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Factorization of Block Triangular Matrix Functions in Wiener Algebras on Ordered Abelian Groups

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Abstract. The notion of Wiener-Hopf type factorization is introduced in the abstract framework of Wiener algebras of matrix-valued functions on connected compact abelian groups. Factorizations of 2×2 block triangular matrix functions with elementary functions on the main diagonal are studied in detail. A conjecture is formulated concerning characterization of dual groups with the property that every invertible matrix function in a Wiener algebra admits a factorization. Applications of factorization are given to systems of difference equations and orthogonal families of functions.

1. Wiener Algebras

Let G be a (multiplicative) connected compact abelian group and let Γ be its (additive) character group. Recall that Γ consists of all continuous homomorphisms of G into the group of unimodular complex numbers. Since G is compact, Γ is discrete. In applications, often Γ is an additive subgroup of $\mathbb{R}$, the group of real numbers, or of $\mathbb{R}^k$, and G is the Bohr compactification of Γ. The group G can be also thought of as the character group of Γ, an observation that will be often used.

The group G has a unique invariant measure ν satisfying $\nu(G) = 1$, while Γ is equipped with the discrete topology and the (translation invariant) counting measure. It is well-known [31] that, because G is connected, Γ can be made into a linearly ordered group. So let $\preceq$ be a linear order such that $(\Gamma, \preceq)$ is an ordered group, i. e., if $x, y, z \in \Gamma$ and $x \preceq y$, then $x + z \preceq y + z$. *Throughout the paper it will be assumed that* Γ *is ordered with a fixed linear order* $\preceq$. The notation $\prec$, $\succeq$, $\succ$,

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max, min (with obvious meaning) will also be used. We put $\Gamma_+ = \{x \in \Gamma : x \succeq 0\}$ and $\Gamma_- = \{x \in \Gamma : x \preceq 0\}$.

For any nonempty set M, let $\ell^1(M)$ stand for the complex Banach space of all complex-valued M-indexed sequences $\boldsymbol{x} = \{x_j\}_{j \in M}$ having at most countably many nonzero terms that are finite with respect to the norm

$$\|\boldsymbol{x}\|_1 = \sum_{j \in M} |x_j|.$$

Then $\ell^1(\Gamma)$ is a commutative Banach algebra with unit element with respect to the convolution product $(\boldsymbol{x} * \boldsymbol{y})_j = \sum_{k \in \Gamma} x_k \, y_{j-k}$. Further, $\ell^1(\Gamma_+)$ and $\ell^1(\Gamma_-)$ are closed subalgebras of $\ell^1(\Gamma)$ containing the unit element.

Given $a = \{a_j\}_{j \in \Gamma} \in \ell^1(\Gamma)$, by the *symbol* of a we mean the complex-valued continuous function $\hat{a}$ on G defined by

$$\hat{a}(g) = \sum_{j \in \Gamma} a_j \langle j, g \rangle, \qquad g \in G, \tag{1}$$

where $\langle j, g \rangle$ stands for the action of the character $j \in \Gamma$ on the group element $g \in G$ (thus, $\langle j, g \rangle$ is a unimodular complex number), or, by Pontryagin duality, of the character $g \in G$ on the group element $j \in \Gamma$. The set $\{j \in \Gamma : a_j \neq 0\}$ will be called the *Fourier spectrum* of $\hat{a}$ given by (1). Since Γ is written additively, we have

$$\langle \alpha + \beta, g \rangle = \langle \alpha, g \rangle \cdot \langle \beta, g \rangle, \qquad \alpha, \beta \in \Gamma, \quad g \in G,$$

$$\langle \alpha, gh \rangle = \langle \alpha, g \rangle \cdot \langle \alpha, h \rangle, \qquad \alpha \in \Gamma, \quad g, h \in G.$$

We will use the shorthand notation e_α for the function

$$e_\alpha(g) = \langle \alpha, g \rangle, \qquad g \in G. \tag{2}$$

Thus, $e_{\alpha+\beta} = e_\alpha e_\beta$, $\alpha, \beta \in \Gamma$.

The entries a_j can be retrieved from the symbol $\hat{a}$ as follows:

$$a_j = \int_G \hat{a}(g) e_j(g) \, d\nu(g), \qquad j \in \Gamma.$$

The set of all symbols of elements $a \in \ell^1(\Gamma)$ forms an algebra $W(G)$ of continuous functions on G. The algebra $W(G)$ (with pointwise multiplication and addition) is isomorphic to $\ell^1(\Gamma)$. Denote by $W(G)_+$ (resp., $W(G)_-$) the algebra of symbols of elements in $\ell^1(\Gamma_+)$ (resp., $\ell^1(\Gamma_-)$).

For every Banach algebra $\mathcal{A}$ with identity element we denote its group of invertible elements by $\mathcal{G}(\mathcal{A})$. We have the following result:

Theorem 1. *Let G be a compact abelian group with the character group Γ, and let $W(G)^{n \times n}$ be the corresponding Wiener algebra of $n \times n$ matrix functions. Then $\hat{A} \in \mathcal{G}(W(G)^{n \times n})$ if and only if $\hat{A}(g) \in \mathcal{G}(\mathbb{C}^{n \times n})$ for every $g \in G$.*

This is an immediate consequence of Theorem A.1 in [21] (also proved in [1]). To rephrase the theorem in terms of this result, in the Banach algebra $\mathcal{A} = W(G)^{n\times n}$ we distinguish the closed Banach subalgebra $\mathcal{Z} = \{\hat{a}\, I_n : \hat{a} \in W(G)^{1\times 1}\}$ contained in the center of $\mathcal{A}$ and the algebra $\mathcal{F}$ of constant $n \times n$ matrix functions. Then the algebraic tensor product $\mathcal{Z} \otimes \mathcal{F}$ is dense in $\mathcal{A}$ and the multiplicative functionals on $\mathcal{Z}$ are the evaluation maps $\hat{a} \mapsto \hat{a}(g)$, where $g \in G$. Further, the multiplicative projectors (in the terminology of [21]) turn out to be the evaluation maps $\hat{A} \mapsto \hat{A}(g)$, where $g \in G$. Thus the theorem is immediate from Theorem A.1 in [21], which completes the proof.

We now consider the discrete abelian subgroup Γ' of Γ and denote its character group by G'. Then we introduce the annihilator

$$\Lambda = \{g \in G : \langle j, g\rangle = 1 \text{ for all } j \in \Gamma'\}, \tag{3}$$

which is a closed subgroup of G and hence a compact group. According to Theorem 2.1.2 in [31], we have $G' \simeq (G/\Lambda)$.

Let us now introduce the natural projection $\pi : G \to (G/\Lambda)$. We observe that the above theorem also applies to $W(G')^{n\times n}$. Given $A \in \ell^1(\Gamma)^{n\times n}$ with its Fourier spectrum restricted to Γ' (i.e., $A_j = 0$ for $j \in \Gamma \setminus \Gamma'$), we have two symbol definitions:

$$\hat{A}_\Gamma(g) = \sum_{j\in\Gamma'} A_j \langle j, g\rangle, \qquad g \in G,$$

$$\hat{A}_{\Gamma'}(g) = \sum_{j\in\Gamma'} A_j \langle j, g\rangle, \qquad g \in G',$$

where we have taken into account that $A_j = 0$ for $j \in \Gamma \setminus \Gamma'$. The latter can be replaced by

$$\hat{A}_{\Gamma'}([g]) = \sum_{j\in\Gamma'} A_j \langle j, g\rangle, \qquad [g] \in (G/\Lambda),$$

where $[g] = \pi(g)$ for $g \in G$. Obviously, $\langle j, g\rangle$ only depends on $[g] = \pi(g)$ if $j \in \Gamma'$. (If $[g_1] = [g_2]$, then $g_1 g_2^{-1} \in \Lambda$ and hence $\langle j, g_1 g_2^{-1}\rangle = 1$ for all $j \in \Gamma'$, which implies the statement.) Thus the two symbol definitions are equivalent in the sense that the value of "the" symbol $\hat{A}$ on $g \in G$ only depends on $[g] = \pi(g)$. The following result is now obtained.

Theorem 2. *Let Γ' be a subgroup of the discrete abelian group Γ, let G and G' be the character groups of Γ and Γ', respectively, and let Λ be defined by* (3)*. If $\hat{A} \in W(G)^{n\times n}$ is an element which has all of its Fourier spectrum within Γ', then $\hat{A} \in \mathcal{G}(W(G')^{n\times n})$ if and only if $\hat{A}(g) \in \mathcal{G}(\mathbb{C}^{n\times n})$ for every $g \in G$.*

For the proof note that according to Theorem 1, we need to show that $\hat{A}_{\Gamma'}([g]) \in \mathcal{G}(\mathbb{C}^{n\times n})$ for every $[g] \in (G/\Lambda)$ if and only if $\hat{A}(g) \in \mathcal{G}(\mathbb{C}^{n\times n})$ for every $g \in G$. By the paragraph preceding Theorem 2, the statement is clear, since under the hypotheses on $\hat{A}$ the values $\hat{A}(g)$ of the symbol $\hat{A}$ depend only on $[g] \in (G/\Lambda)$.

We now consider factorizations, and begin with scalar-valued functions. A linearly ordered group Γ is called *Archimedean* if for every $u, v \in \Gamma$, $u, v \succ 0$, there exists a positive integer n such that $nu \succeq v$.

Proposition 3. *Assume that Γ is Archimedean, or that Γ is finitely generated. If $a \in \ell^1(\Gamma)$ and $\hat{a}(g) \neq 0$ for every $g \in G$, then $\hat{a}$ admits a factorization*

$$\hat{a}(g) = \hat{a}_+(g) e_j(g) \hat{a}_-(g), \qquad g \in G, \tag{4}$$

where $a_+ \in \mathcal{G}(\ell^1(\Gamma_+))$, $a_- \in \mathcal{G}(\ell^1(\Gamma_-))$, and $j \in \Gamma$. The element $j = j(\hat{a})$ is uniquely determined by $\hat{a}$.

Using the terminology of the classical case $\Gamma = \mathbb{Z}$, G the unit circle, we call the element $j(\hat{a})$ the *winding number* of $\hat{a}$. In the case $\Gamma = \mathbb{R}$ and G is the Bohr compactification of $\mathbb{R}$, the winding number is known as the *mean motion.* For the particular case $\Gamma = \mathbb{R}^k$, Proposition 3 was proved in [25] by elementary means; see also [30], where the case when Γ is a subgroup of $\mathbb{R}^k$ is studied. For the case $\Gamma = \mathbb{Z}^d$ ($\mathbb{Z}$ stands for the group of integers) and G the d-torus, Proposition 3 was proved in [11]. The general situation of Proposition 3 is easily reduced to the cases just mentioned in view of Hölder's theorem (see, e.g., [15]) which asserts that every linearly ordered Archimedean abelian group is order isomorphic to a subgroup of $\mathbb{R}$ (with the natural linear order induced from $\mathbb{R}$).

The concept of factorization as in Proposition 3 extends to $n \times n$ matrix functions in $(W(G))^{n\times n}$. A *(left) factorization* of $A(g) \in (W(G))^{n\times n}$ is a representation of the form

$$A(g) = A_+(g)\,(\operatorname{diag}(e_{j_1}(g), \dots, e_{j_n}(g)))\,A_-(g), \quad g \in G, \tag{5}$$

where $A_+ \in \mathcal{G}((W(G)_+)^{n\times n})$, $A_- \in \mathcal{G}((W(G)_-)^{n\times n})$, and $j_1, \dots, j_n \in \Gamma$. Here and elsewhere we use $\operatorname{diag}(x_1, \dots, x_n)$ to denote the $n \times n$ diagonal matrix with $x_1, \dots, x_n$ on the main diagonal, in that order. The elements j_k are uniquely defined (if ordered $j_1 \preceq j_2 \preceq \dots \preceq j_n$); this can be proved by a standard argument (see [18, Theorem VIII.1.1]). The elements $j_1, \dots, j_n$ in (5) are called the *(left) factorization indices* of A. Analogously, by a *right factorization* of $A(g) \in (W(G))^{n\times n}$ we mean a representation of the form

$$A(g) = A_-(g)\,(\operatorname{diag}(e_{j_1}(g), \dots, e_{j_n}(g)))\,A_+(g), \quad g \in G, \tag{6}$$

where $A_+ \in \mathcal{G}((W(G)_+)^{n\times n})$, $A_- \in \mathcal{G}((W(G)_-)^{n\times n})$, and $j_1, \dots, j_n \in \Gamma$. Unless stated otherwise, all notions involving factorization will pertain to left factorization.

If all factorization indices are zero, the factorization is called *canonical.* If a factorization of A exists, the function A is called *factorizable.* For $\Gamma = \mathbb{Z}$ and G the unit circle, the definitions and the results are classical [19], [18], [7]; many results have been generalized to $\Gamma = \mathbb{R}^k$ (see [4] and references there), and Γ a subgroup of $\mathbb{R}^k$ (see [27], [30]).

We postpone development of a comprehensive theory of factorization to another occasion. In this paper we will be concerned mainly with factorization of block triangular 2×2 matrix functions of the form

$$A(g) = \begin{bmatrix} e_\lambda(g) I_p & 0 \\ c_1 e_{\alpha_1}(g) + \cdots + c_m e_{\alpha_m}(g) & e_{-\lambda}(g) I_q \end{bmatrix}, \qquad g \in G, \tag{7}$$

where $\lambda, \alpha_j \in \Gamma$, and $c_j \in \mathbb{C}^{q \times p}$, applications of factorizations of this kind of functions, and related problems. Here and elsewhere, we use I_k (or I if k is clear from context) to denote the $k \times k$ identity matrix, and $0_{p \times q}$ (or 0) to denote the $p \times q$ zero matrix.

We immediately observe, as in the case of $\Gamma = \mathbb{R}$ [4], that by applying elementary row (resp., column) operations that do not change the factorizability property and (in case a factorization exists) the indices, one can eliminate terms in (7) with $\alpha_k \succeq \lambda$ and with $\alpha_k \preceq -\lambda$. In particular, when $\lambda \preceq 0$, the function (7) is factorizable with indices $\pm\lambda$ (p and q times, respectively). Thus, we often assume in the sequel that

$$\lambda \succ 0, \qquad -\lambda \prec \alpha_1 \prec \ldots \prec \alpha_m \prec \lambda. \tag{8}$$

Note also that for A defined by (7), A^{-1} has finite Fourier spectrum.

Recently, there is an interest in special cases of left and right factorizations (5), (6) in which the factors $A_\pm$ and their inverses $A_\pm^{-1}$ have finite Fourier spectrum ([17], [5]). If this happens we say that the factorization is *finite.* The function $A \in (W(G))^{n \times n}$ will be called *finitely factorizable* if it admits a finite factorization.

Proposition 4. *If A is given as in* (7), *and if one of the four functions A_+, A_+^{-1}, A_-, A_-^{-1} in a factorization* (5) *has finite Fourier spectrum, then the factorization* (5) *is finite. An analogous statement applies to factorization* (6).

Proof. Say, A_+ has finite Fourier spectrum. By taking determinants in (5), we easily see that A_+ and A_- have constant nonzero determinants. Thus, A_+^{-1} has finite Fourier spectrum, as follows from the formula

$$A_+^{-1} = \frac{\operatorname{adj}(A_+)}{\det(A_+)},$$

where $\operatorname{adj}(X)$ stands for the algebraic adjoint of a matrix X, and it suffices to show that the Fourier spectrum of A_-^{-1} is finite. To this end, rewrite (5) in the form

$$AA_-^{-1} = A_+ \operatorname{diag}(e_{j_1}, \ldots, e_{j_n})$$

and observe that the right-hand side, and hence also the left-hand side, has finite Fourier spectrum. Using the special structure of A (7), it is easy to see that A_-^{-1} has finite Fourier spectrum. □

To conclude the introduction we describe the contents of the paper. In Section 2 the Portuguese transformation (known in the case of $\Gamma = \mathbb{R}$, see, e.g., [4]) is described in the context of abstract groups. In Section 3 we focus on factorization of matrices of the form (7) whose off-diagonal blocks are binomials ($m = 2$). As we shall see, even in this case factorization does not always exist without additional hypotheses on the order. Problems of factorizability vs. invertibility are treated in

Section 4. There, known examples [22], [4] of nonfactorable functions of the form (7) with off-diagonal trinomials in the real case ($\Gamma = \mathbb{R}$) play an essential role. Applications of factorization to systems of differential equations and to orthogonal functions are given in the last two sections.

2. Portuguese transformation

The Portuguese transformation is the main tool to prove factorizability and compute indices of 2×2 matrix functions of the form (7). In explicit form, under the assumption that the matrices c_k commute, it appeared in [3], although for some particular cases it was employed earlier [34], [32]. Without the commutativity hypotheses, the Portuguese transformation was given in [26] (where it was called the BKST transformation), and a thorough exposition, from the viewpoint of corona theorems, of the Portuguese transformation is found in [4]. All previous works on the Portuguese transformation were restricted to the case $\Gamma = \mathbb{R}$.

In this section, we give formulas for the Portuguese transformation in the abstract setting of ordered abelian groups. Since corona theorems are not generally available in this setting, we will use the algebraic approach employed in [26] which does not utilize results of corona type.

Consider (7), and assume that $p = q$ and that (8) holds. Furthermore, assume that the matrix c_1 is invertible. Then, renaming terms in the off-diagonal entry of (7), and replacing m by $m + 1$, we re-write (7) in the form

$$A(g) = \begin{bmatrix} e_\lambda(g) I_p & 0 \\ a e_{-\nu}(g)(I - \sum_{k=1}^{m} b_k e_{\gamma_k}(g)) & e_{-\lambda}(g) I_p \end{bmatrix}, \qquad g \in G. \tag{9}$$

Thus, $\nu = -\alpha_1$, $\gamma_j = \alpha_{j+1} + \nu$ ($j = 1, 2, \cdots, m$), $0 \prec \gamma_1 \prec \cdots \prec \gamma_m \prec \lambda + \nu$, and $b_1, \dots b_m$ are nonzero $p \times p$ matrices. We make another assumption:

$$(\aleph) \qquad n\gamma_1 \succ \lambda + \nu \ \text{ for some positive integer } n.$$

Denote by $\mathbb{Z}^+$ the set of nonnegative integers, and for any $N = (n_1, \dots, n_m) \in (\mathbb{Z}^+)^m$ define

$$y_N(A) = \sum b_{j_1} b_{j_2} \dots b_{j_w}, \tag{10}$$

where $w = n_1 + \cdots + n_m$, and the sum in (10) is taken over all ordered w-tuples of integers $(j_1, \dots, j_w)$ exactly n_k of which are equal to k, for $k = 1, \dots, m$ (if $N = (0, \dots, 0)$, we let $y_N(A) = I$). Finally, define

$$f(g) = \sum y_{(n_1, \dots, n_m)}(A)(g) a^{-1} e_{-\lambda + n_1\gamma_1 + \cdots + n_m\gamma_m}(g), \qquad g \in G, \tag{11}$$

where the sum is taken over all $(n_1, \dots, n_m) \in (\mathbb{Z}^+)^m$ for which

$$-\nu \prec -\lambda + n_1\gamma_1 + \cdots + n_m\gamma_m \prec \nu. \tag{12}$$

If the set of $(n_1, \dots, n_m)$'s satisfying (12) is empty (for example, this would be the case when $\nu \prec 0$), then we take f to be the zero function. Condition ($\aleph$) guarantees that the sum in (11) is finite.

Theorem 5. *Under the conditions in the previous paragraph, there exist* $A_+ \in \mathcal{G}((W(G)_+)^{n\times n})$ *and* $A_- \in \mathcal{G}((W(G)_-)^{n\times n})$ *with finite Fourier spectra such that the equality*

$$A = A_+ \, B \, A_-$$

holds, where

$$B = \begin{bmatrix} e_\nu I_p & 0 \\ f & e_{-\nu} I_p \end{bmatrix}.$$

In particular, the matrix functions A *and* B *are factorizable (resp., finitely factorizable) only simultaneously, and in case they are factorizable, they have the same factorization indices.*

Theorem 5 is proved by using the calculations given in [26, Section 2]. The formulas for $A_\pm$ (which we will not reproduce) are also given there.

Note that Assumption ($\aleph$) is valid for the function B. Thus, one can apply the Portuguese transformation again to B provided the matrix coefficient of the lowest term in the bottom left corner of B is invertible (analogously to the condition of invertibility of c_1). This condition is always satisfied if $p = 1$. Once the formulas of Theorem 5 are in place, one can repeat without difficulty many results on factorization that depend on the Portuguese transformation. We refer the reader to [4, Chapters 14 and 15] and references therein. Here, we state just one such result.

Theorem 6. *Let* A *have the form* (7), *where* $0 \preceq \alpha_1 \prec \ldots \prec \alpha_m$, *and let the matrix* c_1 *be invertible. Assume that Condition* ($\aleph$) *holds. Then* A *is finitely factorizable, and the factorization indices are* $\pm\alpha_1$ *(*p *times each).*

Observe that the invertibility of c_1 is essential in Theorem 6; in [26] an example was given (with $\Gamma = \mathbb{R}$) of a non-factorizable function of the form (7) with $m = 4$ and $0 \preceq \alpha_1 \prec \alpha_2 \prec \alpha_3 \prec \alpha_4$.

3. Off diagonal binomials

We prove in this section the following result:

Theorem 7. *Let* A *have the form*

$$A(g) = \begin{bmatrix} e_\lambda(g) I_p & 0 \\ c_1 e_\sigma(g) - c_2 e_\mu(g) & e_{-\lambda}(g) I_q \end{bmatrix}, \qquad g \in G, \tag{13}$$

where $\lambda, \sigma, \mu \in \Gamma$, $-\lambda \prec \sigma \prec \mu \prec \lambda$, *and* $c_j \in \mathbb{C}^{q\times p}$ *(the cases when one or both of* c_1 *and* c_2 *are equal to zero are not excluded). Assume that the following condition holds:*

$$n(\mu - \sigma) \succeq \max\{\lambda - \sigma, \lambda + \mu\} \quad \text{for some positive integer } n. \tag{14}$$

Then A *admits a finite factorization.*

For the case $\Gamma = \mathbb{R}$, Theorem 7 was proved in [23] (in this case (14) is automatically satisfied). We adapt the approach of [23] to prove Theorem 7 in full generality.

Proof. Let r and s be the nonnegative integers having the following properties:

$$\lambda - \sigma - (s+1)(\mu - \sigma) \prec 0 \preceq \lambda - \sigma - s(\mu - \sigma), \tag{15}$$
$$\lambda + \mu - (r+1)(\mu - \sigma) \prec 0 \preceq \lambda + \mu - r(\mu - \sigma). \tag{16}$$

In view of (14), s and r are correctly defined.

Consider first the case when $p = q$ and both matrices c_1 and c_2 are invertible. We distinguish between three possibilities:

$$s > r \Leftrightarrow \mu + \lambda - s(\mu - \sigma) \prec 0; \tag{17}$$
$$s < r \Leftrightarrow \lambda - \sigma - r(\mu - \sigma) \prec 0; \tag{18}$$
$$s = r \Leftrightarrow \begin{cases} \lambda - \sigma - (s+1)(\mu - \sigma) \prec 0 \preceq \lambda - \sigma - s(\mu - \sigma), \\ \lambda + \mu - (s+1)(\mu - \sigma) \prec 0 \preceq \lambda + \mu - s(\mu - \sigma). \end{cases} \tag{19}$$

Then the following formulas (which can be verified in a straightforward manner) give a factorization $A = A_+ \Lambda A_-$ of A. In the case that (17) holds:

$$A_+ = \begin{bmatrix} -\sum_{j=0}^{s-1}(c_2^{-1}c_1)^j c_2^{-1} e_{\lambda-\mu-j(\mu-\sigma)} & I \\ I & 0 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} e_\mu I & 0 \\ 0 & e_{-\mu} I \end{bmatrix},$$

$$A_- = \begin{bmatrix} c_1 e_{\sigma-\mu} - c_2 & e_{-\mu-\lambda} I \\ (c_2^{-1}c_1)^s e_{\mu+\lambda-s(\mu-\sigma)} & \sum_{j=0}^{s-1}(c_2^{-1}c_1)^j c_2^{-1} e_{-j(\mu-\sigma)} \end{bmatrix}.$$

Clearly, $A_+ \in \mathcal{G}((W(G)_+)^{2p\times 2p})$, $A_- \in (W(G)_-)^{2p\times 2p}$, and since (use Schur complements)

$$\begin{aligned} \pm\det(A_-) &= (e_{-\mu-\lambda})^p \det\Big((c_2^{-1}c_1)^s e_{\mu+\lambda-s(\mu-\sigma)} \\ &\quad - e_{\mu+\lambda}\left(\sum_{j=0}^{s-1}(c_2^{-1}c_1)^j c_2^{-1} e_{-j(\mu-\sigma)}\right)(c_1 e_{\sigma-\mu} - c_2)\Big) \\ &= (e_{-\mu-\lambda})^p \det(e_{\mu+\lambda} I) = 1 \end{aligned}$$

it follows that $A_-^{-1} \in (W(G)_-)^{2p\times 2p}$. In the case that (18) holds:

$$A_+ = \begin{bmatrix} e_{\lambda-\sigma} I & -c_1^{-1}\sum_{j=0}^{r-1}(c_2 c_1^{-1})^j e_{j(\mu-\sigma)} \\ -c_2 e_{\mu-\sigma} + c_1 & (c_2 c_1^{-1})^r e_{r(\mu-\sigma)-\lambda+\sigma} \end{bmatrix},$$

$$\Lambda = \begin{bmatrix} e_\sigma I & 0 \\ 0 & e_{-\sigma} I \end{bmatrix}, \quad A_- = \begin{bmatrix} I & c_1^{-1}\sum_{j=0}^{r-1}(c_2 c_1^{-1})^j e_{j(\mu-\sigma)-\sigma-\lambda} \\ 0 & I \end{bmatrix}.$$

Again, clearly $A_+ \in (W(G)_+)^{2p\times 2p}$, $A_- \in \mathcal{G}((W(G)_-)^{2p\times 2p})$, and since

$$\begin{aligned}\det(A_+) &= (e_{\lambda-\sigma})^p \det\left(e_{-\lambda+\sigma}(-c_2 e_{\mu-\sigma}+c_1)\left(c_1^{-1}\sum_{j=0}^{r-1}(c_2c_1^{-1})^j e_{j(\mu-\sigma)}\right)\right.\\ &+ \left.(c_2c_1^{-1})^r e_{r(\mu-\sigma)-\lambda+\sigma}\right) = 1\end{aligned}$$

we also have $A_+ \in \mathcal{G}((W(G)_+)^{2p\times 2p})$. In the case that (19) holds:

$$A_+ = \begin{bmatrix} I+\sum_{j=1}^{s}(c_1^{-1}c_2)^j e_{j(\mu-\sigma)} & -\sum_{j=0}^{s-1}(c_2^{-1}c_1)^j c_2^{-1} e_{\lambda-\mu-j(\mu-\sigma)} \\ -c_2(c_1^{-1}c_2)^s e_{s(\mu-\sigma)-\lambda+\mu} & I \end{bmatrix},$$

$$\Lambda = \begin{bmatrix} e_{\lambda-s(\mu-\sigma)}I & 0 \\ 0 & e_{-\lambda+s(\mu-\sigma)}I \end{bmatrix},$$

$$A_- = \begin{bmatrix} (c_2^{-1}c_1)^s & e_{s(\mu-\sigma)-\mu-\lambda}\sum_{j=0}^{s-1}(c_2^{-1}c_1)^j c_2^{-1} e_{-j(\mu-\sigma)} \\ c_1 e_{\sigma+\lambda-s(\mu-\sigma)} & e_{-s(\mu-\sigma)}I+\sum_{j=0}^{s-1}(c_2c_1^{-1})^{s-j} e_{-j(\mu-\sigma)} \end{bmatrix}.$$

It is easily verified that $A_\pm \in (W(G)_\pm)^{2p\times 2p}$ (follows from (19)). Similarly, $A_\pm^{-1} \in (W(G)_\pm)^{2p\times 2p}$. This concludes the proof in the case when $p=q$ and both c_1 and c_2 are invertible.

Consider now the general case. Applying the transformation

$$A \mapsto \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} A \begin{bmatrix} T_1^{-1} & 0 \\ 0 & T_2^{-1} \end{bmatrix},$$

where T_1 and T_2 are invertible matrices of sizes $p\times p$ and $q\times q$, respectively, the matrices c_1 and c_2 are replaced by $T_2c_1T_1^{-1}$ and $T_2c_2T_1^{-1}$, respectively. Now use the well-known canonical form (also known as Kronecker form) for pairs of rectangular matrices under the transformation $(X,Y)\mapsto(TXS,TYS)$, where T and S are invertible matrices of appropriate sizes, see, e.g., [16], or [20, Appendix]. Ignoring the zero blocks in the canonical form of (c_1,c_2) (trivial case), and blocks where both matrices are invertible (this case was taken care of already), we are left with the following situations to consider:

(a) c_1 and c_2 are of size $k\times(k+1)$ (so $q=k$, $p=k+1$) of the form

$$c_1 = \begin{bmatrix} I_k & 0_{k\times 1} \end{bmatrix}, \quad c_2 = \begin{bmatrix} 0_{k\times 1} & I_k \end{bmatrix}.$$

(b) c_1 and c_2 are of size $(k+1)\times k$ (so $q=k+1$, $p=k$) of the form

$$c_1 = \begin{bmatrix} I_k \\ 0_{1\times k} \end{bmatrix}, \quad c_2 = \begin{bmatrix} 0_{1\times k} \\ I_k \end{bmatrix}.$$

(c) c_1 is the $k\times k$ upper triangular nilpotent Jordan block, denoted by V_k, and $c_2 = I_k$ (so $p=q=k$).

(d) $c_1 = I_k$, and $c_2 = V_k$ (so $p=q=k$).

Let J_k be the $k\times k$ matrix with 1's along the top-right to the left-bottom diagonal and zeros in all other positions. If $A(g) = [a_{i,j}(g)]_{i,j=1}^n \in (W(G))^{n\times n}$, then A^* will denote the matrix function defined by $[\overline{a_{j,i}(g)}]_{i,j=1}^n$; clearly, $A^* \in (W(G))^{n\times n}$, and if $A \in (W(G)_\pm)^{n\times n}$, then $A^* \in (W(G)_\mp)^{n\times n}$. The transformation

$$A \mapsto \begin{bmatrix} 0 & J_k \\ J_k & 0 \end{bmatrix} A^* \begin{bmatrix} 0 & J_k \\ J_k & 0 \end{bmatrix}$$

transforms the case (c) to the case (d). The transformation

$$A \mapsto \begin{bmatrix} 0 & J_{k+1} \\ J_k & 0 \end{bmatrix} A^* \begin{bmatrix} 0 & J_k \\ J_{k+1} & 0 \end{bmatrix}$$

transforms the case (b) to the case (a). Thus, it will suffice to consider the cases (a) and (d).

We will need the nonnegative integers s and r introduced by (15) and (16), respectively, as well as two additional nonnegative integers r' and s' defined as follows: $s' = s$ if $n(\mu-\sigma) \neq \lambda - \sigma$ for any positive integer n, and $s' = s-1$ otherwise; $r' = r$ if $n(\mu-\sigma) \neq \lambda+\mu$ for any positive integer n, and $r' = r-1$ otherwise.

Consider the case (d) first:

$$A = \begin{bmatrix} e_\lambda I_k & 0 \\ e_\sigma I_k - e_\mu V_k & e_{-\lambda} I_k \end{bmatrix}. \tag{20}$$

Let

$$B_+ = \begin{bmatrix} I_k - e_{\mu-\sigma}V_k & -e_{\lambda-\sigma}I_k \\ 0 & \sum_{j=0}^{k-1} e_{j(\mu-\sigma)}V_k^j \end{bmatrix}, \quad B_- = \begin{bmatrix} \sum_{j=0}^{r-1} e_{j(\mu-\sigma)-\lambda-\sigma}V_k^j & I_k \\ -I_k & 0 \end{bmatrix}.$$

Clearly, $B_\pm \in \mathcal{G}((W(G)_\pm)^{n\times n})$. Using the property that $V_k^k = 0$, we obtain

$$\begin{aligned} B_+AB_- &= \begin{bmatrix} e_{-\sigma}I_k & 0 \\ -\sum_{j=r}^{k-1} e_{j(\mu-\sigma)-\lambda}V_k^j & e_\sigma I_k \end{bmatrix} \\ &= \begin{cases} \begin{bmatrix} e_{-\sigma}I_k & 0 \\ 0 & e_\sigma I_k \end{bmatrix} & \text{if } r \geq k; \\ \begin{bmatrix} e_{-\sigma}I_r & 0 & 0 \\ 0 & H_{k-r} & 0 \\ 0 & 0 & e_\sigma I_r \end{bmatrix} & \text{if } 1 \leq r < k; \end{cases} \end{aligned}$$

where

$$H_{k-r} = \begin{bmatrix} e_{-\sigma}I_{k-r} & 0 \\ -\sum_{j=0}^{k-r-1} e_{(j+r)(\mu-\sigma)-\lambda}V_{k-r}^j & e_\sigma I_{k-r} \end{bmatrix}.$$

If $s' < r < k$, then since we have $\lambda - \mu \preceq (r-1)(\mu-\sigma)$ (this relation is easily obtained from $(r+1)(\mu-\sigma) \succ \lambda+\mu$), it follows that H_{k-r} admits a factorization of the form

$$H_{k-r} = \begin{bmatrix} I_{k-r} & 0 \\ -\sum_{j=0}^{k-r-1} e_{(j+r-1)(\mu-\sigma)-\lambda+\mu}V_{k-r}^j & I_{k-r} \end{bmatrix} \begin{bmatrix} e_{-\sigma}I_{k-r} & 0 \\ 0 & e_\sigma I_{k-r} \end{bmatrix}.$$

If $r \le \min\{s', k-1\}$, then let

$$C_+ = \begin{bmatrix} \sum_{j=0}^{k-r-1} e_{j(\mu-\sigma)} V_{k-r}^j & e_{\lambda-\sigma-r(\mu-\sigma)} I_{k-r} \\ 0 & I_{k-r} - e_{\mu-\sigma} V_{k-r} \end{bmatrix},$$

$$C_- = \begin{bmatrix} e_{\lambda+\sigma-r(\mu-\sigma)} I_{k-r} & -I_{k-r} \\ I_{k-r} & 0 \end{bmatrix}.$$

We have $C_\pm \in \mathcal{G}((W(G)_\pm)^{(k-r)\times(k-r)})$; to verify that $C_- \in (W(G)_-)^{(k-r)\times(k-r)}$ we need to verify the relation $\lambda + \sigma \preceq r(\mu - \sigma)$, which in turn follows easily from $(r+1)(\mu-\sigma) \succ \lambda + \mu$. Furthermore,

$$C_+ H_{k-r} C_- = \begin{bmatrix} e_{\lambda-r(\mu-\sigma)} I_{k-r} & 0 \\ -e_\mu V_{k-r} & e_{-\lambda+r(\mu-\sigma)} I_{k-r} \end{bmatrix}$$

$$= \begin{cases} \begin{bmatrix} e_{\lambda-r(\mu-\sigma)} & 0 \\ 0 & e_{-\lambda+r(\mu-\sigma)} \end{bmatrix} & \text{if } s' \ge r = k-1; \\ \begin{bmatrix} e_{\lambda-r(\mu-\sigma)} & 0 & 0 \\ 0 & \widetilde{H}_{k-r-1} & 0 \\ 0 & 0 & e_{-\lambda+r(\mu-\sigma)} \end{bmatrix} & \text{if } 1 \le r \le \min\{s', k-2\}, \end{cases} \tag{21}$$

where

$$\widetilde{H}_{k-r-1} = \begin{bmatrix} e_{\lambda-r(\mu-\sigma)} I_{k-r-1} & 0 \\ -e_\mu I_{k-r-1} & e_{-\lambda+r(\mu-\sigma)} I_{k-r-1} \end{bmatrix}. \tag{22}$$

Factorizability of $\widetilde{H}_{k-r-1}$ follows from Theorem 6, or could be verified directly.

Finally, consider the case (a):

$$A = \begin{bmatrix} e_\lambda I_k & 0 & 0 \\ 0 & e_\lambda & 0 \\ e_\sigma I_k - e_\mu V_k & h & e_{-\lambda} I_k \end{bmatrix}, \quad \text{where} \quad h = \begin{bmatrix} 0_{(k-1)\times 1} \\ -e_\mu \end{bmatrix}.$$

Let

$$B_+ = \begin{bmatrix} I_k - e_{\mu-\sigma} V_k & b & -e_{\lambda-\sigma} I_k \\ 0 & 1 & 0 \\ 0 & 0 & \sum_{j=0}^{k-1} e_{j(\mu-\sigma)} V_k^j \end{bmatrix}, \quad \text{where} \quad b = \begin{bmatrix} 0_{(k-1)\times 1} \\ -e_{\mu-\sigma} \end{bmatrix},$$

$$B_- = \begin{bmatrix} \sum_{j=0}^{r-1} e_{j(\mu-\sigma)-\lambda-\sigma} V_k^j & 0 & I_k \\ 0 & 1 & 0 \\ -I_k & 0 & 0 \end{bmatrix}.$$

Clearly, $B_+ \in \mathcal{G}((W(G)_+)^{(2k+1)\times(2k+1)})$ and (because of $(r-1)(\mu-\sigma)-\lambda-\sigma \preceq 0$, which follows from (16)) we have $B_- \in \mathcal{G}((W(G)_-)^{(2k+1)\times(2k+1)})$. A computation shows that

$$\Phi_0 := B_+ A B_- = \begin{bmatrix} e_{-\sigma} I_k & 0 & 0 \\ 0 & e_\lambda & 0 \\ -\sum_{j=r}^{k-1} e_{j(\mu-\sigma)-\lambda} V_k^j & h_k & e_\sigma I_k \end{bmatrix},$$

where

$$(h_k)^T = \begin{bmatrix} -e_{(k-1)(\mu-\sigma)+\mu} & \cdots & -e_{(\mu-\sigma)+\mu} & -e_\mu \end{bmatrix}. \tag{23}$$

Let us define for $j = 0, 1, \dots, \min\{k, s'\} - 1$ the auxiliary matrices

$$R_{+,k-j} = \begin{bmatrix} 1 & 0 & e_{\lambda-\mu-j(\mu-\sigma)} \\ 0 & I_{k-j-1} & h_{k-j-1}e_{-\sigma} \\ 0 & 0 & 1 \end{bmatrix},$$

$$R_{-,k-j} = \begin{bmatrix} e_{\sigma-\mu} & 0 & -1 \\ 0 & I_{k-j-1} & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad R_{k-j} = \begin{bmatrix} e_{\lambda-j(\mu-\sigma)} & 0 \\ h_{k-j} & e_\sigma I_{k-j} \end{bmatrix}.$$

Clearly, $R_{-,k-j} \in \mathcal{G}((W(G)_-)^{(k-j+1)\times(k-j+1)})$. By (15), we easily obtain $R_{+,k-j} \in \mathcal{G}((W(G)_+)^{(k-j+1)\times(k-j+1)})$. We also have the recurrence relations

$$R_{+,k-j}R_{k-j}R_{-,k-j} = \begin{bmatrix} R_{k-j-1} & 0 \\ 0 & e_\mu \end{bmatrix}, \quad R_0 = e_{\lambda-k(\mu-\sigma)}, \tag{24}$$

for $j = 0, \dots, \min\{k, s'\} - 1$.

If $k \le \min\{r, s'\}$, then $\Phi_0 = \operatorname{diag}(e_{-\sigma}I_k, R_k)$, and applying consecutively (24) for $j = 0, \dots, k-1$, we obtain a factorization $A = A_+\Lambda A_-$ with $\Lambda = \operatorname{diag}(e_{-\sigma}I_k, e_{\lambda-k(\mu-\sigma)}, e_\mu I_k)$. If $s' < k \le r$, then again $\Phi_0 = \operatorname{diag}(e_{-\sigma}I_k, R_k)$. Applying (24) for $j = 0, 1, \dots, s'-1$, we reduce R_k to $\operatorname{diag}(R_{k-s'}, e_\mu I_{s'})$, and in view of the inequality $\lambda - \mu \preceq s'(\mu - \sigma)$ the equation

$$R_{k-s'} = \begin{bmatrix} 1 & 0 \\ h_{k-s'}e_{s'(\mu-\sigma)-\lambda} & I_{k-s'} \end{bmatrix} \begin{bmatrix} e_{\lambda-s'(\mu-\sigma)} & 0 \\ 0 & e_\sigma I_{k-s'} \end{bmatrix}$$

represents a factorization of $R_{k-s'}$. If $s' < r < k$, then in view of the inequalities

$$\lambda - \mu \preceq s'(\mu - \sigma) \preceq (r-1)(\mu - \sigma)$$

(which follow from (15)), we have

$$A_+ := \begin{bmatrix} I_k & 0 & 0 \\ 0 & 1 & 0 \\ \sum_{j=r}^{k-1} e_{j(\mu-\sigma)-\lambda+\sigma}V_k^j & 0 & I_k \end{bmatrix} \in \mathcal{G}((W(G)_+)^{(2k+1)\times(2k+1)}),$$

and since $A_+\Phi_0 = \operatorname{diag}(e_{-\sigma}I_k, R_k)$, we have reduced the proof of factorizability of A to that of R_k. Now repeat the arguments given above for the case $k \le \min\{r, s'\}$ to obtain the factorizability of R_k.

It remains to consider the case $r \le \min\{k-1, s'\}$. Then

$$\lambda - \mu \succ (r-1)(\mu - \sigma).$$

We have the recurrence relations (which can be verified using (24)):

$$\Phi_{j+1} = \begin{bmatrix} I_k & 0 & 0 \\ 0 & R_{+,k-j} & 0 \\ 0 & 0 & I_j \end{bmatrix} \Phi_j \begin{bmatrix} I_k & 0 & 0 \\ 0 & R_{-,k-j} & 0 \\ 0 & 0 & I_j \end{bmatrix}, \quad j = 0, 1, \dots, r-2,$$

where

$$\Phi_j = \operatorname{diag}(e_{-\sigma}I_r, F_j, e_\mu I_j), \qquad j = 1, \dots, r-1,$$

and where

$$F_j = \begin{bmatrix} e_{-\sigma} I_{k-r} & 0 & 0 \\ 0 & e_{\lambda - j(\mu-\sigma)} & 0 \\ f_j & h_{k-j} & e_\sigma I_{k-j} \end{bmatrix}, \ f_j = \begin{bmatrix} -\sum_{\ell=0}^{k-r-1} e_{(\ell+r)(\mu-\sigma)-\lambda} V_{k-r}^{\ell} \\ 0_{(r-j)\times(k-r)} \end{bmatrix}.$$

Thus, it suffices to construct a factorization for F_{r-1}. Note that

$$\begin{aligned} F_{r-1} &= \begin{bmatrix} e_{-\sigma} I_{k-r+1} & 0 \\ -\sum_{j=0}^{k-r} e_{(j+r)(\mu-\sigma)-\lambda} V_{k-r+1}^{j} & e_\sigma I_{k-r+1} \end{bmatrix} \\ &\times \begin{bmatrix} I_{k-r} & 0 & 0 \\ 0 & e_{\lambda+\sigma-(r-1)(\mu-\sigma)} & 0 \\ 0 & 0 & I_{k-r+1} \end{bmatrix}. \end{aligned}$$

Letting

$$C_+ = \begin{bmatrix} \sum_{j=0}^{k-r} e_{j(\mu-\sigma)} V_{k-r+1}^{j} & e_{\lambda-\sigma-r(\mu-\sigma)} I_{k-r+1} \\ 0 & I_{k-r+1} - e_{\mu-\sigma} V_{k-r+1} \end{bmatrix},$$

$$C_- = \begin{bmatrix} e_{\lambda+\sigma-r(\mu-\sigma)} I_{k-r} & 0 & -I_{k-r} & 0 \\ 0 & e_{\sigma-\mu} & 0 & -1 \\ I_{k-r} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

we have (in view of (15) and (16))

$$C_\pm \in \mathcal{G}((W(G)_\pm)^{(2k-2r+2)\times(2k-2r+2)}),$$

and we obtain analogously to (21) that

$$\begin{aligned} C_+ F_{r-1} C_- &= \left[\begin{array}{c|cc} e_{\lambda-r(\mu-\sigma)} I_{k-r+1} & 0 & \\ \hline -e_\mu V_{k-r+1} & e_{r(\mu-\sigma)-\lambda} I_{k-r} & 0 \\ & 0 & e_\mu \end{array}\right] \\ &= \operatorname{diag}\left(e_{\lambda-r(\mu-\sigma)}, \widetilde{H}_{k-r}, e_\mu\right), \end{aligned}$$

where $\widetilde{H}_{k-r}$ has the form (22) (with size of blocks $(k-r)\times(k-r)$). The factorizability of $\widetilde{H}_{k-r}$ follows from Theorem 6. □

Using the proof of Theorem 7, one can deduce formulas for the factorization indices, and in particular, necessary and sufficient conditions for canonical factorization. We omit the details, as they are long and cumbersome.

Once formulas for the factors in the proof of Theorem 7 are available, we may use them to derive additional results. For example:

Theorem 8. *Let A be given by (13), where $p = q$, $-\lambda \prec \sigma \prec \mu \prec \lambda$, and $c_j \in \mathbb{C}^{p\times p}$. Assume that one of the two conditions below is satisfied:*

(1) *$n(\mu-\sigma) \prec \lambda - \mu$ for all positive integers n, the matrix c_2 is invertible, and the spectral radius of $c_2^{-1}c_1$ is less than one;*
(2) *$n(\mu-\sigma) \prec \lambda + \sigma$ for all positive integers n, the matrix c_1 is invertible, and the spectral radius of $c_2 c_1^{-1}$ is less than one.*

Then A is factorizable.

Proof. If (1) holds, the factorization is given by the formulas

$$A_+ = \begin{bmatrix} -\sum_{j=0}^{\infty}(c_2^{-1}c_1)^j c_2^{-1} e_{\lambda-\mu-j(\mu-\sigma)} & I \\ I & 0 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} e_\mu I & 0 \\ 0 & e_{-\mu} I \end{bmatrix},$$

$$A_- = \begin{bmatrix} c_1 e_{\sigma-\mu} - c_2 & e_{-\mu-\lambda} I \\ 0 & \sum_{j=0}^{\infty}(c_2^{-1}c_1)^j c_2^{-1} e_{-j(\mu-\sigma)} \end{bmatrix}.$$

If (2) holds, the factorization is given by formulas

$$A_+ = \begin{bmatrix} e_{\lambda-\sigma} I & -c_1^{-1}\sum_{j=0}^{\infty}(c_2 c_1^{-1})^j e_{j(\mu-\sigma)} \\ -c_2 e_{\mu-\sigma} + c_1 & 0 \end{bmatrix},$$

$$\Lambda = \begin{bmatrix} e_\sigma I & 0 \\ 0 & e_{-\sigma} I \end{bmatrix}, \quad A_- = \begin{bmatrix} I & c_1^{-1}\sum_{j=0}^{\infty}(c_2 c_1^{-1})^j e_{j(\mu-\sigma)-\sigma-\lambda} \\ 0 & I \end{bmatrix}. \qquad \square$$

An interesting case occurs when both inequalities in conditions (1) and (2) of Theorem 8 are satisfied. We then have:

Theorem 9. *Let A be given as in Theorem 8, and assume that*

$$n(\mu-\sigma) \prec \min\{\lambda-\mu, \lambda+\sigma\} \quad \text{for all integers } n > 0. \tag{25}$$

If the matrix c_1 is invertible and the spectrum of $c_1^{-1}c_2$ does not intersect the unit circle, or if c_2 is invertible and the spectrum of $c_2^{-1}c_1$ does not intersect the unit circle, then A admits a factorization. Moreover, the factorization indices belong to the set $\{\pm\sigma, \pm\mu\}$.

Proof. Say c_1 is invertible. Using a transformation $c_j \mapsto Sc_jT$, $j = 1, 2$, for suitable invertible S and T, we can assume without loss of generality that $c_1 = I$ and c_2 is in the Jordan form. Now apply the result of Theorem 8 to each Jordan block of c_2 separately, to obtain Theorem 9. □

The condition on the spectrum of $c_1^{-1}c_2$ (or of $c_2^{-1}c_1$) in Theorem 9 is essential, as the following example shows: Let $\Gamma = \mathbb{Z}^2$ with the lexicographical order, G is the 2-torus, and

$$A = \begin{bmatrix} z & 0 \\ 1-w & z^{-1} \end{bmatrix} \tag{26}$$

(here z is the first variable, w is the second variable; so z corresponds to $e_{(1,0)}$, w corresponds to $e_{(0,1)}$, and the lexicographical order is such that $(1,0) \succ (0,n)$ for every integer n). The function (26) is not factorizable. Indeed, if it was, then considering A as a function of z only, with w a parameter, we would obtain that the factorization indices of A are independent of $w \in \mathbb{T}$ (the unit circle), whereas in fact the factorization indices are zeros if $w \neq 1$, and they are ± 1 if $w = 1$:

$$\begin{bmatrix} z & 0 \\ 1-w & z^{-1} \end{bmatrix} = \begin{bmatrix} 1 & (1-w)^{-1}z \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -(1-w)^{-1} \\ 1-w & z^{-1} \end{bmatrix}, \quad w \neq 1$$

is a canonical factorization.

4. Invertibility vs factorizability

Clearly, a necessary condition for the existence of a factorization is that $\det A(g) \neq 0$ for every $g \in G$. For the case $\Gamma = \mathbb{Z}$, G is the unit circle, this condition is well known to be also sufficient for the existence of a factorization. An example (a 2×2 triangular matrix function of the form (7)) shows that in general this condition is not sufficient (see Section 15.1 in [4]). Motivated by this example, we formulate a conjecture.

Conjecture 10. *Assume $n \geq 2$. Every function $A \in \mathcal{G}(W(G)^{n\times n})$ admits a factorization if and only if Γ is (isomorphic to) a subgroup of the additive group of rational numbers $\mathbb{Q}$.*

For $n = 1$ the statement of the conjecture is false (see Proposition 3).

Toward this conjecture, we prove the following result:

Theorem 11. *If Γ is not isomorphic to a subgroup of $\mathbb{Q}$, then there exists a 2×2 matrix function of the form*

$$A(g) = \begin{bmatrix} e_\lambda(g) & 0 \\ c_1 e_{\alpha_1}(g) + c_2 + c_3 e_{\alpha_3}(g) & e_{-\lambda}(g) \end{bmatrix}, \qquad g \in G, \tag{27}$$

where $\lambda, \alpha_1, \alpha_2, \alpha_3 \in \Gamma$, and $c_1, c_2, c_3 \in \mathbb{C}$, which does not admit a finite factorization.

Conversely, if Γ is isomorphic to a subgroup of $\mathbb{Q}$, then every function $A \in \mathcal{G}(W(G)^{n\times n})$ such that its Fourier spectrum is contained in a finitely generated subgroup, admits a factorization.

We need several preliminary results for the proof of Theorem 11. A subset S of $\mathbb{R}^k$ (the vectors in $\mathbb{R}^k$ are understood as column vectors) is called a *halfspace* if it has the following properties (i) - (iv): (i) $\mathbb{R}^k = S \cup (-S)$; (ii) $S \cap (-S) = \{0\}$; (iii) if $x, y \in S$ then $x + y \in S$; (iv) if $x \in S$ and α is a nonnegative real number, then $\alpha x \in S$. Clearly, every halfspace S in $\mathbb{R}^k$ induces a linear order $\preceq$ on $\mathbb{R}^k$ (there are linear orders on $\mathbb{R}^k$ which are not induced by any halfspace): The linear order $\preceq_S$ induced by a halfspace S is defined by the property that $x \preceq_S y$ if and only if $y - x \in S$. A standard example of a halfspace is given by

$$E_k = \{(x_1, \ldots, x_k)^T \in \mathbb{R}^k : x_1 = x_2 = \cdots = x_{j-1} = 0,\ x_j \neq 0 \Rightarrow x_j > 0\} \cup \{0\}, \tag{28}$$

where the superscript ^{T} denotes the transpose.

All halfspaces in $\mathbb{R}^k$ are described as follows:

Lemma 12. *A set $S \subset \mathbb{R}^k$ is a halfspace if and only if there exists a real invertible $k \times k$ matrix A such that*

$$S = AE_k \stackrel{\text{def}}{=} \{Ax \ : \ x \in E_k\}. \tag{29}$$

For a proof see [28], [12], for example.

It is often convenient to extend linear orders with respect to which countable additive subgroups of $\mathbb{R}^k$ are ordered groups, to so-called *term orders* on $\mathbb{R}^k$, i.e., linear orders $\preceq$ on $\mathbb{R}^k$ having the properties $x \preceq y \Rightarrow x + z \preceq y + z$ and $(x \preceq y$ and $c \in \mathbb{R}^+) \Rightarrow cx \preceq cy$ (note that the term orders are in one-to-one correspondence with halfspaces). That this is always possible follows from the next lemma [6]:

Lemma 13. *If $\preceq$ is a linear order on $\mathbb{Z}^k$, then there exists a term order $\preceq_{\mathbb{R}^k}$ on $\mathbb{R}^k$, and an order preserving (with respect to $\preceq$ and $\preceq_{\mathbb{R}^k}$) one-to-one group homomorphism from $\mathbb{Z}^k$ into $\mathbb{R}^k$.*

We now prove the following auxiliary result.

Lemma 14. *Let Γ_0 be a countable subgroup of $\mathbb{R}^k$, and let $\preceq_0$ be a term order on $\mathbb{R}^k$. Then for every selection of two finite subsets Ω_+, Ω_- of Γ_0 such that Ω_+ is positive and Ω_- is negative with respect to $\preceq_0$, there exists a term order $\preceq$ on $\mathbb{R}^k$ with the following properties:*

(a) *Γ_0 is Archimedean with respect to $\preceq$;*
(b) *Ω_+ is positive and Ω_- is negative with respect to $\preceq$.*

Proof. Since term orders are in one-to-one correspondence with halfspaces, for every term order $\preceq_0$ there exists a subspace of $\mathbb{R}^k$ of dimension $k-1$ which divides $\mathbb{R}^k$ in a set of positive points and a set of negative points (with respect to $\preceq_0$). Let $\mathcal{V}$ be the set of all $(k-1)$-dimensional subspaces V of $\mathbb{R}^k$ such that no point of Ω_+ can be continuously connected to a point of Ω_- without crossing V. Then $\mathcal{V}$ is open in the gap topology (see, e.g., [20, Chapter 13] for definition and properties of the gap topology), while the $(k-1)$-dimensional subspaces of $\mathbb{R}^k$ that have an empty intersection with $\Gamma_0 \setminus \{0\}$, form a dense subset of the metric space of linear subspaces of $\mathbb{R}^k$ with the gap topology. Thus there exists $V \in \mathcal{V}$ such that $V \cap \Gamma_0 = \{0\}$. Now let $\preceq$ be a term order on $\mathbb{R}^k$ which orders points first by their signed distance from V (with points in Ω_+ having positive distances to V) and then by applying any term order on V to the orthogonal projections of points onto V. Since no two different points of Γ_0 can have the same signed distance to V, we see that the signed distance from V is an order preserving group homomorphic embedding of $(\Gamma, \preceq|_{\Gamma_0})$ into $\mathbb{R}$. Consequently, Γ_0 is Archimedean with respect to $\preceq$. □

Proposition 15. *Consider $\mathbb{R}^k$ with a term order $\preceq$. Let $\nu, \delta \in \mathbb{R}^k$ be positive with respect to this order, and assume that the subgroup of $\mathbb{R}^k$ generated by ν and δ is not cyclic. Then none of the functions*

$$\begin{bmatrix} e_{\nu+\delta}(g) & 0 \\ c_1 e_{-\nu}(g) + c_2 + c_3 e_\delta(g) & e_{-\nu-\delta}(g) \end{bmatrix}, \qquad g \in G, \quad \mathbb{R}^k \text{ the dual of } G, \tag{30}$$

admit a finite factorization, with respect to $\Gamma = \mathbb{R}^k$ with the term order $\preceq$, for any triple of nonzero complex numbers (c_1, c_2, c_3) satisfying

$$(\log|c_3|)\nu + (\log|c_1|)\delta = (\log|c_2|)(\nu + \delta). \tag{31}$$

Note that (31) holds, in particular, when $|c_3| = |c_2| = |c_1|$. Note also that the subgroup generated by ν and δ is non-cyclic if and only if an equality $m\nu = n\delta$, $m, n \in \mathbb{Z}$ implies $m = n = 0$.

Proof. The case $k = 1$ is known: [22], also [4, Section 15.1]. In fact, in this case a stronger property holds: the matrix function (30) does not admit any factorization, finite or not.

Consider now the general case. Arguing by contradiction, assume that a function of the form (30) admits a finite factorization with respect to $\preceq$:

$$\begin{bmatrix} e_{\nu+\delta} & 0 \\ c_1 e_{-\nu} + c_2 + c_3 e_\delta & e_{-\nu-\delta} \end{bmatrix} = A_+ \begin{bmatrix} e_{\mu_1} & 0 \\ 0 & e_{\mu_2} \end{bmatrix} A_-, \tag{32}$$

where $A_\pm \in \mathcal{G}((W(G)_\pm)^{n\times n})$, $\{\mu_1, \mu_2\} \subset \mathbb{R}^k$, and the Fourier spectrum of $A_\pm$ is finite. By taking determinants in (32) we see that actually $\mu_2 = -\mu_1$ and $\det(A_\pm)$ is a nonzero constant, so without loss of generality we may assume that $\det(A_\pm) = 1$. Let Γ_0 be the (countable) subgroup of $\mathbb{R}^k$ generated by ν, δ, μ_1, and the Fourier spectra of A_+ and A_-, and let G_0 be the group for which Γ_0 is the dual. By Lemma 14 there is a term order $\preceq_0$ on $\mathbb{R}^k$ with respect to which Γ_0 is Archimedean and such that the same formula (32) is a factorization with respect to $\preceq_0$. By Hölder's theorem [31], Γ_0 is $\preceq_0$-order isomorphic to a subgroup of $\mathbb{R}$. Such an isomorphism would transform (32) to a factorization of a function of the form (30) where $k = 1$, a contradiction with the known case [22], [4, Section 15.1]. □

It remains an open question whether functions (30) admit a non-finite factorization in the case $k > 1$.

The next lemma is well-known in abstract group theory, see [24, Chapter VIII, §30], for example.

Lemma 16. *An abstract group Ω is isomorphic to a subgroup of $\mathbb{Q}$ if and only if Ω is torsion-free (every nonzero element has infinite order) and every finitely generated subgroup of Ω is cyclic.*

Proof of Theorem 11. Let Γ be a discrete ordered abelian group that is not isomorphic to a subgroup of $\mathbb{Q}$. By Lemma 16 there exist $\nu_0, \delta_0 \in \Gamma$ that are positive elements and generate the noncyclic subgroup Γ_0 of Γ. Say, $\nu_0 \succ \delta_0 \succ 0$. By Proposition 15 the functions

$$A = \begin{bmatrix} e_{\nu_0+\delta_0} & 0 \\ c_1 e_{-\nu_0} + c_2 + c_3 e_{\delta_0} & e_{-\nu_0-\delta_0} \end{bmatrix}, \tag{33}$$

where

$$c_1, c_2, c_3 \neq 0, \quad (\log|c_3|)\nu + (\log|c_1|)\delta = (\log|c_2|)(\nu+\delta),$$

are not finitely factorizable with respect to Γ_0 (to make Proposition 15 applicable, identify Γ_0 with $\mathbb{Z}^2$ and use Lemma 13).

It remains to prove that the functions (33) are not finitely factorizable with respect to Γ, and with the factors $A_\pm$ having finite spectrum. Suppose one of such functions is:

$$A = A_+ \begin{bmatrix} e_{j_1} & 0 \\ 0 & e_{j_2} \end{bmatrix} A_-, \tag{34}$$

for some c_1, c_2, c_3 as in (33), where $A_\pm \in \mathcal{G}((W(G)_\pm)^{2p\times 2p})$, the Fourier spectra of $A_\pm$ are finite, and $j_1, j_2 \in \Gamma$. Comparing determinants in (33) and in (34), we see that in fact $j_1 + j_2 = 0$.

Let Γ' be a finitely generated subgroup of Γ that contains j_1, ν_0, δ_0, and the Fourier spectra of A_+ and A_-, and let G' be the dual group of Γ'. By Theorems 1 and 2, $A_\pm^{-1} \in W(G')^{2p\times 2p}$. It now follows that $A_\pm \in \mathcal{G}((W(G')_\pm)^{2p\times 2p}$, and (34) is actually a factorization with respect to Γ'. Identifying Γ' with $\mathbb{Z}^m$ for some m, we see that by Lemma 13, Γ' is order isomorphic to a subgroup of $\mathbb{R}^m$, and transforming (34) with the help of such an order isomorphism, we obtain a contradiction with Proposition 15.

The converse statement follows from Lemma 16 and the well-known result that the converse statement holds for $\Gamma = \mathbb{Z}$. Indeed, every finitely generated subgroup of $\mathbb{Q}$ is cyclic. $\square$

5. Systems of difference equations

As an application of factorization we consider systems of difference equations on a general discrete ordered abelian group $(\Gamma, \preceq)$ of the form

$$\sum_{s\in\Gamma_+} A(t-s)x(s) = b(t), \qquad t \in \Gamma_+, \tag{35}$$

where $A(t)$ is an $n\times n$ matrix function whose spectrum is a subset of Γ and for which $\sum_{t\in\Gamma} \|A(t)\| < \infty$, and $b(t)$ is an $n \times 1$ matrix function whose spectrum is a subset of Γ_+ and for which $\sum_{t\in\Gamma_+} \|b(t)\| < \infty$. We seek solutions x which are $n\times 1$ matrix functions whose spectrum is a subset of Γ_+ and for which $\sum_{t\in\Gamma_+} \|x(t)\| < \infty$. Let us now define

$$b(t) = \sum_{s\in\Gamma_+} A(t-s)x(s), \qquad t \in \Gamma_- \setminus \{0\}.$$

Then (35) is true for every $t \in \Gamma$, while $x(t)$ for $t \in \Gamma_+$ and $b(t)$ for $t \in \Gamma_- \setminus \{0\}$ are the unknowns. Putting

$$\hat{x}(g) = \textstyle\sum_{t\in\Gamma_+} x(t)\langle t, g\rangle, \qquad \hat{u}(g) = -\sum_{t\in\Gamma_-\setminus\{0\}} b(t)\langle t, g\rangle,$$

$$\hat{b}(g) = \textstyle\sum_{t\in\Gamma_+} b(t)\langle t, g\rangle, \qquad \hat{A}(g) = \sum_{t\in\Gamma} A(t)\langle t, g\rangle,$$

we obtain

$$\hat{A}(g)\hat{x}(g) + \hat{u}(g) = \hat{b}(g), \qquad g \in G, \tag{36}$$

where $\hat{u}(g)$ belongs to $(W(G))^{n\times n}$ and has only spectrum in $\Gamma_- \setminus \{0\}$.

By standard methods (such as [18] for $\Gamma = \mathbb{Z}$) one can prove that if $\hat{A}(g)$ has a right canonical factorization

$$\hat{A}(g) = \hat{A}_-(g)\hat{A}_+(g), \qquad g \in G,$$

where

$$\hat{A}_+(g)^{-1} = \sum_{t \succeq 0} \alpha_+(t)e_t(g), \quad \hat{A}_-(g)^{-1} = \sum_{t \preceq 0} \alpha_-(t)e_t(g),$$

then (35) has a unique solution which is given by

$$x(t) = \sum_{u \succeq 0} F(t,u)b(u), \qquad t \in \Gamma_+, \tag{37}$$

where

$$F(t,u) = \sum_{0 \preceq v \preceq \min(t,u)} \alpha_+(t-v)\alpha_-(v-u).$$

Let us now consider (35) in the case when $\hat{A}(g)$ has a right, not necessarily right canonical, factorization. In the statement below, we denote by $\sigma(\cdot)$ the Fourier spectrum, and use the shorthand $[\alpha, \beta)$ for the set $\{\gamma \in \Gamma \, : \, \alpha \preceq \gamma \prec \beta\}$, with the obvious modification for the set $(\alpha, \beta]$.

Theorem 17. *Suppose $\hat{A}(g)$ has the right factorization*

$$\hat{A}(g) = \hat{A}_-(g)\mathrm{diag}\,(e_{j_1}(g), \ldots, e_{j_n}(g))\hat{A}_+(g), \qquad g \in G,$$

where $j_1 \preceq \ldots \preceq j_n$. Put $P_r = \mathrm{row}\,(\delta_{ir})_{i=1}^n$, where δ_{ir} is the Kronecker symbol. Then (35) has at least one solution if and only if

$$\sigma\left(P_r\hat{A}_-(g)^{-1}\hat{b}(g)\right) \cap [0, j_r) = \emptyset \text{ whenever } j_r \succ 0. \tag{38}$$

The solutions of the homogeneous version of (36) are exactly those $\hat{x}(g) \in (W(G)_+)^{n\times 1}$ for which

$$\sigma\left(P_r\hat{A}_+(g)\hat{x}(g)\right) \subseteq [j_r, 0) \text{ whenever } j_r \prec 0. \tag{39}$$

Thus all solutions are unique if and only if $0 \preceq j_1 \preceq \ldots \preceq j_n$.

Proof. From (36) we immediately have

$$e_{j_r}(g)P_r\hat{A}_+(g)\hat{x}(g) + P_r\hat{A}_-(g)^{-1}\hat{u}(g) = P_r\hat{A}_-(g)^{-1}\hat{b}(g). \tag{40}$$

Then for $j_r \succ 0$ condition (38) is clear, because the first term on the left-hand side of (39) has its spectrum in $[j_r, \infty)$ and the second term in $(-\infty, 0)$. Thus for $j_r \succ 0$ the right-hand side of (40) should not have any spectrum in $[0, j_r)$ in order for a solution to exist. If (38) holds, then one of the solutions of (36) is delivered by $\hat{x}(g) = \hat{A}_+(g)^{-1}\mathrm{diag}\,(e_{-j_1}(g), \ldots, e_{-j_n}(g))\hat{A}_-(g)^{-1}\hat{b}(g)$.

To prove the uniqueness statement, we write (36) with $\hat{b}(g) \equiv 0$ in the form

$$e_{j_r}(g)P_r\hat{A}_+(g)\hat{x}(g) = -P_r\hat{A}_-(g)^{-1}\hat{u}(g),$$

where the left-hand side has its spectrum in $[j_r, \infty)$ and the right-hand side in $\{\mu \in \Gamma \, : \, \mu \prec 0\}$. □

6. Orthogonal families of functions

In this section we show how factorization can be used to determine orthogonal families of functions. Let $f = \sum_{j\in\Gamma} f_j e_j \in W(G)$, where e_j is defined by (2), be such that

$$f(g) \geq \epsilon > 0 \quad \text{for every } g \in G. \tag{41}$$

We let $L^2(f\nu)$ be the weighted L^2 Hilbert space with respect to the normalized invariant measure ν on G with the weight f.

Theorem 18. *Assume that $f \in W(G)$ satisfies (41). For $\mu \succeq 0$ let*

$$Q_\mu f = \sum_{j\in\Gamma,\ -\mu\preceq j\preceq\mu} f_j e_j.$$

Assume that for every $\mu \succeq 0$ there exists a factorization

$$\begin{bmatrix} Q_\mu f & e_{-\mu} \\ e_\mu & 0 \end{bmatrix} = \begin{bmatrix} a_\mu & b_\mu \\ c_\mu & d_\mu \end{bmatrix} \begin{bmatrix} p_\mu & q_\mu \\ r_\mu & s_\mu \end{bmatrix}, \tag{42}$$

where

$$\begin{bmatrix} a_\mu & b_\mu \\ c_\mu & d_\mu \end{bmatrix} \in \mathcal{G}((W(G)_+)^{2\times 2}), \qquad \begin{bmatrix} p_\mu & q_\mu \\ r_\mu & s_\mu \end{bmatrix} \in \mathcal{G}((W(G)_-)^{2\times 2})$$

with the additional properties that 0 is not in the Fourier spectrum of β_μ and $\alpha_\mu \in \mathcal{G}(W(G)_+)$, where α_μ and β_μ are taken from

$$\begin{bmatrix} \alpha_\mu & \beta_\mu \\ \gamma_\mu & \delta_\mu \end{bmatrix} = \begin{bmatrix} a_\mu & b_\mu \\ c_\mu & d_\mu \end{bmatrix}^{-1}.$$

Then the family

$$\pi_\mu(g) := e_\mu(g)\overline{\alpha_\mu(g)}, \quad g \in G, \qquad \mu \succeq 0$$

forms an orthogonal set in $L^2(f\nu)$. In addition, the Fourier spectrum of π_μ lies in $\{g \in \Gamma \ :\ 0 \preceq g \preceq \mu\}$ and contains μ.

Note that factorization (42) is, upon a row interchange, and except for the additional properties, a canonical factorization of a triangular 2×2 matrix function of the form (7).

Proof. We will use in the proof the fact that $\langle e_\lambda, e_\mu\rangle = 0$ if $\lambda \neq \mu$; this orthogonality is a result of the unitarity of the Fourier transform from $\ell^2(\Gamma)$ onto $L^2(G)$ [31].

From the factorization (42) it follows that

$$\alpha_\mu(Q_\mu f) = p_\mu - e_\mu \beta_\mu(z),$$

which has Fourier spectrum in $\{g \in \Gamma \ :\ g \preceq 0 \text{ or } \mu \prec g\}$. Here we used that 0 is not in the Fourier spectrum of β. Thus $e_\mu(Q_\mu f)\alpha_\mu^*$ has Fourier spectrum in $\{g \in \Gamma \ :\ g \prec 0 \text{ or } \mu \preceq g\}$. Here and elsewhere α_μ^* is a shorthand notation for the function $\overline{\alpha_\mu(g)}$, $g \in G$. Thus for $0 \preceq l \prec \mu$ we have that

$$\langle e_l, e_\mu(Q_\mu f)\alpha_\mu^*\rangle = 0,$$

where $\langle\,,\rangle$ is the inner product in $L^2(\nu)$. In addition, since $\alpha_\mu = e_\mu q_\mu$, it follows that α_μ has Fourier spectrum in $\{g \in \Gamma \;:\; 0 \preceq g \preceq \mu\}$, and since α_μ is invertible it cannot be the zero element in $W(G)$. From the location of the Fourier spectrum of α_μ it follows that $e_\mu(f - (Q_\mu f)(z))\alpha_\mu^*$ has Fourier spectrum in $\{g \in \Gamma \;:\; g \prec 0 \text{ or } \mu \prec g\}$. Thus it follows that

$$0 = \langle e_l, e_\mu(Q_\mu f)\alpha_\mu^*\rangle = \langle e_l, e_\mu f \alpha_\mu^*\rangle,\ 0 \preceq l \prec \mu.$$

Thus $e_\mu\alpha_\mu^*$ is orthogonal to e_l, $0 \preceq l \prec \mu$, in $L^2(f\nu)$. Moreover, the Fourier spectrum of $e_\mu\alpha_\mu^*$ lies in $\{g \in \Gamma \;:\; 0 \preceq g \preceq \mu\}$ and must contain μ (because α_μ is assumed to be invertible). □

In the case that Γ is a subgroup of $\mathbb{R}^k$ the existence of a factorization (42) with the additional properties described in Theorem 18 follows from the results in [29] (combine Theorem 2.22 with Lemma 4.9; see also the proof of Theorem 4.11). The very special case that $\Gamma = \mathbb{Z}$ and $G = \mathbb{T}$, the unit circle, follows of course directly as a particular case. We state the result.

Theorem 19. *Let $f(z) = \sum_{j=-\infty}^{\infty} f_j z^j \in W(\mathbb{T})$ be such that $f(z) > 0$, $z \in \mathbb{T}$. For each nonnegative integer n, put*

$$Q_n f(z) = \sum_{j=-n}^{n} f_j z^j.$$

Then there exist factorizations

$$\begin{bmatrix} Q_n f(z) & z^{-n} \\ z^n & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix}, \tag{43}$$

where

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} := \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}$$

are analytic in $\overline{\mathbb{D}}$, the closure of the unit disk,

$$\begin{bmatrix} p & q \\ r & s \end{bmatrix}, \begin{bmatrix} p & q \\ r & s \end{bmatrix}^{-1}$$

are analytic in $(\mathbb{C} \cup \{\infty\}) \setminus \overline{\mathbb{D}}$, and in addition $\beta(0) = 0$ and α^{-1} is analytic in $\overline{\mathbb{D}}$. Moreover, if we define

$$\pi_n(z) := z^n \overline{\alpha(1/\overline{z})},$$

then the family π_n, $n = 0, 1, 2, \ldots$, consists of orthogonal polynomials with respect to the inner product on $L_2(\mathbb{T})$ with weight $f(z)$.

While the last statement of this theorem is a direct consequence of Theorem 18, it is interesting to make the connection to the results in [13], [14], [9], and [10]. In these papers a machinery has been developed where Riemann-Hilbert problems are used to solve various asymptotics problems in the area of orthogonal polynomials and their applications. This approach has been very successful and continues to be. The book [8] may be consulted for further recent developments in this area.

We end this paper with a proof of Theorem 19 based on results in [13] and [9] specialized to the case of the unit circle.

Alternative proof of Theorem 19. Consider the following Riemann-Hilbert problem: find a 2×2 matrix-valued function $Y(z)$ such that

$$\begin{cases} Y(z), & \text{is analytic in } \mathbb{C} \setminus \mathbb{T}; \\ Y_+(z) = Y_-(z) \begin{bmatrix} 1 & z^{-n} f(z) \\ 0 & 1 \end{bmatrix}, & z \in \mathbb{T} \\ Y(z) \begin{bmatrix} z^{-n} & 0 \\ 0 & z^n \end{bmatrix} = (I + O(\frac{1}{|z|})), & \text{uniformly as } z \to \infty. \end{cases}$$

For the definition of Y_+ and Y_- we require that $Y(z)$ is continuous up to the boundary in each component of $\mathbb{C} \setminus \mathbb{T}$, and we define

$$Y_+(z) = \lim_{w \to z,\ w \in \mathbb{D}} Y(w) \quad \text{and} \quad Y_-(z) = \lim_{w \to z,\ w \in \mathbb{C} \setminus \mathbb{D}} Y(w).$$

By [13], [9] (see also [2, Lemma 4.1], where the result is written out for the case of a circle) it follows that this problem has a unique solution $Y(z) = (Y_{ij})_{i,j=1}^2$, that $Y_{11}(z)$ is the nth monic orthogonal polynomial with respect to the weight $f(e^{i\theta})\frac{d\theta}{2\pi}$ on the unit circle, and that $Y_{21}(z)$ is a polynomial of degree $n-1$. Further note that $\det Y(z)$ has no jump on the circle, and that $\det Y(z) \to 1$ as $z \to \infty$. Thus $\det Y(z) \equiv 1$. Let now

$$R_1(z) = \overline{Y(\frac{1}{\bar{z}})} \begin{bmatrix} z^n & 0 \\ 0 & z^{-n} \end{bmatrix}, z \in \mathbb{D}, \qquad \text{and} \qquad R_2(z) = \overline{Y(\frac{1}{\bar{z}})} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, z \in \mathbb{C} \setminus \overline{\mathbb{D}},$$

and extend them continuously to the boundary (which is possible since Y_+ and Y_- are well-defined). Then for $|z| = 1$ we have that

$$R_2(z) = \overline{Y_+(\frac{1}{\bar{z}})} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \overline{Y_-(\frac{1}{\bar{z}})} \begin{bmatrix} 1 & z^n f(z) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} =$$

$$R_1(z) \begin{bmatrix} z^{-n} & 0 \\ 0 & z^n \end{bmatrix} \begin{bmatrix} 1 & z^n f(z) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = R_1(z) \begin{bmatrix} f(z) & z^{-n} \\ z^n & 0 \end{bmatrix}.$$

Furthermore, as $Y_{21}(z)$ is a polynomial of degree $n-1$, it follows that $(R_1)_{21}(0) = 0$. Next, let

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = R_1(z) \begin{bmatrix} 1 & \sum_{j=1}^{\infty} f_{j+n} z^j \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} p & q \\ r & s \end{bmatrix} = R_2(z) \begin{bmatrix} 1 & 0 \\ -\sum_{j=-\infty}^{-1} f_{j-n} z^j & 1 \end{bmatrix}.$$

Then (43) is satisfied. Moreover, note that $\pi(z) = Y_{11}(z)$. By the results of Szegő [33], π has all its zeros in the open unit disk, so α^{-1} is analytic in $\overline{\mathbb{D}}$. In order to get that $\beta(0) = 0$, one may multiply

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$$

with an appropriate constant matrix on the left (e.g., the inverse of its value at 0). □

Finally, we remark that the formulas for the factors given in the first line of the proof of Theorem 2.18 in [29] are in direct correspondence with the formulas for the solution to the Riemann-Hilbert problem given in [13], [9] (see [2, Lemma 4.1] where it is explicitly written out for the circle case).

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