LU-factorization of Block Toeplitz Matrices

Cornelis Van der Mee

Dipartimento di Matematica, Università di Cagliari

Via Ospedale 72, 09124 Cagliari, Italy. *E-mail: cornelis@bugs.unica.it*

Abstract

We give a review of the theory of factorization of block Toeplitz matrices of the type $T = (T_{i-j})_{i,j\in\mathbb{Z}^d}$, where T_{i-j} are complex $k \times k$ matrices, in the form $T = LDU$, with L and L^{-1} lower block triangular, U and U^{-1} upper block triangular Toeplitz matrices, and D a diagonal matrix function. In particular, it is discussed how decay properties of T_i affect decay properties of L, L^{-1}, U, and U^{-1}. We also discuss factorizations of the type $A = LDU$, where D is no longer diagonal.

1 Introduction

Given a bi-infinite block Toepliz matrix $A = (A_{i-j})_{i,j\in\mathbb{Z}}$, indexed by the integers $i, j \in \mathbb{Z}$ and having complex $k \times k$ matrices as its entries, it is well known how to factorize it in the form

$$A = LDU, \tag{1.1}$$

where $L = (L_{i-j})_{i,j\in\mathbb{Z}}$ is a lower triangular (i.e., $L_i = 0$ for $i < 0$) block Toeplitz matrix with $L_0 = I_k$ (the $k \times k$ unit matrix) having a block lower triangular inverse, $U = (U_{i-j})_{i,j\in\mathbb{Z}}$ is an upper triangular (i.e., $U_i = 0$ for $i > 0$) block Toeplitz matrix matrix with $U_0 = I_k$ and having a block upper triangular inverse, and D is a nonsingular $k \times k$ matrix. Such factorizations are usually studied for A in the Wiener class of block Toeplitz matrices that satisfy

$$\|A\|_{\mathcal{W}} = \sum_{i=-\infty}^{\infty} \|A_i\| < +\infty, \tag{1.2}$$

the norms on the right-hand side being arbitrary $k \times k$ matrix norms, and in that case the factors L and U and their inverses L^{-1} and U^{-1} have finite Wiener norm when the LDU-factorization (1.1) exists. A necessary (but not sufficient) condition for the existence of the factorization (1.1) is that the values of the symbol

$$\hat{A}(z) = \sum_{i=-\infty}^{\infty} z^i A_i, \qquad |z| = 1, \tag{1.3}$$

are nonsingular $k \times k$ matrices.

Three special cases should be mentioned. If A is a banded (i.e., if $A_i = 0$ for $|i| > m$), then the factors L and U, when they exist, are banded as well (i.e., $L_i = 0$ for $i > m$ and $U_i = 0$ for $i < -m$). If A is positive definite (as a bounded linear operator on the Hilbert space $\ell^2(\mathbb{Z})$) or, equivalently, if $\hat{A}(z)$ is positive definite for every z on the unit circle, the factorization (1.1) exists, $U = L^\dagger$ (the conjugate transpose of L) and D is positive definite. In that case, putting $Ł = LD^{1/2}$, we obtain the block Cholesky factorization

$$A = ŁŁ^\dagger \tag{1.4}$$

of A. Finally, if the symbol of A is scalar (i.e., $k = 1$), a necessary and sufficient condition for the existence of the factorization (1.1) is that $z = 0$ has zero winding number with respect to the curve $z \mapsto \hat{A}(z)$. In this case the factorization can be obtained by separating the Fourier expansion of $\log \hat{A}(z)$ in terms analytic inside and outside the unit disks and exponentiating the terms obtained.

The theory of Wiener-Hopf factorization of matrix functions of the form (1.3) with $k \times k$ matrix coefficients A_i satisfying (1.2) is well known from the theoretical point of view. We mention the seminal article by Gohberg and Krein [13] and several textbooks [10, 6, 11]. The scalar case goes back to the article by Krein [20]. In the special case of banded matrices, the symbol A is a trigonometric matrix polynomial and the factorization can be implemented by applying the theory of matrix polynomials [14, 15, 26].

Numerical methods for computing the Cholesky factors of a bi-infinite positive definite block Toeplitz matrix have been developed by various authors. In [16] the relative merits of various methods for the scalar case have been discussed in detail. For banded block Toeplitz matrices, a numerical method based on matrix polynomial factorization theory was developed in [22, 23], one based on band extension was given in [24], and one relying on semidefinite programming was explained in [21].

In this article we are primarily interested in multi-index block Toeplitz matrices, i.e., matrices $A = (A_{i-j})_{i,j \in \mathbb{Z}^d}$ which are indexed by $i, j \in \mathbb{Z}^d$ (the lattice points in $\mathbb{R}^d$) and have complex $k \times k$ matrices as their entries. At first sight, multi-index Toeplitz matrix theory can be developed more or less as in the one-index case. The symbols now are sums of d-variable Fourier series and are continuous $k \times k$ matrix-valued functions on the d-dimensional torus. The usual Banach algebra techniques (see [8, 11] for the scalar case and [3, 11] for the matrix case) can be applied to study the invertibility of bi-infinite multi-index block Toeplitz matrices. The method of writing the logarithm of the symbol as the sum of two series, either of which is then exponentiated, can be applied, exclusively in the scalar case, to obtain LDU-factorizations [17, 7].

Multi-index block Toeplitz factorization theory has several features that make it more challenging than the corresponding one-index theory, both from the functional analytic and the numerical point of view.

First of all, in order to define LDU-factorizations in a meaningful way, one must introduce a linear order $\preceq$ on $\mathbb{Z}^d$ which turns $\mathbb{Z}^d$ into an ordered group. The net effect is that instead of two such orders as for $\mathbb{Z}$ (the natural and the reversed natural order), there are now infinitely many such orders. Using a suitable order, one can now formulate (i) scalar factorizations through separation of logarithms of the symbol and exponentiation [17], (ii) band extension [1, 2], and (iii) the projection method of approximating the solutions of the given bi-infinite block Toeplitz systems by the solutions of finite block Toeplitz systems (extending methods given in [4, 12, 27]). However, the band extension method leads to the approximation of the solution of the original bi-infinite system by the solutions of infinite systems, which makes it as good as useless from the numerical point of view.

Secondly, there is no meaningful multi-variable matrix polynomial theory to assist in the factorization of banded multi-index block Toeplitz matrices. Moreover [28, 29], multi-index Toeplitz matrices having a trigonometric polynomial symbol and having an LDU-factorization, may not have factors whose symbols are nontrivial trigonometric polynomials. Hence, there is no obvious way to generalize the numerical methods developed in [22, 23].

Finally, in order to study the algebraic or exponential decay of the coefficients of the factors and their inverses in the case of algebraic or exponential decay of the coefficients of the given matrix, one can either apply Banach algebra techniques with weighted Wiener algebras [8, 11] or generalize the so-called exponential equivalence of bi-infinite matrices to the multi-index case [16, 17]. In the scalar case ($k = 1$), one does not encounter many problems (see Theorem 2.2 below). However, in the block Toeplitz case ($k \geq 2$) it appears impossible to extend the existing techniques for proving algebraic or exponential decay of the coefficients of the factors and their inverses to the multi-index case.

Numerical methods for evaluating the LDU-factorization in the multi-index case are rare and incompletely developed. A method based on Krein's method of additive logarithmic decomposition is given in [17, 25], exclusively for the scalar case ($k = 1$). A numerical method based on the projection method was formulated in [25]. For $d = 2$ and the lexicographical order on $\mathbb{Z}^2$, a modified band extension method was developed in [9]. Only the first two methods were implemented numerically.

In this article we will discuss the principal results on LDU-factorization in Section 2. In Section 3 we will discuss a more general type of factorization $A = LDU$, where D is allowed to be of a more general form.

2 LDU-factorization

a. Bi-infinite block Toeplitz matrices. By a bi-infinite block Toeplitz matrix, with blocks of order k, we mean a matrix $A = (A_{i-j})_{i,j\in\mathbb{Z}^d}$ whose entries A_{i-j} are complex $k \times k$ matrices. Such a matrix is said to be in the *Wiener class* $\mathcal{W}_k^d$ if

$$\|A\|_{\mathcal{W}_k^d} := \sum_{i\in\mathbb{Z}^d} \|A_i\| < +\infty, \tag{2.1}$$

where $\|\cdot\|$ is an arbitrary $k \times k$ matrix norm. Using multi-index notation, [1] we define its *symbol* by

$$\hat{A}(z) := \sum_{i\in\mathbb{Z}^d} z^i A_i, \qquad z \in \mathbb{T}^d, \tag{2.2}$$

where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. Clearly, the symbol $\hat{A}$ is a continuous complex-valued function on the d-dimensional torus $\mathbb{T}^d$.

Consider a sequence $\boldsymbol{\beta} = (\beta_i)_{i\in\mathbb{Z}^d}$ of weights satisfying the condition $1 \leq \beta_{i+j} \leq \beta_i\beta_j$ for $i, j \in \mathbb{Z}^d$. Then a bi-infinite block Toeplitz matrix A is said to be in $\mathcal{W}_{k,\boldsymbol{\beta}}^d$ if

$$\|A\|_{\mathcal{W}_{k,\boldsymbol{\beta}}^d} := \sum_{i\in\mathbb{Z}^d} \beta_i \|A_i\| < +\infty. \tag{2.3}$$

The following result is well-known ([11] if $d = 1$; [7] if $k = 1$; also [25]).

[1] For $z = (z_1, \ldots, z_d) \in \mathbb{C}^d$ and $i = (i_1, \ldots, i_d) \in \mathbb{Z}^d$ we write $z^i = z_1^{i_1} \ldots z_d^{i_d}$ and $|i| = |i_1| + \ldots + |i_d|$.

Proposition 2.1 *The $\boldsymbol{\beta}$-weighted Wiener class $\mathcal{W}^d_{k,\boldsymbol{\beta}}$ is a Banach algebra with respect to the convolution product*

$$(A * B)_i = \sum_{j\in\mathbb{Z}^d} A_j B_{i-j}, \qquad i \in \mathbb{Z}^d,$$

with involution $A \mapsto A^\dagger$ defined by $(A^\dagger)_i = (A_{-i})^\dagger$, $i \in \mathbb{Z}^d$. Its invertible elements are exactly those $A \in \mathcal{W}^d_{k,\boldsymbol{\beta}}$ for which $\hat{A}(z)$ is a nonsingular $k \times k$ matrix for all $z \in \Omega_{\boldsymbol{\beta}}$, where

$$\Omega_{\boldsymbol{\beta}} := \left\{ z \in \mathbb{C}^d : \sup_{i\in\mathbb{Z}^d} \frac{|z^i|}{\beta_i} < +\infty \right\}.$$

b. *LDU*-factorization of bi-infinite block Toeplitz matrices. Given a block Toeplitz matrix $A = (A_{i-j})_{i,j\in\mathbb{Z}^d}$ of Wiener class, by an *LDU-factorization* of A (with respect to the order $\preceq$) we mean a representation of A in the form

$$A = LDM^\dagger, \tag{2.4}$$

where $L = (L_{i-j})_{i,j\in\mathbb{Z}^d}$, $M = (M_{i-j})_{i,j\in\mathbb{Z}^d}$ and $D = (D_{i-j})_{i,j\in\mathbb{Z}^d}$ are block Toeplitz matrices of Wiener class having the following properties:

a) $L_0 = M_0 = I_k$ (the $k \times k$ unit matrix),

b) $D_i = 0$ for $i \neq 0$ and $L_i = M_i = 0$ for $i \prec 0$, and

c) the inverses L^{-1} and M^{-1} of L and M are block Toeplitz matrices of Wiener class satisfying $[L^{-1}]_i = [M^{-1}]_i = 0$ for $i \prec 0$.

Passing to the respective symbols $\hat{L}$, $\hat{D}(z) \equiv D_0$ and $\hat{M}$, one gets

$$\hat{A}(z) = \hat{L}(z) D_0 \hat{M}(z)^\dagger, \qquad z \in \mathbb{T}^d. \tag{2.5}$$

When A is positive definite on the Hilbert space $\ell^2(\mathbb{Z}^d)$ of square integrable sequences on $\mathbb{Z}^d$ (or, equivalently, if $\hat{A}(z)$ is positive definite for all $z \in \mathbb{T}^d$), A always has an LDU-factorization of the form (2.5) with $L = M$ and D_0 a positive definite $k \times k$ matrix. In that case we put $Ł_i = L_i D_0^{1/2}$ and obtain the *block Cholesky factorization*

$$\hat{A}(z) = \hat{Ł}(z)\hat{Ł}(z)^\dagger, \qquad z \in \mathbb{T}^d. \tag{2.6}$$

When the weight sequence $\boldsymbol{\beta}$ is to be taken into account, the classical argument of exploiting the compactness of Hankel operators [11] fails if $d \geq 2$ and $k \geq 2$. For $k = 1$ one can apply factorization in suitable commutative Banach algebras [7] to establish the following result and its corollary, thus generalizing a well-known result by Krein [20].

Theorem 2.2 *Let $A \in \mathcal{W}^d_{1,\boldsymbol{\beta}}$ be a bi-infinite Toeplitz matrix with scalar elements (i.e., with $k = 1$) for some weight sequence $\boldsymbol{\beta}$, and let $\preceq$ be a linear order which turns $\mathbb{Z}^d$ into an ordered group. Then the following statements are equivalent:*

1) *A has an LDU factorization where the factors and their inverses belong to $\mathcal{W}^d_{1,\boldsymbol{\beta}}$;*
2) *there exists $B \in \mathcal{W}^d_{1,\boldsymbol{\beta}}$ such that $A = \exp(B)$;*
3) *for $s = 1, \ldots, d$ and fixed $(z_1, \ldots, z_{s-1}, z_{s+1}, \ldots, z_d) \in \mathbb{T}^{d-1}$ the curves $z_s \mapsto \hat{A}(z)$ have winding number zero with respect to the origin in the complex plane.*

When these conditions are fullfilled, write $\log(A) = (B_{i-j})_{i,j\in\mathbb{Z}^d}$. Put $L = (L_{i-j})_{i,j\in\mathbb{Z}^d}$, $M = (M_{i-j})_{i,j\in\mathbb{Z}^d}$ and $D = (D_{i-j})_{i,j\in\mathbb{Z}^d}$, where $L_i = B_i$ and $M_i = B_{-i}$ for $i \succ 0$, $L_0 = M_0 = 1$, $L_i = M_i = 0$ for $i \prec 0$, and $D_0 = B_0$ and $D_i = 0$ for $i \neq 0$. Then

$$A = \exp(L)\exp(D)\exp(M^\dagger) = \exp(L)\exp(D)[\exp(M)]^\dagger$$

is an LDU-factorization of A in $\mathcal{W}^d_{1,\boldsymbol{\beta}}$.

Corollary 2.3 *Let $\boldsymbol{\beta} = (\beta_i)_{i\in\mathbb{Z}^d}$ be a weight sequence. Suppose $A \in \mathcal{W}^d_{k,\boldsymbol{\beta}}$ has an LDU-factorization in $\mathcal{W}^d_k$ of the type* (2.4) *and that $\hat{A}(z)$ is a nonsingular $k \times k$ matrix for $z \in \Omega_{\boldsymbol{\beta}}$. Then, IN THE SCALAR CASE $k = 1$, the factors L and $M^\dagger$ and their inverses belong to $\mathcal{W}^d_{k,\boldsymbol{\beta}}$.*

Corollary 2.3 is not true if both $d \geq 2$ and $k \geq 2$. In fact, there exist counterexamples, even for positive definite A.

A bi-infinite block Toeplitz matrix A is called (*finitely*) *banded* if all but finitely many A_i are equal to the zero matrix. A well-known result (Féjer's theorem if A is positive definite) states that, for $d = 1$, the factors L and $M^\dagger$ (resp., the factor Ł) in an LDU-factorization (resp. Cholesky factorization of an arbitrary (resp. positive definite) (finitely) banded block Toeplitz matrix of Wiener class are (finitely) banded themselves. This is no longer the case if $d \geq 2$ [29]. For instance [5, 28], if $d = 2$, $\delta \in (0, \frac{1}{4})$ and

$$\hat{A}(z) = 1 + 2\delta\left[\cos(z_1) + \cos(z_2)\right],$$

then $\hat{A}(z)$ is positive for every $z = (z_1, z_2) \in \mathbb{T}^2$ but cannot be written as the product of two nonconstant trigonometric polynomials in z_1 and z_2. In other words, no matter the choice of the order $\preceq$ in $\mathbb{Z}^2$, the corresponding Toeplitz matrix A has an LDU-factorization (resp., a Cholesky factorization) of the form (2.4) (resp., (2.6)), but its factors L and $M^\dagger$ (resp., the factor Ł) are not (finitely) banded Toeplitz matrices. For $d = 2$, necessary and sufficient conditions to write $\hat{A}(z)$ as the squared absolute value of a stable polynomial in (z_1, z_2) have been given in [9].

3 LU-equivalence and Generalized Factorization

So far we have studied LDU-factorization of $A \in \mathcal{W}^d_{k,\boldsymbol{\beta}}$. As evident from Proposition 2.1, a necessary condition for the existence of such a factorization is that $\hat{A}(z)$ is a nonsingular $k \times k$ matrix for every $z \in \Omega_{\boldsymbol{\beta}}$. Only in the case $k = 1$ we have established necessary AND SUFFICIENT conditions for its existence (see Theorem 2.2). For $k \geq 2$ such necessary and sufficient conditions are very difficult to formulate, especially if $d \geq 2$.

The bi-infinite block Toeplitz matrices $A^{(1)}$ and $A^{(2)}$ in $\mathcal{W}^d_{k,\boldsymbol{\beta}}$ such that $\hat{A}^{(1)}(z)$ and $\hat{A}^{(2)}(z)$ are nonsingular for $z \in \Omega_{\boldsymbol{\beta}}$, are called *$LU$-equivalent* (relative to $\mathcal{W}^d_{k,\boldsymbol{\beta}}$ and the linear order $\preceq$ that makes $\mathbb{Z}^d$ into an ordered group) if there exist $L = (L_{i-j})_{i,j\in\mathbb{Z}^d}$ and $M = (M_{i-j})_{i,j\in\mathbb{Z}^d}$ in $\mathcal{W}^d_{k,\boldsymbol{\beta}}$ such that

a) $L_0 = M_0 = I_k$ (the $k \times k$ unit matrix),

b) $L_i = M_i = 0$ for $i \prec 0$,

c) the inverses L^{-1} and M^{-1} of L and M are block Toeplitz matrices in $\mathcal{W}^d_{k,\boldsymbol{\beta}}$ satisfying $[L^{-1}]_i = [M^{-1}]_i = 0$ for $i \prec 0$, and

d) $A^{(2)} = LA^{(1)}M^\dagger$.

Passing to the respective symbols $\hat{L}$ and $\hat{M}$, one gets

$$\hat{A}^{(2)}(z) = \hat{L}(z)\hat{A}^{(1)}(z)\hat{M}(z)^\dagger, \qquad z \in \mathbb{T}^d.$$

Clearly, LU-equivalence is an equivalence relation for the invertible elements of the Banach algebra $\mathcal{W}^d_{k,\boldsymbol{\beta}}$. It is then natural to ask what are convenient representatives of the various LU-equivalence classes. For $k = 1$ (scalar case), a complete answer to the question is almost immediate from Theorem 2.2: Two invertible elements $\hat{A}^{(1)}$ and $\hat{A}^{(2)}$ are LU-equivalent if and only if for $s = 1, \ldots, d$ and fixed $(z_1, \ldots, z_{s-1}, z_{s+1}, \ldots, z_d) \in \mathbb{T}^{d-1}$ the curves $z_s \mapsto \hat{A}^{(1)}(z)$ and $z_s \mapsto \hat{A}^{(2)}(z)$ have the same winding number with respect to the origin in the complex plane (see slao [7]). Further, given an invertible element A of $\mathcal{W}^d_{1,\boldsymbol{\beta}}$ and letting κ_s stand for the winding number of the curve $z_s \mapsto \hat{A}(z)$ $(s = 1, \ldots, d)$, there exists a factorization of A in the form $A = LDM^\dagger$ where

a) $L_0 = M_0 = D_\kappa = I_k$ (the $k \times k$ unit matrix), $D_i = 0$ for $i \neq \kappa$;

b) $L_i = M_i = 0$ for $i \prec 0$,

c) the inverses L^{-1} and M^{-1} of L and M are block Toeplitz matrices in $\mathcal{W}^d_{k,\boldsymbol{\beta}}$ satisfying $[L^{-1}]_i = [M^{-1}]_i = 0$ for $i \prec 0$.

Passing to the respective symbols $\hat{L}$ and $\hat{M}$, one gets

$$\hat{A}(z) = \hat{L}(z) z_1^{\kappa_1} \dots z_d^{\kappa_d} \hat{M}(z)^\dagger, \qquad z \in \mathbb{T}^d.$$

For $d = 1$ (one-index case), a complete answer to the question has been given by Gohberg and Krein [13, 10]: Given an invertible element $A \in \mathcal{W}^1_{k,\boldsymbol{\beta}}$, there exist unique integers $\ell_1, \dots, \ell_k$, with $\ell_1 \geq \dots \geq \ell_k$ (the so-called partial indices), such that there exists a factorization of A in the form $A = LDM^\dagger$ where

a) $L_0 = M_0 = I_k$ (the $k \times k$ unit matrix), $D_i = 0$ for $i \notin \{\ell_1, \dots, \ell_d\}$, $D_i = \operatorname{diag}(d_1^i, \dots, d_k^i)$ with $d_s^i = 1$ if $\ell_s = i$ and $d_s^i = 0$ if $\ell_s \neq i$;

b) $L_i = M_i = 0$ for $i \prec 0$,

c) the inverses L^{-1} and M^{-1} of L and M are block Toeplitz matrices in $\mathcal{W}^d_{k,\boldsymbol{\beta}}$ satisfying $[L^{-1}]_i = [M^{-1}]_i = 0$ for $i \prec 0$.

Passing to the respective symbols $\hat{L}$ and $\hat{M}$, one gets

$$\hat{A}(z) = \hat{L}(z) \operatorname{diag}(z^{\ell_1}, \dots, z^{\ell_k}) \hat{M}(z)^\dagger, \qquad z \in \mathbb{T}^d.$$

There is no convenient description of the partial indices in terms of A, although $\ell_1 + \dots + \ell_k$ is the winding number of the curve $z \mapsto \det \hat{A}(z)$ with respect to the origin in the complex plane. It is known that all partial indices vanish if A is positive definite.

In the two special cases studied so far ($k = 1$, and $d = 1$), the characterization of the LU-equivalence classes does not depend on the weight sequence $\boldsymbol{\beta}$ nor on the linear order $\preceq$. Further, if one of d and k equals one, each equivalence class contains at least one element having a diagonal matrix symbol. When both $d \geq 2$ and $k \geq 2$, the description of the LU-equivalence classes is ill understood: They may depend on the weight sequence $\boldsymbol{\beta}$ and there may exist equivalence clases that do not contain one single block Toeplitz matrix having a diagonal matrix symbol. These observations can be based on a class of illustrative examples in the factorization theory of almost periodic 2×2 matrix functions [18, 19].

Karlovich and Spitkovsky [18, 19] have studied $2m \times 2m$ matrix functions of the type

$$W(x) = \begin{pmatrix} e^{i\lambda x} I_m & 0 \\ c_{-1}\, e^{-i\nu x} - c_0 + c_1\, e^{i\alpha x} & e^{-i\lambda x}\, I_m \end{pmatrix}, \qquad x \in \mathbb{R}, \tag{3.1}$$

where $\nu, \alpha\lambda \in \mathbb{R}$ with $\beta = (\nu/\alpha) \notin \mathbb{Q}$, $\alpha + \nu = \lambda$, and c_{-1}, c_0 and c_1 are nonsingular $m \times m$ matrices. Now let us consider the additive subgroup G of $\mathbb{R}$ generated by ν and α. Defining the linear order on $\mathbb{Z}^2$ defined by

$$i = (i_1, i_2) \succeq j = (j_1, j_2) \Longleftrightarrow \nu i_1 + \alpha i_2 \geq \nu j_1 + \alpha j_2,$$

the map $i = (i_1, i_2) \mapsto \nu i_1 + \alpha i_2$ is an order preserving group isomorphism from $\mathbb{Z}^2$ onto G. Here we note that the line $y = -\beta x$ does not contain any points of $\mathbb{Z}^2$, apart from the origin. Making the transformation $z_1 = e^{i\nu x}$ and $z_2 = e^{i\alpha x}$, we convert $W(x)$ into the matrix function

$$\hat{A}(z_1, z_2) = \begin{pmatrix} z_1 z_2\, I_m & 0 \\ \frac{1}{z_1}\, c_{-1} - c_0 + z_2\, c_1 & \frac{1}{z_1 z_2}\, I_m \end{pmatrix},$$

which is the symbol of the bi-infinite Toeplitz matrix with nontrivial coefficients

$$\begin{cases} A_{(0,0)} = \begin{pmatrix} 0 & 0 \\ -c_0 & 0 \end{pmatrix}, \; A_{(-1,0)} = \begin{pmatrix} 0 & 0 \\ c_{-1} & 0 \end{pmatrix}, \; A_{(0,1)} = \begin{pmatrix} 0 & 0 \\ c_1 & 0 \end{pmatrix}, \\ A_{(1,1)} = \begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix}, \; A_{(-1,-1)} = \begin{pmatrix} 0 & 0 \\ 0 & I_m \end{pmatrix}. \end{cases} \tag{3.2}$$

In this way a factorization problem for the almost periodic matrix function W given by (3.1) can be converted into an equivalent LDU-factorization problem (with respect to $\preceq$ as above) for the bi-infinite block Toeplitz matrix $A = (A_{i-j})_{i,j \in \mathbb{Z}^2}$ whose nontrivial entries are given by (3.2).

Putting $c_{1,0} = c_0^{-1} c_1$ and $c_{-1,0} = c_0^{-1} c_{-1}$ and considering the weights $\beta_i \equiv 1$, it can be shown ([19], Theorem 5.3) that A has an LDU-factorization (with respect to $\preceq$ as above) if either $\|c_{1,0}\|^{\beta} \|c_{-1,0}\|$ or $\|c_{1,0}^{-1}\|^{\beta} \|c_{-1,0}^{-1}\|$ is strictly less than one. On the other hand, if $m = 1$ and $\left| c_1^{\beta} c_0^{-1-\beta} c_{-1} \right| = 1$, then A is NOT LU-equivalent (with respect to $\preceq$) to any bi-infinite block Toeplitz matrix with diagonal matrix function. In other words, if $\left| c_1^{\beta} c_0^{-1-\beta} c_{-1} \right| = 1$, there do not exist integer pairs $n^{(1)} = (n_1^{(1)}, n_2^{(1)})$ and $n^{(2)} = (n_1^{(2)}, n_2^{(2)})$ such that

$$\hat{A}(z_1, z_2) = \hat{L}(z_1, z_2)\, \mathrm{diag}\left(z_1^{n_1^{(1)}} z_2^{n_2^{(1)}}, z_1^{n_1^{(2)}} z_2^{n_2^{(2)}} \right) \hat{M}(z_1, z_2)^{\dagger}, \quad |z_1| = |z_2| = 1,$$

where $\hat{L}$, $\hat{M}$ and their inverses are lower $\preceq$-triangular.

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