# Conditions for runaway phenomena in kinetic theory revisited


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**Abstract**

Starting from the initial-boundary value problem for the linear Boltzmann equation with a force term and unbounded collision frequency, we consider the abstract problem characterized by the sum of two evolution operators $A$ and $B$. We investigate relations between the semigroup generator $G$ and the operator $A+B$ where $A$ generates a positive contraction semigroup in a very general case. A characterization theorem for $G = \overline{A+B}$ is stated. The results are based on a spectral analysis of $B(\lambda - A)^{-1}$, where the main issue is to study the conditions under which the value 1 belongs to the resolvent, the continuous spectrum or the residual spectrum of $B(\lambda - A)^{-1}$, respectively. Two illustrative examples are discussed.

# 1 Introduction

In this paper we consider a generalization of the initial-boundary value problem for the simplified linear Boltzmann equation

$$\begin{cases} \dfrac{\partial f}{\partial t}(v,t) = -a\dfrac{\partial f}{\partial v} - \nu(v)f(v,t) + (Kf)(v,t), & (v,t) \in \mathbb{R}\times\mathbb{R}_+; \\ f(v,0) = f_0(v), \qquad v \in \mathbb{R}, \end{cases} \tag{1.1}$$

within an abstract framework. Equation (1.1) describes the electron distribution $f(v,t)$ in a weakly ionized host medium as a function of the velocity $v \in \mathbb{R}$ and time $t \geq 0$, $a > 0$ is the electrostatic acceleration, $\nu(v)$ is the collision frequency between an electron and the host medium, and $K$ denotes the scattering operator. The main focus of the research on Eq. (1.1) has so far been the study of conditions under which one of the mutually exclusive phenomena, relaxation to equilibrium and runaway, occurs. In the recent past the runaway phenomenon in transport theory was studied from the mathematical point of view by many authors, (see, e.g., [9, 11, 2, 10, 16, 17]). The asymptotic behaviour of the particle distribution in the runaway case, which corresponds to the generation of a travelling wave in velocity space has been studied in the framework of wave operator theory ([12, 7]).

In many applications, the collision frequency and the operator $K$ are unbounded but balance each other in such a way that the total number of particles is expected to be constant in time. In this case the general theory of the existence and uniqueness of solutions to kinetic equations, as developed by Beals and Protopopescu [8, 13], should be modified. Such a modification was in fact pioneered by Frosali et al. [11], while a way to deal with the intricacies of the unbounded case by extending the Kato-Voigt theorem on semigroup perturbations was suggested by Arlotti [1]. In semiconductor physics, the unbounded case was studied by Majorana and coauthors, basically by approximating the integral kernel of the scattering operator by bounded truncations (cf. [14]), which has led to an existence result without resolving the well-posedness problem. In [3, 4, 5, 6], Banasiak and coauthors applied classical semigroup theory to derive very general existence and uniqueness results for integrable solutions to linear transport equations, thus obtaining, among other things, an alternative derivation of the main result of [14]. Unfortunately, the evolution semigroup corresponding to the physical problem is constructed by a limiting procedure which does not allow one to control the domain of the generator. In [15] a similar approximation argument was used to extend the theory of Beals and Protopopescu to deal with the existence issues in the unbounded case.

When adopting the semigroup approach, the existence of the evolution operator is almost immediate, but the full characterization of the generator requires additional mathematical tools. In general, one can only prove that the generator of the evolution semigroup $\{S(t), t \geq 0\}$ solving the Cauchy problem (1.1) is a closed (but not necessarily the minimal closed) extension of the operator $T$ defined by the right-hand side of (1.1), while proving it to be the minimal extension of $T$ requires additional assumptions. In principle, this allows for a situation in which the Cauchy problem has solutions which cannot be obtained by the usual approximation approach, and therefore has multiple solutions. Such a situation was in fact encountered for a fragmentation model from reaction diffusion theory [5], where multiple solutions were found precisely in the cases where the evolution semigroup $\{S(t); t \geq 0\}$ fails to satisfy the stochasticity requirement

$$\|S(t)f\|_1 = \|f\|_1, \quad \forall t \geq 0,\ \forall f \geq 0\,.$$

In other words, in this case the physical requirement that the total number of particles is conserved in time, fails to be satisfied.

In this article we prove that a necessary and sufficient condition for $S(t)$ to be stochastic is that the generator is the (minimal) closure of $T$. We distinguish between the following three situations: *i)* $T$ itself is the generator, *ii)* the generator is the closure of $T$, in which case $S(t)$ is stochastic, and *iii)* the generator is a proper closed extension of $T$, in which case $S(t)$ is not stochastic. For each of these three cases we give various characterizations in terms of the spectral properties of the operator $B(\lambda - A)^{-1}$. We will not always give complete proofs, but instead present them in a later, more general publication. In the final section we revisit the electron ionization model discussed in [11] and the fragmentation model treated in [5].

# 2 Abstract formulation of the problem

In this section we rewrite the initial value problem (1.1) in the general abstract form

$$\begin{cases} \dfrac{\partial f}{\partial t} &= Af + Bf\,, \quad t > 0 \\ \\ f(0) &= f_0, \end{cases} \tag{2.1}$$

where the unknown function $f$ belongs to the Banach space $X = L^1(\Omega, \mu)$ for some measure space $(\Omega, \mu)$. In the abstract setting, we make the following general assumptions:

**i)** $A = T_0 + T_F + T_A$ where $T_0$ is the free streaming operator, $T_F$ is the external force term, and $T_A$ is the absorption (loss) term;

**ii)** $T_A = -\nu I$, where $I$ is the identity operator and $\nu = \nu(\mu)$ is non-negative and belongs to $L^{1,loc}(\Omega, \mu)$, i.e. $\nu$ is $\mu$-integrable on every bounded Borel subset of $\Omega$;

**iii)** $A$ is the generator of a positive contraction semigroup (a so-called sub-stochastic semigroup) $\{S_0(t); t > 0\}$ on $X$.

Let us denote by $L_\lambda = (\lambda - A)^{-1}$ with $\mathrm{Re}\lambda > 0$ the resolvent operator of $A$. Then

$$L_\lambda f = \int_0^\infty e^{-\lambda t} S_0(t)\, f\, dt, \quad f \in X, \quad \mathrm{Re}\,\lambda > 0\,. \tag{2.2}$$

Let us now assume that

**iv)** $\lambda \|L_\lambda f\|_1 + \|\nu L_\lambda f\|_1 = \|f\|_1, \quad f \geq 0, \quad f \in X.$

Let $X_\nu = L^1(\Omega, \nu(\mu) d\mu)$ with the measure $\nu d\mu$. Then $X_\nu$ is a Banach space if $\nu$ is positive $\mu$-almost everywhere.

Let $B$ be a positive operator mapping $\{f \in X : \nu f \in X\}$ into $X$; in most applications the operator $B$ is the gain collision operator. We assume that $B : X_\nu \to X$ is such that

**v)** $\|Bf\|_1 = \|\nu f\|_1, \quad f \geq 0, \quad f \in X_\nu\,.$

Then $BL_\lambda$ is positive and bounded on $X$ and from *iv)* we have

$$\|BL_\lambda f\|_1 = \|f\|_1 - \lambda \|L_\lambda f\|_1, \quad f \geq 0, \quad f \in X\,. \tag{2.3}$$

When $T$ coincides with $T_A$, we simply have $L_\lambda f = \frac{1}{\nu+\lambda} f$ for all $f \in X$ and $\lambda > 0$. The estimate iv) follows immediately from $\nu L_\lambda f + \lambda L_\lambda f = f$.

**Example 2.1** *(cf. [11]) When $T = T_F + T_A = -a\dfrac{\partial}{\partial v} - \nu(v)$, with $a > 0$ constant and $v \in \mathbb{R}$, the operator $L_\lambda$ takes the form*

$$(L_\lambda f)(v) = \frac{1}{a} \int_{-\infty}^{v} \exp\left( -\frac{1}{a} \int_{v'}^{v} [\nu(v'') + \lambda]\, dv'' \right) f(v') dv'. \tag{2.4}$$

*Estimate iv) easily follows from $\int_{-\infty}^{+\infty} (\nu(v) + \lambda)\, dv = +\infty$ for $\lambda > 0$. In particular, for $\lambda = 0$, we have the identity $\|L_0 f\|_\nu = \|f\|_1$ for $f \geq 0$ whenever $\int_{-\infty}^{+\infty} \nu(v) dv = +\infty$.*

Rewriting formally the resolvent equation for $A+B$ in terms of $L_\lambda$, we obtain

$$f = L_\lambda B f + L_\lambda g\,. \tag{2.5}$$

Modifying the expansion of the resolvent equation given in [11], we study the convergence of the series

$$f = L_\lambda \sum_{n=0}^{\infty} (BL_\lambda)^n\, g, \tag{2.6}$$

where $g \in X$, $f \in X \cap X_\nu$, and $\lambda > 0$. Writing the equality

$$\|L_\lambda f\|_1 = \frac{1}{\lambda}\left\{\|f\|_1 - \|\nu L_\lambda f\|_1\right\}$$

for $\sum_{j=0}^{n} (BL_\lambda)^j f$ instead of $f$ and taking into account the positivity of the operators as well as assumptions iv) and v), we have

$$\left\| L_\lambda \sum_{j=0}^{n} (BL_\lambda)^j \right\|_1 = \frac{1}{\lambda}\left\{ \sum_{j=0}^{n} \|\,(BL_\lambda)^j f\|_1 - \sum_{j=0}^{n} \|\,(BL_\lambda)^{j+1} f\|_1 \right\}.$$

Using the definition

$$T_\lambda^{[n]} f = L_\lambda \sum_{j=0}^{n} (BL_\lambda)^j\ f\,, \tag{2.7}$$

for the $n$-th partial sum of the series in (2.6), we obtain

$$\begin{aligned} \|T_\lambda^{[n]} f\|_1 &= \frac{1}{\lambda}\left\{ \sum_{j=0}^{n} \|\,(BL_\lambda)^j f\|_1 - \sum_{j=1}^{n+1} \|\,(BL_\lambda)^j f\|_1 \right\} \\ &= \frac{1}{\lambda}\left\{ \|f\|_1 - \|\,(BL_\lambda)^{n+1} f\|_1 \right\} \le \frac{1}{\lambda}\|f\|_1, \end{aligned} \tag{2.8}$$

where $f \ge 0$ in $X$, $\lambda > 0$, and $n = 0, 1, 2, \ldots$.

Since the sequence $\|\,(BL_\lambda)^n f\|_1$ is monotonically decreasing, there exists a bounded positive operator $T_\lambda$ on $X$ such that

$$\lim_{n\to\infty} \|T_\lambda f - T_\lambda^{[n]} f\|_1 = 0, \qquad f \in X, \tag{2.9}$$

where the convergence is monotone if $f \ge 0$. This operator satisfies

$$\|T_\lambda f\|_1 = \frac{1}{\lambda}\left\{ \|f\|_1 - \lim_{n\to\infty} \|\,(BL_\lambda)^{n+1} f\|_1 \right\}, \qquad f \ge 0 \text{ in } X, \tag{2.10}$$

where the limit exists.

Using a monotone approximation of $B$ by positive operators $B_m$ which are bounded on $X$, we easily prove

**Lemma 2.1** *$\{T_\lambda : \lambda > 0\}$ is the resolvent of a positive contractive semigroup $\{S(t) : t \geq 0\}$ on $X$.*

We now characterize the generator of the semigroup $S(t)$ when $I - BL_\lambda$ is invertible.

**Theorem 2.1** *Let $G$ denote the generator of the semigroup $S(t)$ whose resolvent is $T_\lambda$ defined by (2.9). Then the following statements are equivalent:*

*1) $\mathcal{D}(G) = \mathcal{D}(A)$ and $G = A + B$.*

*2) $I - BL_\lambda$ is invertible on $X$ for some $\lambda > 0$.*

*3) $I - BL_\lambda$ is invertible on $X$ for all $\operatorname{Re}\lambda > 0$.*

**Proof:** From Lemma 2.1 we have $T_\lambda = (\lambda - G)^{-1}$ for some extension $G$ of $A + B$. After some calculation we get

$$L_\lambda = T_\lambda(I - BL_\lambda), \qquad \lambda > 0, \tag{2.11}$$

which implies that

$$\begin{cases} \ker(I - BL_\lambda) \subseteq \ker L_\lambda = \{0\} \\ \operatorname{Im} L_\lambda \subseteq \operatorname{Im} T_\lambda \ . \end{cases} \tag{2.12}$$

Since $\ker(I - BL_\lambda) = \{0\}$, we see that $\mathcal{D}(G) = \operatorname{Im} T_\lambda$ and $\mathcal{D}(A) = \operatorname{Im} L_\lambda$ coincide if and only if $I - BL_\lambda$ is invertible. ■

In many practical situations, we can easily verify the conditions of Theorem 2.1 by using the following proposition.

**Proposition 2.1** *The equivalent conditions of Theorem 2.1 are satisfied under any of the following hypotheses:*

*a) $BL_\lambda$ is (weakly) compact on $X$.*

*b) $\nu$ is (essentially) bounded.*

*c) The spectral radius of $BL_\lambda$ is strictly less then $1$.*

*If either $a$) or $c$) is satisfied for some $\lambda > 0$, it is satisfied for all $\lambda > 0$.*

**Proof:** Obviously, $c$) implies the invertibility of $I - BL_\lambda$ and hence the conditions of Theorem 2.1. Next, if $a$) is true, then, by the Dunford-Pettis theorem, $(BL_\lambda)^2$ is compact and hence $I - BL_\lambda$ is invertible, which implies the conditions of Theorem 2.1. Next, if $\nu$ is (essentially) bounded, then by assumption $v$), $B$ is bounded on $X$. As a result, $G = A + B$, and the conditions of Theorem 2.1 follow. The final statement of the proposition is immediate from the resolvent identity for $L_\lambda$. ■

# 3 The spectral analysis

In this section we sketch a characterization of the generator $G$ in terms of the spectral properties $BL_\lambda$. Details will be object of a forthcoming paper.

**Theorem 3.1** *The following conditions are equivalent:*

*a)* $G = \overline{A+B}$.

*b)* $\{S(t);\, t > 0\}$ *is a stochastic semigroup, i.e., a positive semigroup of isometries* ($\|S(t)f\|_1 = \|f\|_1,\ f \geq 0$ *in* $X$).

*c)* $\lim_{n\to\infty} \|(BL_\lambda)^n f\|_1 = 0,\ f \in X$, *for all* $\lambda > 0$.

*d)* $1 \in \sigma_c(BL_\lambda) \cup \rho(BL_\lambda)$ *(continuous spectrum plus resolvent set) for all* $\lambda > 0$.

*e)* $\|T_\lambda f\|_1 = \frac{1}{\lambda}\|f\|_1,\ f \geq 0,\ f \in X$, *for all* $\lambda > 0$.

*If any of the conditions c) - e) holds for some* $\lambda > 0$, *it automatically holds for all* $\lambda > 0$.

*Sketch of the proof:* From (2.10) it follows immediatly that c) $\Leftrightarrow$ e) and from the expression of the resolvent it follows that b) $\Leftrightarrow$ e).

To complete the proof it is sufficient to prove that d) $\Rightarrow$ c) $\Rightarrow$ a) $\Rightarrow$ d). This is illustrated by the following diagram:

$$\begin{array}{ccccc} b) & & a) & & \\ \updownarrow & & \uparrow & \searrow & \\ e) & \leftrightarrow & c) & \leftarrow & d) \end{array}$$

More precisely, d) $\Rightarrow$ c) is proved by extending

$$\beta_\lambda(f) = \lim_{n\to\infty} \|(BL_\lambda)^n f\|_1, \quad f \geq 0 \text{ in } X, \tag{3.1}$$

to a bounded (and positive) linear functional on $X$ and writing it in the form $\langle f, \varphi_\lambda\rangle$ for some $\varphi_\lambda \geq 0$ in $L^\infty(\Omega, d\mu)$. A density argument is used to prove that $D(G) \subseteq D(\overline{A+B})$, while a) $\Rightarrow$ d) follows by contradiction. ■

Let us return to the general situation and let us summarize:

$$1 \in \rho(BL_\lambda) \text{ for } \textit{some} \text{ (and hence all) } \lambda > 0 \iff G = A + B$$

$$1 \in \sigma_c(BL_\lambda) \text{ for } \textit{some} \text{ (and hence all) } \lambda > 0 \iff \begin{cases} G = \overline{A+B} \\ A+B \text{ is } \textbf{not} \\ \text{a closed operator} \end{cases}$$

$$1 \in \sigma_r(BL_\lambda) \text{ for } \textit{some} \text{ (and hence all) } \lambda > 0 \iff G \supsetneq \overline{A+B}$$

Let us now see if $T_\lambda$ is the resolvent of the minimal positive evolution semigroup. Let us look for positive bounded operators $\widetilde{T_\lambda}$ on $X$ which satisfy the equation

$$\widetilde{T_\lambda}(I - BL_\lambda) = L_\lambda, \qquad \lambda > 0. \tag{3.2}$$

Then for $f \geq 0$ in $X$ we have

$$\begin{aligned} \widetilde{T_\lambda} f &= L_\lambda f + \widetilde{T_\lambda} BL_\lambda f = L_\lambda f + L_\lambda BL_\lambda f + \widetilde{T_\lambda}(BL_\lambda)^2 f = \dots \\ &= L_\lambda \sum_{j=0}^{n} (BL_\lambda)^j f + \widetilde{T_\lambda}(BL_\lambda)^{n+1} f. \end{aligned}$$

Taking the strong limit in $X$ as $n \to \infty$ we obtain

$$\widetilde{T_\lambda} = T_\lambda + \operatorname*{s-lim}_{n\to\infty} \widetilde{T_\lambda}(BL_\lambda)^n \geq T_\lambda \geq 0 \tag{3.3}$$

(the strong limit in (3.3) must exist and is a positive operator). Consequently, $T_\lambda$ is the *minimal* positive solution of (3.2).

As a result, $\{S(t)f\}_{t>0}$ is the *unique* positive and contractive solution of

$$\begin{cases} u'(t) \simeq (A+B)u(t), \quad t > 0 & \text{(we need to write } u'(t) = (A+B)u(t)) \\ u(0) = f & \text{(with } f \geq 0 \text{ in } X) \end{cases}$$

if the minimal solution is also the maximal solution. This occurs if (and only if) $G = \overline{A+B}$

# 4 Applications

In this section we discuss two applications.

**a. The runaway problem revisited**. Let $a > 0$ and let $\nu \in L^{1,loc}(\mathbb{R}; dv)$ be nonnegative. Introduce the Banach spaces $X = L^1(\mathbb{R}; dv)$ and $X_\nu = L^1(\mathbb{R}; \nu(v)dv)$ with their usual norms.[1] Define

$$T_F = -a\frac{\partial}{\partial v}, \qquad A = T_F + T_A = -a\frac{\partial}{\partial v} - \nu(v)$$

on suitable domains contained in $X = L^1(\mathbb{R}; dv)$, and let

$$(Bf)(v) = \int_{-\infty}^{\infty} k(v, v')\nu(v')f(v')\,dv', \tag{4.4}$$

[1] In fact, $X_\nu$ is a Banach space if $\nu$ is positive a.e.

where $k$ is positive and $\int_{-\infty}^{\infty} k(v,v')\,dv = 1\ \forall v'$. Defining $L_\lambda$ as in (2.4) by

$$(L_\lambda f)(v) = \frac{1}{a}\int_{-\infty}^{v} \exp\left\{-\frac{\lambda}{a}(v-v') - \frac{1}{a}\int_{v'}^{v} \nu(\hat{v})d\hat{v}\right\} f(v')\,dv', \tag{4.5}$$

we have for $f \geq 0$ in $X$:

$$\begin{aligned}\int_{-\infty}^{\infty} & [\lambda+\nu(v)](L_\lambda f)(v)\,dv \\ &= \begin{cases} \displaystyle\int_{-\infty}^{\infty} f(v')\,dv' & \text{if } \mathrm{Re}\,\lambda > 0 \\ \displaystyle\int_{-\infty}^{\infty}\left[1-\exp\left\{-\frac{1}{a}\int_{v'}^{\infty} r(\hat{v})d\hat{v}\right\}\right] f(v')dv' & \text{if } \lambda = 0, \end{cases}\end{aligned}$$

and

$$\begin{aligned}\int_{-\infty}^{\infty} \nu(v)(L_0 f)(v)\,dv &= \int_{-\infty}^{\infty} f(v')\,dv' \quad \text{if } \int_{-\infty}^{\infty}\nu(v)dv = \infty \\ \int_{-\infty}^{\infty} \nu(v)(L_0 f)(v)\,dv &\leq \left[1-\exp\left\{-\frac{\|\nu\|_1}{a}\right\}\right]\|f\|_1 \text{ if } \nu \in L^1(\mathbb{R})\,.\end{aligned}$$

Further

$$(BL_\lambda f)(v) = \int_{-\infty}^{\infty} l_a(v,v'';\lambda)\, f(v'')\,dv''$$

where

$$l_a(v,v'';\lambda) = \frac{1}{a}\int_{v''}^{\infty} k(v,v')\nu(v')\exp\left\{-\frac{\lambda}{a}(v'-v'') - \frac{1}{a}\int_{v''}^{v'}\nu(\hat{v})d\hat{v}\right\} dv'.$$

and, as a result,

$$\|BL_\lambda\| \leq \begin{cases} 1 \text{ if } \nu \notin L^1(\mathbb{R}) \\ 1-\ \exp\left\{-\frac{\|\nu\|_1}{a}\right\} \text{ if } \nu \in L^1(\mathbb{R})\,. \end{cases}$$

Therefore, the spectral radius of $BL_\lambda$ is less than 1 whenever $\nu \in L^1(\mathbb{R})$.

Thus, according to Proposition 2.1, we have $G = A + B$ if at least one of the three conditions (1) $BL_\lambda$ is weakly compact on $X$ for some (and hence) all $\lambda > 0$, (2) $\nu$ is bounded, and (3) $\nu \in L^1(\mathbb{R})$, is satisfied. We do not have any example in which $G \supsetneq \overline{A+B}$ and hence the semigroup $\{S(t) : t \geq 0\}$ is not stochastic.

**b. The fragmentation model revisited**. Given $\alpha \in \mathbb{R}$, we introduce the collision frequency $\nu(x) = x^\alpha$ for $x > 0$, the Banach space $X = L^1(\mathbb{R}^+; x\,dx)$, and the collision operator $B$ defined by

$$(Bf)(x) = 2 \int_x^\infty y^{\alpha-1} f(y)\,dy.$$

Then $(L_\lambda f)(x) = f(x)/(\lambda + x^\alpha)$ for $\lambda > 0$ and hence the operator $BL_\lambda$ given by

$$(BL_\lambda f)(x) = 2 \int_x^\infty \frac{y^{\alpha-1}}{\lambda + y^\alpha} f(y)\,dy$$

is bounded on $X$. For $\alpha \geq 0$ one easily proves that $G = \overline{A+B}$ and hence that the evolution semigroup is stochastic. For $\alpha < 0$ this semigroup is not stochastic, because the adjoint operator $(BL_\lambda)^*$ defined on $X^* = L^\infty(\mathbb{R}^+; x\,dx)$ has a (simple) eigenvector at the eigenvalue 1.

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