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Direct and Inverse Scattering for Skewselfadjoint Hamiltonian Systems

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Abstract. In this article the direct and inverse scattering theory for skewselfadjoint Hamiltonian systems on the line is developed. The inverse scattering problem of recovering the skewselfadjoint matrix potential from the reflection coefficient is solved explicitly using state space methods if bound states are assumed absent.

1. Introduction

Consider the skewselfadjoint Hamiltonian system of differential equations

$$-iJ_{2n}\frac{dX(x,\lambda)}{dx} - V(x)\,X(x,\lambda) = \lambda\,X(x,\lambda), \qquad x \in \mathbb{R}, \tag{1.1}$$

where

$$J_{2n} = \begin{bmatrix} I_n & 0 \\ 0 & -I_n \end{bmatrix}, \qquad V(x) = \begin{bmatrix} 0 & k(x) \\ -k(x)^\dagger & 0 \end{bmatrix}, \tag{1.2}$$

with I_n the identity matrix of order n, the $n \times n$ matrix function k has complex-valued entries belonging to $L^1(\mathbb{R})$, $\lambda \in \mathbb{R}$ is an eigenvalue parameter, and $\dagger$ denotes the matrix conjugate transpose. We call the function V the potential matrix, k the potential and the parameter λ the wavenumber. Note that $V(x)$ is a J_{2n}-selfadjoint $2n \times 2n$ matrix and satisfies

$$J_{2n}\,V(x) = -V(x)\,J_{2n}.$$

We can think of $X(x,\lambda)$ in (1.1) as either a column vector of $2n$ entries or as a $2n \times 2n$ matrix. For $\lambda \in \mathbb{R}$, we define the Jost solution from the left, $F_l(x,\lambda)$, and the Jost solution from the right, $F_r(x,\lambda)$, as the $2n \times 2n$ matrix solutions of (1.1) satisfying the boundary conditions

$$F_l(x,\lambda) = e^{i\lambda J_{2n}x}\left[I_{2n} + o(1)\right], \qquad x \to +\infty, \tag{1.3}$$

$$F_r(x,\lambda) = e^{i\lambda J_{2n}x}\left[I_{2n} + o(1)\right], \qquad x \to -\infty. \tag{1.4}$$

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Using (1.1), (1.3), and (1.4), we obtain

$$F_l(x,\lambda) = e^{i\lambda J_{2n}x} - iJ_{2n}\int_x^\infty dy\, e^{-i\lambda J_{2n}(y-x)}\, V(y)\, F_l(y,\lambda), \tag{1.5}$$

$$F_r(x,\lambda) = e^{i\lambda J_{2n}x} + iJ_{2n}\int_{-\infty}^x dy\, e^{i\lambda J_{2n}(x-y)}\, V(y)\, F_r(y,\lambda). \tag{1.6}$$

For a square matrix function $E(x)$, let us use $||E||_1$ to denote $\int_{-\infty}^\infty dx\, ||E(x)||$, where $||\cdot||$ stands for the matrix norm defined by $||A|| = \sup\{||Av||_2 : \ ||v||_2 = 1\}$ and $||\cdot||_2$ is the Euclidean vector norm. Since the entries of $k(x)$ belong to $L^1(\mathbb{R})$, for each fixed $\lambda \in \mathbb{R}$ it follows by iteration that (1.5) and (1.6) are uniquely solvable and that $||F_l(x,\lambda)||$ and $||F_r(x,\lambda)||$ are bounded above by $e^{||k||_1}$. From (1.3)–(1.6) we get

$$F_l(x,\lambda) = e^{i\lambda J_{2n}x}\left[a_l(\lambda) + o(1)\right], \qquad x \to -\infty, \tag{1.7}$$

$$F_r(x,\lambda) = e^{i\lambda J_{2n}x}\left[a_r(\lambda) + o(1)\right], \qquad x \to +\infty, \tag{1.8}$$

where

$$a_l(\lambda) = I_{2n} - iJ_{2n}\int_{-\infty}^\infty dy\, e^{-i\lambda J_{2n}y}\, V(y)\, F_l(y,\lambda),$$

$$a_r(\lambda) = I_{2n} + iJ_{2n}\int_{-\infty}^\infty dy\, e^{-i\lambda J_{2n}y}\, V(y)\, F_r(y,\lambda).$$

In this article we solve the direct and inverse scattering problem for (1.1), where the inverse scattering problem consists of the determination of the potential $k(x)$ from either of the reflection coefficients $R(\lambda)$ and $L(\lambda)$, which are defined in (3.7) in terms of the matrices $a_l(\lambda)$ and $a_r(\lambda)$, plus suitable bound state data. In this paper we restrict ourselves to inverse scattering problems where there are no bound states.

Shabat [32] and Beals and Coifman [10, 11] considered the $n \times n$ matrix differential operator

$$d\varphi/dx = \lambda \boldsymbol{J}\varphi + q(x)\,\varphi,$$

where $\boldsymbol{J} = \operatorname{diag}\{\alpha_1,\ldots,\alpha_n\}$ with distinct complex α_j and $q(x)$ an $n \times n$ off-diagonal matrix with entries belonging to $L^1(\mathbb{R})$ or more restrictive classes, without requiring $q(x)$ to be selfadjoint. They proved that the inverse problem has a unique solution within a certain class of potentials for an open and dense set of scattering data. The solution of the inverse scattering problem for such linear systems is useful in solving the Cauchy problem for various nonlinear evolution equations. For details and further references, we refer the interested reader to [1, 12] and the references therein.

By putting $Z(x,\lambda) = \frac{1}{\sqrt{2}}[I_{2n} + i\boldsymbol{q}_{2n}]\,X(x,\lambda)$, where

$$\boldsymbol{q}_{2n} = \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix},$$

we can convert (1.1) into the massless Dirac equation of order $2n$. The direct and inverse scattering problems for the Dirac system on the half line were studied in [16]. The interested reader is referred to [16, 22, 23] and the references therein for more information on the Dirac system.

Melik-Adamyan [25, 26, 27], L.A. Sakhnovich [30, 31], and A.L. Sakhnovich [29] have studied the direct and inverse scattering problems for (1.1) on the half line. Alpay and Gohberg [3, 4, 5, 6] have applied state space methods to derive explicit expressions for the solution of the inverse problem for (1.1) on the half line from the general theory in [27] when the scattering data are rational functions and consist of either the spectral function of the differential operator $H = -iJ_{2n}\frac{d}{dx} - V(x)$ or a reflection function. Gohberg et al. [18, 19, 20] have solved a similar inverse problem when the scattering data consist of the spectral function of H and this function is rational.

In this article we develop a direct and inverse scattering theory for (1.1) when $k(x)$ has entries belonging to $L^1(\mathbb{R})$. Working within the framework established by Faddeev [15] and Deift and Trubowitz [14] for the Schrödinger equation on the line, we derive the analyticity and asymptotic properties of the Faddeev matrices and the scattering coefficients, employ them to derive a Riemann-Hilbert problem and various Marchenko integral equations, and recover the potential in terms of the solutions of the Marchenko equations. We prove the J_{2n}-unitarity of the scattering matrix and exploit this property to prove the unique solvability of the Marchenko equations. We also establish the unique canonical Wiener-Hopf factorization of the scattering matrix and show how the potential is obtained once the factors are known. After that, for rational reflection coefficients we present a procedure to compute explicitly the scattering matrix from a reflection coefficient, assuming there are no bound states. When the reflection coefficients are rational, we apply state space methods to solve the Marchenko equations and the inverse problem explicitly.

This article follows its predecessor [2], where the potential matrix $V(x)$ is selfadjoint, the scattering matrix is unitary, and there do not exist bound states. Here we are dealing with a more complicated but physically much more interesting problem, where the potential matrix $V(x)$ is J_{2n}-selfadjoint, the scattering matrix is J_{2n}-unitary, and bound states may exist. As a result, there are notable differences in the proof of many equations in this article from their counterparts in [2]. However, when deriving the Marchenko integral equations and applying state space methods to solve them we assume for simplicity that there are no bound states.

Let us discuss the organization of this article. In Section 2 we introduce the Faddeev matrices, obtain their analyticity properties, and analyze some other properties of the Faddeev matrices and the Jost solutions of (1.1). In Section 3 we define the scattering matrix $\boldsymbol{S}(\lambda)$ in terms of the spatial asymptotics of the Jost solutions, prove the J_{2n}-unitarity of $\boldsymbol{S}(\lambda)$, and obtain various properties of the scattering coefficients. In Section 4 we analyze the Fourier transforms of the Faddeev matrices and the scattering coefficients. We then go on, in Section 5, to derive a Riemann-Hilbert problem for the Faddeev matrices. In Section 6, we convert the Riemann-Hilbert problem into both coupled and uncoupled Marchenko integral equations and prove their unique solvability, assuming there are no bound states. In Section 7 we show how to construct $\boldsymbol{S}(\lambda)$ explicitly when one of the reflection coefficients is a rational function and bound states are absent. Finally, in Section 8 we give an explicit solution of the inverse scattering problem with rational reflection coefficients.

Let us give some definitions. By $\mathbb{C}^+$ and $\mathbb{C}^-$ we denote the open upper half and lower half complex planes, respectively. We will use the notation $L^j(I;\mathbb{C}^{p\times q})$ to denote the Banach space of all complex $p\times q$ matrix functions $z(\alpha)$ whose entries belong to $L^j(I)$, endowed with the norm $\left[\int_I d\alpha\, \|z(\alpha)\|^j\right]$; if $q=1$, we simply write $L^j(I;\mathbb{C}^p)$.

2. Scattering solutions

In this section we introduce the Faddeev matrices and study some of their properties. The results obtained here will be used later to establish various properties of the scattering matrix and to solve the inverse scattering problem by the Marchenko method.

Proposition 2.1. *Let $X(x,\lambda)$ and $Y(x,\lambda)$ be any two solutions of* (1.1)*. Then, for real λ, $X(x,\lambda)^\dagger Y(x,\lambda)$ is independent of x.*

Proof. The result follows by differentiating $X(x,\lambda)^\dagger Y(x,\lambda)$ and using (1.1) together with the selfadjointness of $J_{2n}V(x)$ and J_{2n}. □

Proposition 2.2. *For $\lambda\in\mathbb{R}$, either Jost solution $F_l(x,\lambda)$ or $F_r(x,\lambda)$ forms a fundamental matrix of* (1.1) *and has determinant equal to one. Moreover, the matrices $a_l(\lambda)$ and $a_r(\lambda)$ appearing in* (1.7) *and* (1.8)*, respectively, satisfy*

$$\det a_l(\lambda) = \det a_r(\lambda) = 1. \tag{2.1}$$

Moreover, for $\lambda\in\mathbb{R}$, the Jost solutions satisfy

$$F_l(x,\lambda) = F_r(x,\lambda)\, a_l(\lambda), \tag{2.2}$$

$$F_r(x,\lambda)^\dagger\, F_l(x,\lambda) = a_r(\lambda)^\dagger = a_l(\lambda), \tag{2.3}$$

$$F_l(x,\lambda)^\dagger\, F_l(x,\lambda) = a_l(\lambda)^\dagger\, a_l(\lambda) = I_{2n}, \tag{2.4}$$

$$F_r(x,\lambda)^\dagger\, F_r(x,\lambda) = a_r(\lambda)^\dagger\, a_r(\lambda) = I_{2n}, \tag{2.5}$$

and hence

$$a_l(\lambda)\, a_r(\lambda) = a_r(\lambda)\, a_l(\lambda) = I_{2n}, \tag{2.6}$$

$$a_l(\lambda)^{-1} = a_l(\lambda)^\dagger, \qquad a_r(\lambda)^{-1} = a_r(\lambda)^\dagger. \tag{2.7}$$

In particular, $a_l(\lambda)$ and $a_r(\lambda)$ are unitary matrices.

Proof. From (1.1) it follows from [28] that

$$\frac{d\,[\det F_l(x,\lambda)]}{dx} = \big(\mathrm{tr}\,\{iJ_{2n}\,V(x) + i\lambda J_{2n}\}\big)\big(\det F_l(x,\lambda)\big),$$

where tr denotes the matrix trace. By (1.2), $iJ_{2n}\,V(x)+i\lambda J_{2n}$ has zero trace, and hence $\det F_l(x,\lambda)$ is independent of x and its value can be evaluated as $x\to+\infty$. Thus, we get $\det F_l(x,\lambda) = 1$, from which we also conclude that $F_l(x,\lambda)$ is a fundamental matrix of (1.1). Similarly, we find that $\det F_r(x,\lambda) = 1$ and $F_r(x,\lambda)$ is a fundamental matrix of (1.1). Then, from (1.2), (1.7), and (1.8) we obtain (2.1). Since either of $F_l(x,\lambda)$ and $F_r(x,\lambda)$ is a fundamental matrix of (1.1), with the help of (1.3) and (1.7), we get (2.2). Using Proposition 2.1, we obtain (2.3)–(2.5) by evaluating $F_r(x,\lambda)^\dagger\,F_l(x,\lambda)$, $F_l(x,\lambda)^\dagger\,F_l(x,\lambda)$, and $F_r(x,\lambda)^\dagger\,F_r(x,\lambda)$ as $x\to\pm\infty$. Then (2.6) and (2.7) readily follow. □

In terms of the Jost solutions, we define the Faddeev matrices $M_l(x,\lambda)$ and $M_r(x,\lambda)$ as

$$M_l(x,\lambda) = F_l(x,\lambda)\,e^{-i\lambda J_{2n}x}, \qquad M_r(x,\lambda) = F_r(x,\lambda)\,e^{-i\lambda J_{2n}x}. \tag{2.8}$$

From (1.3) and (1.4) we get

$$M_l(x,\lambda) = I_{2n} + o(1), \qquad x\to+\infty,$$
$$M_r(x,\lambda) = I_{2n} + o(1), \qquad x\to-\infty.$$

Let us partition the Jost solutions and Faddeev matrices into $n\times n$ blocks as follows:

$$F_l(x,\lambda) = \begin{bmatrix} F_{l1}(x,\lambda) & F_{l2}(x,\lambda)\\ F_{l3}(x,\lambda) & F_{l4}(x,\lambda)\end{bmatrix}, \qquad F_r(x,\lambda) = \begin{bmatrix} F_{r1}(x,\lambda) & F_{r2}(x,\lambda)\\ F_{r3}(x,\lambda) & F_{r4}(x,\lambda)\end{bmatrix}, \tag{2.9}$$

$$M_l(x,\lambda) = \begin{bmatrix} M_{l1}(x,\lambda) & M_{l2}(x,\lambda)\\ M_{l3}(x,\lambda) & M_{l4}(x,\lambda)\end{bmatrix}, \quad M_r(x,\lambda) = \begin{bmatrix} M_{r1}(x,\lambda) & M_{r2}(x,\lambda)\\ M_{r3}(x,\lambda) & M_{r4}(x,\lambda)\end{bmatrix}. \tag{2.10}$$

We also define

$$\sigma_\pm(x) = \pm\int_x^{\pm\infty} dy\,||k(y)||. \tag{2.11}$$

Proposition 2.3. *Assume that the entries of $k(x)$ belong to $L^1(\mathbb{R})$. Then:*

(i) *For each fixed $x\in\mathbb{R}$, $\begin{bmatrix} M_{l1}(x,\lambda)\\ M_{l3}(x,\lambda)\end{bmatrix}$ can be extended to a matrix function that is continuous in $\lambda\in\overline{\mathbb{C}^+}$ and analytic in $\lambda\in\mathbb{C}^+$ and tends to $\begin{bmatrix} I_n\\ 0\end{bmatrix}$ as $\lambda\to\infty$ in $\overline{\mathbb{C}^+}$.*

(ii) *For all $\lambda\in\overline{\mathbb{C}^+}$, $M_{l1}(x,\lambda)$ and $M_{l3}(x,\lambda)$ are bounded by $e^{\sigma_+(x)}$ in the norm.*

(iii) *For each fixed* $x \in \mathbb{R}$, $\begin{bmatrix} M_{l2}(x,\lambda) \\ M_{l4}(x,\lambda) \end{bmatrix}$ *can be extended to a matrix function that is continuous in* $\lambda \in \overline{\mathbb{C}^-}$ *and analytic in* $\lambda \in \mathbb{C}^-$ *and tends to* $\begin{bmatrix} 0 \\ I_n \end{bmatrix}$ *as* $\lambda \to \infty$ *in* $\overline{\mathbb{C}^-}$.

(iv) *For all* $\lambda \in \overline{\mathbb{C}^-}$, $M_{l2}(x,\lambda)$ *and* $M_{l4}(x,\lambda)$ *are bounded by* $e^{\sigma_+(x)}$ *in the norm.*

Proof. Using (2.8) in (1.5), we obtain

$$M_l(x,\lambda) = I_{2n} - iJ_{2n} \int_x^\infty dy\, e^{-i\lambda J_{2n}(y-x)}\, V(y)\, M_l(y,\lambda)\, e^{i\lambda J_{2n}(y-x)}. \tag{2.12}$$

Iterating (2.12) once, we get the uncoupled systems

$$M_{l1}(x,\lambda) = I_n - \int_x^\infty dy \int_y^\infty dz\, e^{2i\lambda(z-y)}\, k(y)\, k(z)^\dagger\, M_{l1}(z,\lambda), \tag{2.13}$$

$$\begin{aligned} M_{l2}(x,\lambda) = &-i \int_x^\infty dy\, e^{-2i\lambda(y-x)}\, k(y) \\ &- \int_x^\infty dy \int_y^\infty dz\, e^{-2i\lambda(y-x)}\, k(y)\, k(z)^\dagger\, M_{l2}(z,\lambda), \end{aligned} \tag{2.14}$$

$$\begin{aligned} M_{l3}(x,\lambda) = &-i \int_x^\infty dy\, e^{2i\lambda(y-x)}\, k(y)^\dagger \\ &- \int_x^\infty dy \int_y^\infty dz\, e^{2i\lambda(y-x)}\, k(y)^\dagger\, k(z)\, M_{l3}(z,\lambda), \end{aligned} \tag{2.15}$$

$$M_{l4}(x,\lambda) = I_n - \int_x^\infty dy \int_y^\infty dz\, e^{-2i\lambda(z-y)}\, k(y)^\dagger\, k(z)\, M_{l4}(z,\lambda). \tag{2.16}$$

Iterating the Volterra integral equations (2.13) and (2.15), we prove that the series of iterates converge absolutely and uniformly in $\lambda \in \overline{\mathbb{C}^+}$, and we also get the estimate in (ii). Similarly, we prove that the series of iterates of (2.14) and (2.16) converge absolutely and uniformly in $\lambda \in \overline{\mathbb{C}^-}$ and that the estimate in (iv) holds. To prove the assertions concerning the large-λ limit we first consider $M_{l3}(x,\lambda)$. To deal with the first term on the right-hand side of (2.15) we define

$$\omega(\lambda) = \sup_{x\in\mathbb{R}} \left\| \int_x^\infty dy\, e^{2i\lambda(y-x)}\, k(y)^\dagger \right\|.$$

By approximating $k(y)$ by infinitely differentiable matrix functions of compact support (as in the proof of the Riemann-Lebesgue lemma) it follows that $\omega(\lambda) \to 0$ as $\lambda \to \infty$ in $\overline{\mathbb{C}^+}$. Iterating (2.15) we get $\|M_{l3}(x,\lambda)\| \le \omega(\lambda)\, e^{\sigma_+(x)}$, which implies that $\|M_{l3}(x,\lambda)\| \to 0$ as $\lambda \to \infty$ in $\overline{\mathbb{C}^+}$. Next we consider $M_{l1}(x,\lambda)$. Let $G_{l1}(x,\lambda) = M_{l1}(x,\lambda) - I_n$ and consider the following integral equation for $G_{l1}(x,\lambda)$ which follows from (2.13):

$$G_{l1}(x,\lambda) = H_{l1}(x,\lambda) - \int_x^\infty dy \int_y^\infty dz\, e^{2i\lambda(z-y)}\, k(y)\, k(z)^\dagger\, G_{l1}(z,\lambda),$$

where
$$H_{l1}(x,\lambda) = -\int_x^\infty dy \int_y^\infty dz\, e^{2i\lambda(z-y)}\, k(y)\, k(z)^\dagger.$$
Since $\|H_{l1}(x,\lambda)\| \le \nu(\lambda)\,\sigma_+(x)$, we conclude that $\|G_{l1}(x,\lambda)\| \to 0$ as $\lambda \to \infty$ in $\overline{\mathbb{C}^+}$. This proves the assertion of (i) regarding the limit $\lambda \to \infty$. The proof of the corresponding statement in (iii) is similar. □

We have a similar result for the Faddeev matrix $M_r(x,\lambda)$.

Proposition 2.4. *Assume that the entries of $k(x)$ belong to $L^1(\mathbb{R})$. Then:*

(i) *For each fixed $x \in \mathbb{R}$, $\begin{bmatrix} M_{r1}(x,\lambda) \\ M_{r3}(x,\lambda) \end{bmatrix}$ can be extended to a matrix function that is continuous in $\lambda \in \overline{\mathbb{C}^-}$ and analytic in $\lambda \in \mathbb{C}^-$ and tends to $\begin{bmatrix} I_n \\ 0 \end{bmatrix}$ as $\lambda \to \infty$ in $\overline{\mathbb{C}^-}$.*

(ii) *For all $\lambda \in \overline{\mathbb{C}^-}$, $M_{r1}(x,\lambda)$ and $M_{r3}(x,\lambda)$ are bounded by $e^{\sigma_-(x)}$ in the norm.*

(iii) *For each fixed $x \in \mathbb{R}$, $\begin{bmatrix} M_{r2}(x,\lambda) \\ M_{r4}(x,\lambda) \end{bmatrix}$ can be extended to a matrix function that is continuous in $\lambda \in \overline{\mathbb{C}^+}$ and analytic in $\lambda \in \mathbb{C}^+$ and tends to $\begin{bmatrix} 0 \\ I_n \end{bmatrix}$ as $\lambda \to \infty$ in $\overline{\mathbb{C}^+}$.*

(iv) *For all $\lambda \in \overline{\mathbb{C}^+}$, $M_{r2}(x,\lambda)$ and $M_{r4}(x,\lambda)$ are bounded by $e^{\sigma_-(x)}$ in the norm.*

Proof. Using (2.8) in (1.6), we obtain
$$M_r(x,\lambda) = I_{2n} + iJ_{2n}\int_{-\infty}^x dy\, e^{i\lambda J_{2n}(x-y)}\, V(y)\, M_r(y,\lambda)\, e^{-i\lambda J_{2n}(x-y)}. \tag{2.17}$$
Iterating (2.17) once, we obtain the four systems given by
$$M_{r1}(x,\lambda) = I_n - \int_{-\infty}^x dy \int_{-\infty}^y dz\, e^{-2i\lambda(y-z)}\, k(y)\, k(z)^\dagger\, M_{r1}(z,\lambda), \tag{2.18}$$
$$\begin{aligned} M_{r2}(x,\lambda) = {} & i\int_{-\infty}^x dy\, e^{2i\lambda(x-y)}\, k(y) \\ & - \int_{-\infty}^x dy \int_{-\infty}^y dz\, e^{2i\lambda(x-y)}\, k(y)\, k(z)^\dagger\, M_{r2}(z,\lambda), \end{aligned} \tag{2.19}$$
$$\begin{aligned} M_{r3}(x,\lambda) = {} & i\int_{-\infty}^x dy\, e^{-2i\lambda(x-y)}\, k(y)^\dagger \\ & - \int_{-\infty}^x dy \int_{-\infty}^y dz\, e^{-2i\lambda(x-y)}\, k(y)^\dagger\, k(z)\, M_{r3}(z,\lambda), \end{aligned} \tag{2.20}$$
$$M_{r4}(x,\lambda) = I_n - \int_{-\infty}^x dy \int_{-\infty}^y dz\, e^{2i\lambda(y-z)}\, k(y)^\dagger\, k(z)\, M_{r4}(z,\lambda). \tag{2.21}$$
Iterating (2.18)–(2.21) as in the proof of Proposition 2.3, we complete the proof. □

Let us write

$$a_l(\lambda) = \begin{bmatrix} a_{l1}(\lambda) & a_{l2}(\lambda) \\ a_{l3}(\lambda) & a_{l4}(\lambda) \end{bmatrix}, \qquad a_r(\lambda) = \begin{bmatrix} a_{r1}(\lambda) & a_{r2}(\lambda) \\ a_{r3}(\lambda) & a_{r4}(\lambda) \end{bmatrix}. \tag{2.22}$$

From (1.7), (1.8), and (2.8) we see that

$$\begin{bmatrix} a_{l1}(\lambda) & a_{l2}(\lambda) \\ a_{l3}(\lambda) & a_{l4}(\lambda) \end{bmatrix} = \lim_{x\to-\infty} \begin{bmatrix} M_{l1}(x,\lambda) & e^{-2i\lambda x} M_{l2}(x,\lambda) \\ e^{2i\lambda x} M_{l3}(x,\lambda) & M_{l4}(x,\lambda) \end{bmatrix}, \tag{2.23}$$

$$\begin{bmatrix} a_{r1}(\lambda) & a_{r2}(\lambda) \\ a_{r3}(\lambda) & a_{r4}(\lambda) \end{bmatrix} = \lim_{x\to+\infty} \begin{bmatrix} M_{r1}(x,\lambda) & e^{-2i\lambda x} M_{r2}(x,\lambda) \\ e^{2i\lambda x} M_{r3}(x,\lambda) & M_{r4}(x,\lambda) \end{bmatrix}. \tag{2.24}$$

Using (2.12), (2.17), (2.23), and (2.24) we find the integral representations

$$a_{l1}(\lambda) = I_n - i \int_{-\infty}^{\infty} dy\, k(y)\, M_{l3}(y,\lambda) \tag{2.25}$$

$$a_{l2}(\lambda) = -i \int_{-\infty}^{\infty} dy\, e^{-2i\lambda y}\, k(y)\, M_{l4}(y,\lambda), \tag{2.26}$$

$$a_{l3}(\lambda) = -i \int_{-\infty}^{\infty} dy\, e^{2i\lambda y}\, k(y)^\dagger\, M_{l1}(y,\lambda), \tag{2.27}$$

$$a_{l4}(\lambda) = I_n - i \int_{-\infty}^{\infty} dy\, k(y)^\dagger\, M_{l2}(y,\lambda), \tag{2.28}$$

$$a_{r1}(\lambda) = I_n + i \int_{-\infty}^{\infty} dy\, k(y)\, M_{r3}(y,\lambda), \tag{2.29}$$

$$a_{r2}(\lambda) = i \int_{-\infty}^{\infty} dy\, e^{-2i\lambda y}\, k(y)\, M_{r4}(y,\lambda), \tag{2.30}$$

$$a_{r3}(\lambda) = i \int_{-\infty}^{\infty} dy\, e^{2i\lambda y}\, k(y)^\dagger\, M_{r1}(y,\lambda), \tag{2.31}$$

$$a_{r4}(\lambda) = I_n + i \int_{-\infty}^{\infty} dy\, k(y)^\dagger\, M_{r2}(y,\lambda). \tag{2.32}$$

Proposition 2.5. *Assume that the entries of $k(x)$ belong to $L^1(\mathbb{R})$. Then:*

(i) *The matrices $a_{l1}(\lambda)$ and $a_{r4}(\lambda)$ are continuous in $\lambda \in \overline{\mathbb{C}^+}$ and analytic in $\lambda \in \mathbb{C}^+$ and tend to I_n as $\lambda \to \infty$ in $\overline{\mathbb{C}^+}$.*

(ii) *The matrices $a_{l4}(\lambda)$ and $a_{r1}(\lambda)$ are continuous in $\lambda \in \overline{\mathbb{C}^-}$ and analytic in $\lambda \in \mathbb{C}^-$ and tend to I_n as $\lambda \to \infty$ in $\overline{\mathbb{C}^-}$.*

(iii) *The matrices $a_{l2}(\lambda)$, $a_{l3}(\lambda)$, $a_{r2}(\lambda)$, and $a_{r3}(\lambda)$ are continuous in $\lambda \in \mathbb{R}$ and vanish as $\lambda \to \pm\infty$.*

(iv) *The matrices $a_{l2}(\lambda)$, $a_{l3}(\lambda)$, $a_{r2}(\lambda)$, and $a_{r3}(\lambda)$ satisfy*

$$a_{r2}(\lambda) = a_{l3}(\lambda)^\dagger, \qquad a_{r3}(\lambda) = a_{l2}(\lambda)^\dagger, \qquad \lambda \in \mathbb{R}. \tag{2.33}$$

Proof. Using Propositions 2.3 and 2.4 in (2.25)–(2.32), we get (i), (ii), and (iii). We obtain (iv) from (2.3). □

Using the notations of (2.9), let us form the following matrices:

$$f_+(x,\lambda) = \begin{bmatrix} F_{l1}(x,\lambda) & F_{r2}(x,\lambda) \\ F_{l3}(x,\lambda) & F_{r4}(x,\lambda) \end{bmatrix}, \; f_-(x,\lambda) = \begin{bmatrix} F_{r1}(x,\lambda) & F_{l2}(x,\lambda) \\ F_{r3}(x,\lambda) & F_{l4}(x,\lambda) \end{bmatrix}. \tag{2.34}$$

Let an asterisk denote complex conjugation. From Propositions 2.3 and 2.4, it follows that $f_+(x,\lambda)$ is a solution of (1.1) that is continuous in $\lambda \in \overline{\mathbb{C}^+}$ and analytic in $\lambda \in \mathbb{C}^+$; similarly, $f_-(x,\lambda)$ is a solution of (1.1) that is continuous in $\lambda \in \overline{\mathbb{C}^-}$ and analytic in $\lambda \in \mathbb{C}^-$.

Proposition 2.6. *The $2n \times 2n$ matrix $f_-(x,\lambda^*)^\dagger f_+(x,\lambda)$ is independent of x for all $\lambda \in \overline{\mathbb{C}^+}$. Similarly, $f_+(x,\lambda^*)^\dagger f_-(x,\lambda)$ is independent of x for all $\lambda \in \overline{\mathbb{C}^-}$. We have*

$$f_-(x,\lambda^*)^\dagger f_+(x,\lambda) = \begin{bmatrix} a_{l1}(\lambda) & 0 \\ 0 & a_{r4}(\lambda) \end{bmatrix}, \qquad \lambda \in \overline{\mathbb{C}^+}. \tag{2.35}$$

Furthermore, $a_{l1}(\lambda)^\dagger$ and $a_{r4}(\lambda)^\dagger$ have analytic extensions to $\mathbb{C}^-$, $a_{r1}(\lambda)^\dagger$ and $a_{l4}(\lambda)^\dagger$ have analytic extensions to $\mathbb{C}^+$, and

$$a_{l1}(\lambda) = a_{r1}(\lambda^*)^\dagger, \qquad a_{r4}(\lambda) = a_{l4}(\lambda^*)^\dagger, \qquad \lambda \in \overline{\mathbb{C}^+}, \tag{2.36}$$

$$a_{r1}(\lambda) = a_{l1}(\lambda^*)^\dagger, \qquad a_{l4}(\lambda) = a_{r4}(\lambda^*)^\dagger, \qquad \lambda \in \overline{\mathbb{C}^-}. \tag{2.37}$$

Proof. Using (1.1), one can show that $f_\mp(x,\lambda^*)^\dagger f_\pm(x,\lambda)$ is independent of x for $\lambda \in \overline{\mathbb{C}^\pm}$. Evaluating it as $x \to \pm\infty$ and using (1.7) and (1.8) we get (2.35)–(2.37). □

As in the proof of Proposition 2.2 we find that $\det f_+(x,\lambda)$ is independent of x, and evaluating that determinant as $x \to \pm\infty$ we obtain

$$\det f_+(x,\lambda) = \det a_{l1}(\lambda) = \det a_{r4}(\lambda). \tag{2.38}$$

In analogy with (2.38), we get

$$\det f_-(x,\lambda) = \det a_{r1}(\lambda) = \det a_{l4}(\lambda), \tag{2.39}$$

Using (2.10) and (2.34), let us define

$$m_+(x,\lambda) = \begin{bmatrix} M_{l1}(x,\lambda) & M_{r2}(x,\lambda) \\ M_{l3}(x,\lambda) & M_{r4}(x,\lambda) \end{bmatrix} = f_+(x,\lambda)\, e^{-i\lambda J_{2n} x}, \tag{2.40}$$

$$m_-(x,\lambda) = \begin{bmatrix} M_{r1}(x,\lambda) & M_{l2}(x,\lambda) \\ M_{r3}(x,\lambda) & M_{l4}(x,\lambda) \end{bmatrix} = f_-(x,\lambda)\, e^{-i\lambda J_{2n} x}. \tag{2.41}$$

3. The scattering matrix

In this section we define and analyze the properties of the scattering coefficients of (1.1) when the entries of the potential $k(x)$ belong to $L^1(\mathbb{R})$.

We can write (2.6) as

$$a_{l1}(\lambda)\,a_{r1}(\lambda) + a_{l2}(\lambda)\,a_{r3}(\lambda) = I_n = a_{r1}(\lambda)\,a_{l1}(\lambda) + a_{r2}(\lambda)\,a_{l3}(\lambda), \tag{3.1}$$

$$a_{l1}(\lambda)\,a_{r2}(\lambda) + a_{l2}(\lambda)\,a_{r4}(\lambda) = 0 = a_{r1}(\lambda)\,a_{l2}(\lambda) + a_{r2}(\lambda)\,a_{l4}(\lambda), \tag{3.2}$$

$$a_{l3}(\lambda)\,a_{r1}(\lambda) + a_{l4}(\lambda)\,a_{r3}(\lambda) = 0 = a_{r3}(\lambda)\,a_{l1}(\lambda) + a_{r4}(\lambda)\,a_{l3}(\lambda), \tag{3.3}$$

$$a_{l3}(\lambda)\,a_{r2}(\lambda) + a_{l4}(\lambda)\,a_{r4}(\lambda) = I_n = a_{r3}(\lambda)\,a_{l2}(\lambda) + a_{r4}(\lambda)\,a_{l4}(\lambda). \tag{3.4}$$

For those real λ for which $a_{l1}(\lambda)$ and $a_{r4}(\lambda)$ are nonsingular, let us define the *transmission coefficients* $T_l(\lambda)$ from the left and $T_r(\lambda)$ from the right, and the *reflection coefficients* $R(\lambda)$ from the right and $L(\lambda)$ from the left, as follows:

$$T_l(\lambda) = a_{l1}(\lambda)^{-1}, \qquad T_r(\lambda) = a_{r4}(\lambda)^{-1}, \tag{3.5}$$

$$R(\lambda) = a_{r2}(\lambda)\,a_{r4}(\lambda)^{-1}, \qquad L(\lambda) = a_{l3}(\lambda)\,a_{l1}(\lambda)^{-1}. \tag{3.6}$$

From (3.2), (3.3), and (3.6) we get

$$R(\lambda) = -a_{l1}(\lambda)^{-1}\,a_{l2}(\lambda), \qquad L(\lambda) = -a_{r4}(\lambda)^{-1}\,a_{r3}(\lambda). \tag{3.7}$$

Note that using (2.3) and (3.1)–(3.7), we can express the matrices in (2.22) in terms of the scattering coefficients as follows

$$a_l(\lambda) = \begin{bmatrix} T_l(\lambda)^{-1} & -T_l(\lambda)^{-1}\,R(\lambda) \\ L(\lambda)\,T_l(\lambda)^{-1} & [T_r(\lambda)^\dagger]^{-1} \end{bmatrix}, \tag{3.8}$$

$$a_r(\lambda) = \begin{bmatrix} [T_l(\lambda)^\dagger]^{-1} & R(\lambda)\,T_r(\lambda)^{-1} \\ -T_r(\lambda)^{-1}\,L(\lambda) & T_r(\lambda)^{-1} \end{bmatrix}, \tag{3.9}$$

where the off-diagonal entries can be expressed in terms of $L(\lambda)$ or $R(\lambda)$ by using

$$L(\lambda)\,T_l(\lambda)^{-1} = [R(\lambda)\,T_r(\lambda)^{-1}]^\dagger, \tag{3.10}$$

which is immediate from (2.33).

The *scattering matrix* $\boldsymbol{S}(\lambda)$ associated with (1.1) is defined as follows:

$$\boldsymbol{S}(\lambda) = \begin{bmatrix} T_l(\lambda) & R(\lambda) \\ L(\lambda) & T_r(\lambda) \end{bmatrix}. \tag{3.11}$$

Theorem 3.1. *The scattering matrix $\boldsymbol{S}(\lambda)$ is continuous and J_{2n}-unitary, except at those $\lambda \in \mathbb{R}$ where $a_{l1}(\lambda)$ and $a_{r4}(\lambda)$ are singular. Further, it converges to I_{2n} as $\lambda \to \pm\infty$. Hence the scattering coefficients satisfy*

$$T_l(\lambda)\,T_l(\lambda)^\dagger - R(\lambda)\,R(\lambda)^\dagger = I_n = T_r(\lambda)^\dagger\,T_r(\lambda) - R(\lambda)^\dagger\,R(\lambda), \tag{3.12}$$

$$T_l(\lambda)^\dagger\,T_l(\lambda) - L(\lambda)^\dagger\,L(\lambda) = I_n = T_r(\lambda)\,T_r(\lambda)^\dagger - L(\lambda)\,L(\lambda)^\dagger, \tag{3.13}$$

$$T_r(\lambda)\,R(\lambda)^\dagger - L(\lambda)\,T_l(\lambda)^\dagger = 0 = T_r(\lambda)^\dagger\,L(\lambda) - R(\lambda)^\dagger\,T_l(\lambda). \tag{3.14}$$

Moreover, for those $\lambda \in \mathbb{R}$ where $a_{l1}(\lambda)$ and $a_{r4}(\lambda)$ are nonsingular, we have

$$\det T_l(\lambda) = \det T_r(\lambda), \tag{3.15}$$

$$\det \begin{bmatrix} I_n & -R(\lambda) \\ R(\lambda)^\dagger & I_n \end{bmatrix} = \det \begin{bmatrix} I_n & L(\lambda)^\dagger \\ -L(\lambda) & I_n \end{bmatrix} = |\det T_l(\lambda)|^2, \tag{3.16}$$

$$\det \boldsymbol{S}(\lambda) = \frac{\det T_l(\lambda)}{[\det T_l(\lambda)]^*}. \tag{3.17}$$

Proof. The continuity and the large-λ asymptotics follow from Proposition 2.5. Using (3.5)–(3.7) in (2.7), we get $\boldsymbol{S}(\lambda)\, J_{2n}\, \boldsymbol{S}(\lambda)^\dagger = J_{2n}$, from which (3.12)–(3.14) follow. Furthermore, from (2.38), (3.8), and (3.9) we obtain (3.15). Using (3.10), we can write (3.8) and (3.9) as

$$a_l(\lambda) = \begin{bmatrix} T_l(\lambda)^{-1} & 0 \\ 0 & [T_r(\lambda)^\dagger]^{-1} \end{bmatrix} \begin{bmatrix} I_n & -R(\lambda) \\ R(\lambda)^\dagger & I_n \end{bmatrix}, \tag{3.18}$$

$$a_r(\lambda) = \begin{bmatrix} [T_l(\lambda)^\dagger]^{-1} & 0 \\ 0 & T_r(\lambda)^{-1} \end{bmatrix} \begin{bmatrix} I_n & L(\lambda)^\dagger \\ -L(\lambda) & I_n \end{bmatrix}, \tag{3.19}$$

and hence, using (2.1), (3.15), (3.18), and (3.19), we get (3.16). Using (2.2), (2.34), (3.5), and (3.6) it follows that

$$f_-(x,\lambda) = f_+(x,\lambda)\, J_{2n}\, \boldsymbol{S}(\lambda)\, J_{2n}, \qquad \lambda \in \mathbb{R}. \tag{3.20}$$

Thus, from (3.5), (2.38), (2.39), (3.20), and $\det J_{2n} = (-1)^n$, we obtain (3.17). □

Corollary 3.2. *Suppose $a_{l1}(\lambda)$ and $a_{r4}(\lambda)$ are nonsingular for all $\lambda \in \overline{\mathbb{C}^+}$. Then the transmission coefficients $T_l(\lambda)$ and $T_r(\lambda)$ and their inverses $T_l(\lambda)^{-1}$ and $T_r(\lambda)^{-1}$ are continuous in $\lambda \in \overline{\mathbb{C}^+}$ and analytic in $\lambda \in \mathbb{C}^+$; these four matrices converge to I_n as $\lambda \to \infty$ in $\overline{\mathbb{C}^+}$. Similarly, the matrices $T_l(\lambda^*)^\dagger$ and $T_r(\lambda^*)^\dagger$ and their inverses $[T_l(\lambda^*)^\dagger]^{-1}$ and $[T_r(\lambda^*)^\dagger]^{-1}$ are continuous in $\lambda \in \overline{\mathbb{C}^-}$ and analytic in $\lambda \in \mathbb{C}^-$; these four matrices converge to I_n as $\lambda \to \infty$ in $\overline{\mathbb{C}^-}$.*

In general, $R(\lambda)$ and $L(\lambda)$ do not have analytic continuations off the real axis. In the special case when $k(x)$ vanishes on a half line, we have the following.

Proposition 3.3. *If $k(x)$ is supported in the right half line $\mathbb{R}^+$, then $a_{l3}(\lambda)$ extends to a function that is continuous on $\overline{\mathbb{C}^+}$, is analytic on $\mathbb{C}^+$, and vanishes as $\lambda \to \infty$ in $\overline{\mathbb{C}^+}$. Similarly, if $k(x)$ is supported in the left half line $\mathbb{R}^-$, then $a_{r2}(\lambda)$ extends to a function that is continuous on $\overline{\mathbb{C}^+}$, is analytic on $\mathbb{C}^+$, and vanishes as $\lambda \to \infty$ in $\overline{\mathbb{C}^+}$.*

Proof. If k has support in $\mathbb{R}^+$, then from (2.27) and Proposition 2.3 we see that $a_{l3}(\lambda)$ has an extension that is continuous in $\lambda \in \overline{\mathbb{C}^+}$, is analytic in $\lambda \in \mathbb{C}^+$, and converges to 0 as $\lambda \to \infty$ in $\overline{\mathbb{C}^+}$. In a similar manner, if k is supported in $\mathbb{R}^-$, using (2.30) and Proposition 2.3, we obtain that $a_{r2}(\lambda)$ extends to a function that is continuous on $\overline{\mathbb{C}^+}$, is analytic on $\mathbb{C}^+$, and vanishes as $\lambda \to \infty$ in $\overline{\mathbb{C}^+}$. □

4. Fourier transforms

Let $\mathcal{W}^q$ denote the Wiener algebra of all $q \times q$ matrix functions of the form

$$Z(\lambda) = Z_\infty + \int_{-\infty}^{\infty} d\alpha\, z(\alpha)\, e^{i\lambda\alpha}, \tag{4.1}$$

where $z(\alpha)$ is a $q \times q$ matrix function whose entries belong to $L^1(\mathbb{R})$ and $Z_\infty = Z(\pm\infty)$. Then $\mathcal{W}^q$ is a Banach algebra with a unit element and endowed with the norm

$$\|Z\|_{\mathcal{W}^q} = \|Z_\infty\| + \int_{-\infty}^{\infty} d\alpha\, \|z(\alpha)\|,$$

and its invertible elements are those $Z(\lambda)$ as in (4.1) for which Z_∞ and $Z(\lambda)$ are nonsingular matrices for all $\lambda \in \mathbb{R}$ (see, e.g., [17]). We will use $\mathcal{W}^q_\pm$ to denote the subalgebra of those functions $Z(\lambda)$ for which $z(\alpha)$ has support in $\mathbb{R}^\pm$ and $\mathcal{W}^q_{\pm,0}$ to denote the subalgebra of those functions $Z(\lambda)$ for which $Z_\infty = 0$ and $z(\alpha)$ has support in $\mathbb{R}^\pm$. Then, $\mathcal{W}^q = \mathcal{W}^q_+ \oplus \mathcal{W}^q_{-,0} = \mathcal{W}^q_{+,0} \oplus \mathcal{W}^q_-$.

In this section we prove that the matrix functions $M_l(x,\cdot)$, $M_r(x,\cdot)$, and $\boldsymbol{S}(\cdot)$ belong to $\mathcal{W}^{2n}$, and that $m_\pm(x,\cdot)$ belongs to $\mathcal{W}^{2n}_\pm$. Let us construct the L^1-matrix functions $b_\pm(x,\cdot)$, $B_l(x,\cdot)$, and $B_r(x,\cdot)$ such that

$$m_\pm(x,\lambda) = I_{2n} + \int_0^{\infty} d\alpha\, b_\pm(x,\alpha)\, e^{\pm i\lambda\alpha}, \tag{4.2}$$

$$\begin{cases} M_l(x,\lambda) = I_{2n} + \displaystyle\int_0^{\infty} d\alpha\, B_l(x,\alpha)\, e^{i\lambda J_{2n}\alpha}, \\ M_r(x,\lambda) = I_{2n} + \displaystyle\int_0^{\infty} d\alpha\, B_r(x,\alpha)\, e^{-i\lambda J_{2n}\alpha}. \end{cases} \tag{4.3}$$

Indeed, partitioning the matrix functions $B_l(x,\alpha)$ and $B_r(x,\alpha)$ in (4.3) into $n \times n$ blocks as

$$B_l(x,\alpha) = \begin{bmatrix} B_{l1}(x,\alpha) & B_{l2}(x,\alpha) \\ B_{l3}(x,\alpha) & B_{l4}(x,\alpha) \end{bmatrix},$$
$$B_r(x,\alpha) = \begin{bmatrix} B_{r1}(x,\alpha) & B_{r2}(x,\alpha) \\ B_{r3}(x,\alpha) & B_{r4}(x,\alpha) \end{bmatrix},$$

so that

$$b_+(x,\alpha) = \begin{bmatrix} B_{l1}(x,\alpha) & B_{r2}(x,\alpha) \\ B_{l3}(x,\alpha) & B_{r4}(x,\alpha) \end{bmatrix},$$
$$b_-(x,\alpha) = \begin{bmatrix} B_{r1}(x,\alpha) & B_{l2}(x,\alpha) \\ B_{r3}(x,\alpha) & B_{l4}(x,\alpha) \end{bmatrix}, \tag{4.4}$$

we apply (4.3) to (2.12) and (2.17), and derive the coupled integral equations for $\alpha > 0$

$$B_{l1}(x,\alpha) = -i \int_x^\infty dy\, k(y)\, B_{l3}(y,\alpha), \tag{4.5}$$

$$B_{l2}(x,\alpha) = -\frac{i}{2} k(x+\alpha/2) - i \int_x^{x+\alpha/2} dy\, k(y)\, B_{l4}(y,\alpha+2x-2y), \tag{4.6}$$

$$B_{l3}(x,\alpha) = -\frac{i}{2} k(x+\alpha/2)^\dagger - i \int_x^{x+\alpha/2} dy\, k(y)^\dagger\, B_{l1}(y,\alpha+2x-2y), \tag{4.7}$$

$$B_{l4}(x,\alpha) = -i \int_x^\infty dy\, k(y)^\dagger\, B_{l2}(y,\alpha), \tag{4.8}$$

$$B_{r1}(x,\alpha) = i \int_{-\infty}^x dy\, k(y)\, B_{r3}(y,\alpha), \tag{4.9}$$

$$B_{r2}(x,\alpha) = \frac{i}{2} k(x-\alpha/2) + i \int_{x-\alpha/2}^x dy\, k(y)\, B_{r4}(y,\alpha+2y-2x), \tag{4.10}$$

$$B_{r3}(x,\alpha) = \frac{i}{2} k(x-\alpha/2)^\dagger + i \int_{x-\alpha/2}^x dy\, k(y)^\dagger\, B_{l1}(y,\alpha+2y-2x), \tag{4.11}$$

$$B_{r4}(x,\alpha) = i \int_{-\infty}^x dy\, k(y)^\dagger\, B_{r2}(y,\alpha). \tag{4.12}$$

We first prove that, for each $x \in \mathbb{R}$, the four systems of integral equations (4.5) and (4.7), (4.6) and (4.8), (4.9) and (4.11), (4.10) and (4.12) have unique solutions with entries in $L^1(\mathbb{R}^+)$. Then for the matrix functions $m_\pm(x,\lambda)$, $M_l(x,\lambda)$, and $M_r(x,\lambda)$ defined in (4.2) and (4.3), we derive the integral relations (2.13)–(2.16) and (2.18)–(2.21). In this way we will have proved that $M_l(x,\cdot)$ and $M_r(x,\cdot)$ belong to $\mathcal{W}^{2n}$ and $m_\pm(x,\cdot)$ belongs to $\mathcal{W}^{2n}_\pm$.

Let us introduce the following mixed norm on the $2n \times 2n$ matrix functions $B(x,\alpha)$ depending on $(x,\alpha) \in \mathbb{R} \times \mathbb{R}^+$:

$$\|B(\cdot,\cdot)\|_{\infty,1} = \sup_{x\in\mathbb{R}} \|B(x,\cdot)\|_1. \tag{4.13}$$

The proof of the next result is identical to that of the analogous result in [2].

Theorem 4.1. *Assume that the entries of $k(x)$ belong to $L^1(\mathbb{R})$. Then, for each $x \in \mathbb{R}$, the four pairs of integral equations* (4.5) *and* (4.7), (4.6) *and* (4.8), (4.9) *and* (4.11), (4.10) *and* (4.12) *have unique solutions with finite mixed norm as defined in* (4.13). *Consequently, $m_+(x,\cdot)$ belongs to $\mathcal{W}^{2n}_+$, $m_-(x,\cdot)$ belongs to $\mathcal{W}^{2n}_-$, and $M_l(x,\cdot)$ and $M_r(x,\cdot)$ belong to $\mathcal{W}^{2n}$.*

The integral equations (4.5)–(4.12) allow us to derive the following relations for the potential $k(x)$:

$$k(x) = 2i\, B_{l2}(x,0^+) = -2i\, B_{r2}(x,0^+) = -2i\, B_{l3}(x,0^+)^\dagger = 2i\, B_{r3}(x,0^+)^\dagger. \tag{4.14}$$

Theorem 4.2. *The scattering coefficients* $a_{l2}(\lambda)$, $a_{l3}(\lambda)$, $a_{r2}(\lambda)$, *and* $a_{r3}(\lambda)$ *belong to* $\mathcal{W}^n$, *and vanish as* $\lambda \to \pm\infty$. *The scattering coefficients* $a_{l1}(\lambda)$, $a_{l4}(\lambda)$, $a_{r1}(\lambda)$, *and* $a_{r4}(\lambda)$ *belong to* $\mathcal{W}_+^n$, *and they converge to* I_n *as* $\lambda \to \infty$ *in* $\overline{\mathbb{C}^+}$.

Proof. Using (2.3) and (2.6)–(2.8) we get

$$a_l(\lambda) = e^{-i\lambda J_{2n} x}\, J_{2n} M_r(x,\lambda)^\dagger\, J_{2n}\, M_l(x,\lambda)\, e^{i\lambda J_{2n} x}, \tag{4.15}$$

$$a_r(\lambda) = e^{-i\lambda J_{2n} x}\, J_{2n}\, M_l(x,\lambda)^\dagger\, J_{2n}\, M_r(x,\lambda)\, e^{i\lambda J_{2n} x}. \tag{4.16}$$

From Theorem 4.1 we see that $M_l(x,\lambda)$ and $M_r(x,\lambda)$ belong to $\mathcal{W}^{2n}$. Using (4.15) and (4.16) at $x = 0$, we can show that $a_l(\lambda)$ and $a_r(\lambda)$ are products of elements of $\mathcal{W}^{2n}$ and hence belong to $\mathcal{W}^{2n}$. □

If $a_{l1}(\lambda)$ and $a_{r4}(\lambda)$ are both nonsingular for any $\lambda \in \mathbb{R}$, then Theorem 4.2 and (3.6) show that the reflection coefficients $L(\lambda)$ and $R(\lambda)$ belong to $\mathcal{W}^n$. Theorem 4.2 and (3.5) show that in this case the transmission coefficients $T_l(\lambda)$ and $T_r(\lambda)$ belong to $\mathcal{W}^n$ as well. To prove that $T_l(\lambda)$ and $T_r(\lambda)$ belong to $\mathcal{W}_+^n$, one needs that $a_{l1}(\lambda)$ and $a_{r4}(\lambda)$ are nonsingular for all $\lambda \in \overline{\mathbb{C}^+}$.

5. Wiener-Hopf factorization

Using (2.40), (2.41), and (3.20), we obtain the Riemann-Hilbert problem

$$m_-(x,\lambda) = m_+(x,\lambda)\, \boldsymbol{G}(x,\lambda), \tag{5.1}$$

where $\boldsymbol{G}(x,\lambda)$ is the unitarily dilated scattering matrix given by

$$\boldsymbol{G}(x,\lambda) = e^{i\lambda J_{2n} x}\, J_{2n}\, \boldsymbol{S}(\lambda)\, J_{2n}\, e^{-i\lambda J_{2n} x} = \begin{bmatrix} T_l(\lambda) & -R(\lambda)\, e^{2i\lambda x} \\ -L(\lambda)\, e^{-2i\lambda x} & T_r(\lambda) \end{bmatrix}. \tag{5.2}$$

Here $\boldsymbol{G}(x,\lambda)$ is a J_{2n}-unitary matrix which is defined for those $\lambda \in \mathbb{R}$ where $a_{l1}(\lambda)$ and $a_{r4}(\lambda)$ are nonsingular.

Equation (5.1) can in principle be used to compute the potential from a reflection matrix. To do so, we first construct the scattering matrix $\boldsymbol{S}(\lambda)$ in terms of $L(\lambda)$ or $R(\lambda)$ alone. Indeed, given $R(\lambda)$ for $\lambda \in \mathbb{R}$ and assuming it to be continuous for $\lambda \in \mathbb{R}$, we first obtain the matrix function $T_{l0}(\lambda)$ which is continuous on $\overline{\mathbb{C}^+}$, is analytic on $\mathbb{C}^+$ and tends to I_n as $\lambda \to \infty$ in $\overline{\mathbb{C}^+}$, by performing the Wiener-Hopf factorization

$$T_{l0}(\lambda)\, T_{l0}(\lambda)^\dagger = I_n + R(\lambda)\, R(\lambda)^\dagger, \qquad \lambda \in \mathbb{R}, \tag{5.3}$$

in agreement with (3.12). In a similar way, the matrix function $T_{r0}(\lambda)$ which is continuous on $\overline{\mathbb{C}^+}$, is analytic on $\mathbb{C}^+$ and tends to I_n as $\lambda \to \infty$ in $\overline{\mathbb{C}^+}$, is constructed by performing the Wiener-Hopf factorization

$$T_{r0}(\lambda)^\dagger\, T_{r0}(\lambda) = I_n + R(\lambda)^\dagger\, R(\lambda), \qquad \lambda \in \mathbb{R}, \tag{5.4}$$

in agreement with (3.12). We then define the matrix function

$$L_0(\lambda) = T_{r0}(\lambda)\, R(\lambda)^\dagger\, [T_{l0}(\lambda)^\dagger]^{-1}, \qquad \lambda \in \mathbb{R}. \tag{5.5}$$

Let $T_{l0}(\lambda)$ and $T_{r0}(\lambda)$ be the $n \times n$ matrix functions that are continuous in $\overline{\mathbb{C}^+}$, are analytic in $\mathbb{C}^+$, are nonsingular in $\overline{\mathbb{C}^+}$, and tend to I_n as $\lambda \to \infty$ in $\overline{\mathbb{C}^+}$, such that (5.3) and (5.4) are satisfied, and let $L_0(\lambda)$ be the corresponding right-hand side of (5.5). Then $T_{l0}(\lambda)$, $T_{r0}(\lambda)$, $R(\lambda)$, and $L_0(\lambda)$ are the scattering coefficients if $a_{l1}(\lambda)$ and $a_{r4}(\lambda)$ are nonsingular for $\lambda \in \overline{\mathbb{C}^+}$ (i.e., in the absence of bound states). When both $a_{l1}(\lambda)$ and $a_{r4}(\lambda)$ are nonsingular for $\lambda \in \mathbb{R}$ and at least one of $a_{l1}(\lambda)$ and $a_{r4}(\lambda)$ is singular for some $\lambda \in \mathbb{C}^+$ (i.e., in the presence of bound states), there are rational matrix functions $B_l(\lambda)$ and $B_r(\lambda)$ that are unitary for $\lambda \in \mathbb{R}$, tend to I_n as $\lambda \to \infty$, and are analytic in $\mathbb{C}^-$, such that

$$T_l(\lambda) = T_{l0}(\lambda)B_l(\lambda), \qquad T_r(\lambda) = B_r(\lambda)T_{r0}(\lambda). \tag{5.6}$$

Then we easily see that

$$L(\lambda) = B_r(\lambda)L_0(\lambda)B_l(\lambda). \tag{5.7}$$

Moreover, (3.15) implies that

$$\det B_l(\lambda) = \det B_r(\lambda) = \prod_{j=1}^{\mathcal{N}} \left(\frac{\lambda + i\kappa_j}{\lambda - i\kappa_j}\right)^{p_j}, \tag{5.8}$$

where $i\kappa_1, \ldots, i\kappa_{\mathcal{N}}$ are the distinct poles of the transmission coefficients $T_l(\lambda)$ and $T_r(\lambda)$ in $\mathbb{C}^+$ and $p_1, \ldots, p_{\mathcal{N}}$ are the respective poles orders of $\det B_l(\lambda)$. Thus $T_l(\lambda)$ and $T_r(\lambda)$ necessarily have the same poles in $\mathbb{C}^+$ and their determinants have the same pole orders. However, except in the case $n = 1$ where $B_l(\lambda)$ and $B_r(\lambda)$ are scalar functions that both coincide with the right-hand side of (5.8), the transmission coefficients may be different and have different sets of partial pole orders at the same pole in $\mathbb{C}^+$.

6. The Marchenko method

In order to establish the connection between the Riemann-Hilbert problem (5.1) and the Marchenko integral equations, *we assume throughout this section that $a_{l1}(\lambda)$ and $a_{r4}(\lambda)$ are nonsingular for all $\lambda \in \mathbb{R}$*. This allows one to express the scattering coefficients in terms of their Fourier transforms as

$$R(\lambda) = \int_{-\infty}^{\infty} d\alpha\, \hat{R}(\alpha)\, e^{-i\lambda\alpha}, \qquad L(\lambda) = \int_{-\infty}^{\infty} d\alpha\, \hat{L}(\alpha)\, e^{-i\lambda\alpha}, \tag{6.1}$$

$$T_l(\lambda) = I_n + \int_{-\infty}^{\infty} d\alpha\, \nu_l(\alpha)\, e^{i\lambda\alpha}, \qquad T_r(\lambda) = I_n + \int_{-\infty}^{\infty} d\alpha\, \nu_r(\alpha)\, e^{i\lambda\alpha}. \tag{6.2}$$

Note that by Theorem 4.2, $\nu_l(\alpha)$ and $\nu_r(\alpha)$ vanish for $\alpha < 0$ and their entries belong to $L^1(\mathbb{R}^+)$, while the entries of $\hat{R}(\cdot)$ and $\hat{L}(\cdot)$ belong to $L^1(\mathbb{R})$. Let us define

$$g(x,\alpha) = \begin{bmatrix} 0 & -\hat{R}(2x+\alpha) \\ -\hat{L}(-2x+\alpha) & 0 \end{bmatrix}, \qquad \alpha > 0. \tag{6.3}$$

Theorem 6.1. *Suppose $a_{l1}(\lambda)$ and $a_{r4}(\lambda)$ are nonsingular for all $\lambda \in \overline{\mathbb{C}^+}$. Then for each $x \in \mathbb{R}$ the matrices $b_-(x,\cdot)$ and $b_+(x,\cdot)$ defined in* (4.4) *satisfy the $2n \times 2n$ systems of coupled Marchenko equations*

$$b_-(x,\alpha) = g(x,\alpha) + \int_0^\infty d\beta\, b_+(x,\beta)\, g(x,\alpha+\beta), \tag{6.4}$$

$$b_+(x,\alpha) = J_{2n}\, g(x,\alpha)^\dagger\, J_{2n} + \int_0^\infty d\beta\, b_-(x,\beta)\, J_{2n}\, g(x,\alpha+\beta)^\dagger\, J_{2n}, \tag{6.5}$$

where $\alpha > 0$.

Proof. Using (4.2), (5.1), and the fact that $b_+(x,\alpha) = b_-(x,\alpha) = 0$ for $\alpha < 0$, we get

$$m_+(x,\lambda)\,[\boldsymbol{G}(x,\lambda) - I_{2n}] = \int_{-\infty}^\infty d\alpha\,[b_-(x,\alpha) - b_+(x,-\alpha)]\, e^{-i\lambda\alpha}, \qquad \lambda \in \mathbb{R}. \tag{6.6}$$

Furthermore, from (5.2) we conclude that under the above assumptions

$$\boldsymbol{G}(x,\lambda) - I_{2n} = \int_{-\infty}^\infty d\alpha\, \boldsymbol{H}(\alpha)\, e^{i\lambda\alpha}, \qquad \lambda \in \mathbb{R}, \tag{6.7}$$

where

$$\boldsymbol{H}(\alpha) = \begin{bmatrix} \nu_l(\alpha) & -\hat{R}(2x-\alpha) \\ -\hat{L}(-2x-\alpha) & \nu_r(\alpha) \end{bmatrix}, \qquad \alpha \in \mathbb{R}. \tag{6.8}$$

The hypotheses of Theorem 6.1 imply that $\nu_l(\alpha)$ and $\nu_r(\alpha)$ are supported on $\alpha \in \mathbb{R}^+$. Upon writing

$$m_+(x,\lambda)\,[\boldsymbol{G}(x,\lambda) - I_{2n}] = [\boldsymbol{G}(x,\lambda) - I_{2n}] + [m_+(x,\lambda) - I_{2n}]\,[\boldsymbol{G}(x,\lambda) - I_{2n}],$$

by using (6.6) on the left-hand side, (4.2), (6.1)–(6.3), (6.7), and (6.8) on the right-hand side, together with the convolution theorem, we obtain (6.4). Similarly, using

$$\begin{aligned} m_-(x,\lambda)\, J_{2n}\,[\boldsymbol{G}(x,\lambda)^\dagger - I_{2n}]\, J_{2n} &= J_{2n}\,[\boldsymbol{G}(x,\lambda)^\dagger - I_{2n}]\, J_{2n} \\ &\quad + [m_-(x,\lambda) - I_{2n}]\, J_{2n}\,[\boldsymbol{G}(x,\lambda)^\dagger - I_{2n}]\, J_{2n}, \end{aligned}$$

we obtain (6.5). □

Using (6.4) in (6.5) and vice versa, we can uncouple these $2n \times 2n$ systems. Using the notations in (4.4), this leads to the uncoupled $n \times n$ Marchenko equations for $\alpha > 0$ given by

$$\begin{aligned} B_{l2}(x,\alpha) &= -\hat{R}(\alpha+2x) \\ &\quad - \int_0^\infty d\beta \int_0^\infty d\gamma\, B_{l2}(x,\gamma)\, \hat{R}(\beta+\gamma+2x)^\dagger\, \hat{R}(\alpha+\beta+2x), \end{aligned} \tag{6.9}$$

$$\begin{aligned} B_{l3}(x,\alpha) &= \hat{R}(\alpha+2x)^\dagger \\ &\quad - \int_0^\infty d\beta \int_0^\infty d\gamma\, B_{l3}(x,\gamma)\, \hat{R}(\beta+\gamma+2x)\, \hat{R}(\alpha+\beta+2x)^\dagger, \end{aligned} \tag{6.10}$$

$$B_{r2}(x,\alpha) = \hat{L}(\alpha - 2x)^\dagger - \int_0^\infty d\beta \int_0^\infty d\gamma\, B_{r2}(x,\gamma)\, \hat{L}(\beta+\gamma-2x)\, \hat{L}(\alpha+\beta-2x)^\dagger, \tag{6.11}$$

$$B_{r3}(x,\alpha) = -\hat{L}(\alpha - 2x) - \int_0^\infty d\beta \int_0^\infty d\gamma\, B_{r3}(x,\gamma)\, \hat{L}(\beta+\gamma-2x)^\dagger\, \hat{L}(\alpha+\beta-2x), \tag{6.12}$$

$$B_{l1}(x,\alpha) = -\int_0^\infty d\beta\, \hat{R}(\beta+2x)\, \hat{R}(\alpha+\beta+2x)^\dagger - \int_0^\infty d\beta \int_0^\infty d\gamma\, B_{l1}(x,\gamma)\, \hat{R}(\beta+\gamma+2x)\, \hat{R}(\alpha+\beta+2x)^\dagger, \tag{6.13}$$

$$B_{l4}(x,\alpha) = -\int_0^\infty d\beta\, \hat{R}(\beta+2x)^\dagger\, \hat{R}(\alpha+\beta+2x) - \int_0^\infty d\beta \int_0^\infty d\gamma\, B_{l4}(x,\gamma)\, \hat{R}(\beta+\gamma+2x)^\dagger\, \hat{R}(\alpha+\beta+2x), \tag{6.14}$$

$$B_{r1}(x,\alpha) = -\int_0^\infty d\beta\, \hat{L}(\beta-2x)^\dagger\, \hat{L}(\alpha+\beta-2x) - \int_0^\infty d\beta \int_0^\infty d\gamma\, B_{r1}(x,\gamma)\, \hat{L}(\beta+\gamma-2x)^\dagger\, \hat{L}(\alpha+\beta-2x), \tag{6.15}$$

$$B_{r4}(x,\alpha) = -\int_0^\infty d\beta\, \hat{L}(\beta-2x)\, \hat{L}(\alpha+\beta-2x)^\dagger - \int_0^\infty d\beta \int_0^\infty d\gamma\, B_{r4}(x,\gamma)\, \hat{L}(\beta+\gamma-2x)\, \hat{L}(\alpha+\beta-2x)^\dagger. \tag{6.16}$$

Theorem 6.2. *The coupled system of Marchenko integral equations* (6.4) *and* (6.5) *is uniquely solvable in* $L^1(\mathbb{R}^+;\mathbb{C}^{2n\times 2n})$. *The integral operator in each of the eight uncoupled Marchenko equations* (6.9)–(6.16) *is selfadjoint, and each of these eight equations is uniquely solvable in* $L^1(\mathbb{R}^+;\mathbb{C}^{n\times n})$.

Proof. The selfadjointness of the integral operators in (6.9)–(6.16) is clear. From (3.12), (3.13), and Corollary 3.2 it follows that

$$\sup_{\lambda\in\mathbb{R}} \|L(\lambda)\| < +\infty, \qquad \sup_{\lambda\in\mathbb{R}} \|R(\lambda)\| < +\infty. \tag{6.17}$$

Now observe that, for fixed $x \in \mathbb{R}$, the action of the integral operators with kernels $\hat{R}(\alpha+\beta+2x)$, $\hat{R}(\alpha+\beta+2x)^\dagger$, $\hat{L}(\alpha+\beta-2x)$, and $\hat{L}(\alpha+\beta-2x)^\dagger$ on $L^2(\mathbb{R}^+;\mathbb{C}^n)$ can be interpreted as follows: one imbeds $L^2(\mathbb{R}^+;\mathbb{C}^n)$ into $L^2(\mathbb{R};\mathbb{C}^n)$ isometrically, applies the sign flip $h(\beta) \mapsto h(-\beta)$, implements a convolution with an L^1-matrix function, and then projects orthogonally onto $L^2(\mathbb{R}^+;\mathbb{C}^n)$. Since the Fourier transforms of these matrix functions are bounded in $\lambda \in \mathbb{R}$, also these integral operators are bounded in $L^2(\mathbb{R};\mathbb{C}^n)$. Since the adjoints of all of these systems can be written in the form

$$(I + K^\dagger K)B = C$$

for some bounded operator K on a Hilbert function space, the system of equations (6.4) and (6.5) as well as each of the eight equations (6.9)–(6.16) are uniquely solvable on the direct sum of a suitable number of copies of $L^2(\mathbb{R}^+)$. Since, as a result of the integrability of $\hat{L}(\cdot)$ and $\hat{R}(\cdot)$, the integral operators are compact on both L^2 and L^1 (cf. Lemma XII 2.4 of [17], the proof for the L^2-case there can easily be adapted to cover the L^1-case), the system of equations (6.4) and (6.5) as well as each of the eight equations (6.9)–(6.16) are uniquely solvable on the direct sum of a suitable number of copies of $L^1(\mathbb{R}^+)$. □

From (4.14), we see that we can recover the potential $k(x)$ by solving any one of the four Marchenko equations (6.9)–(6.12).

The unique solvability of the Marchenko equations (6.9)–(6.16) has a number of other consequences. For example, if $R(\lambda)$ is analytic on $\mathbb{C}^+$, then $\hat{R}(\alpha)$ is supported on $\mathbb{R}^-$ and hence the right-hand sides in (6.9), (6.10), (6.13), and (6.14) vanish for $x > 0$. Since these equations are uniquely solvable, their solutions vanish as well and therefore $k(x) = 0$ for $x > 0$. On the other hand, if $L(\lambda)$ is analytic on $\mathbb{C}^+$, then $\hat{L}(\alpha)$ is supported on $\mathbb{R}^-$, and hence the right-hand sides in (6.11), (6.12), (6.15), and (6.16) vanish for $x < 0$. Since these equations are uniquely solvable, their solutions vanish as well, and therefore $k(x) = 0$ for $x < 0$. We have thus proved the converse of Proposition 3.3.

It remains to prove that the potential $k(x)$ obtained by the Marchenko method has entries in $L^1(\mathbb{R})$. To do so, we modify the inversion procedure as follows. We solve one of the Marchenko equations (6.9) and (6.10) for $x > 0$ and then employ (4.14) to compute $k(x)$ for $x > 0$. By the same token, we solve one of (6.11) and (6.12) for $x < 0$ and then use (4.14) to find $k(x)$ for $x < 0$. In fact, this procedure will be implemented in the case of rational reflection coefficients in Section 8.

We first derive the following partial characterization result.

Theorem 6.3. *Let $R(\lambda)$ be a matrix function in $\mathcal{W}^n$ such that*

$$\sup_{\lambda\in\mathbb{R}} \|R(\lambda)\| < +\infty, \qquad \int_0^\infty d\alpha \left(\|\hat{R}(\alpha)\| + \alpha\,\|\hat{R}(\alpha)\|^2\right) < +\infty, \tag{6.18}$$

where $\hat{R}(\alpha)$ is defined in (6.1). *Then, for $x > 0$, the unique solutions $B_{l2}(x,\alpha)$ and $B_{l3}(x,\alpha)$ of* (6.9) *and* (6.10), *respectively, satisfy*

$$\int_0^\infty dx\,\|B_{lj}(x,0^+)\| < +\infty, \qquad j = 2,3.$$

In particular, the entries of $k(x) = 2i\,B_{l2}(x,0^+)$ *and* $k(x) = -2i\,B_{l3}(x,0^+)^\dagger$ *belong to* $L^1(\mathbb{R}^+)$. *Similarly, let* $L(\lambda)$ *be a matrix function in* $\mathcal{W}^n$ *such that*

$$\sup_{\lambda\in\mathbb{R}} \|L(\lambda)\| < +\infty, \qquad \int_{-\infty}^{0} d\alpha\, \left(\|\hat{L}(\alpha)\| - \alpha\,\|\hat{L}(\alpha)\|^2\right) < +\infty, \tag{6.19}$$

where $\hat{L}(\alpha)$ *is defined in* (6.1). *Then for* $x<0$ *the unique solutions* $B_{r2}(x,\alpha)$ *and* $B_{r3}(x,\alpha)$ *of* (6.11) *and* (6.12), *respectively, satisfy*

$$\int_{-\infty}^{0} dx\,\|B_{rj}(x,0^+)\| < +\infty, \qquad j=2,3.$$

In particular, the entries of $k(x)=-2i\,B_{r2}(x,0^+)$ *and* $k(x)=2i\,B_{r3}(x,0^+)^\dagger$ *belong to* $L^1(\mathbb{R}^-)$.

Proof. We only prove the theorem for $x>0$, as the proof for $x<0$ is similar. Put

$$\hat{R}_\Delta(\alpha) = \begin{bmatrix} 0 & -\hat{R}(\alpha) \\ \hat{R}(\alpha)^\dagger & 0 \end{bmatrix},$$

and consider the integral equation

$$B_l(x,\alpha) - \int_0^\infty d\beta\, B_l(x,\beta)\,\hat{R}_\Delta(2x+\alpha+\beta) = \hat{R}_\Delta(2x+\alpha), \tag{6.20}$$

where $\alpha>0$. This integral equation, which follows directly from (4.4) and (6.3)–(6.5), has a unique solution in $L^1(\mathbb{R}^+;\mathbb{C}^{2n\times 2n})$ which coincides with the matrix function $B_l(x,\alpha)$ in (4.3). Moreover, the once iterated integral equation (6.20) has the form

$$(I+K^\dagger K)B = C$$

on $L^2(\mathbb{R}^+;\mathbb{C}^{2n\times 2n})$, which makes (6.20) uniquely solvable in $L^2(\mathbb{R}^+;\mathbb{C}^{2n\times 2n})$. Using the unique solvability of the equation obtained by taking the adjoint of the matrices on either side of (6.20) we obtain

$$\begin{aligned} ||B_l(x,\alpha)|| &\le \|\hat{R}_\Delta(2x+\alpha)\| + \left[\int_{2x+\alpha}^\infty d\gamma\,\|\hat{R}_\Delta(\gamma)\|^2\right]^{\frac12}\left[\int_0^\infty d\beta\,\|B_l(x,\beta)\|^2\right]^{\frac12} \\ &\le \|\hat{R}_\Delta(2x+\alpha)\| + C\left[\int_{2x+\alpha}^\infty d\gamma\,\|\hat{R}_\Delta(\gamma)\|^2\right]^{\frac12}\left[\int_{2x}^\infty d\beta\,\|\hat{R}_\Delta(\beta)\|^2\right]^{\frac12} \\ &\le \|\hat{R}_\Delta(2x+\alpha)\| + C\int_{2x}^\infty d\gamma\,\|\hat{R}_\Delta(\gamma)\|^2, \end{aligned}$$

where C is some constant, and therefore

$$\int_0^\infty dx\,\|B_l(x,\alpha)\| \le \int_0^\infty d\beta\,\|\hat{R}_\Delta(\beta)\| + \int_0^\infty dy\,y\|\hat{R}_\Delta(y)\|^2, \tag{6.21}$$

which is finite. □

The natural conditions under which one would expect to be able to reconstruct a potential with L^1-entries for $x > 0$ are $R \in \mathcal{W}^n$ and

$$\sup_{\lambda \in \mathbb{R}} \|R(\lambda)\| < +\infty, \qquad \lim_{\lambda \to \pm\infty} \|R(\lambda)\| = 0. \tag{6.22}$$

However, evaluating the first iterate of (6.20) as $\alpha \to 0^+$, we get

$$B_l^{(1)}(x, 0^+) = \int_{2x}^{\infty} d\beta\, \hat{R}_\Delta(\beta)^2 = \begin{bmatrix} \int_{2x}^{\infty} d\beta\, \hat{R}(\beta)\, \hat{R}(\beta)^\dagger & 0 \\ 0 & \int_{2x}^{\infty} d\beta\, \hat{R}(\beta)^\dagger\, \hat{R}(\beta) \end{bmatrix},$$

which strongly suggests that condition (6.18) is probably indispensable if the integral $\int_0^\infty dx\, \|B_l^{(1)}(x, 0^+)\|$ is to be finite.

7. Construction of the scattering matrix

Throughout this section we assume that $R(\lambda)$ is a rational matrix function satisfying (6.22). We recall that then $R \in \mathcal{W}^n$ by the comments following the proof of Theorem 6.3. From the theory of transfer functions [7], since $R(\lambda) \to 0$ as $\lambda \to \pm\infty$, it follows that $R(\lambda)$ can be represented in the form

$$R(\lambda) = i\,\mathcal{C}(\lambda - i\mathcal{A})^{-1}\mathcal{B}, \qquad \lambda \in \mathbb{C}, \tag{7.1}$$

where $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are independent of λ and belong to $\mathbb{C}^{p\times p}$, $\mathbb{C}^{p\times n}$, and $\mathbb{C}^{n\times p}$, respectively, for some positive integer p. Here it is assumed that the order p of $\mathcal{A}$ is minimal, i.e., the realization (7.1) is minimal and hence unique up to similarity (cf. Theorems 6.1.4 and 6.1.5 in [24]).

Our goal is to construct $\boldsymbol{S}(\lambda)$ in terms of the matrices $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ given in (7.1). Since $R(\lambda)$ is continuous for $\lambda \in \mathbb{R}$, from the minimality of the realization given in (7.1) it follows that $\mathcal{A}$ does not have any eigenvalues on the imaginary axis (cf. Theorem 6.2.2 of [24]). Using (7.1) in (5.3) and (5.4), we obtain

$$T_l(\lambda)\, T_l(\lambda^*)^\dagger = I_n + i \begin{bmatrix} \mathcal{C} & 0 \end{bmatrix} (\lambda - i\mathcal{K}_l)^{-1} \begin{bmatrix} 0 \\ \mathcal{C}^\dagger \end{bmatrix}, \tag{7.2}$$

$$T_r(\lambda^*)^\dagger\, T_r(\lambda) = I_n + i \begin{bmatrix} 0 & \mathcal{B}^\dagger \end{bmatrix} (\lambda - i\mathcal{K}_r)^{-1} \begin{bmatrix} \mathcal{B} \\ 0 \end{bmatrix}, \tag{7.3}$$

where

$$\mathcal{K}_l = \begin{bmatrix} \mathcal{A} & -\mathcal{B}\mathcal{B}^\dagger \\ 0 & -\mathcal{A}^\dagger \end{bmatrix}, \qquad \mathcal{K}_r = \begin{bmatrix} \mathcal{A} & 0 \\ -\mathcal{C}^\dagger\mathcal{C} & -\mathcal{A}^\dagger \end{bmatrix}. \tag{7.4}$$

Then $\mathcal{K}_l$ and $\mathcal{K}_r$ both have the set $\sigma(\mathcal{A}) \cup \{-\lambda^* : \lambda \in \sigma(\mathcal{A})\}$ as their spectrum ($\sigma(\mathcal{A})$ standing for the spectrum of $\mathcal{A}$), even though they need not have the same Jordan normal form. Note that the inverses of the right-hand sides in (7.2) and (7.3) can be written as

$$\left[T_l(\lambda^*)^\dagger\right]^{-1} T_l(\lambda)^{-1} = I_n - i \begin{bmatrix} \mathcal{C} & 0 \end{bmatrix} (\lambda - i\mathcal{E})^{-1} \begin{bmatrix} 0 \\ \mathcal{C}^\dagger \end{bmatrix}, \tag{7.5}$$

$$T_r(\lambda)^{-1}\left[T_r(\lambda^*)^\dagger\right]^{-1}=I_n-i\begin{bmatrix}0 & \mathcal{B}^\dagger\end{bmatrix}(\lambda-i\mathcal{E})^{-1}\begin{bmatrix}\mathcal{B}\\ 0\end{bmatrix}, \tag{7.6}$$

where $\mathcal{E}$ is the "state characteristic matrix" given by

$$\mathcal{E}=\begin{bmatrix}\mathcal{A} & -\mathcal{B}\mathcal{B}^\dagger\\ -\mathcal{C}^\dagger\mathcal{C} & -\mathcal{A}^\dagger\end{bmatrix}, \tag{7.7}$$

which, apart from some factors $i=\sqrt{-1}$, has been used in [21]. We note that $\mathcal{K}_l$, $\mathcal{K}_r$, and $\mathcal{E}$ do not have eigenvalues on the imaginary axis. This follows from the invertibility of $I_n+R(\lambda)\,R(\lambda)^\dagger$ and Corollary 2.7 in [7]; for $\mathcal{K}_l$ and $\mathcal{K}_r$ this also follows immediately from the special form of the matrices $\mathcal{K}_l$ and $\mathcal{K}_r$ in (7.4) and the fact that $\mathcal{A}$ has no eigenvalues on the imaginary axis. Hence the matrices $(\lambda-i\mathcal{K}_l)^{-1}$, $(\lambda-i\mathcal{K}_r)^{-1}$, and $(\lambda-i\mathcal{E})^{-1}$ in (7.2), (7.3), (7.5), and (7.6) all exist for $\lambda\in\mathbb{R}$.

The following result is essential for obtaining explicit expressions for the factors $T_l(\lambda)$ and $T_r(\lambda)$ and their inverses.

Proposition 7.1. *Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be the matrices in the minimal realization given by* (7.1) *and consider the quadratic matrix equations*

$$\mathcal{A}\mathcal{X}+\mathcal{X}\mathcal{A}^\dagger=\mathcal{B}\mathcal{B}^\dagger-\mathcal{X}\mathcal{C}^\dagger\mathcal{C}\mathcal{X}, \tag{7.8}$$

$$\mathcal{A}^\dagger\mathcal{Y}+\mathcal{Y}\mathcal{A}=-\mathcal{C}^\dagger\mathcal{C}+\mathcal{Y}\mathcal{B}\mathcal{B}^\dagger\mathcal{Y}. \tag{7.9}$$

Then the spectrum of the matrix $\mathcal{E}$ given in (7.7) *is symmetric about the imaginary axis. Moreover, the spectral subspace $\mathcal{M}$ of $\mathcal{E}$ corresponding to its eigenvalues in the right half-plane is of the form*

$$\mathcal{M}=\left\{\begin{bmatrix}\mathcal{X}\\ I_p\end{bmatrix}u:\ u\in\mathbb{C}^p\right\}, \tag{7.10}$$

where $\mathcal{X}$ is a hermitian solution of (7.8), *and the spectral subspace $\mathcal{L}$ of $\mathcal{E}$ corresponding to its eigenvalues in the left half-plane is of the form*

$$\mathcal{L}=\left\{\begin{bmatrix}I_p\\ \mathcal{Y}\end{bmatrix}u:\ u\in\mathbb{C}^p\right\}, \tag{7.11}$$

where $\mathcal{Y}$ is a hermitian solution of (7.9). *The hermitian matrices $\mathcal{X}$ and $\mathcal{Y}$ are unique.*

Proof. The symmetry of the spectrum of $\mathcal{E}$ about the imaginary axis follows from the similarity $J_{2p}\boldsymbol{q}_{2p}\mathcal{E}\boldsymbol{q}_{2p}J_{2p}=-\mathcal{E}^\dagger$, where $\boldsymbol{q}_{2p}$ is defined by

$$\boldsymbol{q}_{2p}=\begin{bmatrix}0 & I_p\\ I_p & 0\end{bmatrix}.$$

The remaining assertions follow from Theorem 7.6.1 in [24] applied to the matrix $J_{2p}\mathcal{E}J_{2p}$ (to comply with the condition $D\geq 0$ there) and the Ξ-neutrality, where $\Xi=iJ_{2p}\boldsymbol{q}_{2p}$, of the spectral subspaces $\mathcal{M}$ and $\mathcal{L}$. Note that the spectral subspaces $\mathcal{M}$ and $\mathcal{L}$ both have dimension p, which is the order of $\mathcal{A}$, because $\mathcal{E}$ has no eigenvalues on the imaginary axis. Also note that the controllibility condition of Theorem 7.6.1 of [24] is satisfied as a result of the minimality of the realization in (7.1). Indeed, $\mathcal{C}^\dagger\mathcal{C}\mathcal{A}^j w = 0$ for $j \geq 0$ implies $\|\mathcal{C}\mathcal{A}^j w\|^2 = 0$ for $j \geq 0$, then $\mathcal{C}\mathcal{A}^j w = 0$ for $j \geq 0$, and then $w = 0$; similarly, one proves the other controllability statement. □

The nonlinear equations (7.8) and (7.9) are called state characteristic equations in [21] and (continuous algebraic) Riccati equations elsewhere in the literature (e.g., [24]). Since in the literature the term "hermitian" (instead of "selfadjoint") seems to have some tradition when referring to solutions of Riccati equations, we will use this terminology here.

The matrices $\mathcal{X}$ and $\mathcal{Y}$ used in Proposition 7.1 allow us to block diagonalize the matrix $\mathcal{E}$. Since the subspaces $\mathcal{L}$ and $\mathcal{M}$ have dimension p and $\mathcal{M} \cap \mathcal{L} = \{0\}$, the matrix Σ defined by

$$\Sigma = \begin{bmatrix} I_p & \mathcal{X} \\ \mathcal{Y} & I_p \end{bmatrix} \tag{7.12}$$

is nonsingular. Hence, both $I_p - \mathcal{X}\mathcal{Y}$ and $I_p - \mathcal{Y}\mathcal{X}$ are nonsingular, and

$$\Sigma^{-1} = \begin{bmatrix} (I_p - \mathcal{X}\mathcal{Y})^{-1} & -(I_p - \mathcal{X}\mathcal{Y})^{-1}\mathcal{X} \\ -(I_p - \mathcal{Y}\mathcal{X})^{-1}\mathcal{Y} & (I_p - \mathcal{Y}\mathcal{X})^{-1} \end{bmatrix}. \tag{7.13}$$

Theorem 7.2. *Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be the matrices in the minimal realization given by* (7.1) *and let $\mathcal{X}$ and $\mathcal{Y}$ be as in Proposition* 7.1. *Then*

$$\Sigma^{-1}\mathcal{E}\Sigma = \begin{bmatrix} \mathcal{E}_r & 0 \\ 0 & -\mathcal{E}_l^\dagger \end{bmatrix}, \tag{7.14}$$

where

$$\mathcal{E}_r = \mathcal{A} - \mathcal{B}\mathcal{B}^\dagger\mathcal{Y}, \qquad \mathcal{E}_l = \mathcal{A} + \mathcal{X}\mathcal{C}^\dagger\mathcal{C}. \tag{7.15}$$

Moreover, the matrices $\mathcal{E}_r$ and $\mathcal{E}_l$ have all their eigenvalues in the left half-plane and are related via the similarity transformation

$$\mathcal{E}_r = (I_p - \mathcal{X}\mathcal{Y})^{-1}\mathcal{E}_l\,(I_p - \mathcal{X}\mathcal{Y}). \tag{7.16}$$

Proof. The relations (7.14)–(7.16) follow by direct computation using (7.7)–(7.9), (7.12), and (7.13). The assertions about the spectra of $\mathcal{E}_r$ and $\mathcal{E}_l$ follow from (7.14) and Proposition 7.1 which imply that $\mathcal{E}|_{\mathcal{L}}$ is similar to $\mathcal{E}_r$ and $\mathcal{E}|_{\mathcal{M}}$ is similar to $-\mathcal{E}_l^\dagger$. □

In the following we also need representations of the form (7.10) and (7.11) for certain invariant subspaces of $\mathcal{K}_l$ and $\mathcal{K}_r$.

Proposition 7.3. *Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be the matrices in the minimal realization given by* (7.1). *Then the spectrum of $\mathcal{K}_l$ ($\mathcal{K}_r$) is symmetric about the imaginary axis. Moreover, the invariant spectral subspaces of $\mathcal{K}_l$ and $\mathcal{K}_r$ corresponding to the left and right half-planes all have dimension p. In the case of $\mathcal{K}_r$ both of the invariant subspaces are of the form*

$$\left\{ \begin{bmatrix} \widetilde{\mathcal{X}} \\ I_p \end{bmatrix} u : u \in \mathbb{C}^p \right\}, \tag{7.17}$$

where $\widetilde{\mathcal{X}}$ is a solution of the Riccati equation

$$\mathcal{A}\widetilde{\mathcal{X}} + \widetilde{\mathcal{X}}\mathcal{A}^\dagger = -\widetilde{\mathcal{X}}\mathcal{C}^\dagger\mathcal{C}\widetilde{\mathcal{X}}. \tag{7.18}$$

In the case of $\mathcal{K}_l$ both of the invariant subspaces are of the form

$$\left\{ \begin{bmatrix} I_p \\ \widetilde{\mathcal{Y}} \end{bmatrix} u : u \in \mathbb{C}^p \right\}, \tag{7.19}$$

where $\widetilde{\mathcal{Y}}$ is a hermitian solution of the Riccati equation

$$\mathcal{A}^\dagger\widetilde{\mathcal{Y}} + \widetilde{\mathcal{Y}}\mathcal{A} = \widetilde{\mathcal{Y}}\mathcal{B}\mathcal{B}^\dagger\widetilde{\mathcal{Y}}. \tag{7.20}$$

Proof. Apply Theorem 7.2.4 of [24] to $J_{2p}\mathcal{K}_l J_{2p}$ and $\boldsymbol{q}_{2p}\mathcal{K}_r\boldsymbol{q}_{2p}$. The symmetry of the spectrum about the imaginary axis follows as in the proof of Proposition 7.1 for $\mathcal{E}$. □

Before we can apply the main factorization result from [7] to (7.2) and (7.3), we need the following proposition based on the positive selfadjointness of the matrix functions in (7.2) and (7.3) for all $\lambda \in \mathbb{R}$.

Proposition 7.4. *Let $\mathcal{M}$ (resp. $\mathcal{L}$) be the invariant subspace of the matrix $\mathcal{E}$ given in (7.7) corresponding to the eigenvalues in the right and left half-plane, respectively, and let $\mathcal{N}$ (resp. $\mathcal{V}$) be the invariant subspace of $\mathcal{K}_r$ (resp. $\mathcal{K}_l$) corresponding to its eigenvalues in the right and left half-plane, respectively. Then*

$$\mathcal{L} \oplus \mathcal{N} = \mathbb{C}^{2p}, \qquad \mathcal{M} \oplus \mathcal{V} = \mathbb{C}^{2p}.$$

Proof. The above decompositions then follow from Theorem I 1.5 of [7], due to the existence of left and right canonical factorizations of a positive selfadjoint matrix function with respect to the imaginary line. □

Now let Π be the projection such that

$$\text{Im}\,\Pi = \mathcal{L}, \qquad \text{Ker}\,\Pi = \mathcal{N}, \tag{7.21}$$

and let $\mathcal{Q}$ be the projection such that

$$\text{Im}\,\mathcal{Q} = \mathcal{V}, \qquad \text{Ker}\,\mathcal{Q} = \mathcal{M}. \tag{7.22}$$

Applying Theorem 1.5 of [7] we can express the transmission coefficients in terms of the matrices appearing in (7.1) and the projections Π and $\mathcal{Q}$ as follows:

$$T_r(\lambda^*)^\dagger = I_n + i\begin{bmatrix} 0 & \mathcal{B}^\dagger \end{bmatrix}(\lambda - i\mathcal{K}_r)^{-1}(I_{2p} - \Pi)\begin{bmatrix} \mathcal{B} \\ 0 \end{bmatrix}, \tag{7.23}$$

$$T_r(\lambda) = I_n + i\begin{bmatrix} 0 & \mathcal{B}^\dagger \end{bmatrix}\Pi(\lambda - i\mathcal{K}_r)^{-1}\begin{bmatrix} \mathcal{B} \\ 0 \end{bmatrix}, \tag{7.24}$$

$$T_r(\lambda)^{-1} = I_n - i\begin{bmatrix} 0 & \mathcal{B}^\dagger \end{bmatrix}(\lambda - i\mathcal{E})^{-1}\Pi\begin{bmatrix} \mathcal{B} \\ 0 \end{bmatrix}, \tag{7.25}$$

$$[T_r(\lambda^*)^\dagger]^{-1} = I_n - i\begin{bmatrix} 0 & \mathcal{B}^\dagger \end{bmatrix}(I_{2p} - \Pi)(\lambda - i\mathcal{E})^{-1}\begin{bmatrix} \mathcal{B} \\ 0 \end{bmatrix}, \tag{7.26}$$

$$T_l(\lambda) = I_n + i \begin{bmatrix} \mathcal{C} & 0 \end{bmatrix} (\lambda - i\mathcal{K}_l)^{-1} \mathcal{Q} \begin{bmatrix} 0 \\ \mathcal{C}^\dagger \end{bmatrix}, \tag{7.27}$$

$$T_l(\lambda^*)^\dagger = I_n + i \begin{bmatrix} \mathcal{C} & 0 \end{bmatrix} (I_{2p} - \mathcal{Q}) (\lambda - i\mathcal{K}_l)^{-1} \begin{bmatrix} 0 \\ \mathcal{C}^\dagger \end{bmatrix}, \tag{7.28}$$

$$[T_l(\lambda^*)^\dagger]^{-1} = I_n - i \begin{bmatrix} \mathcal{C} & 0 \end{bmatrix} (\lambda - i\mathcal{E})^{-1} (I_{2p} - \mathcal{Q}) \begin{bmatrix} 0 \\ \mathcal{C}^\dagger \end{bmatrix}, \tag{7.29}$$

$$T_l(\lambda)^{-1} = I_n - i \begin{bmatrix} \mathcal{C} & 0 \end{bmatrix} \mathcal{Q} (\lambda - i\mathcal{E})^{-1} \begin{bmatrix} 0 \\ \mathcal{C}^\dagger \end{bmatrix}. \tag{7.30}$$

With the expressions (7.23)–(7.30) we have accomplished the desired canonical factorizations of the matrix functions on the right-hand sides of (7.2), (7.3), (7.5), and (7.6). Our next goal is to find more explicit representations for the projections Π and $\mathcal{Q}$ and for the invariant subspaces $\mathcal{N}$ and $\mathcal{V}$.

Proposition 7.5. *Let $\mathcal{X}$ and $\mathcal{Y}$ be as in Proposition* 7.1 *and let $\widetilde{\mathcal{X}} = \widetilde{\mathcal{X}}_+$ and $\widetilde{\mathcal{Y}} = \widetilde{\mathcal{Y}}_-$ be as in Proposition* 7.3, *where the subscript $+$ (resp. $-$) indicates that the spectral subspaces given in* (7.17) *and* (7.19) *are those associated with the right* (*left*) *half-plane. Then the invariant subspaces $\mathcal{N}$ and $\mathcal{V}$ and the projections Π and $\mathcal{Q}$ can be written as*

$$\mathcal{N} = \left\{ \begin{bmatrix} \widetilde{\mathcal{X}}_+ \\ I_p \end{bmatrix} u : u \in \mathbb{C}^p \right\}, \qquad \mathcal{V} = \left\{ \begin{bmatrix} I_p \\ \widetilde{\mathcal{Y}}_- \end{bmatrix} u : u \in \mathbb{C}^p \right\}, \tag{7.31}$$

$$\Pi = \begin{bmatrix} (I_p - \widetilde{\mathcal{X}}_+\mathcal{Y})^{-1} & -(I_p - \widetilde{\mathcal{X}}_+\mathcal{Y})^{-1}\widetilde{\mathcal{X}}_+ \\ \mathcal{Y}(I_p - \widetilde{\mathcal{X}}_+\mathcal{Y})^{-1} & -\mathcal{Y}(I_p - \widetilde{\mathcal{X}}_+\mathcal{Y})^{-1}\widetilde{\mathcal{X}}_+ \end{bmatrix}, \tag{7.32}$$

$$\mathcal{Q} = \begin{bmatrix} (I_p - \mathcal{X}\widetilde{\mathcal{Y}}_-)^{-1} & -(I_p - \mathcal{X}\widetilde{\mathcal{Y}}_-)^{-1}\mathcal{X} \\ \widetilde{\mathcal{Y}}_-(I_p - \mathcal{X}\widetilde{\mathcal{Y}}_-)^{-1} & -\widetilde{\mathcal{Y}}_-(I_p - \mathcal{X}\widetilde{\mathcal{Y}}_-)^{-1}\mathcal{X} \end{bmatrix}. \tag{7.33}$$

Furthermore, if $\mathcal{A}$ has all its eigenvalues in the set $\Gamma^\#$, then $\widetilde{\mathcal{X}}_+ = 0$ and

$$\mathcal{N} = \{0\} \oplus \mathbb{C}^p, \qquad \mathcal{V} = \mathbb{C}^p \oplus \{0\}, \tag{7.34}$$

$$\Pi = \begin{bmatrix} I_p & 0 \\ \mathcal{Y} & 0 \end{bmatrix}, \qquad \mathcal{Q} = \begin{bmatrix} I_p & -\mathcal{X} \\ 0 & 0 \end{bmatrix}. \tag{7.35}$$

Proof. First, (7.31) is immediate from (7.17), (7.19), (7.21), and (7.22). Then (7.32) and (7.33) follow from (7.21), (7.22), and Proposition 7.1. If $\mathcal{A}$ has all its eigenvalues in the right half-plane, then $\widetilde{\mathcal{X}}_+ = \widetilde{\mathcal{Y}}_- = 0$, by the particular form of $\mathcal{K}_l$ and $\mathcal{K}_r$ in (7.4), and so (7.34) and (7.35) follow from (7.31)–(7.33). □

In order to find more explicit expressions for Π and $\mathcal{Q}$ when $\mathcal{A}$ has at least one eigenvalue in Γ, we employ suitable similarity transformations which bring the images of $\mathcal{K}_l$ and $\mathcal{K}_r$ in a form amenable to the same treatment as if $\mathcal{A}$ had only eigenvalues in the left half-plane. To set up these similarity transformations it is convenient to choose a basis such that $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are partitioned as

$$\mathcal{A} = \begin{bmatrix} \mathcal{A}_- & 0 \\ 0 & \mathcal{A}_+ \end{bmatrix}, \qquad \mathcal{B} = \begin{bmatrix} \mathcal{B}_- \\ \mathcal{B}_+ \end{bmatrix}, \qquad \mathcal{C} = \begin{bmatrix} \mathcal{C}_- & \mathcal{C}_+ \end{bmatrix}. \tag{7.36}$$

Here $\mathcal{A}_+$ ($\mathcal{A}_-$) has all its eigenvalues in right (left) half-plane and we denote its order by p_+ (p_-), so that $p_+ + p_- = p$. Moreover, $\mathcal{B}_+$, $\mathcal{B}_-$, $\mathcal{C}_+$, and $\mathcal{C}_-$ are $p_+ \times n$, $p_- \times n$, $n \times p_+$, and $n \times p_-$ matrices, respectively. Now put

$$\Phi_l = \begin{bmatrix} I_{p_-} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{p_+} \\ 0 & 0 & I_{p_-} & 0 \\ 0 & -I_{p_+} & 0 & P_1 \end{bmatrix}, \qquad \Phi_r = \begin{bmatrix} I_{p_-} & 0 & 0 & 0 \\ 0 & P_2 & 0 & I_{p_+} \\ 0 & 0 & I_{p_-} & 0 \\ 0 & -I_{p_+} & 0 & 0 \end{bmatrix}, \tag{7.37}$$

where P_1 and P_2 are the unique solutions of the equations (cf. Theorem I 4.1 of [17], Theorem VII 2.4 of [13])

$$\mathcal{A}_+ P_1 + P_1 \mathcal{A}_+^\dagger = \mathcal{B}_+ \mathcal{B}_+^\dagger, \tag{7.38}$$

$$P_2 \mathcal{A}_+ + \mathcal{A}_+^\dagger P_2 = \mathcal{C}_+^\dagger \mathcal{C}_+. \tag{7.39}$$

In fact, we have

$$P_1 = \int_0^\infty dt\, e^{-t\mathcal{A}_+} \mathcal{B}_+ \mathcal{B}_+^\dagger e^{-t\mathcal{A}_+^\dagger}, \qquad P_2 = \int_0^\infty dt\, e^{-t\mathcal{A}_+^\dagger} \mathcal{C}_+^\dagger \mathcal{C}_+ e^{-t\mathcal{A}_+}, \tag{7.40}$$

so that P_1 and P_2 are positive selfadjoint. Then, we easily compute

$$\Omega_1 = \Phi_l \mathcal{K}_l \Phi_l^{-1} = \begin{bmatrix} \Omega_3 & -\Omega_5 \Omega_5^\dagger \\ 0 & -\Omega_3^\dagger \end{bmatrix}, \ \Omega_2 = \Phi_r \mathcal{K}_r \Phi_r^{-1} = \begin{bmatrix} \Omega_4 & 0 \\ -\Omega_6^\dagger \Omega_6 & -\Omega_4^\dagger \end{bmatrix}, \tag{7.41}$$

where

$$\Omega_3 = \begin{bmatrix} \mathcal{A}_- & -\mathcal{B}_- \mathcal{B}_+^\dagger \\ 0 & -\mathcal{A}_+^\dagger \end{bmatrix}, \ \Omega_4 = \begin{bmatrix} \mathcal{A}_- & 0 \\ -\mathcal{C}_+^\dagger \mathcal{C}_- & -\mathcal{A}_+^\dagger \end{bmatrix}, \ \Omega_5 = \begin{bmatrix} \mathcal{B}_- \\ 0 \end{bmatrix}, \ \Omega_6 = \begin{bmatrix} \mathcal{C}_- & 0 \end{bmatrix}.$$

Note that all the eigenvalues of Ω_1 and Ω_2 lie in the open left half-plane. Therefore, in analogy to (7.34), (7.35), and Proposition 7.1, the projection operators $\mathcal{Q}$ and Π are such that

$$\operatorname{Im} \mathcal{Q} = \Phi_l^{-1} [\mathbb{C}^p \oplus \{0\}], \qquad \operatorname{Ker} \mathcal{Q} = \operatorname{Im} \begin{bmatrix} \mathcal{X} \\ I_p \end{bmatrix}, \tag{7.42}$$

$$\operatorname{Ker} \Pi = \Phi_r^{-1} [\{0\} \oplus \mathbb{C}^p], \qquad \operatorname{Im} \Pi = \operatorname{Im} \begin{bmatrix} I_p \\ \mathcal{Y} \end{bmatrix}. \tag{7.43}$$

Let us partition the inverses of Φ_l and Φ_r defined in (7.37) into $p \times p$ blocks as

$$\Phi_l^{-1} = \begin{bmatrix} \Lambda_{l1} & \Lambda_{l2} \\ \Lambda_{l3} & \Lambda_{l4} \end{bmatrix}, \qquad \Phi_r^{-1} = \begin{bmatrix} \Lambda_{r1} & \Lambda_{r2} \\ \Lambda_{r3} & \Lambda_{r4} \end{bmatrix}. \tag{7.44}$$

Note that

$$\Phi_l^{-1} = \boldsymbol{q}_{2p} \Phi_l \boldsymbol{q}_{2p}, \qquad \Phi_r^{-1} = \boldsymbol{q}_{2p} \Phi_r \boldsymbol{q}_{2p}, \tag{7.45}$$

so that the entries of Φ_l^{-1} and Φ_r^{-1} are readily available from (7.3).

Proposition 7.6. *Suppose a basis is chosen such that the matrices $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ in the realization* (7.1) *have the form indicated in* (7.36). *Then the matrices $\widetilde{\mathcal{X}}_+$, $\widetilde{\mathcal{Y}}_-$, Π, and $\mathcal{Q}$ in Proposition* 7.5 *can be expressed as*

$$\widetilde{\mathcal{Y}}_- = \Lambda_{l3}\Lambda_{l1}^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & P_1^{-1} \end{bmatrix}, \qquad \widetilde{\mathcal{X}}_+ = \Lambda_{r2}\Lambda_{r4}^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & -P_2^{-1} \end{bmatrix}, \tag{7.46}$$

$$\mathcal{Q} = \begin{bmatrix} \Lambda_{l1}(\Lambda_{l1} - \mathcal{X}\Lambda_{l3})^{-1} & -\Lambda_{l1}(\Lambda_{l1} - \mathcal{X}\Lambda_{l3})^{-1}\mathcal{X} \\ \Lambda_{l3}(\Lambda_{l1} - \mathcal{X}\Lambda_{l3})^{-1} & -\Lambda_{l3}(\Lambda_{l1} - \mathcal{X}\Lambda_{l3})^{-1}\mathcal{X} \end{bmatrix}, \tag{7.47}$$

$$I_{2p} - \Pi = \begin{bmatrix} -\Lambda_{r2}(\Lambda_{r4} - \mathcal{Y}\Lambda_{r2})^{-1}\mathcal{Y} & \Lambda_{r2}(\Lambda_{r4} - \mathcal{Y}\Lambda_{r2})^{-1} \\ -\Lambda_{r4}(\Lambda_{r4} - \mathcal{Y}\Lambda_{r2})^{-1}\mathcal{Y} & \Lambda_{r4}(\Lambda_{r4} - \mathcal{Y}\Lambda_{r2})^{-1} \end{bmatrix}. \tag{7.48}$$

Proof. It follows from (7.37), (7.42), (7.44), and (7.45) that

$$\mathcal{V} = \operatorname{Im}\mathcal{Q} = \left\{ \begin{bmatrix} \Lambda_{l1} \\ \Lambda_{l3} \end{bmatrix} u : u \in \mathbb{C}^p \right\}.$$

Now (7.31), (7.37), (7.44), and (7.45) imply (7.46) for $\widetilde{\mathcal{Y}}_-$. Similarly, by (7.31), (7.37), and (7.43)–(7.45), we have

$$\mathcal{N} = \operatorname{Ker}\Pi = \left\{ \begin{bmatrix} \Lambda_{r2} \\ \Lambda_{r4} \end{bmatrix} u : u \in \mathbb{C}^p \right\},$$

and so, by comparison with (7.31), we obtain (7.46) for $\widetilde{\mathcal{X}}_+$. Then (7.47) and (7.48) follow on using (7.46) in (7.32) and (7.33). In the derivation of (7.48) we have also used the identity $(I_p - \widetilde{\mathcal{X}}_+\mathcal{Y})^{-1}\widetilde{\mathcal{X}}_+ = \widetilde{\mathcal{X}}_+(I_p - \mathcal{Y}\widetilde{\mathcal{X}}_+)^{-1}$. □

Note that in (7.48) we have stated the result for $I_{2p} - \Pi$ rather than Π because we will only need the former. By using (7.36) and (7.37) one easily verifies that $\widetilde{\mathcal{X}}_+$ and $\widetilde{\mathcal{Y}}_-$ given in (7.46) satisfy (7.18) and (7.20), respectively.

In order to use the results of Proposition 7.6 in (7.23)–(7.30) we need some additional notation. We decompose the solution $\mathcal{X}$ of (7.8) as

$$\mathcal{X} = \begin{bmatrix} \mathcal{X}_1 & \mathcal{X}_2 \\ \mathcal{X}_3 & \mathcal{X}_4 \end{bmatrix},$$

so that $\mathcal{X}_1$ and $\mathcal{X}_4$ are selfadjoint and have orders p_- and p_+, respectively, and $\mathcal{X}_2^\dagger = \mathcal{X}_3$. We denote by P_3 the unique solution of the equation

$$\Omega_4 P_3 + P_3 \mathcal{E}_l^\dagger = -\Lambda_{r4}\mathcal{B}\mathcal{B}^\dagger, \tag{7.49}$$

which is given by

$$P_3 = \int_0^\infty dt\, e^{t\Omega_4}\Lambda_{r4}\mathcal{B}\mathcal{B}^\dagger e^{t\mathcal{E}_l^\dagger}, \tag{7.50}$$

and we define P_4 to be the unique (and generally nonsquare matrix) solution of the equation

$$\Omega_4 P_4 + P_4 \mathcal{A}_-^\dagger = -\Lambda_{r4}\mathcal{B}\mathcal{B}_-^\dagger,$$

given by

$$P_4 = \int_0^\infty dt\, e^{t\Omega_4}\Lambda_{r4}\mathcal{B}\mathcal{B}_-^\dagger\, e^{t\mathcal{A}_-^\dagger}. \tag{7.51}$$

Note that in contrast to the solutions P_1 and P_2 of (7.38) and (7.39), respectively, the matrices P_3 and P_4 are in general not selfadjoint. Furthermore, we let

$$\mathcal{J}_l = (\Lambda_{l1} - \Lambda_{l3}\mathcal{X})^{-1}\Lambda_{l1}, \qquad \mathcal{J}_r = \mathcal{Y}(\Lambda_{r4} - \Lambda_{r2}\mathcal{Y})^{-1}, \tag{7.52}$$

and introduce the matrices

$$\widetilde{\mathcal{A}} = \begin{bmatrix} \widetilde{\mathcal{A}}_+ & 0 \\ 0 & \widetilde{\mathcal{A}}_- \end{bmatrix}, \qquad \widetilde{\mathcal{B}} = \begin{bmatrix} \widetilde{\mathcal{B}}_+ \\ \widetilde{\mathcal{B}}_- \end{bmatrix}, \qquad \widetilde{\mathcal{C}} = \begin{bmatrix} \widetilde{\mathbb{C}}_+ & \widetilde{\mathcal{C}}_- \end{bmatrix}, \tag{7.53}$$

where

$$\widetilde{\mathcal{A}}_+ = \begin{bmatrix} -\mathcal{E}_l^\dagger & 0 \\ 0 & -\mathcal{A}_-^\dagger \end{bmatrix}, \qquad \widetilde{\mathcal{A}}_- = \begin{bmatrix} -\mathcal{A}_\Gamma^\dagger & 0 \\ -\Lambda_{r4}\mathcal{B}\mathcal{B}_+^\dagger & \Omega_4 \end{bmatrix}, \tag{7.54}$$

$$\widetilde{\mathcal{B}}_+ = \begin{bmatrix} \mathcal{J}_l\,\mathcal{C}^\dagger \\ 0 \end{bmatrix}, \qquad \widetilde{\mathcal{B}}_- = \begin{bmatrix} (P_1 - \mathcal{X}_4)^{-1}(\mathcal{X}_3\,\mathcal{C}_-^\dagger + \mathcal{X}_4\,\mathcal{C}_+^\dagger) \\ -P_3\mathcal{J}_l\,\mathcal{C}^\dagger \end{bmatrix}, \tag{7.55}$$

$$\widetilde{\mathbb{C}}_+ = \begin{bmatrix} -\mathcal{B}^\dagger(I_p + \mathcal{J}_rP_3) & \mathcal{B}^\dagger\mathcal{J}_rP_4 + \mathcal{B}_-^\dagger \end{bmatrix}, \qquad \widetilde{\mathcal{C}}_- = \begin{bmatrix} \mathcal{B}_+^\dagger & -\mathcal{B}^\dagger\mathcal{J}_r \end{bmatrix}. \tag{7.56}$$

We mention that $\widetilde{\mathcal{A}}$, $\widetilde{\mathcal{B}}$, and $\widetilde{\mathcal{C}}$ are $3p\times 3p$, $3p\times n$, and $n\times 3p$ matrices, respectively. Moreover, $\widetilde{\mathcal{A}}_+$, $\widetilde{\mathcal{A}}_-$, $\widetilde{\mathcal{B}}_+$, $\widetilde{\mathcal{B}}_-$, $\widetilde{\mathcal{C}}_+$ and $\widetilde{\mathcal{C}}_-$ are $(p_-+p)\times(p_-+p)$, $(p_++p)\times(p_++p)$, $(p_-+p)\times n$, $(p_++p)\times n$, $n\times(p_-+p)$, and $n\times(p_++p)$ matrices, respectively.

Next we present the main result of this section, expressing the scattering matrix in terms of the quantities defined above in connection with the similarity transformations induced by Φ_l and Φ_r.

Theorem 7.7. *Let $R(\lambda)$ be a rational reflection coefficient satisfying* (6.22). *Then the remaining entries of the scattering matrix* (3.11) *are given by*

$$T_l(\lambda) = I_n - i\,\mathcal{C}\Lambda_{l1}(\lambda - i\Omega_3)^{-1}(\Lambda_{l1} - \mathcal{X}\Lambda_{l3})^{-1}\mathcal{X}\mathcal{C}^\dagger, \tag{7.57}$$

$$T_r(\lambda) = I_n + i\,\mathcal{B}^\dagger\mathcal{J}_r(\lambda - i\Omega_4)^{-1}\Lambda_{r4}\mathcal{B}, \tag{7.58}$$

$$L(\lambda) = i\,\widetilde{\mathcal{C}}(\lambda - i\widetilde{\mathcal{A}})^{-1}\widetilde{\mathcal{B}}. \tag{7.59}$$

In the special case when $\mathcal{A}$ has all its eigenvalues in the left half-plane, these expressions simplify to

$$T_l(\lambda) = I_n - i\,\mathcal{C}(\lambda - i\mathcal{A})^{-1}\mathcal{X}\mathcal{C}^\dagger, \tag{7.60}$$

$$T_r(\lambda) = I_n + i\,\mathcal{B}^\dagger\mathcal{Y}(\lambda - i\mathcal{A})^{-1}\mathcal{B}, \tag{7.61}$$

$$L(\lambda) = -i\,\mathcal{B}^\dagger\mathcal{Y}(\lambda - i\mathcal{A})^{-1}\mathcal{X}\mathcal{C}^\dagger - i\,\mathcal{B}^\dagger(I_p - \mathcal{Y}\mathcal{X})(\lambda + i\mathcal{E}_l^\dagger)^{-1}\mathcal{C}^\dagger. \tag{7.62}$$

Proof. Using (7.27), (7.37), (7.41), (7.44), (7.45), (7.47), and the equality

$$\Phi_l\mathcal{Q}\begin{bmatrix} 0 \\ \mathcal{C}^\dagger \end{bmatrix} = -\begin{bmatrix} I_p \\ 0 \end{bmatrix}(\Lambda_{l1} - \mathcal{X}\Lambda_{l3})^{-1}\mathcal{X}\mathcal{C}^\dagger,$$

we obtain (7.57). From (7.23), (7.37), (7.41), (7.44), (7.45), (7.48), as well as the identity

$$\Phi_r(I_{2p} - \Pi)\begin{bmatrix} \mathcal{B} \\ 0 \end{bmatrix} = -\begin{bmatrix} 0 \\ I_p \end{bmatrix}(\Lambda_{r4} - \mathcal{Y}\Lambda_{r2})^{-1}\mathcal{Y}\mathcal{B},$$

it follows that

$$T_r(\lambda^*)^\dagger = I_n - i\,\mathcal{B}^\dagger\Lambda_{r4}(\lambda + i\Omega_4^\dagger)^{-1}(\Lambda_{r4} - \mathcal{Y}\Lambda_{r2})^{-1}\mathcal{Y}\mathcal{B}.$$

Now (7.58) follows by taking the adjoint and using (7.52). Note that Λ_{r2}, Λ_{r4}, and $\mathcal{Y}$ are hermitian.

With the help of (7.14), (7.26), (7.47), and (7.52), we derive

$$\left[T_l(\lambda^*)^\dagger\right]^{-1} = I_n - i\,\mathcal{C}\mathcal{X}(\lambda + i\mathcal{E}_l^\dagger)^{-1}\mathcal{J}_l\,\mathcal{C}^\dagger. \tag{7.63}$$

From (7.1), (7.63), and using

$$I_p - \mathcal{J}_l = -\Lambda_{l3}(\Lambda_{l1} - \mathcal{X}\Lambda_{l3})^{-1}\mathcal{X},$$
$$-i\,\mathcal{C}^\dagger\mathcal{C}\mathcal{X} = (\lambda + i\mathcal{A}^\dagger) - (\lambda + i\mathcal{E}_l^\dagger),$$

we get

$$\begin{aligned} R(\lambda^*)^\dagger[T_l(\lambda^*)^\dagger]^{-1} = {} & i\mathcal{B}^\dagger(\lambda + i\mathcal{A}^\dagger)^{-1}\Lambda_{l3}(\Lambda_{l1} - \mathcal{X}\Lambda_{l3})^{-1}\mathcal{X}\mathcal{C}^\dagger \\ & - i\mathcal{B}^\dagger(\lambda + i\mathcal{E}_l^\dagger)^{-1}\mathcal{J}_l\,\mathcal{C}^\dagger. \end{aligned} \tag{7.64}$$

Using (5.5), (7.57), (7.64), and some standard results on realizations (Chapter 1 of [7]), we obtain

$$L(\lambda) = i\,\Omega_8(\lambda - i\Omega_7)^{-1}\Omega_9, \tag{7.65}$$

where

$$\Omega_7 = \begin{bmatrix} \Omega_4 & \Lambda_{r4}\mathcal{B}\mathcal{B}^\dagger & -\Lambda_{r4}\mathcal{B}\mathcal{B}^\dagger \\ 0 & -\mathcal{A}^\dagger & 0 \\ 0 & 0 & -\mathcal{E}_l^\dagger \end{bmatrix},$$

$$\Omega_8 = \mathcal{B}^\dagger\begin{bmatrix}\mathcal{J}_r & I_p & -I_p\end{bmatrix}, \qquad \Omega_9 = \begin{bmatrix} 0 \\ \Lambda_{l3}(\Lambda_{l1} - \mathcal{X}\Lambda_{l3})^{-1}\mathcal{X} \\ \mathcal{J}_l \end{bmatrix}\mathcal{C}^\dagger.$$

To bring $L(\lambda)$ into the form (7.59) we use a similarity transformation. Let

$$\Psi = \begin{bmatrix} 0 & 0 & 0 & I_p \\ 0 & I_{p_-} & 0 & 0 \\ 0 & 0 & I_{p_+} & 0 \\ -I_p & P_4 & 0 & P_3 \end{bmatrix}, \qquad \Psi^{-1} = \begin{bmatrix} P_3 & P_4 & 0 & -I_p \\ 0 & I_{p_-} & 0 & 0 \\ 0 & 0 & I_{p_+} & 0 \\ I_p & 0 & 0 & 0 \end{bmatrix},$$

where P_3 and P_4 have been defined in (7.50) and (7.51). Then it is straightforward to verify that

$$\widetilde{\mathcal{A}} = \Psi\Omega_7\Psi^{-1}, \qquad \widetilde{\mathcal{C}} = \Omega_8\Psi^{-1}, \qquad \widetilde{\mathcal{B}} = \Psi\Omega_9, \tag{7.66}$$

where $\widetilde{\mathcal{A}}$, $\widetilde{\mathcal{B}}$, and $\widetilde{\mathcal{C}}$ are the matrices defined in (7.53). Using (7.66) in (7.65), together with the fact that for one of the blocks of Ω_9 we can write

$$\begin{aligned} \Lambda_{l3}(\Lambda_{l1} - \mathcal{X}\Lambda_{l3})^{-1}\mathcal{X} &= \begin{bmatrix} 0 & 0 \\ 0 & I_{p_+} \end{bmatrix}\begin{bmatrix} I_{p_-} & -\mathcal{X}_2 \\ 0 & P_1 - \mathcal{X}_4 \end{bmatrix}^{-1}\begin{bmatrix} \mathcal{X}_1 & \mathcal{X}_2 \\ \mathcal{X}_3 & \mathcal{X}_4 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ (P_1 - \mathcal{X}_4)^{-1}\mathcal{X}_3 & (P_1 - \mathcal{X}_4)^{-1}\mathcal{X}_4 \end{bmatrix}, \end{aligned}$$

we obtain (7.59) with the matrices (7.54)–(7.56). The expressions (7.60)–(7.62) can be obtained from (5.5) and (7.23)–(7.30) by using the special forms (7.35) for Π and $\mathcal{Q}$, or by obvious reductions from (7.57)–(7.59). The details are omitted. □

8. Inverse problem with rational scattering matrices

Let $R(\lambda)$ have the form (7.1) for certain matrices $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, where $\mathcal{A}$ has minimal order and hence does not have zero or purely imaginary eigenvalues. Then

$$R(\lambda) = i\,\mathcal{C}(\lambda - i\mathcal{A})^{-1}\mathcal{B} = -\int_{-\infty}^{\infty} dt\, e^{-i\lambda t}\,\mathcal{C}\,E(t;-\mathcal{A})\,\mathcal{B}, \tag{8.1}$$

where

$$E(t;-\mathcal{A}) = \begin{cases} e^{-t\mathcal{A}}P_{\mathcal{A}}^{(+)} = \dfrac{1}{2\pi i}\displaystyle\int_{\Gamma_+} dz\, e^{-tz}\,(z-\mathcal{A})^{-1}, & t>0, \\ -e^{-t\mathcal{A}}P_{\mathcal{A}}^{(-)} = -\dfrac{1}{2\pi i}\displaystyle\int_{\Gamma_-} dz\, e^{-tz}\,(z-\mathcal{A})^{-1}, & t<0, \end{cases} \tag{8.2}$$

is the bisemigroup generated by $\mathcal{A}$ (cf. [8, 9]). Here Γ_+ and Γ_- are the positively oriented simple Jordan contours in the right and left half-planes enclosing all of the eigenvalues of $\mathcal{A}$ in the open right and left half-planes, respectively, and $P_{\mathcal{A}}^{(+)}$ and $P_{\mathcal{A}}^{(-)}$ are the spectral projections of $\mathcal{A}$ corresponding to its eigenvalues in the right and left half-planes, respectively.

Our strategy for reconstructing $k(x)$ from $R(\lambda)$ is as follows. When $x>0$ we will solve the Marchenko equation (6.10) by using $R(\lambda)$ as the input, and when $x<0$ we will solve the Marchenko equation (6.11) by using $L(\lambda)$ as given in (7.59). Then we use (4.14) to determine $k(x)$. Throughout we assume the absence of poles of $T_l(\lambda)$, $T_r(\lambda)$, $T_l(\lambda)^{-1}$, and $T_r(\lambda)^{-1}$ in the upper half-plane.

First consider (6.10) with

$$\hat{R}(t) = -\mathcal{C}\,E(t;-\mathcal{A})\,\mathcal{B}, \qquad \hat{R}(t)^\dagger = -\mathcal{B}^\dagger\,E(t;-\mathcal{A}^\dagger)\,\mathcal{C}^\dagger,$$

which are obtained from (6.1) and (8.1). Introducing the positive selfadjoint $p\times p$ matrices

$$\mathcal{D}_1 = \int_0^\infty dt\, E(t;-\mathcal{A})\,\mathcal{B}\,\mathcal{B}^\dagger E(t;-\mathcal{A}^\dagger),\;\; \mathcal{D}_2 = \int_0^\infty dt\, E(t;-\mathcal{A}^\dagger)\,\mathcal{C}^\dagger\,\mathcal{C}\,E(t;-\mathcal{A}),$$

and assuming $x>0$, we obtain for the hermitian integral kernel in (6.10)

$$\int_0^\infty d\beta\, \hat{R}(\gamma+\beta+2x)\,\hat{R}(\alpha+\beta+2x)^\dagger = \mathcal{C}\,E(\gamma+2x;-\mathcal{A})\,\mathcal{D}_1\,E(\alpha+2x;-\mathcal{A}^\dagger)\,\mathcal{C}^\dagger.$$

The unique solution of the separable integral equation (6.10) is then given by

$$B_{l3}(x,\alpha) = -\mathcal{B}^\dagger\left[I_p + E(2x;-\mathcal{A}^\dagger)\,\mathcal{D}_2\,E(2x;-\mathcal{A})\,\mathcal{D}_1\right]^{-1}E(\alpha+2x;-\mathcal{A}^\dagger)\,\mathcal{C}^\dagger, \tag{8.3}$$

where the inverse exists because of the unique solvability of (6.10). For later use we note that, by (7.36) and (7.40), we have

$$\mathcal{D}_1 = \begin{bmatrix} 0 & 0 \\ 0 & P_1 \end{bmatrix}, \qquad \mathcal{D}_2 = \begin{bmatrix} 0 & 0 \\ 0 & P_2 \end{bmatrix}. \tag{8.4}$$

When $x < 0$ we start from $L(\lambda)$ as given in (7.59), so that

$$\hat{L}(t) = -\widetilde{\mathcal{C}}\, E(t; -\widetilde{\mathcal{A}})\, \widetilde{\mathcal{B}}, \qquad \hat{L}(t)^\dagger = -\widetilde{\mathcal{B}}^\dagger\, E(t; -\widetilde{\mathcal{A}}^\dagger)\, \widetilde{\mathcal{C}}^\dagger.$$

Proceeding as in the derivation of (8.3), we obtain

$$B_{r2}(x,\alpha) = -\widetilde{\mathcal{B}}^\dagger \left[I_{3p} + E(-2x; -\widetilde{\mathcal{A}}^\dagger)\, \mathcal{D}_4\, E(-2x; -\widetilde{\mathcal{A}})\, \mathcal{D}_3\right]^{-1} E(\alpha - 2x; -\widetilde{\mathcal{A}}^\dagger)\, \widetilde{\mathcal{C}}^\dagger, \tag{8.5}$$

where the inverse exists because of the unique solvability of (6.11). Here $\mathcal{D}_3$ and $\mathcal{D}_4$ are the positive selfadjoint matrices given by

$$\mathcal{D}_3 = \int_0^\infty dt\, E(t; -\widetilde{\mathcal{A}})\, \widetilde{\mathcal{B}}\, \widetilde{\mathcal{B}}^\dagger\, E(t; -\widetilde{\mathcal{A}}^\dagger), \quad \mathcal{D}_4 = \int_0^\infty dt\, E(t; -\widetilde{\mathcal{A}}^\dagger)\, \widetilde{\mathcal{C}}^\dagger\, \widetilde{\mathcal{C}}\, E(t; -\widetilde{\mathcal{A}}),$$

which, by means of (7.53)–(7.56), can be written as

$$\mathcal{D}_3 = \begin{bmatrix} P_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \mathcal{D}_4 = \begin{bmatrix} P_6 & P_7 & 0 & 0 \\ P_8 & P_9 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where

$$P_5 = \int_0^\infty dt\, e^{t\mathcal{E}_l^\dagger}\, \mathcal{J}_l \mathcal{C}^\dagger \mathcal{C} \mathcal{J}_l^\dagger\, e^{t\mathcal{E}_l}, \quad P_6 = \int_0^\infty dt\, e^{t\mathcal{E}_l} (I_p + P_3^\dagger \mathcal{J}_r^\dagger) \mathcal{B}\mathcal{B}^\dagger (I_p + \mathcal{J}_r P_3) e^{t\mathcal{E}_l^\dagger}, \tag{8.6}$$

and P_7, P_8, and P_9 are irrelevant because they will not contribute to $k(x)$, as we will see.

Now we are ready to prove the main result of this section. Again we first state the general result and then specialize it to the particular case when $\mathcal{A}$ has all its eigenvalues in the left half-plane or, equivalently, when $R(\lambda)$ is analytic in $\mathbb{C}^+$.

Theorem 8.1. *Suppose that $R(\lambda)$ satisfies (6.22) and is given by the minimal representation (7.1) in a basis where (7.36) holds. Then the matrix potential $k(x)$ in (1.2) is given by*

$$k(x) = \begin{cases} 2i\, \mathcal{C}_+\, e^{-2x\mathcal{A}_+} \left[I_{p_+} + P_1\, e^{-2x\mathcal{A}_+^\dagger}\, P_2\, e^{-2x\mathcal{A}_+}\right]^{-1} \mathcal{B}_+, & x > 0, \\ 2i\, \mathcal{C}\mathcal{J}_l^\dagger \left[I_p + e^{-2x\mathcal{E}_l}\, P_6\, e^{-2x\mathcal{E}_l^\dagger} P_5\right]^{-1} e^{-2x\mathcal{E}_l} (I_p + P_3^\dagger \mathcal{J}_r^\dagger) \mathcal{B}, & x < 0. \end{cases} \tag{8.7}$$

If $R(\lambda)$ is analytic in $\mathbb{C}^+$, then

$$k(x) = \begin{cases} 0, & x > 0, \\ 2i\, \mathcal{C}\left[I_p - e^{-2x\mathcal{E}_l} \mathcal{X}\, e^{-2x\mathcal{E}_r^\dagger}\, \mathcal{Y}\right]^{-1} e^{-2x\mathcal{E}_l}\, (I_p - \mathcal{X}\mathcal{Y})\, \mathcal{B}, & x < 0, \end{cases} \tag{8.8}$$

where $\mathcal{X}$ and $\mathcal{Y}$ are the unique solutions of (7.8) *and* (7.9)*, respectively. Moreover, if $R(\lambda)$ is analytic in $\mathbb{C}^+$, then the jump in the potential at $x = 0$ is given by*

$$k(0^+) - k(0^-) = -k(0^-) = -2i\mathcal{C}\mathcal{B}. \tag{8.9}$$

Proof. The representation (8.7) for $k(x)$ is a direct consequence of (4.14) and (8.3)–(8.6). Thus we need only establish the simplifications that occur when $R(\lambda)$ is analytic in $\mathbb{C}^+$. In this case $\mathcal{A}$ has all its eigenvalues in the left half-plane so that from (8.2) we get $E(t; -\mathcal{A}) = 0$ for $t > 0$. Thus $k(x) = 0$ for $x > 0$. For $x < 0$, starting with (7.36), we can simplify the expression in (8.7) by deleting the blocks associated with the spectrum of $\mathcal{A}$ in the right half-plane. This reduction is implemented by the following substitutions: $\Lambda_{l1} \mapsto I_p$, $\Lambda_{l3} \mapsto 0$, and hence

$$\mathcal{J}_l^\dagger \mapsto I_p, \tag{8.10}$$

by (7.52). Similarly, $\Lambda_{r2} \mapsto 0$, $\Lambda_{r4} \mapsto I_p$, and hence $\mathcal{J}_r^\dagger \mapsto \mathcal{Y}$. Since $\Omega_4 \mapsto \mathcal{A}$, the solution to (7.49) becomes $P_3 = -\mathcal{X}$ and thus

$$I_p + P_3^\dagger \mathcal{J}_r^\dagger \mapsto I_p - \mathcal{X}\mathcal{Y}. \tag{8.11}$$

Furthermore we can compute P_5 and P_6 in (8.6). We observe that P_5 and P_6 are solutions to the following Riccati equations:

$$P_5\mathcal{E}_l + \mathcal{E}_l^\dagger P_5 = -\mathcal{C}^\dagger\mathcal{C}, \tag{8.12}$$

$$P_6\mathcal{E}_l^\dagger + \mathcal{E}_l P_6 = -(I_p - \mathcal{X}\mathcal{Y})\mathcal{B}\mathcal{B}^\dagger(I_p - \mathcal{Y}\mathcal{X}). \tag{8.13}$$

First note the identity

$$\mathcal{E}_l^\dagger\mathcal{Y} + \mathcal{Y}\mathcal{E}_r = -\mathcal{C}^\dagger\mathcal{C}(I_p - \mathcal{X}\mathcal{Y}), \tag{8.14}$$

which follows from (7.9) and (7.15). On multiplying (8.14) from the right by $(I_p - \mathcal{X}\mathcal{Y})^{-1}$, using (7.16), and comparing the result with (8.12), we find that

$$P_5 = \mathcal{Y}(I_p - \mathcal{X}\mathcal{Y})^{-1} = (I_p - \mathcal{Y}\mathcal{X})^{-1}\mathcal{Y}. \tag{8.15}$$

Similarly, on multiplying the identity

$$\mathcal{E}_r\mathcal{X} + \mathcal{X}\mathcal{E}_l^\dagger = \mathcal{B}\mathcal{B}^\dagger(I_p - \mathcal{Y}\mathcal{X}),$$

which follows from (7.8) and (7.15), from the left by $I_p - \mathcal{X}\mathcal{Y}$, using (7.16), and comparing the result with (8.13), we obtain

$$P_6 = -\mathcal{X}(I_p - \mathcal{Y}\mathcal{X}). \tag{8.16}$$

Since, by (7.16) and its adjoint,

$$(I_p - \mathcal{Y}\mathcal{X})e^{-2x\mathcal{E}_l^\dagger} = e^{-2x\mathcal{E}_r^\dagger}(I_p - \mathcal{Y}\mathcal{X}),$$

the result (8.8) for $x < 0$ follows from inserting (8.10), (8.11), (8.15), (8.16) in (8.7). Finally, letting $x \to 0$ from below gives $k(0^-) = -2i\mathcal{C}\mathcal{B}$, and hence (8.9) follows. □

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References

[1] M.J. Ablowitz, P.A. and Clarkson: "Solitons, nonlinear evolution equations and inverse scattering," London Math. Soc. Lecture Note Series 149, Cambridge Univ. Press, Cambridge, 1991.

[2] T. Aktosun, M. Klaus, and C. Van der Mee: "Direct and inverse scattering for selfadjoint Hamiltonian systems on the line," Integral Equations and Operator Theory 38, 129–178 (2000).

[3] D. Alpay and I. Gohberg: "Inverse spectral problems for differential operators with rational scattering matrix functions," J. Differential Equations 118, 1–19 (1995).

[4] D. Alpay and I. Gohberg: Inverse scattering problem for differential operators with rational scattering matrix functions. In: Böttcher, A. and Gohberg, I. (eds.): "Singular integral operators and related topics (Tel Aviv, 1995)," Birkhäuser OT 90, Basel and Boston, 1996, pp. 1–18.

[5] D. Alpay and I. Gohberg: Potential associated with rational weights. In: Gohberg, I. and Lyubich, Yu. (eds.): "New results in operator theory and its applications," Birkhäuser OT 98, Basel and Boston, 1997, pp. 23–40.

[6] D. Alpay and I. Gohberg: "State space method for inverse spectral problems," Progr. Systems Control Theory 22, 1–16 (1997).

[7] H. Bart, I. Gohberg, and M.A. Kaashoek: "Minimal factorization of matrix and operator functions," Birkhäuser OT 1, Basel and Boston, 1979.

[8] H. Bart, I. Gohberg, and M.A. Kaashoek: "Exponentially dichotomous operators and inverse Fourier transforms," Report 8511/M, Econometric Institute, Erasmus University of Rotterdam, 1985.

[9] H. Bart, I. Gohberg, and M.A. Kaashoek: "Wiener-Hopf factorization, inverse Fourier transforms and exponentially dichotomous operators," J. Funct. Anal. 68, 1–42 (1986).

[10] R. Beals and R.R. Coifman: "Scattering and inverse scattering for first order systems," Comm. Pure Appl. Math. 37, 39–90 (1984).

[11] R. Beals and R.R. Coifman: "Scattering and inverse scattering for first-order systems: II," Inverse Problems 3, 577–593 (1987).

[12] R. Beals, P. Deift, and C. Tomei: "Direct and inverse scattering on the line," Math. Surveys and Monographs, 28, Amer. Math. Soc., Providence, 1988.

[13] R. Bhatia: "Matrix analysis," Graduate Texts in Mathematics 169, Springer, New York, 1997.

[14] P. Deift and E. Trubowitz: "Inverse scattering on the line," Comm. Pure Appl. Math. 32, 121–251 (1979).

[15] L.D. Faddeev: "Properties of the S-matrix of the one-dimensional Schrödinger equation," Amer. Math. Soc. Transl. (Ser. 2), 65, 139–166 (1967) [Trudy Mat. Inst. Steklova 73, 314–336 (1964) (Russian)].

[16] M.G. Gasymov: "The inverse scattering problem for a system of Dirac equations of order $2n$," Trans. Moscow Math. Soc. 19, 41–119 (1968) [*Trudy Moscov. Mat. Obšč.* 19, 41–112 (1968) (Russian)].

[17] I. Gohberg, S. Goldberg, and M.A. Kaashoek: "Classes of linear operators. Vol. I," Birkhäuser OT 49, Basel and Boston, 1990.

[18] I. Gohberg, M.A. Kaashoek, and A.L. Sakhnovich: "Canonical systems with rational spectral densities: Explicit formulas and applications," Math. Nachr. 149, 93–125 (1998).

[19] I. Gohberg, M.A. Kaashoek, and A.L. Sakhnovich: "Pseudo-canonical systems with rational Weyl functions: Explicit formulas and applications," J. Differential Equations 146, 375–398 (1998).
[20] I. Gohberg, M.A. Kaashoek, and A.L. Sakhnovich: Canonical systems on the line with rational spectral densities: Explicit formulas. In: Adamayan, V.M., Gohberg, I., Gorbachuk, M.L., and Gorbachuk, V.I. (eds.): "Differential Operators and Related Topics (Odessa, 1997), Vol. I," Birkhäuser OT 117, Basel and Boston, 2000, pp. 127–139.
[21] I. Gohberg and S. Rubinstein: Proper contractions and their unitary minimal completions. In: Gohberg, I. (ed.): "Topics in interpolation theory of rational matrix-valued functions," Birkhäuser OT 33, Basel and Boston, 1988, pp. 223–247.
[22] B. Grébert: "Inverse scattering for the Dirac operator on the real line," Inverse Problems 8, 787–807 (1992).
[23] D.B. Hinton, A.K. Jordan, M. Klaus, J.K. and Shaw: "Inverse scattering on the line for a Dirac system," J. Math. Phys. 32, 3015–3030 (1991).
[24] P. Lancaster and L. Rodman: "Algebraic Riccati equations," Oxford University Press, New York, 1995.
[25] F.È. Melik-Adamyan: "On the properties of the S-matrix of canonical differential equations on the whole axis," Akad. Nauk Armjan. SSR Dokl. 58, 199–205 (1974) [Russian].
[26] F.È. Melik-Adamyan: "Canonical differential operators in a Hilbert space," Izv. Akad. Nauk Armjan. SSR Ser. Mat. 12, 10–31, 85 (1977) [Russian].
[27] F.È. Melik-Adamyan: "On a class of canonical differential operators," Soviet J. Contemporary Math. Anal. 24, 48–69 (1989) [Izv. Akad. Nauk Armyan. SSR Ser. Mat. 24, 570–592, 620 (1989) (Russian)].
[28] I.G. Petrovski: "Ordinary differential equations," Prentice-Hall, Englewood Cliffs, 1966.
[29] A.L. Sakhnovich: "Nonlinear Schrödinger equation on a semi-axis and an inverse problem associated with it," Ukr. Math. J. 42, 316–323 (1990) [Ukrain. Mat. Zh. 42, 356–363 (1990) (Russian)].
[30] L.A. Sakhnovich: "Factorization problems and operator identities," Russ. Math. Surv. 41, 1–64 (1986) [Usp. Mat. Nauk 41, 4–55, 240 (1986) (Russian)].
[31] L.A. Sakhnovich: "The method of operator identities and problems in analysis," St. Petersburg Math. J. 5, 1–69 (1994) [Algebra i Analiz 5, 3–80 (1993) (Russian)].
[32] A.B. Shabat: "An inverse scattering problem," Differential Equations 15, 1299–1307 (1980) [Differ. Urav. 15, 1824–1834, 1918 (1979) (Russian)].

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