STEINER PROBLEM – See **Steiner tree problem**.

MSC1991: 05C35, 51M16

STEINNESS – The property of a manifold or domain to be Stein (cf. **Stein manifold**; **Stein space**).

MSC1991: 32E10

STEP HYPERBOLIC CROSS – A summation domain of multiple **Fourier series**. Like a **hyperbolic cross**, it is used for good approximation in the space of functions with bounded mixed derivative (in L_p).

Let $f(x)$ be an integrable periodic function of n variables defined on $\mathbf{T}^n$. It has a Fourier series expansion $\sum_{\mathbf{k}} c_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}$, $\mathbf{k} = (k_1, \ldots, k_n)$, $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{k}\cdot\mathbf{x} = k_1x_1 + \cdots + k_nx_n$. Unlike in the one-dimensional case, there is no natural ordering of the Fourier coefficients, so the choice of the order of summation is of great importance.

Let $\mathbf{r} = (r_1, \ldots, r_n) \in \mathbf{R}^n$ with all coordinates positive, $r_j > 0$. Let

$$\Delta_{\mathbf{m}}(f) = \sum_{\substack{2^{m_j-1}<|k_j|\leq 2^{m_j}\\ j=1,\ldots,n}} c_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}$$

be a dyadic 'block' of the Fourier series. The *step hyperbolic partial sums*

$$\sum_{|\mathbf{m}\cdot\mathbf{r}|\leq N} \Delta_{\mathbf{m}}(f)$$

where introduced by B. Mityagin [2] for problems in **approximation theory**. They have approximately the same number of harmonics as a hyperbolic cross, but structurally they fit the Marcinkiewicz multiplier theorem (cf. also **Interpolation of operators**). It implies that the operator of taking step hyperbolic partial Fourier sums is bounded in each L^p, $1 < p < \infty$. This means that step hyperbolic partial sums give the best approximation among all hyperbolic cross trigonometric polynomials in L_p, $1 < p < \infty$. In the limit cases $p = 1$ and $p = \infty$, the **Lebesgue constants** of step hyperbolic partial sums have only logarithmic growth, while for hyperbolic partial Fourier sums they grow as a power of N.

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MSC1991: 42B05, 42B08

STOKES PARAMETERS – To characterize the radiance (intensity) or flux and state of polarization of a beam of electromagnetic radiation (cf. also **Electromagnetism**) one can use four real parameters which have the same physical dimension. These so-called Stokes parameters were first introduced by G.C. Stokes [7] in 1852. It took about a hundred years before Stokes parameters were used on a large scale in optics and theories of light scattering by molecules and small particles. (See, e.g., [1], [2], [3], [4], [5], [6], [8].)

To define the Stokes parameters, I, Q, U, and V, one first considers a monochromatic beam of electromagnetic radiation. One defines two orthogonal unit vectors $\mathbf{l}$ and $\mathbf{r}$ such that the direction of propagation of the beam is the direction of the vector product $\mathbf{r}\times\mathbf{l}$. The components of the electric field vectors at a point, O, in the beam can be written as

$$\xi_l = \xi_l^0 \sin(\omega t - \varepsilon_l), \qquad \xi_r = \xi_r^0 \sin(\omega t - \varepsilon_r), \tag{1}$$

where ω is the circular frequency, t is time, and ξ_l^0 and ξ_r^0 are (non-negative) amplitudes. One now defines the Stokes parameters by

$$I = [\xi_l^0]^2 + [\xi_r^0]^2, \tag{2}$$

$$Q = [\xi_l^0]^2 - [\xi_r^0]^2, \tag{3}$$

$$U = 2\xi_l^0\xi_r^0 \cos(\varepsilon_l - \varepsilon_r), \tag{4}$$

$$V = 2\xi_l^0\xi_r^0 \sin(\varepsilon_l - \varepsilon_r). \tag{5}$$

The end point of the electric vector at a point, O, in the beam describes an ellipse, the so-called *polarization ellipse*, whose ellipticity and orientation with respect to $\mathbf{l}$ and $\mathbf{r}$ follow from Q, U and V. If $V > 0$, the electric vector at O moves clockwise, as viewed by an observer looking in the direction of propagation. Clearly, the following relation holds:

$$I = (Q^2 + U^2 + V^2)^{1/2}, \tag{6}$$

where $V = 0$ for linearly polarized radiation and $Q = U = 0$ for circularly polarized radiation.

In general, electromagnetic waves are not exactly monochromatic, but the amplitudes ξ_l^0 and ξ_r, as well as the phase differences $\varepsilon_l - \varepsilon_r$, may vary slowly in time. In this case the Stokes parameters are defined as before, with one exception, namely time averages must be taken on the right-hand sides of (2)–(5). The polarization may now be partial and the beam can be decomposed in a completely unpolarized and a completely polarized beam. The orientation and shape of the polarization ellipse of the latter beam is again given by Q, U and V. The identity of (6) is now replaced by the inequality

$$I \geq (Q^2 + U^2 + V^2)^{1/2} \tag{7}$$

and the ratio $p = [Q^2 + U^2 + V^2]^{1/2}/I$ is called the *degree of polarization*. For completely polarized radiation $p = 1$, for partially polarized radiation $0 < p < 1$, and for unpolarized (natural) radiation $p = 0$.

The Stokes parameters can be combined into a column vector with elements I, Q, U, and V, called a *Stokes vector*. Stokes vectors of constituent beams are added to obtain the Stokes vector of a composite beam if no interference effects occur. Optical devices and processes like scattering and absorption can be described by real 4×4 (Mueller) matrices that transform the Stokes vectors of primary beams into those of secondary beams.

The Stokes parameters as defined above are one of many possible representations of polarized radiation, several of which are only slight modifications of each other (see, e.g., [1]). Stokes parameters are also used in quantum mechanics in connection with polarization of elementary particles.

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MSC 1991: 78A40

STRONG STIELTJES MOMENT PROBLEM – The strong Stieltjes moment problem for a given sequence $\{c_n\}_{n=-\infty}^{\infty}$ of real numbers is concerned with finding real-valued, bounded, monotone non-decreasing functions $\psi(t)$ with infinitely many points of increase for $0 \leq t < \infty$ such that

$$c_n = \int_0^\infty t^n \, d\psi(t), \quad n = 0, \pm 1, \pm 2, \ldots. \tag{1}$$

This problem, which generalizes the *classical Stieltjes moment problem* (where the given sequence is $\{c_n\}_{n=0}^{\infty}$; cf. also **Kreĭn condition**), was first studied by W.B. Jones, W.J. Thron and H. Waadeland [3].

Let $\Lambda_{p,q}$ be the complex linear space spanned by the set of functions $\{z^j\}_{j=p}^{q}$ with $p \leq q$, and define $\Lambda_{2m} = \Lambda_{-m,m}$ and $\Lambda_{2m+1} = \Lambda_{-(m+1),m}$ for $m = 0, 1, \ldots$, and $\Lambda = \cup_{n=0}^{\infty} \Lambda_n$. An element of Λ is called a *Laurent polynomial*. For a given sequence $\{c_n\}_{n=-\infty}^{\infty}$, a necessary and sufficient condition for the strong Stieltjes moment problem to be solvable is that the **linear operator** M defined on the base elements z^n of Λ by

$$M[z^n] = c_n, \quad n = 0, \pm 1, \pm 2, \ldots, \tag{2}$$

is *positive* on $(0, \infty)$, i.e. for any $L \in \Lambda$ such that $L(z) \geq 0$ for $z \in (0, \infty)$ and $L(z) \not\equiv 0$, then $M[L] > 0$. An equivalent condition is that if

$$H_0^{(m)} = 1, \quad H_k^{(m)} = \det(c_{m+i+j})_{i,j=0}^{k-1} \tag{3}$$

for $m = 0, \pm 1, \pm 2, \ldots$, $k = 1, 2, \ldots$, are the Hankel determinants associated with $\{c_k\}$ (cf. also **Hankel matrix**), then

$$H_k^{(m)} > 0, \quad m = 0, \pm 1, \pm 2, \ldots, \quad k = 1, 2, \ldots. \tag{4}$$

Orthogonal Laurent polynomials $\{Q_n(z) \in \Lambda_n : n = 0, 1, \ldots\}$ may be defined with respect to the **inner product** $\langle P, Q \rangle \equiv M[P(z)Q(z)]$ and are given by:

$$Q_{2n}(z) = \frac{1}{H_{2n}^{(-2n)}} \begin{vmatrix} c_{-2n} & \cdots & c_{-1} & z^{-n} \\ \vdots & & \vdots & \vdots \\ c_{-1} & \cdots & c_{2n-2} & z^{n-1} \\ c_0 & \cdots & c_{2n-1} & z^n \end{vmatrix}, \tag{5}$$

$$n = 1, 2, \ldots,$$

and

$$Q_{2n+1}(z) = \frac{-1}{H_{2n+1}^{(-2n)}} \begin{vmatrix} c_{-2n-1} & \cdots & c_{-1} & z^{-n-1} \\ \vdots & & \vdots & \vdots \\ c_{-1} & \cdots & c_{2n-1} & z^{n-1} \\ c_0 & \cdots & c_{2n} & z^n \end{vmatrix}, \tag{6}$$

$$n = 0, 1, \ldots,$$

and $Q_0(z) = 1$. Corresponding *associated orthogonal Laurent polynomials* $\{P_n\}$ are defined by

$$P_n = M\left[\frac{Q_n(t) - Q_n(z)}{t - z}\right], \quad n = 0, 1, \ldots. \tag{7}$$

The rational functions $(-z)P_n(-z)/Q_n(-z)$ are the convergents of the positive *T-fraction* [5],

$$\frac{F_1 z}{1 + G_1 z} + \frac{F_2 z}{1 + G_2 z} + \frac{F_3}{1 + G_3 z} + \cdots \tag{8}$$

$$(F_n > 0, \; G_n > 0),$$

where

$$F_n = \frac{H_n^{(-n)} H_{n-2}^{(-n+3)}}{H_{n-1}^{(-n+2)} H_{n-1}^{(-n+1)}},$$

$$G_n = \frac{H_n^{(-n)} H_{n-1}^{(-n+2)}}{H_n^{(-n+1)} H_{n-1}^{(-n+1)}},$$