &\leq \frac{\delta\|hu\|_1 + \|f\|_1 + \|g_-\|_1}{1-\kappa} \\ &\leq \frac{\|f\|_1 + \|g_-\|_1}{1-\delta-\kappa}.\end{aligned} \tag{12}$$

Applying (7) to Eqs. (1)-(2) we find

$$\|hu\|_1 + \|u_+\|_1 \leq \|Ku_+\|_1 + \|g_-\|_1 + \|Ju\|_1 + \|f\|_1, \tag{13}$$

where the equality sign occurs if J and K are positive operators, $f \geq 0$ and $g_- \geq 0$. Hence

$$(1-\delta)\|hu\|_1 + (1-\kappa)\|u_+\|_1 \leq \|f\|_1 + \|g_-\|_1, \tag{14}$$

which suggests that the restriction to $\delta, \kappa \in [0, 1)$ with $\delta + \kappa < 1$ is not necessary.

Let us now extend the above estimates for $W(f, g_-)$ to the case where $\delta < 1$ and $\kappa < 1$, without assuming that $\delta + \kappa < 1$. Now choose $\kappa_0, \kappa_1 \in [0, 1)$ such that $\kappa = \kappa_0 + \kappa_1$ and $\delta + \kappa_0 < 1$, and let $u = V(f, g_-)$ denote the solution of Eqs. (1)-(2) with K replaced by $(\kappa_0/\kappa)K$ and $J = J$. Replacing K by $(\kappa_0/\kappa)K$ and observing that the latter boundary operator has norm κ_0 and that $\delta + \kappa_0 < 1$, we obtain from Eq. (12) the bound

$$\|hV(f,g_-)\|_1 + \|V(f,g_-)_+\|_1 \leq \frac{\|f\|_1 + \|g_-\|_1}{1-\delta-\kappa_0}.$$

Now observe that the solution of Eqs. (1)-(2) has the form

$$u = V(f, (\kappa_1/\kappa)Ku_+ + g_-) = V(0, (\kappa_1/\kappa)Ku_+) + V(f, g_-).$$

Since Eq. (14) (applied for $(\kappa_0/\kappa)K$ instead of K, and hence with κ_0 taking the place of κ) implies that

$$\left\|V(0, \frac{\kappa_1}{\kappa}Ku_+)_+\right\|_1 \leq \frac{(\kappa_1/\kappa)\|Ku_+\|_1}{1-\kappa_0} \leq \frac{\kappa_1}{1-\kappa_0}\|u_+\|_1,$$

and since $(\kappa_1/(1-\kappa_0)) < 1$, a contraction argument yields the existence of the solution u of Eqs. (1)-(2). As a result of Eq. (14), we now obtain the estimate

$$\|hW(f,g_-)\|_1 + \|W(f,g_-)_+\|_1 \leq \frac{\|f\|_1 + \|g_-\|_1}{1-\delta} + \frac{\|f\|_1 + \|g_-\|_1}{1-\kappa}, \tag{15}$$

valid under the hypothesis that $\delta, \kappa \in [0,1)$. □

Using the monotonicity argument of Beals and Protopopescu [3] and Sec. XI.5 of Greenberg et al.,[9] we obtain

Theorem 3.2. *The boundary value problem (1)-(2) has a unique solution* $u \in L^1(\Sigma; hd\mu)$ *for every* $f \in L^1(\Sigma; d\mu)$ *and* $g_- \in L^1(\Sigma_-; d\mu_-)$, *provided* J *and* K *are positive operators and*

$$\|Ju\|_1 \leq \delta \|hu\|_1, \qquad \|Ku_+\|_1 \leq \kappa \|u_+\|_1,$$

for certain $\delta \in [0,1)$ *and* $\kappa \in [0,1]$. *Finally,* $u \in L^1(\Sigma; d\mu)$ *whenever* h *is essentially bounded away from zero.*

In general, under the conditions of Theorem 3.2 the solution u of Eqs. (1)-(2) need not satisfy $u_\pm \in L^1(\Sigma_\pm; d\mu_\pm)$ if $\delta = 1$.

4. Closed Loop Integral Curves

To illustrate the pitfalls of having X with closed loop integral curves, we consider the vector field

$$X = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} \tag{16}$$

on $\Sigma = \mathbb{R}^2$ equipped with the Lebesgue measure and the nonnegative measurable function h on $\mathbb{R}^2$. Defining $(x,y) = \sqrt{x^2+y^2}\,(\cos s, \sin s)$ and $z(x,y) = (\sqrt{x^2+y^2}, 0)$, we see that any solution of Eq. (7) must satisfy (8), where

$$\begin{aligned} u(z,0) = \exp\left[-\int_0^{2\pi} h(z,\hat\sigma)\,d\hat\sigma\right] u(z,0) \\ + \int_0^{2\pi} \exp\left[-\int_\sigma^{2\pi} h(z,\hat\sigma)\,d\hat\sigma\right] f(z,\sigma)\,d\sigma, \end{aligned}$$

allowing one to compute $u(z,0)$ uniquely from $f \in L^1(\mathbb{R}^2)$, provided $h(z,\sigma) \not\equiv 0$ on the integral curve passing through $(z,0)$. Integrating along integral curves, one obtains the estimate

$$\|hu\|_1 \leq \|f\|_1, \tag{17}$$

where equality holds whenever $f \geq 0$. Thus Eq. (7) has a unique solution $u \in L^1(R^2, hdxdy)$ for every $f \in L^1(\mathbb{R}^2; dxdy)$, unless h vanishes a.e. on some annulus about the origin.

Instead of defining the vector field X on $\mathbb{R}^2$, we can also define it on $\Sigma = \mathbb{R}^2 \setminus \{(x,0) : x \geq 0\}$ endowed with Lebesgue measure. Then every integral curve is a circle about the origin with left and right endpoint on the "cut" $\Sigma_{\pm} = \{(x,0) : x \geq 0\}$, which is equipped with the measure $d\mu_{\pm}(x) = x\,dx$. Using the "periodic" boundary operator $K : L^1(\Sigma_+; d\mu_+) \to L^1(\Sigma_-; d\mu_-)$ which acts as the identity on $L^1(\mathbb{R}^+; x\,dx)$, we convert the problem described in the preceding paragraph into a problem as treated in Sec. 3, where $g_- = 0$ and $J = 0$.

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