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A Sturm–Liouville Inverse Spectral Problem with Boundary Conditions Depending on the Spectral Parameter*

C. van der Mee and V. N. Pivovarchik

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Abstract. We consider a boundary value problem generated by the Sturm-Liouville equation on a finite interval. Both the equation and the boundary conditions depend quadratically on the spectral parameter. This boundary value problem occurs in the theory of small vibrations of a damped string. The inverse problem, i.e., the problem of recovering the equation and the boundary conditions from the given spectrum, is solved.

Key words: Sturm–Liouville problem, damped string, spectral parameter-dependent boundary conditions, eigenvalues, asymptotics.

In [1] we considered the boundary value problem

$$y'' + \lambda^2 y - i\lambda p y - qy = 0, \tag{1}$$

$$y(0) = 0, \tag{2}$$

$$y'(a) + (-m\lambda^2 + i\alpha\lambda + \beta)y(a) = 0, \tag{3}$$

where λ is the spectral parameter, $a > 0$, $p > 0$, $m > 0$, $\alpha > 0$, $\beta \in \mathbb{R}$, and $q(x)$ is a real function belonging to the Sobolev space $W_2^2(0,a)$. The problem of small transverse vibrations of an inhomogeneous smooth string in a medium with viscous damping (here p is proportional to the viscous damping coefficient) can be reduced by a Liouville transformation (cf. [2]) to system (1)–(3), where the left endpoint is fixed and the right endpoint bears a lumped mass proportional to m and can move with viscous damping proportional to α in the direction orthogonal to the equilibrium position of the string (cf. [1] and system (5)–(7) below). The spectrum of this problem consists only of normal eigenvalues accumulating at infinity.

In [1], the inverse problem of recovering the sextuple $\{a, q, p, m, \alpha, \beta\}$ from the spectrum of problem (1)–(3) was solved under the restrictive condition that there are no purely imaginary eigenvalues, which corresponds to the weak damping of the corresponding quadratic operator pencil (cf. [3]). For $m = p = 0$, the inverse problem with purely imaginary eigenvalues taken into account was solved in [4]. In the present article, using the method of [4], we obtain the complete solution of the inverse problem for system (1)–(3) with $m, p > 0$.

Definition 1. A sequence of complex numbers $\{\lambda_k\}_{0\neq k\in\mathbb{Z}}$ is said to be *properly numbered* if

1) $\operatorname{Re}\lambda_k \geqslant \operatorname{Re}\lambda_l$ for all $k > l$;

2) $\lambda_{-k} = -\overline{\lambda}_k$ for all λ_k that are not purely imaginary.

3) Each complex number occurs in the sequence at most finitely many times.

Otherwise, the numbering is arbitrary.

Definition 2. Let κ be a nonnegative integer. A properly numbered sequence $\{\lambda_k\}_{0\neq k\in\mathbb{Z}}$ is said to belong to $\mathcal{SHB}_\kappa^-$ if

1) All but κ terms of the sequence lie in the open upper half-plane.

2) Each term in the closed lower half-plane is purely imaginary and occurs only once. If $\kappa \geqslant 1$, we denote these terms by $\lambda_{-j} = -i\,|\lambda_{-j}|$ $(j = 1,\dots,\kappa$. We assume that $|\lambda_{-j}| < |\lambda_{-(j+1)}|$ $(j = 1,\dots,\kappa-1)$.

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3) If $\kappa \geqslant 1$, then the numbers $i\,|\lambda_{-j}|$ $(j = 1, \ldots, \kappa)$ (with the exception of λ_{-1} if it is zero) are not terms of the sequence.

4) If $\kappa \geqslant 2$, then the number of terms in each of the intervals $(i\,|\lambda_{-j}|, i\,|\lambda_{-(j+1)}|)$ $(j = 1, \ldots, \kappa - 1)$ is odd.

5) If $|\lambda_{-1}| > 0$, then the interval $(0, i\,|\lambda_{-1}|)$ contains no terms at all or an even number of terms.

6) If $\kappa \geqslant 1$, then the interval $(i\,|\lambda_{-\kappa}|, i\infty)$ contains an odd number of terms.

7) If $\kappa = 0$, then the sequence has an even or zero number of positive imaginary terms.

Definition 3. By $\mathcal{B}_{\mp}$ we denote the class of sets $\{a, q, p, m, \alpha, \beta\}$ such that $a > 0$, $p > 0$, $m > 0$, $\pm(\alpha - pm) > 0$, $\beta \in \mathbb{R}$, and q is a real function in $L_2(0, a)$ with the property that the self-adjoint operator A defined by

$$Af = -f'' + qf, \qquad D(A) = \{f \in W_2^2(0, a) : f'(a) + \beta f(a) = 0,\ f(0) = 0\},$$

is strictly positive. Moreover, if q belongs to the Sobolev space $W_2^2(0, a)$, then we say that the sextuple $\{a, q, p, m, \alpha, \beta\}$ belongs to $\mathcal{B}_{\mp}^0$. Furthermore, if $\{a, q, p, m, \alpha, \beta\}$ belongs to $\mathcal{B}_+$ (resp., $\mathcal{B}_+^0$) and $\alpha > 0$, then we say that $\{a, q, p, m, \alpha, \beta\}$ belongs to $\widehat{\mathcal{B}}_+$ (resp., $\widehat{\mathcal{B}}_+^0$).

In the framework of the direct problem, we obtain the following two results, which strengthen Theorem 3.1 in [1].

Theorem 4. *Let $\{a, q, m, p, \alpha, \beta\} \in \mathcal{B}_-^0$. Then the spectrum of problem (1)–(3) satisfies the following conditions:*

1) $\{\lambda_k\}_{0 \neq k \in \mathbb{Z}} \in \mathcal{SHB}_0^-$,

2) $\{\lambda_k - (ip/2)\}_{0 \neq k \in \mathbb{Z}} \in \mathcal{SHB}_\kappa^-$ *for some* $\kappa \geqslant 0$,

3) *we have the following equation (asymptotically as $k \to \infty$):*

$$\lambda_k = \frac{\pi(k-1)}{a} + \frac{ip}{2} + \frac{p_0}{k-1} + \frac{ip_1}{(k-1)^2} + \frac{p_2}{k^3} + \frac{b_k}{k^3}, \tag{4}$$

where $p_0, p_2 \in \mathbb{R}$, $p_1 > 0$, and $\sum_{0 \neq k \in \mathbb{Z}} |b_k|^2 < \infty$.

We note that the integer κ in 2) is equal to the number of nonpositive eigenvalues of the self-adjoint operator

$$A_1 f = -f'' + \left(q - \frac{p^2}{4}\right) f,$$
$$D(A_1) = \left\{f \in W_2^2(0, a) : f'(a) + \left(\beta - \frac{\alpha p}{2} + \frac{mp^2}{4}\right) f(a) = 0,\ f(0) = 0\right\}.$$

Theorem 5. *Let $\{a, q, m, p, \alpha, \beta\} \in \mathcal{B}_+^0$. Then assertion 3) of Theorem 4 is true with $p_1 < 0$ and assertion 2) is replaced by the following assertion:*

2′) $\{(ip/2) - \lambda_k\}_{0 \neq k \in \mathbb{Z}} \in \mathcal{SHB}_\kappa^-$, *where κ is the number of nonpositive eigenvalues of the operator A_1.*

Moreover, if $\{a, q, m, p, \alpha, \beta\} \in \widehat{\mathcal{B}}_+^0$, then assertion 1) of Theorem 4 also holds.

Let us now consider the inverse problem of reconstructing the sextuple $\{a, q, m, p, \alpha, \beta\}$ from the spectrum of problem (1)–(3).

Theorem 6. *Let a sequence $\{\lambda_k\}_{0 \neq k \in \mathbb{Z}}$ of complex numbers satisfy the following conditions:*

1) $\{\lambda_k\}_{0 \neq k \in \mathbb{Z}} \in \mathcal{SHB}_0^-$,

2) $\{\lambda_k - (ip/2)\}_{0 \neq k \in \mathbb{Z}} \in \mathcal{SHB}_\kappa^-$ *for some* $\kappa \geqslant 0$,

3) *condition 3) in Theorem 4 holds.*

Then there exists a unique sextuple $\{a, q, p, m, \alpha, \beta\} \in \mathcal{B}_-$ such that $\{\lambda_k\}_{0 \neq k \in \mathbb{Z}}$ is the spectrum of problem (1)–(3) generated by $\{a, q, p, m, \alpha, \beta\}$.

Theorem 7. *Let a sequence $\{\lambda_k\}_{0 \neq k \in \mathbb{Z}}$ satisfy the condition that $\{-\lambda_k + (ip/2)\}_{0 \neq k \in \mathbb{Z}} \in \mathcal{SHB}_\kappa^-$ for some $\kappa \geqslant 0$ as well as Eq. (4). Then there exists a unique sextuple $\{a, q, p, m, \alpha, \beta\} \in \mathcal{B}_+$ such that $\{\lambda_k\}_{0 \neq k \in \mathbb{Z}}$ is the spectrum of the problem (1)–(3) generated by the sextuple $\{a, q, p, m, \alpha, \beta\}$.*

We set

$$a=\lim_{k\to\infty}\frac{\pi n}{\lambda_n},\quad p=-2i\lim_{k\to\infty}\left(\lambda_n-\frac{\pi(n-1)}{a}\right),\quad \chi(\lambda)=\lim_{n\to\infty}\prod_{1\leqslant|k|\leqslant n}\left(1-\frac{\lambda}{\lambda_k}\right).$$

Let us define

$$B_0=\lim_{n\to\infty}\left(\frac{a}{2\pi n}\chi(\theta_n)\right),\quad \text{where } \theta_n=\frac{ip}{2}+\sqrt{\left(\frac{(4n-3)\pi}{2a}\right)^2-\frac{p^2}{4}}\,.$$

Then $B_0\neq 0$. We also set

$$B_1=-i\lim_{n\to\infty}\left(\frac{1}{B_0}\chi(\theta_n)-\frac{\pi(4n-3)}{2a}\right).$$

Theorem 8. *Suppose that a sequence* $\{\lambda_k\}_{0\neq k\in\mathbb{Z}}$ *satisfies the assumptions of Theorem* 7 *and* $B_1<p$. *Then there exists a unique sextuple* $\{a,q,p,m,\alpha,\beta\}\in\widehat{\mathcal{B}}_+$ *such that* $\{\lambda_k\}_{0\neq k\in\mathbb{Z}}$ *is the spectrum of the problem* (1)–(3) *generated by the sextuple* $\{a,q,p,m,\alpha,\beta\}$.

By way of application, consider corollaries of Theorems 4 and 6. By S_l, where $l>0$, we denote the class of data $\{A(s),p,\nu,\mu\}$ that satisfy the following conditions: $A(s)\in W_2^2(0,l)$ is a real-valued function, $A(s)\geqslant\varepsilon>0$, and $p>0$, $\mu>0$, and $\nu>p\mu$ are constants. Let S_l^0 be the subset of S_l such that $A(s)\in W_2^4(0,l)$.

Corollary 9. *Let* $\{A(s),p,\mu,\nu\}\in S_l^0$. *Then the spectrum of the problem*

$$\frac{d}{ds}\left(A(s)\,\frac{dv}{ds}\right)+\lambda^2v-i\lambda pv=0, \tag{5}$$

$$v(0)=0, \tag{6}$$

$$v'(l)+i\lambda\nu v(l)-\lambda^2\mu v(l)=0 \tag{7}$$

satisfies the assertions of Theorem 4.

Corollary 10. *Let a sequence* $\{\lambda_k\}_{0\neq k\in\mathbb{Z}}$ *of complex numbers satisfy the conditions of Theorem* 6. *Then for every* $l>0$ *there exists a unique quadruple* $\{A(s),p,\mu,\nu\}\in S_l$ *such that* $\{\lambda_k\}_{0\neq k\in\mathbb{Z}}$ *is the spectrum of problem* (5)–(7).

In the proofs, the results of [5–7] are used.

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Dipartimento di Matematica, Università di Cagliari, Italy
e-mail: cornelis@bugs.unica.it

Odessa State Academy of Civil Engineering and Architecture
e-mail: v.pivovarchik@paco.net, vnp@dtp.odessa.ua

Translated by C. van der Mee and V. N. Pivovarchik