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Some Properties of the Eigenvalues of a Schrödinger Equation with Energy-dependent Potential

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ABSTRACT. In this article, we relate the imaginary parts of normal eigenvalues in the upper and lower half-planes for a quadratic operator pencil with (unbounded) selfadjoint coefficients and apply the result obtained to a generalized 1-D Schrödinger equation.

1. Introduction

The direct and inverse scattering problems for the generalized 1-D Schrödinger equation

$$\psi''(k,x) + (k^2 + m^2)\psi(k,x) = [ikP(x) + Q(x)]\psi(k,x), \qquad x \in \mathbb{R}, \tag{1.1}$$

where $m \geq 0$ is a constant, $P(x)$ and $Q(x)$ are real potentials in $L^1(\mathbb{R};(1+|x|)dx)$ and $P(x) \leq 0$, have been studied extensively, both in the case $m = 0$ [**4, 1, 2**] and in the case $m > 0$ [**5, 10, 6**]. Its bound states, which are most easily studied as the normal eigenvalues of the Hamiltonian operator pencil

$$L(k) = k^2 I + ikB - A \tag{1.2}$$

in the k-complex plane, where A is selfadjoint and bounded below and B is nonnegative selfadjoint, have been studied at various occasions, both within an abstract framework [**9**] and for a specific class of generalized Schrödinger equations [**7, 8, 1**]. One of the most striking results is that the eigenvalue spectrum in the upper half k-plane is purely imaginary and (algebraically and geometrically) simple. In this article we compare the eigenvalue spectra of such a pencil in the upper half k-plane $\mathbb{C}^+$ and the lower half k-plane $\mathbb{C}^-$ and relate the imaginary parts of corresponding pairs of one eigenvalue in $\mathbb{C}^+$ and one eigenvalue in $\mathbb{C}^-$. As we will see in Section

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3, the results involving the generalized Schrödinger equation (1.1) then arise as an application in which the operator pencil is given by the 1-D Schrödinger operators

$$L(k) = \frac{d^2}{dx^2} + (k^2 + m^2) - [ikP + Q] = k^2 - ikP - \left(-\frac{d^2}{dx^2} + Q - m^2\right), \tag{1.3}$$

where $m \geq 0$ is a constant, $P(x)$ and $Q(x)$ are real potentials in $L^1(\mathbb{R}; (1+|x|)dx) \cap L^2(\mathbb{R})$, $P(x)$ does not vanish on a set of positive measure, and $P(x) \leq 0$. In terms of (1.2) we here have $B = -P$ and $A = -(d^2/dx^2) + Q - m^2$.

To prove the main result and the auxiliary lemmas, we will draw on a general theory of Hamiltonian operator pencils [**9**] whenever possible. This will be done in Section 2 under the following assumptions on the coefficients of the abstract operator pencil (1.2):

(i) B is nonnegative selfadjoint,
(ii) A is selfadjoint and bounded below with essential spectrum contained in $[-m^2, \infty)$ for some $m \geq 0$,
(iii) the domains of A and B satisfy $\mathcal{D}(A) \subset \mathcal{D}(B)$, and
(iv) $B(\lambda - A)^{-1}$ is a compact operator for all λ in the (connected) resolvent set of A.

In the final Section 3 we will apply the main result, Theorem 2.8, to the generalized Schrödinger equation (1.1).

2. Main result involving the Operator Pencil

By definition, for all $k \in \mathbb{C}$, as the domain of $L(k)$ we take the domain $\mathcal{D}(A)$ of the coefficient A.

DEFINITION 2.1. The set of values $k \in \mathbb{C}$ such that $L(k)^{-1}$ exists as a closed bounded linear operator on $L^2(\mathbb{R})$ is called the *resolvent set* $\rho(L)$ of the pencil $L(k)$. We denote by $\sigma(L)$ the *spectrum* of $L(k)$, i.e., the set $\sigma(L) = \mathbb{C}\backslash\rho(L)$. A number $k_0 \in \mathbb{C}$ is said to be an *eigenvalue* of $L(k)$ if there exists a nonzero vector y_0 (called an eigenvector) such that $L(k_0)y_0 = 0$. The vectors $y_1, y_2, \cdots, y_{r-1}$ are called corresponding *associated eigenvectors* if

$$\sum_{s=0}^{n} \frac{1}{s!}\frac{\partial^s}{\partial k^s} L(k)\,|_{k=k_0}\, y_{n-s} = 0, \qquad n = 1, \cdots, r-1. \tag{2.1}$$

The number r is called the length of the chain composed of the eigenvector and its associated eigenvectors. The *geometric multiplicity* of an eigenvalue is defined to be the maximal number of corresponding linearly independent eigenvectors. Its *algebraic multiplicity* is defined as the maximal value of the sum of the lengths of chains corresponding to linearly independent eigenvectors. An eigenvalue is said to be *isolated* if it has a deleted neighborhood contained in the resolvent set. An isolated eigenvalue k_0 of finite algebraic multiplicity is said to be *normal* if the image $\mathrm{Im}L(k_0)$ is closed. We denote by $\sigma_0(L)$ the set of normal eigenvalues of $L(k)$. The set $\sigma_{ess}(L) = \sigma(L)\backslash\sigma_0(L)$ is called the *essential spectrum* of $L(k)$.

LEMMA 2.2. *The essential spectrum* $\sigma_{ess}(L(k))$ *is a subset of* $\mathbb{R} \cup [-im, im]$.

PROOF. Put $W(k) = L(k)(k^2 I - A)^{-1}$. Then $W(k)$ is a meromorphic operator function on the set $\Omega = \{k \in \mathbb{C} : k^2 \notin \sigma_{ess}(A)\}$, where the principal parts of the Laurent series of $W(k)$ around each pole in Ω is an operator of finite rank. The lemma now follows from $\sigma_{ess}(A) \subset [-m^2, \infty)$. □

LEMMA 2.3. *The part of the spectrum of $L(k)$ in $\mathbb{C}^+$ is a subset of the imaginary axis.*

PROOF. In view of Lemma 2.2, it suffices to prove this lemma for normal eigenvalues only. For this case the proof of Lemma 2.3 of [**9**] can be repeated in full. □

THEOREM 2.4. *The algebraic and geometric multiplicities of each normal eigenvalue of $L(k)$ in $\mathbb{C}^+$ coincide and, when taking multiplicities into account, the number of such eigenvalues coincides with the number of eigenvalues of $A + mB$ in $(-\infty, -m^2)$.*

PROOF. The first statement on the normal eigenvalues of $L(k)$ in $\mathbb{C}^+$ follows from Lemma 2.4 of [**9**]. Next, substituting $k = \lambda + im$ into (1.2) we obtain

$$L_1(\lambda) = L(\lambda + im) = \lambda^2 I + i\lambda(B + 2mI) - (A + mB + m^2 I). \tag{2.2}$$

Since the coefficient $B + 2mI$ is nonnegative selfadjoint, one can apply Theorem 2.1 of [**9**] to prove the second statement. □

Let us introduce the following auxiliary operator pencil:

$$L_2(k, \eta) = k^2 I + ik\eta B - A, \tag{2.3}$$

where $\eta \in \mathbb{C}$ is a parameter. It is clear that $L_2(k, 1) = L(k)$ and $L_2(k, 0) = k^2 I - A$.

LEMMA 2.5. *Let B be bounded with $Ker B = \{0\}$, and let $\eta_0 \in \mathbb{R}$. Suppose $k = i\tau_0$ is a nonzero purely imaginary normal eigenvalue of the pencil $L_2(k, \eta_0)$. Consider $L_2(i\tau_0, \eta) = -\tau_0^2 I - \tau_0\eta B - A$ as a linear pencil with spectral parameter η. Then η_0 is a normal eigenvalue of $L_2(i\tau_0, \eta)$.*

PROOF. First of all, $i\tau_0 \notin \Omega = \{k \in \mathbb{C} : k^2 \in \sigma_{ess}(A)\}$, i.e., $i\tau_0$ does not belong to the essential spectrum of the pencil $L_2(k, \eta)$ for any fixed $\eta \in \mathbb{C}$. Now choose τ_1 such that $-\tau_1^2 \notin \sigma(A)$ and $|\tau_1^2 - \tau_0^2| < 1$; this is possible, because $-\tau_0^2$ is either in the resolvent set of A or an isolated eigenvalue of A. Put

$$Z(\eta) = L_2(i\tau_0, \eta)(-\tau_1^2 I - A)^{-1} = \left[1 + \tau_1^2 - \tau_0^2\right] I - \eta\tau_0 B(-\tau_1^2 I - A)^{-1}.$$

Then $Z(\eta)$ is a Fredholm operator of index zero that depends analytically on the parameter $\eta \in \mathbb{C}$. Moreover, the eigenvalues η of $Z(\eta)$ coincide with the eigenvalues η of $L_2(i\tau_0, \eta)$. Hence [**3**], these eigenvalues are normal. Finally, $1/\eta_0$ is also an eigenvalue of the selfadjoint operator $[1 + \tau_1^2 - \tau_0^2]^{-1}\tau_0 B^{1/2}(-\tau_1^2 I - A)^{-1}B^{1/2}$ and therefore $\eta_0 \in \mathbb{R}$. □

We need two more auxiliary results.

LEMMA 2.6. *Let B be bounded with $Ker B = \{0\}$, and let η_0 be a normal eigenvalue of multiplicity d for the linear pencil $\eta\tau_0 B - \tau_0^2 I - A$, where $\tau_0 \in \mathbb{R}$. Then this eigenvalue may be considered as d eigenvalues of the k-parameter dependent linear operator pencil $L_2(k, \eta))$ which are coinciding at $k = i\tau_0$ and are analytic functions of k:*

$$\eta^{(r)}(k) = \eta_0 + \sum_{s=1}^{\infty} b_s^{(r)} (k - i\tau_0)^s, \qquad r = 1, \ldots, d, \tag{2.4}$$

where $b_1^{(r)} \in \mathbb{R}\backslash\{0\}$ for all $r \in \{1, \ldots, d\}$.

PROOF. This follows from a general result of [**3**] if we take into account that there are no associated eigenvectors corresponding to η_0. □

LEMMA 2.7. *Let B be bounded with $Ker B = \{0\}$. For $\tau_0 > m$, let $-i\tau_0$ be a normal eigenvalue of geometric multiplicity z for the operator polynomial $L_2(k, \eta_0)$ ($\eta_0 \in [0,1]$). Then in some neighborhood of $(-i\tau_0, \eta_0)$ the eigenvalues are given by the formula*

$$k_J^{(r)}(\eta) = -i\tau_0 + \sum_{s=1}^{\infty} \beta_s^{(r)} \left((\eta - \eta_0)_J^{\frac{1}{t_r}} \right)^s, \qquad r = 1, \cdots, z, \tag{2.5}$$

where each $\beta_1^{(r)} \neq 0$ is real or purely imaginary and $(\eta - \eta_0)_J^{\frac{1}{t_r}}$ $(J = 1, \cdots, t_r)$ stands for the branches of the r-th root.

PROOF. We obtain (2.5) by inversion of (2.4). □

THEOREM 2.8. *Let B be bounded with $Ker B = \{0\}$ and let $k_1, \ldots, k_\ell$ be the normal eigenvalues of $L(k)$ in $(im, i\infty)$. Then one can find corresponding eigenvalues $k_{-1}, \ldots, k_{-\ell}$ in $(-i\infty, -im)$ such that*

$$Im\ (k_j + k_{-j}) \leq 0. \tag{2.6}$$

PROOF. The eigenvalues of $L_2(k, \eta)$ are piecewise analytic functions of η. They may lose analyticity only when they coincide. This follows from the result of [**9**] mentioned above. The eigenvalues located on $(im, i\infty)$ are analytic functions of $\eta > 0$ (see [**9**]) and move downwards along the imaginary axis as η increases. For $\eta \geq 0$ sufficiently small, we identify $k_{-j}(\eta)$ as the eigenvalue satisfying the conditions $k_{-j}(0) = -k_j(0)$, where $\mathrm{Im}\, k_j(0) > m$ and $\mathrm{Re}\, k_j(0) = 0$. Here it is possible that $k_j(0) = k_{j'}(0)$. For sufficiently small $\eta > 0$ we have $\mathrm{Im}\, k_{-j}(\eta) < -m$ and $\mathrm{Re}\, k_{-j}(\eta) = 0$, due to the symmetry of the problem, Lemma 2.7 and the fact that the normal eigenvalues of the pencil in $\mathbb{C}^-$ do not have associated eigenvectors when $\eta = 0$. Recalling $ik_{-j}(\eta)$ to be real for sufficiently small positive η, it is easy to derive (see [**9**]) the following formula for the derivative:

$$k'_{-j}(\eta) = \frac{ik_{-j}(\eta)\,(-By_{-j}(\eta), y_{-j}(\eta))}{2k_{-j}(\eta)\,||y_{-j}(\eta)||^2 + i\eta\,(By_{-j}(\eta), y_{-j}(\eta))}. \tag{2.7}$$

This formula implies $\mathrm{Im}\, k'_{-j}(\eta) \leq 0$ and $\mathrm{Re}\, k'_{-j}(\eta) = 0$ for $\eta \geq 0$ small enough. Hence, our theorem is true for $\eta \geq 0$ small enough.

As $\eta > 0$ increases, $k'_{-j}(\eta)$ can change its sign only if the denominator in the right-hand side of (2.7) vanishes, i.e., if eigenvalues coalesce. If such a coalescence takes place on the interval $(-i\infty, -im)$, then the eigenvalues involved behave according to the formula (2.5). Such a coalescence (which involves a purely imaginary eigenvalue moving downwards) on the interval $(-i\infty, -im)$ is of one of the following three types. For the first type, t_r in (2.5) is odd. In this case we identify the eigenvalue moving downwards along the imaginary axis after the coalescence as the one which moved downwards along the imaginary axis before the coalescence took place. By a coalescence of the second type we mean one which has t_r even and β_1 purely imaginary ($\beta_1 \neq 0$) in (2.5). After such a coalescence two new purely imaginary eigenvalues appear which are moving in opposite directions along the imaginary axis; such a coalescence cannot violate Theorem 2.8. The third type of coalescence has t_r even and $\beta_1 \neq 0$ real in (2.5). Let $k_{-j}(\eta)$ be part of such a coalescence at $\eta = \eta_0 \in (0,1]$. Then a coalescence of the second type indeed occurred at some $\eta \in (0, \eta_0)$ in some point $k_\times \in (-i\infty, k_{-j}(\eta_0))$ on the imaginary axis. In this case the eigenvalue that has arisen after this coalescence and is moving downwards, is identified as $k_{-j}(\eta)$. □

3. Application to a Generalized Schrödinger Equation

In this section we apply the results of Section 2 to the generalized Schrödinger operator given in (1.3), where $B = -P$ and $A = -(d^2/dx^2) + Q - m^2$.

Let $\hat{\psi}(k) = (F\psi)(k) = \int_{-\infty}^{\infty} dx\, e^{ikx}\psi(x)$ stand for the Fourier transform of ψ, and let $A_0 = -(d^2/dx^2)$ have as its domain

$$\mathcal{D}(A_0) = \{\psi \in L^2(\mathbb{R}) : (1 + k^2)\hat{\psi} \in L^2(\mathbb{R})\}.$$

The following result is likely to be known. For convenience, we include a proof.

PROPOSITION 3.1. *Let $T \in L^2(\mathbb{R})$. Then the operator $T(\xi^2 I + A_0)^{-1}$ extends to a compact operator on $L^2(\mathbb{R})$ for every $\xi > 0$.*

PROOF. We easily compute

$$(FT(\xi^2 I + A_0)^{-1}\psi)(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\hat{k}\left[\int_{-\infty}^{\infty} dx\, e^{i(k-\hat{k})x}T(x)\right]\frac{\hat{\psi}(\hat{k})}{\xi^2 + \hat{k}^2}.$$

Thus, $FT(\xi^2 I + A_0)^{-1}F^{-1}$ is an integral operator on $L^2(\mathbb{R})$ of Hilbert-Schmidt type for every $\xi > 0$, which implies the compactness of $T(\xi^2 I + A_0)^{-1}$. □

Choosing $\xi_0 > 0$ such that $(-\infty, -\xi_0^2]$ is contained in the resolvent set of $A_1 = A_0 + m^2 I$, we easily find the compactness of the difference

$$(\xi^2 I + A_1)^{-1} - (\xi^2 I + A_0)^{-1}, \qquad \xi \geq \xi_0,$$

if $Q \in L^2(\mathbb{R})$. Indeed, this is clear from Proposition 3.1 and the resolvent identity

$$(\xi^2 I + A_1)^{-1} - (\xi^2 I + A_0)^{-1} = -(\xi^2 I + A_1)^{-1}Q(\xi^2 I + A_0)^{-1}.$$

It is now clear that the theory of Section 2 can be applied to (1.3) if $P(x)$ and $Q(x)$ satisfy the following hypotheses:

1. $P(x)$ is bounded and nonpositive, does not vanish on a set of positive measure, and belongs to $L^1(\mathbb{R}; (1 + |x|)dx) \cap L^2(\mathbb{R})$.
2. $Q(x)$ is real and belongs to $L^1(\mathbb{R}; (1 + |x|)dx) \cap L^2(\mathbb{R})$.

The assumption on the set of zeros of $P(x)$ is necessary to ensure that $B = -P$ has the property $\operatorname{Ker} B = \{0\}$ required by Lemma 2.5 and subsequent results. The compactness of $B(\lambda I - A)^{-1}$ then follows from Proposition 3.1 and the paragraph following its proof.

Theorem 2.8 allows one to relate the eigenvalues of the generalized Schrödinger operator pencil $L(k)$ in $\mathbb{C}^+$ given by (1.3) to the eigenvalues of the generalized Schrödinger operator pencil

$$\tilde{L}(k) = \frac{d^2}{dx^2} + (k^2 + m^2) - [-ikP + Q] = k^2 + ikP - \left(-\frac{d^2}{dx^2} + Q - m^2\right), \tag{3.1}$$

obtained from $L(k)$ by replacing $P(x)$ with $-P(x)$. Then k is an eigenvalue of $\tilde{L}(k)$ whenever $-k$ is an eigenvalue of $L(k)$. Further, if $P(x) \leq 0$ and the (algebraically and geometrically simple) eigenvalues in $\mathbb{C}^+$ are numbered according to increasing imaginary part (i.e., $0 < -ik_1 < -ik_2 < \ldots < -ik_\ell$ with finite ℓ) and the eigenvalues $\tilde{k}_j$ of $\tilde{L}(k)$ in $\mathbb{C}^+$ are numbered in the same way (i.e., $0 < \operatorname{Im}\tilde{k}_1 \leq \operatorname{Im}\tilde{k}_2 \leq \ldots \leq \operatorname{Im}\tilde{k}_{\tilde{\ell}}$, where $\tilde{\ell} \geq \ell$), then there exist integers $n_1 < n_2 < \ldots < n_\ell$ such that

$$\operatorname{Re}\tilde{k}_{n_j} = 0, \quad \operatorname{Im}\tilde{k}_{n_j} \geq \operatorname{Im}k_j, \qquad j = 1, \ldots, \ell.$$

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