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Perturbation results for exponentially dichotomous operators on general Banach spaces

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Abstract

Some perturbation results for exponentially dichotomous operators are applied to prove the existence of stable and anti-stable solutions of Riccati equations associated to block operators on general Banach spaces, both for compact perturbations and for bisemigroups made up of immediately norm continuous semigroups.

Keywords: Block operator; Exponential dichotomy; Semigroup perturbation; Riccati equation

1. Introduction

Exponentially dichotomous operators, i.e., direct sums $A_0 \dotplus (-A_1)$ in which A_0 and A_1 are generators of exponentially decaying C_0-semigroups, have been introduced to derive representations of solutions of vector-valued convolution equations on intervals of the real line [2,3]. They also appear in the study of abstract kinetic equations [13], Pritchard–Salamon systems [15], and block operators of Hamiltonian type [18–20]. In all of these cases it is of major interest to study suitable perturbations of exponentially dichotomous operators and, in fact, to prove that certain perturbations of them are still exponentially dichotomous. Among these perturbations one finds additive perturbations by a bounded linear operator [18–20],

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multiplicative perturbations by a compact perturbation of the identity [13], and additive perturbations by an unbounded (but $A_0 \dot{+} (-A_1)$ bounded [15] or self-adjoint [19,20]) linear operator, and in all of these cases only Hilbert space operators were considered. Only in [21] one finds a multiplicative perturbation analysis in L^p-spaces, while in [13] it is indicated how the Hilbert space theory developed there is to be modified in order to arrive at a Banach space theory.

In [18] an interesting new element has been introduced in the perturbation theory of exponentially dichotomous operators. Specifying the results for 2×2 block operators where the unperturbed exponentially dichotomous operator is block diagonal and the additive perturbation is block off-diagonal, a stable solution of an operator Riccati equation and an antistable solution of another operator Riccati equation showed up. However, a direct application of the perturbation result of [2], which requires a condition on the domain of the square of the perturbed exponentially dichotomous operator, would limit the generality of the result in an artificial way. In [18] this was circumvented by putting extra conditions on the exponentially dichotomous operator, which are met for instance if it generates an analytic bisemigroup.

Algebraic Riccati equations play an important role in optimal control theory. We refer to the books [4,14,17] for a good background. In a finite dimensional setting the method of choice to find the stable and antistable solutions of algebraic Riccati equations is to find the stable and antistable invariant subspaces of 2×2 block operators constructed from the coefficients of the Riccati equations. This approach is outlined in detail in [17]. In contrast, in the infinite dimensional case the method of choice for proving existence of a solution to the algebraic Riccati equation seems to be to show that a Riccati differential equation on a finite interval has a solution, and then to let the length of the interval increase. See, for instance [6,22]. In [18] the invariant subspace approach to finding the stable solution of an algebraic Riccati equation was extended to the infinite dimensional setting for a specific class of systems.

In this article we shall generalize the results of [18] but adopt the basic strategy of [13,21] to derive perturbation results. Introducing bisemigroups as in [2] as $e^{-tA_0} \dot{+} 0$ for $t > 0$ and $0 \dot{+} (-e^{tA_1})$ for $t < 0$, the bisemigroup generated by the additive perturbation is sought as the solution of a full-line vector-valued convolution equation whose right-hand side and convolution kernel contain the bisemigroup generated by the unperturbed exponentially dichotomous operator as well as the bounded perturbation. As in [13], the Bochner–Phillips theorem [5,12] on the invertibility in noncommutative Banach function algebras of Wiener type is applied to get the basic perturbation result under the natural condition that the perturbed exponentially dichotomous operator does not have any spectrum in a sufficiently narrow strip around the imaginary axis. Next, the perturbation theorem is related to the left and right canonical factorizability of the fractional linear map composed of the unperturbed and perturbed exponentially dichotomous operators, which by itself leads to a generalization of the main result of [18]. Therefore, a specialization of our results to block operators leads to a close connection between the existence of a stable (anti-stable, resp.) solution of an operator Riccati equation having the block operator as its Hamiltonian matrix (see [17] for similar results in the matrix case) and the left (right, resp.) canonical factorizability of the above fractional linear function. For small additive perturbations we thus obtain existence results of solutions of Riccati equations, even in the general Banach space setting where such results are a rare commodity.

Let us introduce some notations. We let $\mathbb{R}^{\pm}$ stand for the right (left, resp.) half-line, including the point at zero. For two complex Banach spaces $\mathscr{X}$ and $\mathscr{Y}$, we let $\mathscr{L}(\mathscr{X},\mathscr{Y})$ stand for the Banach spaces of all bounded linear operators from $\mathscr{X}$ into $\mathscr{Y}$. We write $\mathscr{L}(\mathscr{X})$ instead of $\mathscr{L}(\mathscr{X},\mathscr{X})$.

Let $\mathscr{X}$ be a complex Banach space and E an interval of the real line $\mathbb{R}$. Then $L^p(E;\mathscr{X})$ denotes the Banach space of all strongly measurable functions $\phi : E \to \mathscr{X}$ such that $\|\phi(\cdot)\|_{\mathscr{X}} \in L^p(E)$, endowed with the L^p-norm, and $C_0(E;\mathscr{X})$ stands for the Banach space of all bounded continuous functions $\phi : E \to \mathscr{X}$ which vanish at infinity if E is unbounded, endowed with the supremum norm. In particular, $C_0(\mathbb{R}^-;\mathscr{X}) \dot{+} C_0(\mathbb{R}^+;\mathscr{X})$ is the Banach space of all bounded continuous functions $\phi : \mathbb{R} \to \mathscr{X}$ which vanish at $\pm\infty$ and may have a jump discontinuity in zero.

2. Bisemigroups and their perturbations

2.1. Preliminaries on semigroups

A C_0-semigroup $(T(t))_{t\geqslant 0}$ on a complex Banach space $\mathscr{X}$ is called *uniformly exponentially stable* if

$$\|T(t)\| \leqslant M e^{-\varepsilon t}, \quad t \geqslant 0, \tag{2.1}$$

for certain $M, \varepsilon > 0$. It is *eventually norm continuous* if there exists $t_0 > 0$ such that $T(t)$ is norm continuous for $t \geqslant t_0$, and *immediately norm continuous* if $T(t)$ is norm continuous for $t > 0$. Analytic semigroups, immediately compact semigroups (i.e., $T(t)$ is a compact operator for $t > 0$), and immediately differentiable semigroups are all special cases of immediately norm continuous semigroups [8, Chapter II, (4.26)].

The following result appears in Section II 4.20 of [8].

Theorem 1. *Let $\mathscr{X}$ be a complex Hilbert space and let $(T(t))_{t\geqslant 0}$ be a uniformly exponentially stable C_0-semigroup on $\mathscr{X}$. Then $(T(t))_{t\geqslant 0}$ is immediately norm continuous if and only if the resolvent $(\lambda - A)^{-1}$ of its infinitesimal generator A vanishes in the norm as $\lambda \to \infty$ along the imaginary line.*

A closed linear operator A densely defined on a complex Banach space $\mathscr{X}$ is called *sectorial* if there exists a δ with $0 < \delta \leqslant (\pi/2)$ such that the sector

$$\Sigma_{\frac{\pi}{2}+\delta} = \left\{ \lambda \in \mathbb{C} : |\arg \lambda| < \frac{\pi}{2} + \delta \right\} \Big\backslash \{0\}$$

is contained in the resolvent set of A, and if for each $\varepsilon \in (0, \delta)$ there exists $M_\varepsilon \geqslant 1$ such that

$$||(\lambda - A)^{-1}|| \leqslant \frac{M_\varepsilon}{|\lambda|}, \quad 0 \neq \lambda \in \overline{\Sigma_{\frac{\pi}{2}+\delta-\varepsilon}}.$$

According to [8, Theorem II 4.6] the sectorial operators are exactly the generators of bounded analytic semigroups.

2.2. Main results

A closed and densely defined linear operator $-S$ on a Banach space $\mathscr{X}$ is called *exponentially dichotomous* [2] if for some projection P commuting with S, the restrictions of S to $\operatorname{Im} P$ and of $-S$ to $\operatorname{Ker} P$ are the infinitesimal generators of exponentially decaying C_0-semigroups. We then define the *bisemigroup* generated by $-S$ as

$$E(t; -S) = \begin{cases} e^{-tS}(I - P), & t > 0, \\ -e^{-tS}P, & t < 0. \end{cases}$$

Its *separating projection* P is given by $P = -E(0^-; -S) = I_{\mathscr{X}} - E(0^+; -S)$. One easily verifies the existence of $\varepsilon > 0$ such that $\{\lambda \in \mathbb{C} : |\operatorname{Re} \lambda| \leqslant \varepsilon\}$ is contained in the resolvent set $\rho(S)$ of S and for every $x \in \mathscr{X}$

$$(\lambda - S)^{-1}x = -\int_{-\infty}^{\infty} e^{\lambda t} E(t; -S)x \, dt, \quad |\operatorname{Re} \lambda| \leqslant \varepsilon. \tag{2.2}$$

As a result, for every $x \in \mathscr{X}$ we have $||(\lambda - S)^{-1}x|| \to 0$ as $\lambda \to \infty$ in $\{\lambda \in \mathbb{C} : |\operatorname{Re} \lambda| \leqslant \varepsilon'\}$ for some $\varepsilon' \in (0, \varepsilon]$. We call the restrictions of e^{-tS} to $\operatorname{Ker} P$ and of e^{tS} to $\operatorname{Im} P$ the *constituent semigroups* of the exponentially dichotomous operator $-S$.

Observe that $\{x \in \mathscr{X} : (\lambda - S)^{-1}x$ is analytic for $\operatorname{Re} \lambda < 0\} = \operatorname{Ker} P$, and $\{x \in \mathscr{X} : (\lambda - S)^{-1}x$ is analytic for $\operatorname{Re} \lambda > 0\} = \operatorname{Im} P$.

We have the following perturbation results.

Theorem 2. *Let $-S_0$ be exponentially dichotomous, Γ a compact operator, and $-S = -S_0 + \Gamma$, where $\mathscr{D}(S) = \mathscr{D}(S_0)$. Suppose the imaginary axis is contained in the resolvent set of S. Then $-S$ is exponentially dichotomous. Moreover, $E(t; -S) - E(t; -S_0)$ is a compact operator, also in the limits as $t \to 0^\pm$.*

Proof. There exists an $\varepsilon > 0$ such that

$$\int_{-\infty}^{\infty} e^{\varepsilon|t|} ||E(t; -S_0)|| \, dt < \infty. \tag{2.3}$$

Using the resolvent identity

$$(\lambda - S)^{-1} - (\lambda - S_0)^{-1} = -(\lambda - S_0)^{-1}\Gamma(\lambda - S)^{-1}, \quad |\operatorname{Re} \lambda| \leqslant \varepsilon,$$

for some $\varepsilon > 0$, we obtain the convolution integral equation

$$E(t;-S)x - \int_{-\infty}^{\infty} E(t-\tau;-S_0)\Gamma E(\tau;-S)x\,d\tau = E(t;-S_0)x, \tag{2.4}$$

where $x \in \mathscr{H}$ and $0 \neq t \in \mathbb{R}$. In (2.4), the convolution kernel $E(\cdot;-S_0)\Gamma$ is continuous in the norm except for a jump discontinuity in $t=0$, as a result of the strong continuity (except for the jump) of $E(\cdot;-S_0)$ and the compactness of Γ. Further, (2.3) implies that $e^{\varepsilon|\cdot|}E(\cdot;-S_0)\Gamma$ is Bochner integrable.

The symbol of the convolution integral equation (2.4), which equals $I_{\mathscr{H}} + (\lambda - S_0)^{-1}\Gamma = (\lambda - S_0)^{-1}(\lambda - S)$, tends to $I_{\mathscr{H}}$ in the norm as $\lambda \to \infty$ in the strip $|\mathrm{Re}\,\lambda| \leqslant \varepsilon$ because $(\lambda - S_0)^{-1}x \to 0$ for all x and Γ is compact. Moreover, it is a compact perturbation of the identity which, by definition, only takes invertible values on the imaginary axis. Thus there exists $\varepsilon_0 \in (0,\varepsilon]$ such that the symbol only takes invertible values on the strip $|\mathrm{Re}\,\lambda| \leqslant \varepsilon_0$. By the Bochner–Phillips theorem [5], the convolution equation (2.4) has a unique solution $u(\cdot;x) = E(\cdot;-S)x$ with the following properties:

(1) $E(\cdot;-S)$ is strongly continuous, except for a jump discontinuity at $t=0$,
(2) $\int_{-\infty}^{\infty} e^{\varepsilon_0|t|}\|E(t;-S)\|\,dt < \infty$; hence $E(\cdot;-S)$ is exponentially decaying,
(3) $E(t;-S) - E(t;-S_0)$ is a compact operator, also in the limits as $t \to 0^{\pm}$,
(4) identity (2.2) holds.

As a result [2], $-S$ is exponentially dichotomous. □

Theorem 3. *Let $-S_0$ be exponentially dichotomous with immediately norm continuous constituent semigroups, let Γ be a bounded operator, and let $-S = -S_0 + \Gamma$, where $\mathscr{D}(S) = \mathscr{D}(S_0)$. Suppose a strip $\{\lambda \in \mathbb{C} : |\mathrm{Re}\,\lambda| \leqslant \varepsilon\}$ is contained in the resolvent set of S. Then $-S$ is exponentially dichotomous with immediately norm continuous constituent semigroups.*

Proof. There exists $\varepsilon > 0$ such that (2.3) is true. Using the resolvent identity we again derive (2.4), where $x \in \mathscr{H}$ and $0 \neq t \in \mathbb{R}$. In (2.4), the convolution kernel $E(\cdot;-S_0)\Gamma$ is continuous in the norm except for a jump discontinuity in $t=0$, as a result of the immediate norm continuity of the constituent semigroups of $-S_0$. Further, (2.3) implies that $e^{\varepsilon|\cdot|}E(\cdot;-S_0)\Gamma$ is Bochner integrable.

The symbol of the convolution integral equation (2.4), which equals $I_{\mathscr{H}} + (\lambda - S_0)^{-1}\Gamma = (\lambda - S_0)^{-1}(\lambda - S)$, tends to $I_{\mathscr{H}}$ in the norm as $\lambda \to \infty$ in the strip $|\mathrm{Re}\,\lambda| \leqslant \varepsilon$, because, by Theorem 1, $(\lambda - S_0)^{-1}$ tends to zero in the norm as $\lambda \to \infty$ in the strip. Moreover, since there is a strip $|\mathrm{Re}\,\lambda| \leqslant \varepsilon$ contained in the resolvent of S, the symbol only takes invertible values within this strip. By the Bochner–Phillips theorem [5], the convolution equation (2.4) has a unique solution $u(\cdot;x) = E(\cdot;-S)x$ with the following properties:

(1) $E(\cdot;-S)$ is strongly continuous, except for a jump discontinuity at $t=0$,

(2) $\int_{-\infty}^{\infty} e^{\varepsilon_0|t|}||E(t;-S)||\,dt<\infty$ for some $\varepsilon_0>0$; hence $E(\cdot;-S)$ is exponentially decaying,
(3) $E(t;-S)-E(t;-S_0)$ is a bounded linear operator which depends continuously on $t\in\mathbb{R}$, and
(4) identity (2.2) holds.

Let us look more closely at item (3). First of all, $||E(t;-S)||\leqslant Me^{-\varepsilon_0|t|}$ for all $0\neq t\in\mathbb{R}$. Hence, (2.4) implies that $F(t)=E(t;-S)-E(t;-S_0)$ satisfies

$$||F(t)-F(s)||\leqslant M||\Gamma||\int_{-\infty}^{\infty}||E(t-\tau;-S_0)-E(s-\tau;-S_0)||\,d\varepsilon,$$

where $d\varepsilon=e^{-\varepsilon_0|\tau|}\,d\tau$. Fix $t\neq 0$ and let s be close to t (and on the same side of 0). Let $I(t,s;\varepsilon)$ stand for the real interval of length 2ε and midpoint $\frac{1}{2}(t+s)$. Then $t,s\in I(t,s;\varepsilon)$ whenever $|t-s|<2\varepsilon$). We have

$$\begin{aligned}||F(t)-F(s)||&\leqslant M||\Gamma||\int_{-\infty}^{\infty}||E(t-\tau;-S_0)-E(s-\tau;-S_0)||\,d\varepsilon\\&=M||\Gamma||\left(\int_{I(t,s;\varepsilon)}+\int_{\mathbb{R}\setminus I(t,s;\varepsilon)}\right)||E(t-\tau;-S_0)-E(s-\tau;-S_0)||\,d\varepsilon.\end{aligned}$$

Applying uniform continuity we select δ such that

$$||E(x;-S_0)-E(y;-S_0)||<\varepsilon$$

whenever $|x-y|<\delta$ and $|x|,|y|>\frac{1}{2}\varepsilon$.

Now take ε so small that the distance of 0 to $I(t,s;\varepsilon)$ is at least $\frac{1}{2}\varepsilon$. For $|t-s|\leqslant\min(\delta,2\varepsilon)$ we have

$$\begin{aligned}&\int_{\mathbb{R}\setminus I(t,s;\varepsilon)}||E(t-\tau;-S_0)-E(s-\tau;-S_0)||\,d\varepsilon\\&\quad\leqslant\int_{\mathbb{R}\setminus I(t,s;\varepsilon)}\varepsilon e^{-\varepsilon_0|\tau|}\,d\tau\leqslant\int_{-\infty}^{\infty}\varepsilon e^{-\varepsilon_0|\tau|}\,d\tau=\frac{\varepsilon}{2\varepsilon_0}\end{aligned}$$

(since $|(t-\tau)-(s-\tau)|=|t-s|<\delta$ by assumption). Now choose M_0 such that

$$||E(t;-S_0)||\leqslant M_0$$

for $0\neq t\in\mathbb{R}$. Then

$$\begin{aligned}&\int_{I(t,s;\varepsilon)}||E(t-\tau;-S_0)-E(s-\tau;-S_0)||\,d\varepsilon\\&\quad\leqslant\int_{I(t,s;\varepsilon)}2M_0e^{-\varepsilon_0|\tau|}\,d\tau=4M_0\varepsilon.\end{aligned}$$

Then finally

$$||F(t) - F(s)|| \leqslant M||\Gamma||\varepsilon\left(4M_0 + \frac{1}{2\varepsilon_0}\right),$$

which implies the norm continuity of $F(t)$ in $t \in \mathbb{R}$.

As a result of items (1)–(4) [2], $-S$ is exponentially dichotomous and its constituent semigroups are norm continuous. □

Corollary 4. *Let $-S_0$ be exponentially dichotomous with resolvent $(\lambda - S_0)^{-1}$ vanishing in the norm as $\lambda \to \infty$ along the imaginary line, let Γ be a bounded operator, and let $-S = -S_0 + \Gamma$, where $\mathscr{D}(S) = \mathscr{D}(S_0)$, defined on the complex Hilbert space $\mathscr{X}$. Suppose a strip $\{\lambda \in \mathbb{C} : |\mathrm{Re}\,\lambda| \leqslant \varepsilon\}$ is contained in the resolvent set of S. Then $-S$ is exponentially dichotomous and $||(\lambda - S)^{-1}|| \to 0$ as $\lambda \to \infty$ along the imaginary line.*

Proof. Theorem 1 implies that the constituent semigroups of $-S_0$ are immediately norm continuous. Thus Theorem 3 implies that $-S$ is exponentially dichotomous with immediately norm continuous constituent semigroups. Then in the norm topology we have

$$(\lambda - S)^{-1} = -\int_{-\infty}^{\infty} e^{\lambda t} E(t; -S)\, dt, \quad \mathrm{Re}\,\lambda = 0.$$

Thus the Riemann–Lebesgue lemma implies that $||(\lambda - S)^{-1}|| \to 0$ as $\lambda \to \infty$ along the imaginary line. □

3. Canonical factorization and matching of subspaces

Let $-S_0$ be exponentially dichotomous and Γ a bounded operator on a complex Banach space $\mathscr{X}$, and let $-S = -S_0 + \Gamma$, where $\mathscr{D}(S) = \mathscr{D}(S_0)$ and $\{\lambda \in \mathbb{C} : |\mathrm{Re}\,\lambda| \leqslant \varepsilon\} \subset \rho(S)$ for some $\varepsilon > 0$. Then $-S$ is exponentially dichotomous if either $-S_0$ has immediately norm continuous constituent semigroups or Γ is a compact operator (cf. Theorems 2 and 3). The bisemigroup generated by $-S$ is the unique solution of the convolution equation (2.4). In this section we consider the analogous vector-valued Wiener–Hopf integral equation

$$\phi(t) - \int_0^{\infty} E(t - \tau; -S_0)\Gamma\phi(\tau)\, d\tau = g(t), \tag{3.1}$$

where $t > 0$.

Suppose W is a continuous function from the extended imaginary axis $i(\mathbb{R} \cup \{\infty\})$ into $\mathscr{L}(\mathscr{X})$. Then by a *left canonical (Wiener–Hopf) factorization* of W we mean a representation of W of the form

$$W(\lambda) = W_+(\lambda) W_-(\lambda), \quad \mathrm{Re}\,\lambda = 0, \tag{3.2}$$

in which $W_{\pm}(\pm\lambda)$ is continuous on the closed right half-plane (the point at ∞ included), is analytic on the open right half-plane, and takes only invertible values for λ in the closed right half-plane (the point at infinity included). Obviously, such an operator function only takes invertible values on the extended imaginary axis. By a *right canonical* (*Wiener–Hopf*) *factorization* we mean a representation of W of the form

$$W(\lambda) = W_-(\lambda)W_+(\lambda), \quad \operatorname{Re}\lambda = 0, \tag{3.3}$$

where $W_{\pm}(\lambda)$ are as above.

We first need the following crucial lemma.

Lemma 5. *Let S_0 be an exponentially dichotomous operator on a complex Banach space $\mathscr{X}$. Then the operator $\mathbb{L}$ defined by*

$$(\mathbb{L}\chi)(t) = \int_{-\infty}^{\infty} E(t-\tau;-S_0)\chi(\tau)\,d\tau, \quad t>0 \tag{3.4}$$

is bounded on $L^p(\mathbb{R};\mathscr{X})$ $(1\leqslant p<\infty)$, $C_0(\mathbb{R};\mathscr{X})$, and $C_0(\mathbb{R}^-;\mathscr{X})\dot{+}C_0(\mathbb{R}^+;\mathscr{X})$.

Proof. Certainly, for every $t\in\mathbb{R}$ the function $\tau\mapsto E(t-\tau;-S_0)\chi(\tau)$ is strongly measurable in $\tau\in\mathbb{R}$ if χ is a measurable $\mathscr{X}$-valued step function. In this case we easily prove that $\mathbb{L}\chi\in L^p(\mathbb{R};\mathscr{X})$ $(1\leqslant p\leqslant\infty)$ and that

$$\|\mathbb{L}\chi\|_{L^p(\mathbb{R};\mathscr{X})}\leqslant C\|\chi\|_{L^p(\mathbb{R};\mathscr{X})},$$

where

$$C = \int_{-\infty}^{\infty} \|E(\tau;-S_0)\|_{\mathscr{L}(\mathscr{X})}\,d\tau.$$

Since for $1\leqslant p<\infty$ the measurable step functions are dense in $L^p(\mathbb{R};\mathscr{X})$ (cf. [7]), we obtain the lemma for $L^p(\mathbb{R};\mathscr{X})$ $(1\leqslant p<\infty)$.

Now note that the integral in (3.4) is a Bochner integral (cf. [7]) if $\chi\in C_0(\mathbb{R}^-;\mathscr{X})\dot{+}C_0(\mathbb{R}^+;\mathscr{X})$. The Theorem of Dominated Convergence for Bochner integrals then implies that the vector function $\mathbb{L}\chi\in C_0(\mathbb{R}^-;\mathscr{X})\dot{+}C_0(\mathbb{R}^+;\mathscr{X})$ whenever $\chi\in C_0(\mathbb{R}^-;\mathscr{X})\dot{+}C_0(\mathbb{R}^+;\mathscr{X})$. □

If $\mathscr{X}$ is a Hilbert space and $p=2$, there is an alternative proof of Lemma 5 (cf. [9]). Using that the Fourier transform $\mathscr{F}$ is a unitary operator on $L^2(\mathbb{R};\mathscr{X})$, we easily see that $\mathscr{F}\mathbb{L}\mathscr{F}^{-1}$ is the premultiplication by the bounded operator function $L(\lambda) = i(\lambda+iS_0)^{-1}$ (cf. (2.2)), which settles the boundedness of $\mathbb{L}$ in this particular case.

We have the following fundamental result. Similar results in various different contexts exist in the finite dimensional case [1], for equations with symbols analytic in a strip and at infinity [3], for extended Pritchard–Salamon realizations [15], and for abstract kinetic equations [13].

By $E(\mathbb{R}^+;\mathscr{X})$ we mean any of the spaces $L^p(\mathbb{R}^+;\mathscr{X})$ $(1\leqslant p<\infty)$ or $C_0(\mathbb{R}^+;\mathscr{X})$.

Theorem 6. *Suppose $\mathscr{X}$ is a complex Banach space. Let $-S_0$ be exponentially dichotomous, let Γ be a bounded operator, and let $-S = -S_0 + \Gamma$, where $\mathscr{D}(S) = \mathscr{D}(S_0)$, have the property that $\{\lambda \in \mathbb{C} : |\mathrm{Re}\,\lambda| < \varepsilon\} \subset \rho(S)$ for some $\varepsilon > 0$. Assume that either the constituent semigroups of $-S_0$ are immediately norm continuous or that Γ is a compact operator. Let P_0 and P stand for the separating projections of $-S_0$ and $-S$, respectively. Then the following statements are equivalent*:

(a) *The operator function*

$$W(\lambda) = (\lambda - S_0)^{-1}(\lambda - S) = I_{\mathscr{X}} + (\lambda - S_0)^{-1}\Gamma, \quad |\mathrm{Re}\,\lambda| \leqslant \varepsilon, \tag{3.5}$$

has a left canonical factorization with respect to the imaginary axis.

(b) *We have the decomposition*

$$\mathrm{Ker}\, P \dotplus \mathrm{Im}\, P_0 = \mathscr{X}. \tag{3.6}$$

(c) *For some* (*and hence every*) $E(\mathbb{R}^+; \mathscr{X})$, *the vector-valued Wiener–Hopf equation*

$$\phi(t) - \int_0^\infty E(t-\tau; -S_0)\Gamma\phi(\tau)\, d\tau = g(t), \quad t > 0 \tag{3.7}$$

is uniquely solvable in $E(\mathbb{R}^+; \mathscr{X})$ for any $g \in E(\mathbb{R}^+; \mathscr{X})$.

(d) *For some* (*and hence every*) $E(\mathbb{R}^+; \mathscr{X})$, *the vector-valued Wiener–Hopf equation*

$$\psi(t) - \int_0^\infty \Gamma E(t-\tau; -S_0)\psi(\tau)\, d\tau = h(t), \quad t > 0 \tag{3.8}$$

is uniquely solvable in $E(\mathbb{R}^+; \mathscr{X})$ for any $g \in E(\mathbb{R}^+; \mathscr{X})$.

(e) *Consider $\Gamma_1 \in \mathscr{L}(\mathscr{X}_0, \mathscr{X})$ and $\Gamma_2 \in \mathscr{L}(\mathscr{X}, \mathscr{X}_0)$ such that $\Gamma = \Gamma_1\Gamma_2$. Then for some* (*and hence every*) $E(\mathbb{R}^+; \mathscr{X}_0)$, *the vector-valued Wiener–Hopf equation*

$$\varphi(t) - \int_0^\infty \Gamma_2 E(t-\tau; -S_0)\Gamma_1\varphi(\tau)\, d\tau = f(t), \quad t > 0 \tag{3.9}$$

is uniquely solvable in $E(\mathbb{R}^+; \mathscr{X}_0)$ for any $g \in E(\mathbb{R}^+; \mathscr{X}_0)$.

Proof. We first note that $-S$ is exponentially dichotomous, as a result of Theorems 2 and 3.

(c) $\Leftrightarrow$ (d) $\Leftrightarrow$ (e): It follows immediately from Lemma 5 that the operator $\mathbb{L}_+$ defined by

$$(\mathbb{L}_+\chi)(t) = \int_0^\infty E(t-\tau; -S_0)\chi(\tau)\, d\tau, \quad t > 0 \tag{3.10}$$

is bounded on all of the spaces $E(\mathbb{R}^+; \mathscr{X})$. Further, (3.7)–(3.9) can be written in the concise form

$$\phi - \mathbb{L}_+\Gamma\phi = g, \tag{3.7a}$$

$$\psi - \Gamma\mathbb{L}_+\psi = h, \tag{3.8a}$$

$$\varphi - \Gamma_2\mathbb{L}_+\Gamma_1\varphi = f. \tag{3.9a}$$

A simple Schur complement argument then yields the equivalence of parts (c)–(e) in each of the spaces $E(\mathbb{R}^+;\mathscr{X})$.

(b) $\Rightarrow$ (a): Suppose decomposition (3.6) is true. Let $\mathscr{P}$ denote the projection of $\mathscr{X}$ onto Ker P along Im P_0. Then (cf. [1])

$$W(\lambda) = [I_{\mathscr{X}} + (\lambda - S_0)^{-1}(I - \mathscr{P})\Gamma][I_{\mathscr{X}} + \mathscr{P}(\lambda - S_0)^{-1}\Gamma] \tag{3.11}$$

is a left canonical factorization of $W(\cdot)$. Indeed,

$$[I_{\mathscr{X}} + (\lambda - S_0)^{-1}(I - \mathscr{P})\Gamma]^{-1} = I_{\mathscr{X}} - (I - \mathscr{P})(\lambda - S)^{-1}\Gamma, \tag{3.12}$$

$$[I_{\mathscr{X}} + \mathscr{P}(\lambda - S_0)^{-1}\Gamma]^{-1} = I_{\mathscr{X}} - (\lambda - S)^{-1}\mathscr{P}\Gamma. \tag{3.13}$$

The norm continuity of the factors as $\lambda \to \infty$ along the imaginary axis follows from the fact that $\|(\lambda - S_0)^{-1}\| \to 0$ as $\lambda \to \infty$ along the imaginary axis, in case $-S_0$ is the infinitesimal generator of a uniformly exponentially stable immediately norm continuous semigroup [8, Corollary II 4.19], and from $\|(\lambda - S_0)^{-1}\Gamma\| \to 0$ as $\lambda \to \infty$ along the imaginary axis in case Γ is compact. The norm continuity of the inverses of the factors as $\lambda \to \infty$ along the imaginary axis follows in the same way, using Theorems 2 and 3.

(a) $\Rightarrow$ (c): Suppose the operator function $W(\cdot)$ in (3.5) has a left canonical factorization $W = W_- W_+$ with respect to the imaginary line and let $\gamma_\pm \in L^1(\mathbb{R}^+;\mathscr{L}(\mathscr{X}))$ be such that

$$W_+(\lambda)^{-1} = I_{\mathscr{X}} + \int_0^\infty e^{\lambda t}\gamma_+(t)\,dt,$$
$$W_-(\lambda)^{-1} = I_{\mathscr{X}} + \int_{-\infty}^0 e^{\lambda t}\gamma_-(-t)\,dt.$$

Then standard methods (cf. [10, Section I.8], also [11, Chapter XIII]) show that

$$\phi(t) = g(t) + \int_0^\infty \gamma(t,\tau)g(\tau)\,d\tau,$$

where

$$\gamma(t,\tau) = \begin{cases} \gamma_+(t-\tau) + \int_0^\tau \gamma_+(t-\alpha)\gamma_-(\tau-\alpha)\,d\alpha, & 0 \leqslant \tau < t < \infty, \\ \gamma_-(\tau - t) + \int_0^t \gamma_+(t-\alpha)\gamma_-(\tau-\alpha)\,d\alpha, & 0 \leqslant t < \tau < \infty, \end{cases}$$

represents the unique solution of (3.7) in $L^p(\mathbb{R}^+;\mathscr{X})$ for each $g \in L^p(\mathbb{R}^+;\mathscr{X})$. It is evident that this solution belongs to $C_0(\mathbb{R}^+;\mathscr{X})$ whenever $g \in C_0(\mathbb{R}^+;\mathscr{X})$.

(c) $\Rightarrow$ (b): Now suppose (3.7) has a unique solution $\phi \in C_0(\mathbb{R}^+; \mathscr{X})$ for every $g \in C_0(\mathbb{R}^+; \mathscr{X})$. Consider the solution $\phi(\cdot; x)$ of (3.7) at $t = 0^+$ if $g(t) = E(t; -S_0)x$ for $t > 0$ and $x \in \mathscr{X}$. Let us define $\mathscr{P}x = \phi(0^+; x)$, i.e.,

$$\mathscr{P}x = [(I - \mathbb{L}_+ \Gamma)^{-1} E(\cdot; -S_0)x](t = 0^+),$$

where $x \in \mathscr{X}$. For $u \geqslant 0$, $t > 0$ and $x \in \mathscr{X}$ we now compute

$$\begin{aligned}
&\phi(t+u; x) - \int_0^\infty E(t-\tau; -S_0)\Gamma\phi(\tau+u; x)\, d\tau \\
&\quad = \phi(t+u; x) - \int_u^\infty E(t+u-\tau; -S_0)\Gamma\phi(\tau; x)\, d\tau \\
&\quad = E(t+u; -S_0)x + \int_0^u E(t+u-\tau; -S_0)\Gamma\phi(\tau; x)\, d\tau \\
&\quad = E(t; -S_0)\left[E(u; -S_0)x + \int_0^u E(u-\tau; -S_0)\Gamma\phi(\tau; x)\, d\tau\right] \\
&\quad = E(t; -S_0)(I - P_0)\left[E(u; -S_0)x + \int_0^u E(u-\tau; -S_0)\Gamma\phi(\tau; x)\, d\tau\right].
\end{aligned}$$

Now note that $(I - P_0)\int_u^\infty E(u-\tau; -S_0)\Gamma\phi(\tau; x)\, d\tau = 0$, to see that

$$\begin{aligned}
&\phi(t+u; x) - \int_0^\infty E(t-\tau; -S_0)\Gamma\phi(\tau+u; x)\, d\tau \\
&\quad = E(t; -S_0)\left[E(u; -S_0)x + \int_0^\infty E(u-\tau; -S_0)\Gamma\phi(\tau; x)\, d\tau\right] \\
&\quad = E(t; -S_0)\phi(u; x).
\end{aligned}$$

Hence

$$\phi(t+u; x) = \phi(t; \phi(u; x)).$$

Thus for every $t \geqslant 0$ there exists $\mathscr{P}_t \in \mathscr{L}(\mathscr{X})$, which is strongly continuous in t, such that $\mathscr{P}_t \mathscr{P}_u = \mathscr{P}_{t+u}$ for $t, u \geqslant 0$ and $\mathscr{P}_0 = \mathscr{P}$. Hence $\mathscr{P}$ is a projection. Further, $\mathscr{P}x = 0$ iff $\mathscr{P}_t x = 0$ for all $t \geqslant 0$ iff $\phi(t; x) \equiv 0$ for all $t \geqslant 0$ iff $E(t; -S_0)x = 0$ for all $t \geqslant 0$ iff $(I - P_0)x = 0$, so that $\operatorname{Ker} \mathscr{P} = \operatorname{Im} P_0$.

If $y \in \mathscr{D}(S_0) = \mathscr{D}(S)$ (so that $E(t; -S)y \in \mathscr{D}(S)$), we compute for $t > 0$

$$\begin{aligned}
&E(t; -S)y - \int_0^\infty E(t-\tau; -S_0)\Gamma E(\tau; -S_0)y\, d\tau \\
&\quad = E(t; -S)y - \left(\int_0^t + \int_t^\infty\right) \frac{\partial}{\partial\tau}\{E(t-\tau; -S_0)E(\tau; -S)y\}\, d\tau \\
&\quad = E(t; -S)y - E(0^+; -S_0)E(t; -S)y
\end{aligned}$$

$$\begin{aligned} &+ E(0^-; -S_0)E(t; -S)y + E(t; -S_0)E(0^+; -S)y \\ &= E(t; -S_0)E(0^+; -S)y \\ &= E(t; -S_0)(I - P)y. \end{aligned}$$

Hence for all $y \in \mathscr{D}(S_0) = \mathscr{D}(S)$

$$\phi(t; (I - P)y) = E(t; -S)y. \tag{3.14}$$

By continuous extension, it is clear that (3.14) holds for every $y \in \mathscr{X}$. The latter implies $I - P = \mathscr{P}(I - P)$, so that $\operatorname{Ker} P \subset \operatorname{Im} \mathscr{P}$.

To finish the proof, it remains to show that $\operatorname{Im} \mathscr{P} \subset \operatorname{Ker} P$. For $z \in \mathscr{D}(S_0') \subset \mathscr{X}'$, we have after some calculations

$$\frac{d}{dt} \langle \phi(t; x), z \rangle + \langle \phi(t; x), S_0' z \rangle = \langle \Gamma \phi(t; x), z \rangle,$$

so that

$$\frac{d}{dt} \langle \phi(t; x), z \rangle = -\langle \phi(t; x), S' z \rangle.$$

Laplace transforming the latter expression for $\operatorname{Re} \lambda < 0$ we get

$$\begin{aligned} 0 &= \int_0^\infty e^{\lambda t} \left\{ \frac{d}{dt} \langle \phi(t; x), z \rangle + \langle \phi(t; x), S' z \rangle \right\} dt \\ &= [e^{\lambda t} \langle \phi(t; x), z \rangle]_{t=0}^\infty - \int_0^\infty e^{\lambda t} \langle \phi(t; x), (\lambda - S') z \rangle \, dt \\ &= -\langle \mathscr{P} x, z \rangle - \langle \hat{\phi}(\lambda; x), (\lambda - S') z \rangle. \end{aligned}$$

It now follows that the map $z \mapsto \langle \hat{\phi}(\lambda; x), (\lambda - S') z \rangle$ is a bounded linear functional on $\mathscr{X}'$ (and in fact belongs to the canonical image of $\mathscr{X}$ in $\mathscr{X}''$). Hence $\hat{\phi}(\lambda; x) \in \mathscr{D}(S'') \cap \mathscr{X} = \mathscr{D}(S)$, and

$$\langle \mathscr{P} x, z \rangle = -\langle (\lambda - S) \hat{\phi}(\lambda; x), z \rangle, \quad z \in \mathscr{D}(S').$$

Since $\mathscr{D}(S')$ is dense in $\mathscr{X}'$, we have

$$\mathscr{P} x = (\lambda - S) \hat{\phi}(\lambda; x),$$

whence

$$\hat{\phi}(\lambda; x) = (\lambda - S)^{-1} \mathscr{P} x, \quad \operatorname{Re} \lambda < 0.$$

Now recall that S is exponentially dichotomous. Then the analyticity of $\hat{\phi}(\lambda; x)$ for $\operatorname{Re} \lambda < 0$ (which follows from the fact that $\phi(\cdot; x) \in C_0(\mathbb{R}^+; \mathscr{X})$) implies that

$\mathscr{P}x \in \operatorname{Im}(I-P)$. In other words, $\operatorname{Im}\mathscr{P} \subset \operatorname{Ker} P$, which we set out to prove. This proves part (b). □

The following analogous result is easily derived from Theorem 6 by applying Theorem 6 to the operators $-S_0$, $-\Gamma$ and $-S$ rather than to the operators S_0, Γ and S.

By $E(\mathbb{R}^-;\mathscr{X})$ we mean any of the spaces $L^p(\mathbb{R}^-;\mathscr{X})$ $(1 \leqslant p < \infty)$ or $C_0(\mathbb{R}^-;\mathscr{X})$.

Theorem 7. *Suppose $\mathscr{X}$ is a complex Banach space. Let $-S_0$ be exponentially dichotomous, let Γ be a bounded operator, and let $-S = -S_0 + \Gamma$, where $\mathscr{D}(S) = \mathscr{D}(S_0)$, have the property that $\{\lambda \in \mathbb{C} : |\operatorname{Re}\lambda| < \varepsilon\} \subset \rho(S)$ for some $\varepsilon > 0$. Assume that either the constituent semigroups of $-S_0$ are immediately norm continuous or that Γ is a compact operator. Let P_0 and P stand for the separating projections of $-S_0$ and $-S$, respectively. Then the following statements are equivalent*:

(a) *The operator function*

$$W(\lambda) = (\lambda - S_0)^{-1}(\lambda - S) = I_{\mathscr{X}} + (\lambda - S_0)^{-1}\Gamma, \quad |\operatorname{Re}\lambda| \leqslant \varepsilon, \tag{3.15}$$

has a right canonical factorization with respect to the imaginary axis.

(b) *We have the decomposition*

$$\operatorname{Ker} P_0 \dot{+} \operatorname{Im} P = \mathscr{X}. \tag{3.16}$$

(c) *For some* (*and hence every*) $E(\mathbb{R}^-;\mathscr{X})$, *the vector-valued Wiener–Hopf equation*

$$\phi(t) - \int_{-\infty}^{0} E(t-\tau; -S_0)\Gamma\phi(\tau)\, d\tau = g(t), \quad t < 0 \tag{3.17}$$

is uniquely solvable in $E(\mathbb{R}^-;\mathscr{X})$ for any $g \in E(\mathbb{R}^-;\mathscr{X})$.

(d) *For some* (*and hence every*) $E(\mathbb{R}^-;\mathscr{X})$, *the vector-valued Wiener–Hopf equation*

$$\psi(t) - \int_{-\infty}^{0} \Gamma E(t-\tau; -S_0)\psi(\tau)\, d\tau = h(t), \quad t < 0 \tag{3.18}$$

is uniquely solvable in $E(\mathbb{R}^-;\mathscr{X})$ for any $g \in E(\mathbb{R}^-;\mathscr{X})$.

(e) *Consider $\Gamma_1 \in \mathscr{L}(\mathscr{H}_0, \mathscr{H})$ and $\Gamma_2 \in \mathscr{L}(\mathscr{H}, \mathscr{H}_0)$ such that $\Gamma = \Gamma_1\Gamma_2$. Then for some* (*and hence every*) $E(\mathbb{R}^-;\mathscr{X}_0)$, *the vector-valued Wiener–Hopf equation*

$$\varphi(t) - \int_{-\infty}^{0} \Gamma_2 E(t-\tau; -S_0)\Gamma_1\varphi(\tau)\, d\tau = f(t), \quad t < 0 \tag{3.19}$$

is uniquely solvable in $E(\mathbb{R}^-;\mathscr{X}_0)$ for any $g \in E(\mathbb{R}^-;\mathscr{X}_0)$.

We recall that by $E(\mathbb{R}^\pm;\mathscr{H})$ we mean any of the spaces $L^p(\mathbb{R}^\pm;\mathscr{H})$ $(1 \leqslant p < \infty)$ or $C_0(\mathbb{R}^\pm;\mathscr{H})$.

Corollary 8. *Let $\mathscr{H}$ be a complex Hilbert space. Suppose $-S_0$ is exponentially dichotomous, Γ is a bounded operator, and $-S = -S_0 + \Gamma$, where $\mathscr{D}(S) = \mathscr{D}(S_0)$, have the property that $\{\lambda \in \mathbb{C} : |\mathrm{Re}\, \lambda| < \varepsilon\} \cap \sigma(S) = \emptyset$ for some $\varepsilon > 0$. Assume that either the constituent semigroups of $-S_0$ are immediately norm continuous or that Γ is a compact operator. Let P_0 and P be the separating projections of $-S_0$ and $-S$, respectively. Suppose there exists, for some complex Hilbert space $\mathscr{H}_0$, $\Gamma_1 \in \mathscr{L}(\mathscr{H}_0, \mathscr{H})$ and $\Gamma_2 \in \mathscr{L}(\mathscr{H}; \mathscr{H}_0)$ with $\Gamma = \Gamma_1 \Gamma_2$ such that one of the following conditions hold*:

1. *We have*

$$\sup_{\mathrm{Re}\, \lambda = 0} ||\Gamma_2(\lambda - S_0)^{-1}\Gamma_1|| < 1. \tag{3.20}$$

2. *There exists $\delta > 0$ such that*

$$\langle [I_{\mathscr{H}_0} + \Gamma_2(\lambda - S_0)^{-1}\Gamma_1]x, x\rangle \geqslant \delta ||x||^2 \tag{3.21}$$

   for every $x \in \mathscr{H}_0$.

Then all of the following statements are true:

(a) *The operator function $W(\cdot)$ in* (3.5) *has a left and a right canonical factorization with respect to the imaginary axis.*
(b) *We have the decompositions* (3.6) *and* (3.16).
(c) *For some* (*and hence every*) $E(\mathbb{R}^{\pm}; \mathscr{H})$, *the vector-valued Wiener–Hopf equation* (3.7) ((3.17), *respectively*) *is uniquely solvable in $E(\mathbb{R}^{\pm}; \mathscr{H})$ for any $g \in E(\mathbb{R}^{\pm}; \mathscr{H})$.*
(d) *For some* (*and hence every*) $E(\mathbb{R}^{\pm}; \mathscr{H})$, *the vector-valued Wiener–Hopf equation* (3.8) ((3.18), *respectively*) *is uniquely solvable in $E(\mathbb{R}^{\pm}; \mathscr{H})$ for any $g \in E(\mathbb{R}^{\pm}; \mathscr{H})$.*
(e) *For some* (*and hence every*) $E(\mathbb{R}^{\pm}; \mathscr{H}_0)$, *the vector-valued Wiener–Hopf equation* (3.9) ((3.19), *respectively*) *is uniquely solvable in $E(\mathbb{R}^{\pm}; \mathscr{H}_0)$ for any $g \in E(\mathbb{R}^{\pm}; \mathscr{H}_0)$.*

Proof. It suffices to prove part (e) of Theorem 6 for $p = 2$. Since the Fourier transform is (up to a constant factor) a unitary operator mapping $L^2(\mathbb{R}; \mathscr{H}_0)$ onto itself (which is true, since $\mathscr{H}_0$ is a Hilbert space), it is easy to see that on $L^2(\mathbb{R}^+; \mathscr{H}_0)$

$$||\mathbb{L}_+|| \leqslant \sup_{\mathrm{Re}\, \lambda = 0} ||\Gamma_2(\lambda - S_0)^{-1}\Gamma_1|| < 1.$$

Hence (3.9) is uniquely solvable in $L^2(\mathbb{R}^+; \mathscr{H}_0)$ for every $f \in L^2(\mathbb{R}^+; \mathscr{H}_0)$ whenever condition 1 holds. It is easily seen that condition 2 implies that condition 1 is true, because (3.21) amounts to requiring that there is a $c > 0$ for which

$$||c(I_{\mathscr{H}_0} + \Gamma_2(\lambda - S_0)^{-1}\Gamma_1) - I_{\mathscr{H}_0}|| < 1.$$

Then $cW(\cdot)$ has a left canonical factorization with respect to the imaginary line and hence so does $W(\cdot)$, which in turn implies all five statements. □

4. Block operators

Suppose $-S_0$ is exponentially dichotomous and Γ is a bounded linear operator on a complex Banach space $\mathscr{X}$. Define S by $-S = -S_0 + \Gamma$, and put

$$\mathscr{X}^{\pm} = \operatorname{Im} E(0^{\pm}; -S_0),$$

i.e., $\mathscr{X}^+ = \operatorname{Im}(I - P_0) = \operatorname{Ker} P_0$ and $X^- = \operatorname{Im} P_0$. Assuming that $\Gamma[\mathscr{X}^{\pm}] \subset \mathscr{X}^{\mp}$, we have the following block decompositions of S_0 and S with respect to the direct sum $\mathscr{X} = \mathscr{X}^+ \dot{+} \mathscr{X}^-$:

$$S_0 = \begin{pmatrix} A_0 & 0 \\ 0 & -A_1 \end{pmatrix}, \quad S = \begin{pmatrix} A_0 & -D \\ -Q & -A_1 \end{pmatrix}, \tag{4.1}$$

where $Q : \mathscr{X}^+ \to \mathscr{X}^-$ and $D : \mathscr{X}^- \to \mathscr{X}^+$ are bounded. Then S written in the form (4.1) is called a *block operator*. In this section we shall reformulate Theorem 6 in terms of solutions of Riccati equations and specialize the result obtained to the Hilbert space setting.

4.1. Riccati equations

In this subsection we relate the equivalent conditions (a)–(e) of Theorems 6 and 7 to the existence of certain bounded solutions of operator Riccati equations. These solutions are generated as angular operators pertaining to one of decompositions (3.6) and (3.16), an idea going back to [1].

Theorem 9. *Suppose $\mathscr{X}$ is a complex Banach space. Let $-S_0$ be exponentially dichotomous, Γ a bounded operator satisfying $\Gamma[\mathscr{X}^{\pm}] \subset \mathscr{X}^{\mp}$ and let $-S = -S_0 + \Gamma$, where $\mathscr{D}(S) = \mathscr{D}(S_0)$, have the property that $\{\lambda \in \mathbb{C} : |\operatorname{Re} \lambda| < \varepsilon\} \subset \rho(S)$ for some $\varepsilon > 0$. Assume that either the constituent semigroups of $-S_0$ are immediately norm continuous or that Γ is a compact operator, and let P_0 and P stand for the separating projections of $-S_0$ and $-S$, respectively. Then there exists a bounded linear operator Π_+ from $\mathscr{X}^-$ into $\mathscr{X}^+$ which maps $\mathscr{D}(A_1)$ into $\mathscr{D}(A_0)$, has the property that $B_1 = A_1 + Q\Pi_+$ generates an exponentially stable semigroup on $\mathscr{X}^-$, and satisfies the Riccati equation*

$$A_0 \Pi_+ x + \Pi_+ A_1 x - Dx + \Pi_+ Q \Pi_+ x = 0, \quad x \in \mathscr{D}(A_1), \tag{4.2}$$

if and only if the equivalent statements (a)–(e) *of Theorem* 6 *are true. Analogously, there exists a bounded linear operator Π_- from $\mathscr{X}^+$ into $\mathscr{X}^-$ which maps $\mathscr{D}(A_0)$ into $\mathscr{D}(A_1)$, has the property that $B_0 = A_0 - D\Pi_-$ generates an exponentially stable semigroup on $\mathscr{X}^+$, and satisfies the Riccati equation*

$$\Pi_- A_0 x + A_1 \Pi_- x - \Pi_- D \Pi_- x + Qx = 0, \quad x \in \mathscr{D}(A_0) \tag{4.3}$$

if and only if the equivalent statements (a)–(e) *of Theorem* 7 *are true.*

Proof. Suppose the equivalent conditions (a)–(e) of Theorem 6 are satisfied. Then we have decomposition (3.6), where $P_0 = -E(0^-; -S_0)$ and $P = -E(0^-; -S)$. Let $\mathscr{P}$ be the projection of $\mathscr{X}$ onto $\operatorname{Ker} P$ along $\operatorname{Im} P_0$. Since

$$P_0 = \begin{pmatrix} I_{\mathscr{X}^+} & 0 \\ 0 & 0 \end{pmatrix}, \quad I - P_0 = \begin{pmatrix} 0 & 0 \\ 0 & I_{\mathscr{X}^-} \end{pmatrix},$$

there exists an angular operator $\Pi_+ \in \mathscr{L}(\mathscr{X}^-, \mathscr{X}^+)$ such that

$$\mathscr{P} = \begin{pmatrix} 0 & \Pi_+ \\ 0 & I_{\mathscr{X}^-} \end{pmatrix}. \tag{4.4}$$

Because $\operatorname{Im} \mathscr{P}$ is an S-invariant subspace of $\mathscr{X}$, there exists a linear operator B_1 defined on a dense domain in $\mathscr{X}^-$ such that

$$\begin{pmatrix} A_0 & -D \\ -Q & -A_1 \end{pmatrix} \begin{pmatrix} \Pi_+ \\ I_{\mathscr{X}^-} \end{pmatrix} = \begin{pmatrix} \Pi_+ \\ I_{\mathscr{X}^-} \end{pmatrix} (-B_1). \tag{4.5}$$

Then $B_1 = A_1 + Q\Pi_+$ with $\mathscr{D}(B_1) = \mathscr{D}(A_1)$, $\Pi_+[\mathscr{D}(A_1)] \subset \mathscr{D}(A_0)$, and the Riccati equation (4.2) holds. Conversely, let Π_+ be a bounded linear operator from $\mathscr{X}^-$ into $\mathscr{X}^+$ which maps $\mathscr{D}(A_1)$ into $\mathscr{D}(A_0)$ and satisfies the Riccati equation (4.2). Put $B_1 = A_1 + Q\Pi_+$, where $\mathscr{D}(B_1) = \mathscr{D}(A_1)$. Then (4.5) is true and the operator $\mathscr{P}$ defined by (4.4) is a bounded projection on $\mathscr{X}$ whose range is a closed complement of $\operatorname{Im} P_0$. As a result, we have found decomposition (3.6).

In the same way we prove that the decomposition (3.16) is valid if and only if there exists a bounded linear operator Π_- from $\mathscr{X}^+$ into $\mathscr{X}^-$ which maps $\mathscr{D}(A_0)$ into $\mathscr{D}(A_1)$ and satisfies the Riccati equation (4.3). Indeed, the projection $\mathscr{Q}$ of $\mathscr{X}$ onto $\operatorname{Im} P$ along $\operatorname{Ker} P_0$ is given by

$$\mathscr{Q} = \begin{pmatrix} I_{\mathscr{X}^+} & 0 \\ \Pi_- & 0 \end{pmatrix}, \tag{4.6}$$

while (4.5) is replaced by

$$\begin{pmatrix} A_0 & -D \\ -Q & -A_1 \end{pmatrix} \begin{pmatrix} I_{\mathscr{X}^+} \\ \Pi_- \end{pmatrix} = \begin{pmatrix} I_{\mathscr{X}^+} \\ \Pi_- \end{pmatrix} B_0, \tag{4.7}$$

where $B_0 = A_0 - D\Pi_-$ with $\mathscr{D}(B_0) = \mathscr{D}(A_0)$. □

The proof of Theorem 9 shows that if both of the direct sum decompositions (3.6) and (3.16) exist, then $\mathscr{P} + \mathscr{Q}$ is a boundedly invertible operator on $\mathscr{X}$ such that

$$\operatorname{Im} P_0 \dot{+} \operatorname{Ker} P = \operatorname{Im} P \dot{+} \operatorname{Ker} P_0 = \mathscr{X}, \tag{4.8}$$

which makes S similar to the direct sum $B_0 \dot{+} (-B_1)$, where B_0 and B_1 both have their spectrum in the open left half-plane. In this case the inverse of $\mathscr{P} + \mathscr{Q}$ is given by

$$[\mathscr{P} + \mathscr{Q}]^{-1} = P_0 P + (I - P_0)(I - P). \tag{4.9}$$

4.2. Expressions for the Wiener–Hopf factors

Substituting expression (4.4) into expressions (3.11)–(3.13) for the left Wiener–Hopf factors of the symbol W and utilizing the Riccati equation (4.2) in the form

$$\begin{aligned} &-\Pi_+(\lambda+A_1)^{-1}+(\lambda-A_0)^{-1}\Pi_+ +(\lambda-A_0)^{-1}\Pi_+Q\Pi_+(\lambda+A_1)^{-1}\\ &=(\lambda-A_0)^{-1}D(\lambda+A_1)^{-1}, \end{aligned}$$

we obtain the following expressions for the factors:

$$W(\lambda)=\begin{pmatrix} I_{\mathscr{X}^+} & (\lambda-A_0)^{-1}D\\ 0 & I_{\mathscr{X}^-}\end{pmatrix}\begin{pmatrix} W^l(\lambda) & 0\\ 0 & I_{\mathscr{X}^-}\end{pmatrix}\begin{pmatrix} I_{\mathscr{X}^+} & 0\\ (\lambda+A_1)^{-1}Q & I_{\mathscr{X}^-}\end{pmatrix},$$

where

$$\begin{aligned} W^l(\lambda)&=[I_{\mathscr{X}^+}-(\lambda-A_0)^{-1}\Pi_+Q][I_{\mathscr{X}^+}+\Pi_+(\lambda+A_1)^{-1}Q],\\ W^l(\lambda)^{-1}&=[I_{\mathscr{X}^+}-\Pi_+(\lambda+B_1)^{-1}Q][I_{\mathscr{X}^+}+(\lambda-\tilde{B}_0)^{-1}\Pi_+Q]. \end{aligned}$$

Here $\tilde{B}_0=A_0+\Pi_+Q$ and $B_1=A_1+Q\Pi_+$ are generators of exponentially stable semigroups on $\mathscr{X}^+$.

On the other hand, if one replaces $\mathscr{P}$ by $\mathscr{Q}$ in (3.11)–(3.13), one obtains the expressions of the right Wiener–Hopf factors of the symbol W. Substituting expression (4.6) into these expressions for the right Wiener–Hopf factors and utilizing the Riccati equation (4.3) in the form

$$\begin{aligned} &-\Pi_-(\lambda-A_0)^{-1}+(\lambda+A_1)^{-1}\Pi_- +(\lambda+A_1)^{-1}\Pi_-D\Pi_-(\lambda-A_0)^{-1}\\ &=(\lambda+A_1)^{-1}Q(\lambda-A_0)^{-1}, \end{aligned}$$

we obtain the following expressions for the factors:

$$W(\lambda)=\begin{pmatrix} I_{\mathscr{X}^+} & 0\\ (\lambda+A_1)^{-1}Q & I_{\mathscr{X}^-}\end{pmatrix}\begin{pmatrix} I_{\mathscr{X}^+} & 0\\ 0 & W^r(\lambda)\end{pmatrix}\begin{pmatrix} I_{\mathscr{X}^+} & (\lambda-A_0)^{-1}D\\ 0 & I_{\mathscr{X}^-}\end{pmatrix},$$

where

$$\begin{aligned} W^r(\lambda)&=[I_{\mathscr{X}^-}-(\lambda+A_1)^{-1}\Pi_-D][I_{\mathscr{X}^-}+\Pi_-(\lambda-A_0)^{-1}D],\\ W^r(\lambda)^{-1}&=[I_{\mathscr{X}^-}-\Pi_-(\lambda-B_0)^{-1}D][I_{\mathscr{X}^-}+(\lambda+\tilde{B}_1)^{-1}\Pi_-D]. \end{aligned}$$

Here $B_0=A_0-D\Pi_-$ and $\tilde{B}_1=A_1-\Pi_-D$ are generators of exponentially stable semigroups on $\mathscr{X}^-$.

4.3. Perturbation results if one of D and Q is compact

We restrict ourselves to the case in which D is compact. The case in which Q is compact, can be reduced to it by replacing S_0 and Γ by $-S_0$ and $-\Gamma$ and considering the last two operators as block operators with respect to the decompositions $\mathscr{X}^- \dot{+} \mathscr{X}^+ = \mathscr{X}$, which yields

$$-S_0 = \begin{pmatrix} A_1 & 0 \\ 0 & -A_0 \end{pmatrix}, \quad -\Gamma = \begin{pmatrix} 0 & -Q \\ -D & 0 \end{pmatrix}.$$

Solving the Lyapunov equation

$$ZA_0 + A_1 Z = -Q \quad \text{where } Z \in \mathscr{L}(\mathscr{X}^+, \mathscr{X}^-) \text{ and } Z[\mathscr{D}(A_0)] \subset \mathscr{D}(A_1), \tag{4.10}$$

which allows the unique solution represented by

$$Zx = \int_0^\infty e^{-tA_1} Q e^{-tA_0} x \, dt, \quad x \in \mathscr{X}^+, \tag{4.11}$$

we obtain the two identities

$$\begin{pmatrix} I_{\mathscr{X}^+} & 0 \\ Z & I_{\mathscr{X}^-} \end{pmatrix} \begin{pmatrix} A_0 & 0 \\ Q & -A_1 \end{pmatrix} \begin{pmatrix} I_{\mathscr{X}^+} & 0 \\ -Z & I_{\mathscr{X}^-} \end{pmatrix} = \begin{pmatrix} A_0 & 0 \\ 0 & -A_1 \end{pmatrix}, \tag{4.12}$$

$$\begin{pmatrix} I_{\mathscr{X}^+} & 0 \\ Z & I_{\mathscr{X}^-} \end{pmatrix} \begin{pmatrix} A_0 & D \\ Q & -A_1 \end{pmatrix} \begin{pmatrix} I_{\mathscr{X}^+} & 0 \\ -Z & I_{\mathscr{X}^-} \end{pmatrix} = \begin{pmatrix} A_0 - DZ & D \\ -ZDZ & -A_1 + ZD \end{pmatrix}. \tag{4.13}$$

Thus the right-hand sides of (4.12) and (4.13) differ by a compact operator, while the right-hand side of (4.12) obviously is exponentially dichotomous.

We now have the following result.

Theorem 10. *Suppose $\mathscr{X}$ is a complex Banach space. Let $-S_0$ be exponentially dichotomous, Γ a bounded operator, and $-S = -S_0 + \Gamma$, such that the block decomposition* (4.1) *is true. Suppose D is compact, and S does not have imaginary eigenvalues. Then there exists a bounded linear operator Π_+ from $\mathscr{X}^-$ into $\mathscr{X}^+$ which maps $\mathscr{D}(A_1)$ into $\mathscr{D}(A_0)$, has the property that $B_1 = A_1 + Q\Pi_+$ generates an exponentially stable semigroup on $\mathscr{X}^-$, and satisfies the Riccati equation* (4.2) *if and only if the equivalent statements* (a)–(e) *of Theorem* 6 *are true. Analogously, there exists a bounded linear operator Π_- from $\mathscr{X}^+$ into $\mathscr{X}^-$ which maps $\mathscr{D}(A_0)$ into $\mathscr{D}(A_1)$, has the property that $B_0 = A_0 - D\Pi_-$ generates an exponentially stable semigroup on $\mathscr{X}^+$, and satisfies the Riccati equation* (4.3) *if and only if the equivalent statements* (a)–(e) *of Theorem* 7 *are true.*

Proof. What we have to check is the existence of a vertical strip around the imaginary axis that is free of spectrum of S. However, there exists $\varepsilon > 0$ such that

$\{\lambda \in \mathbb{C} : |\mathrm{Re}\,\lambda| \leqslant \varepsilon\} \subset \rho(S_0)$, while $(\lambda - S_0)^{-1}\tilde{\Gamma}$, where $\tilde{\Gamma}$ is the operator defined by

$$\tilde{\Gamma} = \begin{pmatrix} -DZ & D \\ -ZDZ & ZD \end{pmatrix} = \begin{pmatrix} I_{\mathscr{X}^+} & 0 \\ 0 & Z \end{pmatrix} \begin{pmatrix} D & D \\ D & D \end{pmatrix} \begin{pmatrix} -Z & 0 \\ 0 & I_{\mathscr{X}^-} \end{pmatrix} \tag{4.14}$$

vanishes in the operator norm as $\lambda \to \infty$ within this strip. This follows from the compactness of $\tilde{\Gamma}$, which again follows from the compactness of D. Hence there exists $\delta > 0$ such that

$$\{\lambda \in \mathbb{C} : |\mathrm{Re}\,\lambda| \leqslant \varepsilon,\ |\mathrm{Im}\,\lambda| > \delta\} \subset \rho\left(\begin{pmatrix} A_0 - DZ & D \\ -ZDZ & -A_1 + ZD \end{pmatrix} \right) = \rho(S).$$

Using the compactness of the imaginary interval $i[-\delta, \delta]$ one finds $\varepsilon_0 > 0$ (with $0 < \varepsilon_0 \leqslant \varepsilon$) such that

$$\{\lambda \in \mathbb{C} : |\mathrm{Re}\,\lambda| \leqslant \varepsilon_0,\ |\mathrm{Im}\,\lambda| > \delta\} \subset \rho(S),$$

which completes the proof. □

4.4. *The Hilbert space setting*

Let us now consider the Hilbert space setting. Let A_0 be the infinitesimal generator of a uniformly exponentially stable C_0-semigroup on a complex Hilbert space $\mathscr{H}$, and let D and Q be positive semidefinite (and bounded) selfadjoint operators on $\mathscr{H}$. Putting $\hat{\mathscr{H}} = \mathscr{H} \oplus \mathscr{H}$, we consider the block operator

$$S = \begin{pmatrix} A_0 & -D \\ -Q & -A_0^* \end{pmatrix}. \tag{4.15}$$

Such matrices were studied in [18] under the condition that for some $\varepsilon > 0$ and $\beta > (1/2)$ we have $\|(\lambda - A_0)^{-1}\| \leqslant \mathrm{const.}(1 + |\lambda|)^{-\beta}$. Some results in [18] were obtained under the strengthened condition that A_0 is μ-sectorial in the sense of [16]. We will consider $-S$ as a bounded perturbation of $-S_0 = (-A_0) \oplus A_0^*$. In other words, $-S = -S_0 + \Gamma$, where

$$\Gamma = \begin{pmatrix} 0 & D \\ Q & 0 \end{pmatrix}. \tag{4.16}$$

Next, consider the factorization

$$\Gamma = \Gamma_1 \Gamma_2, \quad \Gamma_1 = \begin{pmatrix} 0 & D^{1/2} \\ Q^{1/2} & 0 \end{pmatrix}, \quad \Gamma_2 = \begin{pmatrix} Q^{1/2} & 0 \\ 0 & D^{1/2} \end{pmatrix}. \tag{4.17}$$

Then

$$I_{\hat{\mathscr{H}}} + \Gamma_2(\lambda - S_0)^{-1}\Gamma_1 = \begin{pmatrix} I_{\mathscr{H}} & Q^{1/2}(\lambda - A_0)^{-1}D^{1/2} \\ D^{1/2}(\lambda + A_0^*)^{-1}Q^{1/2} & I_{\mathscr{H}} \end{pmatrix}, \quad (4.18)$$

which has $I_{\hat{\mathscr{H}}}$ as its real part if λ is purely imaginary. Thus the expression in (4.18) has a left and a right canonical factorization with respect to the imaginary axis. As a result of Theorem 10 we have

$$\operatorname{Im} P_0 \dotplus \operatorname{Ker} P = \operatorname{Im} P \dotplus \operatorname{Ker} P_0 = \mathscr{X}, \quad (4.19)$$

provided either $-A_0$ (and hence also $-A_0^*$) generates an immediately strongly continuous semigroup on $\mathscr{H}$ or one of D and Q is a compact operator. Furthermore, the Riccati equations (4.2) and (4.3) both have bounded solutions.

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