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Semi-infinite multi-index perturbed block Toeplitz systems ☆

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Abstract

In this article Banach algebra techniques are employed to study the numerical solution of linear systems with a semi-infinite multi-index suitably perturbed $k \times k$ block Toeplitz matrix. Decay properties of their solutions are studied by using suitably weighted Wiener algebras. A projection-type method for their numerical solution is introduced. Numerical results are presented illustrating both the accuracy of the method for certain perturbed Toeplitz systems and the need for generalizing the theoretical framework to other such systems.

Keywords: Infinite linear system; Multi-index Toeplitz system

1. Introduction

Let $\mathbb{Z}^d$ be the set of points in d-dimensional space with integer coordinates, and let $\preceq$ be a linear order on $\mathbb{Z}^d$ compatible with addition, i.e., satisfying $i + l \preceq j + l$ whenever $i, j, l \in \mathbb{Z}^d$ and $i \preceq j$. Then by a bi-infinite matrix we mean a matrix $A = (A_{ij})_{i,j\in\mathbb{Z}^d}$ indexed by $i, j \in \mathbb{Z}^d$ and by a semi-infinite matrix we mean a matrix $A = (A_{ij})_{i,j\in\mathbb{Z}_+^d}$ indexed by i, j in the set $\mathbb{Z}_+^d = \{i \in \mathbb{Z}^d\colon i \succeq 0\}$ of $\preceq$-nonnegative elements of $\mathbb{Z}^d$. As entries A_{ij} we allow for real or complex scalars or $k \times k$ matrices.

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A bi- or semi-infinite matrix A is called *block Toeplitz* if its entries are real or complex $k \times k$ matrices A_{ij} satisfying $A_{ij} = A_{i-j}$, and *block Hankel* if its entries are real or complex $k \times k$ matrices A_{ij} satisfying $A_{ij} = A_{i+j}$. When $k = 1$, a block Toeplitz or a block Hankel is simply called a Toeplitz or a Hankel matrix, respectively.

Given admissible weights $\boldsymbol{\beta} = (\beta_i)_{i \in \mathbb{Z}^d}$ (i.e., weights β_i satisfying $1 \leqslant \beta_{i+j} \leqslant \beta_i \beta_j$ for $i, j \in \mathbb{Z}^d$), we study semi-infinite block Toeplitz linear systems of the form

$$\sum_{j \in \mathbb{Z}_+^d} T_{i-j} x_j = b_i, \quad i \in \mathbb{Z}_+^d, \tag{1.1}$$

where $\|(T_i)_{i \in \mathbb{Z}_+^d}\|_{\ell^1_{k,\boldsymbol{\beta}}(\mathbb{Z}_+^d)} < \infty$, $\|(b_i)_{i \in \mathbb{Z}_+^d}\|_{\ell^p_{k,\boldsymbol{\beta}}(\mathbb{Z}_+^d)} < \infty$ for some $1 \leqslant p < \infty$, and the solution is to satisfy $\|(x_i)_{i \in \mathbb{Z}_+^d}\|_{\ell^p_{k,\boldsymbol{\beta}}(\mathbb{Z}_+^d)} < \infty$. Here

$$\|(c_i)_{i \in M}\|_{\ell^p_{k,\boldsymbol{\beta}}(M)} = \begin{cases} \left[\sum_{i \in M} (\beta_i \|c_i\|)^p\right]^{1/p}, & 1 \leqslant p < \infty, \\ \sup_{i \in M} \beta_i \|c_i\|, & p = \infty, \end{cases} \tag{1.2}$$

M being a nonempty subset of $\mathbb{Z}^d$ and $\|\cdot\|$ a fixed norm in $\mathbb{C}^k$. We also study semi-infinite linear systems of the type

$$\sum_{j \in \mathbb{Z}_+^d} A_{ij} x_j = b_i, \quad i \in \mathbb{Z}_+^d, \tag{1.3}$$

that are "perturbations" of the above block Toeplitz systems in a sense that will be specified later.

In the one-index case ($d = 1$) the system (1.1) has been studied by many authors. Among them, Krein [21] and Gohberg and Krein [14] (also [4,11,12]) have given a fairly complete theory of (block) Toeplitz systems of the type (1.1), based on the Wiener–Hopf factorization of the symbol $\hat{T}(z) = \sum_{i \in \mathbb{Z}} z^i T_i$ ($|z| = 1$). The symbol can be factorized in various ways, either by additive decomposition of its logarithm (but only if $k = 1$; see [21]), by applying matrix polynomial theory [15,16], and by band extension methods [12]. All three methods have been implemented numerically.

In the one-index case ($d = 1$), perturbed systems of the form (1.3) have not been studied as systematically as block Toeplitz systems, so that the theory is far from satisfactory from the numerical point of view. A useful concept in this respect is the so-called equivalence of bi- and semi-infinite matrices A and B introduced in [18,22], namely $A \sim B$ iff for certain $c > 0$ and $\lambda < 1$ one has $\|A_{ij} - B_{ij}\| \leqslant c\lambda^{i+j}$ for all $i, j \geqslant 0$, which allows one, under rather general conditions, to prove that the solution $(x_i)_{i \in \mathbb{Z}_+}$ of Eq. (1.3) is exponentially decaying if its right-hand side $(b_i)_{i \in \mathbb{Z}_+}$ is exponentially decaying and the coefficient matrix A is equivalent to a (block) Toeplitz matrix T with exponentially decaying entries T_i. Further, in [23] a numerical method for solving systems of the form (1.3) has been developed, based on its decomposition in a finite system and a semi-infinite (block) Toeplitz system.

In the multi-index case ($d \geqslant 2$), linear systems of the type (1.1) have not been studied as systematically as in the one-index ($d = 1$) case, but the spectral factorizations that can lead to their solution have been studied extensively, albeit mostly without taking weights into account. Additive decomposition of the logarithm of the symbol $\hat{T}(z) = \sum_{i \in \mathbb{Z}^d} z^i T_i$ defined for z in the d-dimensional torus $\mathbb{T}^d$ is the obvious strategy in the scalar case ($k = 1$) [19,24]. The band extension method has been generalized to the multi-index case in [2,25] (also [24]), but cannot be implemented numerically, since it leads to an infinite linear system. There is no known analogue of matrix polynomial theory in the multi-index case. On the other hand, the projection method, where the solution of the infinite system is approximated by solutions of finite systems while the displacement structure of the system matrix is largely lost, can be generalized to the multi-index case [24] for positive definite block Toeplitz systems. We wish to mention that linear systems of the type (1.1), where the summation runs through the lattice points in a cone in $\mathbb{R}^d$ rather than through the lattice points in a half-space, have been studied since Kozak's seminal article [20]. However, we do not consider such equations, but for details we refer to [4,6].

Perturbed semi-infinite block Toeplitz systems have a multitude of applications. The orthonormalization of the (multi-)integer translates of a function $\phi: \mathbb{R}^d \to \mathbb{R}^k$ (often a box spline or a rapidly decreasing function) leads to a perturbed semi-infinite d-index block Toeplitz system if the orthonormalization is implemented on a half-space [19]. The finite difference scheme for certain linear PDE (in particular, the Poisson equation) on a half-space leads to a perturbed semi-infinite multi-index block Toeplitz system, the perturbation stemming from boundary effects.

The solution method we propose in this paper, is primarily based on results obtained by applying existing Banach algebra techniques to numerical linear algebra. First we define the Banach algebra $\mathscr{W}^d_{k,\boldsymbol{\beta}}$ of block Toeplitz matrices $T = (T_{i-j})_{i,j \in \mathbb{Z}^d}$ such that $\sum_{i \in \mathbb{Z}^d} \beta_i \|T_i\| < \infty$. This is a classical Banach algebra (see [9,10,12] if $k = 1$; [3,12,17] for any $k \geqslant 1$) in which T is an invertible element if and only if its symbol $\hat{T}(z) = \sum_{i \in \mathbb{Z}^d} z^i T_i$ is nonsingular for all z in a suitable set $\Omega_{\boldsymbol{\beta}}$ containing the d-dimensional torus $\mathbb{T}^d$.[1] Another Banach algebra we need to use is the algebra $\mathbf{W}^d_{k,\boldsymbol{\beta}}$ of all bi-infinite matrices A_{ij} with complex $k \times k$ matrix entries such that

$$\sum_{i \in \mathbb{Z}^d} \beta_i \sup_{l \in \mathbb{Z}^d} \|A_{l,l-i}\| < \infty,$$

which obviously extends $\mathscr{W}^d_{k,\boldsymbol{\beta}}$. For $\beta_i \equiv 1$ this algebra has been used before in [13]. Although the invertibility of its elements could be examined, it is still too large for our purposes. For this reason it is more appropriate to introduce the Banach algebra

[1] For $z = (z_1, \ldots, z_d) \in \mathbb{C}^d$ and $i = (i_1, \ldots, i_d) \in \mathbb{Z}^d$ we write $z^i = z_1^{i_1}, \ldots, z_d^{i_d}$ and $\|i\|_1 = |i_1| + \cdots + |i_d|$.

$\mathbb{W}^d_{k,\boldsymbol{\beta}}$, intermediate between $\mathscr{W}^d_{k,\boldsymbol{\beta}}$ and $\mathbf{W}^d_{k,\boldsymbol{\beta}}$, containing all bi-infinite matrices A_{ij} with complex $k \times k$ matrix entries which are suitable "perturbations" of Toeplitz matrices. In fact, any $A \in \mathbb{W}^d_{k,\boldsymbol{\beta}}$ can be written as the (unique) sum of a block Toeplitz matrix $T \in \mathscr{W}^d_{k,\boldsymbol{\beta}}$ and a bi-infinite matrix Z in $\mathbb{W}^d_{k,\boldsymbol{\beta}}$ that acts as a compact operator on all of the Banach spaces $\ell^p_{k,\boldsymbol{\beta}}(\mathbb{Z}^d)$ of vectors $(c_i)_{i\in\mathbb{Z}^d}$ for which $\|(c_i)_{i\in\mathbb{Z}^d}\|_{\ell^p_{k,\boldsymbol{\beta}}(\mathbb{Z}^d)} < \infty$, for $1 \leq p \leqslant \infty$.

As a result, decay properties of the solution of Eq. (1.3) can be proved in a trivial way. In particular, we have the following result, very important from the numerical point of view: If T is invertible in both $\mathscr{W}^d_{k,\boldsymbol{\varepsilon}}$ ($\varepsilon_i \equiv 1$ being the trivial weights) and $\mathscr{W}^d_{k,\boldsymbol{\beta}}$ (something one can readily verify using the symbol of T) and if A is invertible in $\mathbf{W}^d_{k,\boldsymbol{\varepsilon}}$ (thus for the trivial weights), then A is invertible in $\mathbf{W}^d_{k,\boldsymbol{\beta}}$; in other words, the solution of the bi-infinite system

$$\sum_{j\in\mathbb{Z}^d} A_{ij}x_j = b_i, \quad i \in \mathbb{Z}^d,$$

belongs to $\ell^p_{k,\boldsymbol{\beta}}(\mathbb{Z}^d)$ if its right-hand side $(b_i)_{i\in\mathbb{Z}^d}$ belongs to $\ell^p_{k,\boldsymbol{\beta}}(\mathbb{Z}^d)$ and the bi-infinite matrix A is invertible in both $\mathbb{W}^d_{k,\boldsymbol{\varepsilon}}$ and $\mathbb{W}^d_{k,\boldsymbol{\beta}}$. Other definitions and related properties of other Banach algebras we need to introduce, as well as a generalization of the notion of equivalence will be the topic of Section 2.

Next, in Section 3 various methods for solving Eq. (1.3) for $d \geqslant 2$ are discussed briefly, all but one relying on the Wiener–Hopf factorization of the symbol. Unfortunately, apart from the additive decomposition of the logarithm of the symbol [19] and the projection method, there are no numerical methods leading to a finite system. Furthermore, so far, only for $k = 1$ the necessary spectral factorization results for admissible weights have been proved [8]. In Sections 4 and 5 we apply the method to the numerical solution of perturbed semi-infinite Toeplitz systems arising from the orthonormalization of the multi-integer translates of a fixed function (splines in the case of Section 4 and a multidimensional normal distribution function in the case of Section 5) on a half-space.

2. Preliminaries

2.1. Auxiliary Banach algebras and Banach spaces

2.1.1. Banach algebras of block Toeplitz matrices

Given admissible weights $\boldsymbol{\beta} = (\beta_i)_{i\in\mathbb{Z}^d}$, by $\mathscr{W}^d_{k,\boldsymbol{\beta}}$ we define the Banach algebra of bi-infinite block Toeplitz matrices $A = (A_{i-j})_{i,j\in\mathbb{Z}^d}$ such that

$$\|A\|_{\mathscr{W}^d_{k,\boldsymbol{\beta}}} := \sum_{i\in\mathbb{Z}^d} \beta_i \|A_i\| < +\infty, \tag{2.1}$$

and by $\mathscr{H}^d_{k,\boldsymbol{\beta}}$ we define the Banach space of bi-infinite block Hankel matrices $H = (H_{i+j})_{i,j\in\mathbb{Z}^d}$ such that

$$\|H\|_{\mathscr{H}^d_{k,\boldsymbol{\beta}}} := \sum_{i\in\mathbb{Z}^d} \beta_i \|H_i\| < +\infty. \tag{2.2}$$

Proposition 2.1. *An element $A = (A_{i-j})_{i,j\in\mathbb{Z}^d} \in \mathscr{W}^d_{k,\boldsymbol{\beta}}$ is invertible in $\mathscr{W}^d_{k,\boldsymbol{\beta}}$ if and only if for every z in the set*

$$\Omega_{\boldsymbol{\beta}} := \left\{ z \in \mathbb{C}^d : \sup_{i\in\mathbb{Z}^d} \frac{|z^i|}{\beta_i} < +\infty \right\} \tag{2.3}$$

the symbol

$$\hat{A}(z) = \sum_{i\in\mathbb{Z}^d} z^i A_i \tag{2.4}$$

is a nonsingular $k \times k$ matrix.

Proof. Since the algebraic tensor product $\mathscr{W}^d_{1,\boldsymbol{\beta}} \otimes \mathbb{C}^{k\times k}$, consisting of the finite sums $\sum_j a_j T_j$ where $a_j \in \mathscr{W}^d_{1,\boldsymbol{\beta}}$ and T_j is a $k \times k$ matrix, is dense in $\mathscr{W}^d_{k,\boldsymbol{\beta}}$, it is sufficient to prove Proposition 2.1 in the scalar $k = 1$ case [17, Appendix]. The latter follows directly from standard Gelfand theory [9,10], because the multiplicative linear functionals on the commutative Banach algebra $\mathscr{W}^d_{1,\boldsymbol{\beta}}$ are exactly the evaluations $A \to \hat{A}(z)$ of the associated discrete Fourier transform where $z \in \Omega_{\boldsymbol{\beta}}$. □

Putting $\boldsymbol{\beta}^\# = (\beta_{-i})_{i\in\mathbb{Z}^d}$, we have

Proposition 2.2. *If $A \in \mathscr{W}^d_{k,\boldsymbol{\beta}}$, $H \in \mathscr{H}^d_{k,\boldsymbol{\beta}}$ and $B \in \mathscr{W}^d_{k,\boldsymbol{\beta}^\#}$, then $AHB \in \mathscr{H}^d_{k,\boldsymbol{\beta}}$ and*

$$\|AHB\|_{\mathscr{H}^d_{k,\boldsymbol{\beta}}} \leqslant \|A\|_{\mathscr{W}^d_{k,\boldsymbol{\beta}}} \|H\|_{\mathscr{H}^d_{k,\boldsymbol{\beta}}} \|B\|_{\mathscr{W}^d_{k,\boldsymbol{\beta}^\#}}. \tag{2.5}$$

Proof. One easily verifies that

$$\beta_{i+j} \leqslant \beta_{i-l}\beta_{l+r}\beta_{-(r-j)},$$

and that AHB is a block Hankel matrix. Therefore, for each i,

$$\sum_{j\in\mathbb{Z}^d} \beta_{i+j}\|(AHB)_{i+j}\| \leqslant \sum_{l\in\mathbb{Z}^d} \beta_{i-l}\|A_{i-l}\| \sum_{r\in\mathbb{Z}^d} \beta_{l+r}\|H_{l+r}\| \times \sum_{j\in\mathbb{Z}^d} \beta_{-(r-j)}\|B_{r-j}\|,$$

which completes the proof. □

2.1.2. Banach algebras of perturbed block Toeplitz matrices

By $\mathbf{W}^d_{k,\boldsymbol{\beta}}$ we define the Banach algebra of all bi-infinite matrices $A = (A_{ij})_{i,j\in\mathbb{Z}^d}$ with $k \times k$ matrix entries A_{ij} such that $\|A_{ij}\| \leqslant a_{i-j}$ for some $\mathbf{a} = (a_{i-j})_{i,j\in\mathbb{Z}^d}$ in $\mathscr{W}^d_{1,\boldsymbol{\beta}}$; then we say that the bi-infinite matrices $A \in \mathbf{W}^d_{k,\boldsymbol{\beta}}$ are *$\boldsymbol{\beta}$-decaying*. Further, by $\mathbf{H}^d_{k,\boldsymbol{\beta}}$ we denote the Banach space of all bi-infinite matrices $H = (H_{ij})_{i,j\in\mathbb{Z}^d}$ with $k \times k$ matrix entries H_{ij} such that $\|H_{ij}\| \leqslant h_{i+j}$ for some $\mathbf{h} = (h_{i+j})_{i,j\in\mathbb{Z}^d}$ in $\mathscr{H}^d_{1,\boldsymbol{\beta}}$. As the norms in $\mathbf{W}^d_{k,\boldsymbol{\beta}}$ and $\mathbf{H}^d_{k,\boldsymbol{\beta}}$ we define

$$\|A\|_{\mathbf{W}^d_{k,\boldsymbol{\beta}}} := \inf\left\{\sum_{i\in\mathbb{Z}^d} \beta_i a_i \colon \|A_{ij}\| \leqslant a_{i-j} \text{ for } \mathbf{a} = (a_{i-j})_{i,j\in\mathbb{Z}^d} \in \mathscr{W}^d_{1,\boldsymbol{\beta}}\right\}, \tag{2.6}$$

and

$$\|H\|_{\mathbf{H}^d_{k,\boldsymbol{\beta}}} := \inf\left\{\sum_{i\in\mathbb{Z}^d} \beta_i h_i \colon \|H_{ij}\| \leqslant h_{i+j} \text{ for } \mathbf{h} = (h_{i+j})_{i,j\in\mathbb{Z}^d} \in \mathscr{H}^d_{1,\boldsymbol{\beta}}\right\}, \tag{2.7}$$

respectively. Then $\mathbf{W}^d_{k,\boldsymbol{\beta}}$ is a Banach algebra, so that multiples, sums, differences and products of $\boldsymbol{\beta}$-decaying bi-infinite matrices are $\boldsymbol{\beta}$-decaying. Further, $\mathbf{H}^d_{k,\boldsymbol{\beta}}$ is a Banach space. Moreover, $\mathbf{W}^d_{k,\boldsymbol{\beta}}$ and $\mathbf{H}^d_{k,\boldsymbol{\beta}}$ contain $\mathscr{W}^d_{k,\boldsymbol{\beta}}$ and $\mathscr{H}^d_{k,\boldsymbol{\beta}}$, respectively.

Proposition 2.3. *If $A \in \mathbf{W}^d_{k,\boldsymbol{\beta}}$, $H \in \mathbf{H}^d_{k,\boldsymbol{\beta}}$ and $B \in \mathbf{W}^d_{k,\boldsymbol{\beta}^\#}$, then $AHB \in \mathbf{H}^d_{k,\boldsymbol{\beta}}$ and*

$$\|AHB\|_{\mathbf{H}^d_{k,\boldsymbol{\beta}}} \leqslant \|A\|_{\mathbf{W}^d_{k,\boldsymbol{\beta}}} \|H\|_{\mathbf{H}^d_{k,\boldsymbol{\beta}}} \|B\|_{\mathbf{W}^d_{k,\boldsymbol{\beta}^\#}}. \tag{2.8}$$

Proof. One easily checks that

$$\|(AHB)_{i,j}\| \leqslant \sum_{l,r\in\mathbb{Z}^d} \|A_{i,l}\|\,\|H_{l,r}\|\,\|B_{r,j}\| \leqslant \sum_{l,r\in\mathbb{Z}^d} a_{i-l} h_{l+r} b_{r-j}.$$

The rest of the proof follows the reasoning of the proof of Proposition 2.2. □

2.1.3. Banach algebras of compactly perturbed block Toeplitz matrices

Let $\overline{\mathbb{W}}^d_{k,\boldsymbol{\beta}}$ denote the Banach algebra (without unit element) of those $Z \in \mathbf{W}^d_{k,\boldsymbol{\beta}}$ for which

$$\lim_{\|l\|_1\to\infty} \|Z_{l,l-i}\| = 0, \quad i \in \mathbb{Z}^d.$$

Using this closed subalgebra of $\mathbf{W}^d_{k,\boldsymbol{\beta}}$, by $\mathbb{W}^d_{k,\boldsymbol{\beta}}$ we denote the Banach algebra consisting of those bi-infinite matrices $A \in \mathbf{W}^d_{k,\boldsymbol{\beta}}$ that allow the representation $A =$

$T + Z$ where $T \in \mathscr{W}^d_{k,\boldsymbol{\beta}}$ and $Z \in \overline{\mathbb{W}}^d_{k,\boldsymbol{\beta}}$, equipped with the norm inherited from $\mathbf{W}^d_{k,\boldsymbol{\beta}}$. Clearly, $\mathbb{W}^d_{k,\boldsymbol{\beta}}$ is a closed subalgebra of $\mathbf{W}^d_{k,\boldsymbol{\beta}}$ containing the unit element. Also, the block Toeplitz component T and the remainder term Z are uniquely determined by $A \in \mathbb{W}^d_{k,\boldsymbol{\beta}}$. Thus, the projection $\Pi : \mathbb{W}^d_{k,\boldsymbol{\beta}} \to \mathscr{W}^d_{k,\boldsymbol{\beta}}$ defined by $(\Pi(A))_{ij} = T_{i-j}$ is well-defined, onto and contractive. In other words, Π is the projection of $\mathbb{W}^d_{k,\boldsymbol{\beta}}$ onto $\mathscr{W}^d_{k,\boldsymbol{\beta}}$ along $\overline{\mathbb{W}}^d_{k,\boldsymbol{\beta}}$.

The following result shows that every bi-infinite matrix $A \in \mathbb{W}^d_{k,\boldsymbol{\beta}}$ is a compact perturbation of a block Toeplitz matrix in $\mathscr{W}^d_{k,\boldsymbol{\beta}}$.

Theorem 2.4. *Let $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ be two sets of admissible weights such that $1 \leqslant \gamma_i \leqslant \beta_i$ for every $i \in \mathbb{Z}^d$. Then the Banach algebra $\overline{\mathbb{W}}^d_{k,\boldsymbol{\beta}}$ consists of bi-infinite matrices that are compact operators on each of the Banach spaces $\ell^p_{k,\gamma}(\mathbb{Z}^d)$ of sequences $(x_i)_{i\in\mathbb{Z}^d}$ with terms in $\mathbb{C}^k$ that are bounded with respect to the norm*

$$\left\|(x_i)_{i\in\mathbb{Z}^d}\right\|_{\ell^p_{k,\gamma}(\mathbb{Z}^d)} = \begin{cases} \left[\sum_{i\in\mathbb{Z}^d} (\gamma_i \|x_i\|_2)^p\right]^{1/p}, & 1 \leqslant p < \infty, \\ \sup_{i\in\mathbb{Z}^d} \gamma_i \|x_i\|_2, & p = \infty, \end{cases} \tag{2.9}$$

where $\|\cdot\|_2$ stands for the Euclidean norm on $\mathbb{C}^k$.

Proof. It is easy to prove that the norm in $\mathbf{W}^d_{k,\gamma}$ is an upper bound for the norm of the matrix as a linear operator acting on vectors of $\ell^p_{k,\gamma}(\mathbb{Z}^d)$ if $p = 1$ or $p = \infty$. By interpolation, this is also true for arbitrary $p \in (1, \infty)$. Since the weights satisfy $1 \leqslant \gamma_i \leqslant \beta_i$ for every $i \in \mathbb{Z}^d$, the norm in $\mathbf{W}^d_{k,\boldsymbol{\beta}}$ is an upper bound for the norm of the matrix as a linear operator acting on vectors of $\ell^p_{k,\gamma}(\mathbb{Z}^d)$, $1 \leqslant p \leqslant \infty$. Since the matrices in $\overline{\mathbb{W}}^d_{k,\boldsymbol{\beta}}$ can be approximated in the norm of $\mathbf{W}^d_{k,\boldsymbol{\beta}}$ by matrices having only finitely many nonzero elements (by first approximating them by a matrix having only finitely many nontrivial diagonals), such matrices are compact operators when acting on $\ell^p_{k,\gamma}(\mathbb{Z}^d)$, $1 \leqslant p \leqslant \infty$. □

2.2. Equivalence

Given admissible weights $\boldsymbol{\beta} = (\beta_i)_{i\in\mathbb{Z}^d}$, two bi-infinite matrices A and B having $k \times k$ entries are called ***β*-*equivalent***, and we write $A \sim_{\boldsymbol{\beta}} B$, if $A - B$ belongs to the Banach space $\mathbf{H}^d_{k,\boldsymbol{\beta}}$. In other words, A and B are $\boldsymbol{\beta}$-equivalent if and only if

$$\|A_{ij} - B_{ij}\| \leqslant h_{i+j}, \quad i, j \in \mathbb{Z}^d, \tag{2.10}$$

for some sequence of positive numbers h_i such that $\sum_{i\in\mathbb{Z}^d} \beta_i h_i < +\infty$. Moreover, Proposition 2.3 allows one to derive the following two results. We recall that $\boldsymbol{\beta}^{\#} = (\beta_{-i})_{i\in\mathbb{Z}^d}$.

Theorem 2.5. *Let A and B be $\boldsymbol{\beta}$-decaying, C and D be $\boldsymbol{\beta}^{\#}$-decaying, A and B be $\boldsymbol{\beta}$-equivalent and C and D be $\boldsymbol{\beta}^{\#}$-equivalent. Then AC and BD are $\boldsymbol{\beta}$-equivalent.*

Proof. Applying (2.8) to the identity

$$AC - BD = (A - B)C + B(C - D) = A(C - D) + (A - B)D$$

allows one to conclude that $AC - BD \in \mathbf{H}^d_{k,\boldsymbol{\beta}}$. □

Theorem 2.6. *Let A and B be $\boldsymbol{\beta}$-equivalent, A^{-1} be $\boldsymbol{\beta}$-decaying and B^{-1} be $\beta^{\#}$-decaying. Then A^{-1} and B^{-1} are $\boldsymbol{\beta}$-equivalent.*

Proof. Apply (2.8) to the identity $A^{-1} - B^{-1} = -A^{-1}(A - B)B^{-1}$ and conclude that $A^{-1} - B^{-1} \in \mathbf{H}^d_{k,\boldsymbol{\beta}}$. □

Let $\preceq$ be a linear order on $\mathbb{Z}^d$ such that $i + l \preceq j + l$ whenever $i \preceq j$ and $l \in \mathbb{Z}^d$. Then a bi-infinite matrix A having $k \times k$ entries is called *lower triangular* (resp., *upper triangular*) if $A_{ij} = 0$ for $j \succ i$ (resp., $j \prec i$). Thus the notion of a lower or upper triangular matrix depends on the linear order $\preceq$.

Proposition 2.7. *Let $\boldsymbol{\beta}$ be a set of admissible weights such that $\beta_0 = 1$ and β_s is increasing in s for $s \succeq 0$, and let $\boldsymbol{\varepsilon} = (\varepsilon_s)_{s\in\mathbb{Z}^d}$ be the trivial weights $\varepsilon_s \equiv 1$. Suppose A and B are $\boldsymbol{\beta}$-decaying and $\boldsymbol{\beta}$-equivalent lower triangular matrices, A has an $\boldsymbol{\varepsilon}$-decaying lower triangular inverse A^{-1}, and B has a $\boldsymbol{\beta}^{\#}$-decaying inverse B^{-1}. Then A^{-1} and B^{-1} are $\tilde{\boldsymbol{\beta}}$-equivalent, where $\tilde{\beta}_s = 1$ for $s \prec 0$ and $\tilde{\beta}_s = \beta_s$ for $s \succeq 0$.*

Proof. We first assume that B is the identity matrix. Then there exist scalar sequences $(h_r)_{r\in\mathbb{Z}^d}$ and $(t_r)_{r\in\mathbb{Z}^d}$, with $t_s = 0$ for $s \prec 0$, such that

$$\begin{cases} \|(I - A)_{ij}\| \leqslant h_{i+j}, & \|(A^{-1})_{ij}\| \leqslant t_{i-j}, \\ \sum_{s\in\mathbb{Z}^d} \beta_s h_s < +\infty, & \sum_{s\succeq 0} t_s < +\infty, \end{cases}$$

where $(I - A)_{ij} = (A^{-1})_{ij} = 0$ if $i \prec j$. We now estimate

$$\|(A^{-1} - I)_{ij}\| \leqslant \sum_{j\preceq l\preceq i} \|(I - A)_{il}\| \, \|(A^{-1})_{lj}\| \leqslant \sum_{j\preceq l\preceq i} h_{i+l} t_{l-j} \leqslant u_{i+j},$$

where $u_r = \sum_{l\succeq 0} h_{r+l} t_l$. We now have

$$\begin{aligned} \sum_{s\in\mathbb{Z}^d} \tilde{\beta}_s u_s \quad &\leqslant \sum_{s\prec 0} u_s + \sum_{s\succeq 0,\ l\in\mathbb{Z}^d} \tilde{\beta}_{s+l} h_{s+l} \tilde{\beta}_{-l} t_l \\ &\leqslant \sum_{s\prec 0} u_s + \left(\sum_{l\in\mathbb{Z}^d} \beta_l h_l\right)\left(\sum_{l\succeq 0} t_l\right) < \infty, \end{aligned}$$

which proves that $A^{-1} \sim_{\tilde{\boldsymbol{\beta}}} B^{-1}$.

Let us now take B as in the statement of the proposition. Then Theorem 2.5 implies that $AB^{-1} \sim_{\boldsymbol{\beta}} I$. Further, $(AB^{-1})^{-1} = BA^{-1}$ is lower triangular and $\boldsymbol{\varepsilon}$-decaying. Hence, $BA^{-1} \sim_{\tilde{\boldsymbol{\beta}}} I$, implying $A^{-1} \sim_{\tilde{\boldsymbol{\beta}}} B^{-1}$. □

3. Solving semi-infinite multi-index systems

Let us first consider the semi-infinite linear system (1.1), where the block Toeplitz matrix $T = (T_{i-j})_{i,j\in\mathbb{Z}^d}$ belongs to $\mathscr{W}^d_{k,\boldsymbol{\beta}}$. Then, using a well-known method for $d = 1$ by Krein [21] (also [11,14]), we can prove that, given $1 \leqslant p \leqslant \infty$, this system has a unique solution in $\ell^p_{k,\boldsymbol{\beta}}(\mathbb{Z}^d_+)$ if the symbol $\hat{T}(z) = \sum_{i\in\mathbb{Z}^d} z^i T_i$ is nonsingular for every $z \in \mathbb{T}^d$, the d-dimensional torus, and allows the factorization

$$\hat{T}(z)^{-1} = \Gamma_+(z)\Gamma_-(z), \tag{3.1}$$

where

$$\Gamma_+(z) = \sum_{j\succeq 0} z^j \Gamma^{(1)}_j, \quad \Gamma_-(z) = \sum_{j\succeq 0} z^{-j}\Gamma^{(2)}_j, \tag{3.2}$$

and $(\Gamma^{(s)}_{i-j})_{i,j\in\mathbb{Z}^d}$ belong to $\mathscr{W}^d_{k,\boldsymbol{\beta}}$ $(s = 1, 2)$.[2]

Let us define $\ell^p_{k,\boldsymbol{\beta}}(\mathbb{Z}^d_+)$ as in (2.9), but with $i \in \mathbb{Z}^d_+$. Then the unique solution $(x_i)_{i\succeq 0}$ in $\ell^p_{k,\boldsymbol{\beta}}(\mathbb{Z}^d_+)$ is given by

$$x_i = \sum_{j\succeq 0} \Gamma_{ij} b_j, \quad i \succeq 0, \tag{3.3}$$

where

$$\Gamma_{ij} = \sum_{0\preceq h\preceq \min(i,j)} \Gamma^{(1)}_{i-h}\Gamma^{(2)}_{j-h}. \tag{3.4}$$

Note that, in contrast to the one-index case ($d = 1$), the sum at the right-hand side of (3.4) could be infinite. From [23] it is clear that if $d = 1$, the system (1.1) is uniquely solvable in $\ell^p_{k,\boldsymbol{\varepsilon}}(\mathbb{Z}^d)$ (where $\varepsilon_i \equiv 1$) and $T \in \mathscr{W}^d_{k,\boldsymbol{\beta}}$, then it is uniquely solvable in $\ell^p_{k,\boldsymbol{\beta}}(\mathbb{Z}^d)$.

Let us now consider the semi-infinite linear system (1.3), where the system matrix is the semi-infinite restriction of a matrix $A = (A_{ij})_{i,j\in\mathbb{Z}^d}$ belonging to $\mathbb{W}^d_{k,\boldsymbol{\beta}}$. Then there exist matrices $T = (T_{i-j})_{i,j\in\mathbb{Z}^d} \in \mathscr{W}^d_{k,\boldsymbol{\beta}}$ and $Z = (Z_{ij})_{i,j\in\mathbb{Z}^d} \in \overline{\mathbb{W}}^d_{k,\boldsymbol{\beta}}$ such that

$$A_{ij} = T_{i-j} + Z_{ij}, \quad i, j \in \mathbb{Z}^d. \tag{3.5}$$

According to Theorem 2.4, Z is a compact operator on $\ell^p_{k,\boldsymbol{\gamma}}(\mathbb{Z}^d)$ $(1 \leqslant p \leqslant \infty)$ for any set of admissible weights $\boldsymbol{\gamma}$ satisfying $1 \leqslant \gamma_i \leqslant \beta_i$ $(i \in \mathbb{Z}^d)$.

[2] Here we put $\Gamma^{(1)}_i = \Gamma^{(2)}_i = 0$ if $i \prec 0$.

We now have the following important but rather obvious result.

Proposition 3.1. *Let $\varepsilon_i \equiv 1$ for all $i \in \mathbb{Z}^d$, and let $A \in \mathbb{W}^d_{k,\boldsymbol{\beta}}$ have the representation* (3.5). *If A is invertible in $\mathbf{W}^d_{k,\boldsymbol{\varepsilon}}$ and its block Toeplitz component T is invertible in both $\mathscr{W}^d_{k,\boldsymbol{\varepsilon}}$ and $\mathscr{W}^d_{k,\boldsymbol{\beta}}$, then A is boundedly invertible on $\ell^p_{k,\boldsymbol{\beta}}(\mathbb{Z}^d)$.*

Proof. We note that $A = T + Z$, where

a. A is invertible in $\mathbb{W}^d_{k,\boldsymbol{\varepsilon}}$, so that A is bounded and boundedly invertible on $\ell^p_{k,\boldsymbol{\varepsilon}}(\mathbb{Z}^d)$.
b. T is invertible in both $\mathscr{W}^d_{k,\boldsymbol{\varepsilon}}$ and $\mathscr{W}^d_{k,\boldsymbol{\beta}}$, and hence in both $\mathbb{W}^d_{k,\boldsymbol{\varepsilon}}$ and $\mathbb{W}^d_{k,\boldsymbol{\beta}}$ (hence T is boundedly invertible on both $\ell^p_{k,\boldsymbol{\varepsilon}}(\mathbb{Z}^d)$ and $\ell^p_{k,\boldsymbol{\beta}}(\mathbb{Z}^d)$).
c. Z is a compact operator on both $\ell^p_{k,\boldsymbol{\varepsilon}}(\mathbb{Z}^d)$ and $\ell^p_{k,\boldsymbol{\beta}}(\mathbb{Z}^d)$.
d. $\ell^p_{k,\boldsymbol{\beta}}(\mathbb{Z}^d)$ is continuously and densely embedded in $\ell^p_{k,\boldsymbol{\varepsilon}}(\mathbb{Z}^d)$.

Therefore, A is boundedly invertible on $\ell^p_{k,\boldsymbol{\beta}}(\mathbb{Z}^d)$. □

We now consider the semi-infinite case where the matrices A, T and Z are restricted to their (i, j)-entries with $i, j \succeq 0$. We then replace A, T and Z by their compressions to semi-infinite matrices. This compression keeps Z compact on $\ell^p_{k,\boldsymbol{\gamma}}(\mathbb{Z}^d_+)$ for any set of admissible weights $\boldsymbol{\gamma}$ with $1 \leqslant \gamma_i \leqslant \beta_i$ $(i \in \mathbb{Z}^d)$. We then have the following result whose proof is omitted, as it is analogous to that of Proposition 3.1.

Theorem 3.2. *Let $\varepsilon_i \equiv 1$ for all $i \in \mathbb{Z}^d$, and let $A \in \mathbb{W}^d_{k,\boldsymbol{\beta}}$ have the representation* (3.5). *If the semi-infinite system* (1.3) *is uniquely solvable in $\ell^p_{k,\boldsymbol{\varepsilon}}(\mathbb{Z}^d_+)$ and the semi-infinite block Toeplitz system* (1.1) *is uniquely solvable in both $\ell^p_{k,\boldsymbol{\varepsilon}}(\mathbb{Z}^d_+)$ and $\ell^p_{k,\boldsymbol{\beta}}(\mathbb{Z}^d_+)$, then the semi-infinite system* (1.3) *is uniquely solvable in $\ell^p_{k,\boldsymbol{\beta}}(\mathbb{Z}^d_+)$.*

We now propose a method for solving the semi-infinite system (1.3) which generalizes the method introduced in [23] for $d = 1$. Suppose $A \in \mathbb{W}^d_{k,\boldsymbol{\beta}}$ has the form (3.5) with $T \in \mathscr{W}^d_{k,\boldsymbol{\beta}}$ and $Z \in \overline{\mathbb{W}}^d_{k,\boldsymbol{\beta}}$, where $\|Z_{ij}\|$ is "sufficiently small" for all $(i, j) \in [\mathbb{Z}^d_+ \times \mathbb{Z}^d_+] \setminus [E \times E]$ for some **finite** set $E \subset \mathbb{Z}^d_+$. Put $F = \mathbb{Z}^d_+ \setminus E$ and let us partition (1.3) as follows:

$$A\mathbf{x} = \begin{bmatrix} A_{EE} & A_{EF} \\ A_{FE} & A_{FF} \end{bmatrix} \begin{bmatrix} \mathbf{x}_E \\ \mathbf{x}_F \end{bmatrix} = \begin{bmatrix} \mathbf{b}_E \\ \mathbf{b}_F \end{bmatrix} = \mathbf{b}, \tag{3.6}$$

where $(x_E)_i = x_i$ for $i \in E$ and $(x_F)_i = x_i$ for $i \in F$. Now let the semi-infinite systems (1.1) and (1.3) both be uniquely solvable on $\ell^p_{k,\boldsymbol{\beta}}(\mathbb{Z}^d_+)$, and let us solve the semi-infinite Toeplitz system $T\tilde{\mathbf{x}} = \mathbf{b}$ for $\tilde{\mathbf{x}}$ and write $\tilde{\mathbf{x}}_F$ for its part with indices supported in F. Then $\tilde{\mathbf{x}}_F$ can be viewed as an approximation of the part of the solution of (1.3) whose index is supported on F. When A_{EE} is invertible, [3] we obtain as the E-portion of the approximate solution of the semi-infinite system (1.3):

$$\hat{\mathbf{x}}_E = A_{EE}^{-1}\left[\mathbf{b}_E - A_{EF}\,\tilde{\mathbf{x}}_F\right]. \tag{3.7}$$

It is well-known that the so-called projection method [5,11,13] converges in the one-index case $d = 1$ [5,13] and for positive definite systems [11]. In [24], the projection method for multi-index positive definite block Toeplitz systems has been used to compute numerically the spectral factorization of a banded two-index Toeplitz matrix.

Let us first apply the projection method in abstract Banach spaces to only one directed family of projections (see [13] for its application to a more general situation). Letting Ω stand for an infinite subset of $\mathbb{R}$ satisfying $\inf\Omega > -\infty$ and $\sup\Omega = +\infty$, suppose X is a complex Banach space and $(P_\tau)_{\tau\in\Omega}$ is a family of bounded projections on X. Assume that

$$\lim_{\tau\to\infty} \|P_\tau x - x\|_X = 0, \quad x \in X.$$

Given $A \in \mathscr{L}(X)$, A_τ will denote the unique linear operator defined on $\operatorname{Im} P_\tau$ such that

$$A_\tau x = P_\tau A x, \quad x \in \operatorname{Im} P_\tau.$$

We say that the *projection method relative to* $\{P_\tau\}_{\tau\in\Omega}$ *is applicable to* A if, for some $\tau_0 \in \Omega$ and all $\tau \geqslant \tau_0$ in Ω, and for each $y \in Y$ the equation

$$P_\tau A P_\tau x = P_\tau y$$

has a unique solution $x_\tau \in \operatorname{Im} P_\tau$ and the vectors x_τ converge strongly to a solution of $Ax = y$ as $\tau \to \infty$. It is well-known (cf. Section II.2 in [11]) that the projection method relative to $\{P_\tau\}_{\tau\in\Omega}$ is applicable to A if and only if A is invertible, there exists $\tau_0 \in \Omega$ such that A_τ is invertible for all $\tau \geqslant \tau_0$ in Ω, and

$$\sup_{\tau\geqslant\tau_0} \|A_\tau^{-1}\| < \infty.$$

The following result is well-known (see, e.g., Theorem II.3.1 of [11] or Corollary 7.17 of [4]).

Proposition 3.3. *Let* $(E_n)_{n\in\mathbb{N}}$ *be an increasing sequence of finite subsets of* $\mathbb{Z}_+^d$ *with union* $\mathbb{Z}_+^d$, *and put* $F_n = \mathbb{Z}_+^d \setminus E_n$ $(n \in \mathbb{N})$. *Let* π_n: $\ell_{k,\boldsymbol{\beta}}^p(\mathbb{Z}_+^d) \to \ell_{k,\boldsymbol{\beta}}^p(E_n)$ *be the natural projection and* j_n: $\ell_{k,\boldsymbol{\beta}}^p(E_n) \to \ell_{k,\boldsymbol{\beta}}^p(\mathbb{Z}_+^d)$ *the natural embedding defined by*

$$(\pi_n\mathbf{x})_i = x_i,\ \ i \in E_n; \quad (j_n\,\mathbf{x})_i = \begin{cases} x_i, & i \in E_n, \\ 0, & i \in F_n. \end{cases} \tag{3.8}$$

[3] Note that we are dealing with a finite matrix, since E is finite.

Let $1 \leqslant p < \infty$, *and assume that the semi-infinite systems* (1.1) *and* (1.3) *are uniquely solvable on* $\ell^p_{k,\boldsymbol{\beta}}(\mathbb{Z}^d_+)$ *and that the projection method relative to* $\{\pi_n j_n\}_{n\in\mathbb{N}}$ *is applicable to* $T = (T_{i-j})_{i,j\in\mathbb{Z}^d_+}$. *Then for n large enough the linear system*

$$\pi_n A j_n \mathbf{x}_n = \left(\sum_{j\in E_n} A_{ij} x_{nj} \right)_{i\in E_n} = \pi_n \mathbf{b} \tag{3.9}$$

is uniquely solvable and its solution $\mathbf{x}_n$ *has the property that* $j_n\mathbf{x}_n$ *converges to the unique solution* $\mathbf{x} = (x_i)_{i\succeq 0}$ *of* (1.3) *in the norm of* $\ell^p_{k,\boldsymbol{\beta}}(\mathbb{Z}^d_+)$.

We now prove the convergence of our main numerical method.

Theorem 3.4. *Let* $1 \leqslant p < \infty$, $T \in \mathscr{W}^d_{k,\boldsymbol{\beta}}$, *and* $A \in \mathbf{W}^d_{k,\boldsymbol{\beta}}$. *Suppose the semi-infinite systems* (1.1) *and* (1.3) *are uniquely solvable for* $\mathbf{x} = (x_i)_{i\in\mathbb{Z}^d_+}$ *in* $\ell^p_{k,\boldsymbol{\beta}}(\mathbb{Z}^d_+)$ *for any* $\mathbf{b} = (b_i)_{i\in\mathbb{Z}^d_+}$ *in* $\ell^p_{k,\boldsymbol{\beta}^\#}(\mathbb{Z}^d_+)$. *Assume also that the projection method relative to* $\{\pi_n j_n\}_{n\in\mathbb{N}}$ *is applicable to* $T = (T_{i-j})_{i,j\in\mathbb{Z}^d_+}$. *Further, let the corresponding inverse operators* T^{-1} *and* A^{-1} *satisfy* $A^{-1} \sim_{\boldsymbol{\beta}} T^{-1}$. *Then for every* $\varepsilon > 0$ *and* $\mathbf{b} \in \ell^p_{k,\boldsymbol{\beta}^\#}(\mathbb{Z}^d_+)$ *there exists a finite subset E of* $\mathbb{Z}^d_+$ *such that*

$$\|\mathbf{x}_E - \hat{\mathbf{x}}_E\|_{\ell^p_{k,\boldsymbol{\beta}}(E)} + \|\mathbf{x}_F - \tilde{\mathbf{x}}_F\|_{\ell^p_{k,\boldsymbol{\beta}}(F)} < \varepsilon,$$

where $F = \mathbb{Z}^d_+ \setminus E$.

Proof. Let E be a finite subset of $\mathbb{Z}^d_+$ and let $F = \mathbb{Z}^d_+ \setminus E$. Also, let $(h_i)_{i\succeq 0}$ and $(t_i)_{i\in\mathbb{Z}^d}$ be sequences of nonnegative numbers such that

$$\begin{cases} \|(A^{-1})_{ij} - (T^{-1})_{ij}\| \leqslant h_{i+j}, & \sum_{i\succeq 0} \beta_i h_i < +\infty, \\ \|A_{ij}\| \leqslant t_{i-j}, & \sum_{i\in\mathbb{Z}^d} \beta_i t_i < +\infty. \end{cases}$$

Now put $\delta\mathbf{x}_F = \mathbf{x}_F - \tilde{\mathbf{x}}_F$. Then $\delta\mathbf{x}_F = P_F(A^{-1} - T^{-1})\mathbf{b}$, where

$$(P_F\mathbf{y})_i = \begin{cases} 0, & i \in E, \\ y_i, & i \in F. \end{cases}$$

We now easily estimate (using $\beta^\#_i = \beta_{-i}$)

$$\begin{aligned} \|\delta\mathbf{x}_F\|_{\ell^\infty_{k,\boldsymbol{\beta}}(F)} &\leqslant \sup_{i\in F} \beta_i \sum_{j\succeq 0} h_{i+j}\|b_j\| \\ &\leqslant \left(\sup_{j\succeq 0} \sum_{s\in j+F} \beta_s h_s \right) \|b\|_{\ell^\infty_{k,\boldsymbol{\beta}^\#}(\mathbb{Z}^d_+)}, \end{aligned}$$

$$\|\delta\mathbf{x}_F\|_{\ell^1_{k,\boldsymbol{\beta}}(F)} \leqslant \sum_{i\in F}\beta_i\sum_{j\succeq 0}h_{i+j}\|b_j\|$$
$$\leqslant \left(\sup_{j\succeq 0}\sum_{s\in j+F}\beta_s h_s\right)\|b\|_{\ell^1_{k,\boldsymbol{\beta}^\#}(\mathbb{Z}^d_+)},$$

and hence, by interpolation,

$$\|\delta\mathbf{x}_F\|_{\ell^p_{k,\boldsymbol{\beta}}(F)} \leqslant \left(\sup_{j\succeq 0}\sum_{s\in j+F}\beta_s h_s\right)\|b\|_{\ell^p_{k,\boldsymbol{\beta}^\#}(\mathbb{Z}^d_+)}, \quad 1\leqslant p\leqslant\infty. \tag{3.10}$$

Next, put $\delta\mathbf{x}_E = \mathbf{x}_E - \hat{\mathbf{x}}_E$. Then

$$\delta\mathbf{x}_E = -A^{-1}_{EE}A_{EF}\,\delta\mathbf{x}_F. \tag{3.11}$$

In analogy to (3.10) we prove the estimate

$$\|A_{EF}\,\delta\mathbf{x}_F\|_{\ell^p_{k,\boldsymbol{\beta}}(E)} \leqslant \left(\sup_{j\in F}\sum_{s\in E-j}\beta_s t_s\right)\|\delta\mathbf{x}_F\|_{\ell^p_{k,\boldsymbol{\beta}}(F)}, \quad 1\leqslant p\leqslant\infty. \tag{3.12}$$

Let us now take an increasing sequence of finite subsets E_n of $\mathbb{Z}^d_+$ whose union is $\mathbb{Z}^d_+$, and let us put $F_n = \mathbb{Z}^d_+ \setminus E_n$. Then we make the following observations:

a. There exists $N \in \mathbb{N}$ such that for $n \geqslant N$ the inverses $A_{E_nE_n}$ exist and the set of their norms $A^{-1}_{E_nE_n}$ is bounded. Thus there exists a constant M such that $\|A^{-1}_{E_nE_n}\| \leqslant M$. This follows directly from Proposition 3.3.
b. The upper bounds $\sup_{j\in F}\sum_{s\in E-j}\beta_s t_s$ appearing in the right-hand side of (3.12) are bounded above by the finite constant $\sum_{i\in\mathbb{Z}^d}\beta_i t_i$.
c. For any increasing sequence of finite subsets $E_n = \mathbb{Z}^d_+ \setminus F_n$ of $\mathbb{Z}^d_+$ whose union is $\mathbb{Z}^d_+$, the number $\sup_{j\succeq 0}\sum_{s\in j+F_n}\beta_s h_s$ vanishes as $n\to\infty$.

As a result, the right-hand side of (3.12) has an n-independent upper bound if $E = E_n$, while the right-hand side of (3.10) vanishes as $n\to\infty$ when taking $E = E_n$. Using (3.11), one completes the proof. □

In applications we usually have "even" weights, i.e., weights for which $\beta_{-i} = \beta_i$ for $i \in \mathbb{Z}^d$. In that case $\boldsymbol{\beta}^\# = \boldsymbol{\beta}$, which leads to a considerable simplification of Theorem 3.4.

4. Two examples

Let us now consider two numerical examples of some interest in different fields of Mathematics, and especially in Approximation Theory.

In the first example, already studied in [19], we consider the bivariate linear box spline

$$\phi(\mathbf{x}) = \begin{cases} 1 - x_2, & x_1 \in [0,1),\ x_2 \in [x_1, 1), \\ 1 - x_1, & x_1 \in [0,1),\ x_2 \in [0, x_1), \\ 1 - x_1 + x_2, & x_1 \in [0,1),\ x_2 \in (-1 + x_1, 0), \\ \phi(-\mathbf{x}), & x_1 \in (-1,0),\ x_2 \in (-1, 1 + x_1), \\ 0, & \text{otherwise}, \end{cases} \tag{4.1}$$

and, for $j = (j_1, j_2) \in \mathbb{Z}^2$, we introduce its integer translates $\phi_j(\mathbf{x}) := \phi(\mathbf{x} - j)$.

Let $T = (T_{i-j})_{i,j\in\mathbb{Z}^2}$ be the Gram matrix associated to these translates, that is

$$T_{ij} = \int_{\mathbb{R}^2} \phi_i(\mathbf{x})\phi_j(\mathbf{x})\,\mathrm{d}\mathbf{x} = T_{i-j}, \quad i, j \in \mathbb{Z}^2,$$

where

$$\begin{cases} T_{(0,0)} = \frac{1}{2}, \\ T_{(1,0)} = T_{(-1,0)} = T_{(0,1)} = T_{(0,-1)} = T_{(1,1)} = T_{(-1,-1)} = \frac{1}{12}, \\ T_{(i_1,i_2)} = 0 \ \text{ otherwise}. \end{cases}$$

Then, for $\mathbf{z} = (\mathrm{e}^{\mathrm{i}\theta_1}, \mathrm{e}^{\mathrm{i}\theta_2}) \in \mathbb{T}^2$, the symbol of T is strictly positive, as

$$\hat{T}(\mathbf{z}) = \tfrac{1}{6}\Big((3 + \cos\theta_1 + \cos\theta_2 + \cos(\theta_1 + \theta_2))\Big) \geqslant \tfrac{1}{4},$$

so that T is positive definite. Moreover, since it is banded as $T_{ij} = 0$ for $|i - j| > 2$, it is in the Banach algebra $\mathscr{W}^2_{1,\boldsymbol{\beta}}$ for any choice of the set of admissible weights $\boldsymbol{\beta}$.

Our second example involves the Gram matrix associated to the integer translates of the multivariate Gaussian distribution. Let Σ be a positive definite real $d \times d$ matrix, and let

$$\phi(\mathbf{x}) = \pi^{-d/2}(\det\Sigma)^{1/2}\mathrm{e}^{-(\Sigma\mathbf{x},\mathbf{x})} = \pi^{-d/2}(\det\Sigma)^{1/2}\exp\left(-\sum_{i,j=1}^{d}\Sigma_{ij}x_ix_j\right),$$

where $\mathbf{x} = (x_i)_{i=1}^d \in \mathbb{R}^d$ and $(\mathbf{x}, \mathbf{y})$ denotes the Euclidean inner product in $\mathbb{R}^d$. Then, for $\mathbf{t} \in \mathbb{R}^d$, let

$$\kappa_\phi(\mathbf{t}) = \int_{\mathbb{R}^d} \phi(\mathbf{x})\phi(\mathbf{x} - \mathbf{t})\,d\mathbf{x} = (2\pi)^{-d/2}(\det\Sigma)^{1/2}\mathrm{e}^{-\frac{1}{2}(\Sigma\mathbf{t},\mathbf{t})}. \tag{4.2}$$

In particular, for $\mathbf{t}, \mathbf{s} \in \mathbb{R}^d$ we have

$$k_\phi(\mathbf{t}, \mathbf{s}) = \int_{\mathbb{R}^d} \phi(\mathbf{x} - \mathbf{t})\phi(\mathbf{x} - \mathbf{s})\,d\mathbf{x} = \kappa_\phi(\mathbf{t} - \mathbf{s}) = \kappa_\phi(\mathbf{s} - \mathbf{t}).$$

To construct a numerical example, for the sake of simplicity we take $d = 2$ and let $\Sigma = I$, the identity matrix of dimension 2. Then, we consider the Gram matrix of the integer translates of the bivariate Gaussian distribution

$$T_{ij} = T_{i-j} = k_\phi(i, j) = \frac{1}{2\pi}\mathrm{e}^{-\frac{1}{2}(i-j,i-j)}, \quad i, j \in \mathbb{Z}^2. \tag{4.3}$$

To better illustrate the numerical performance of the projection method, for each of the Toeplitz matrices T introduced in the above two examples we introduce the matrix $B = T + P$, obtained by adding to the matrix T a perturbation matrix P having compact support, that is

$$P_{ij} = \begin{cases} \frac{1}{5}, & \|i\|_\infty < 5,\ \|j\|_\infty < 5, \\ 0, & \text{otherwise}, \end{cases}$$

where $\|i\|_\infty = \max_{k=1,\dots,d} |i_k|$. Because of the finite precision of floating point systems, such a matrix is numerically representative of any perturbation having exponential decay with respect to $|i + j|$, i.e., $\boldsymbol{\beta}$-decaying with weights $\beta_\ell = \theta^{|\ell|}$, $\ell \in \mathbb{Z}^2$, for a given $\theta > 1$.

Let us now denote by $\preceq$ the usual lexicographical ordering on $\mathbb{Z}^2$, i.e., $i \succeq j$ whenever $i_1 > j_1$ or $i_1 = j_1$ and $i_2 \geqslant j_2$, and let $\mathbb{Z}^2_+ = \{i \in \mathbb{Z}^2 : i \succeq 0\}$; this linear order satisfies the requirements given at the beginning of Section 1. Numerical experiments have been performed considering linear systems of the type (1.3), where the matrix A is the semi-infinite restriction

$$A_{ij} = B_{ij}, \quad i, j \in \mathbb{Z}^d_+,$$

of the perturbed matrices $B = T + P$, and T is the Gram matrix of box-splines and Gaussians functions, respectively. The right hand side has been computed by taking $\mathbf{b} = A\mathbf{x}$, with

$$x_i = \left(\tfrac{7}{8}\right)^{\sqrt{i_1^2+i_2^2}}, \quad i \in \mathbb{Z}^2_+.$$

The system is solved with the algorithm discussed in Section 3. Namely, in correspondance of a chosen *projection parameter* p, we consider the finite set

$$E = E_p = \{i \in \mathbb{Z}^2_+ : \|i\|_\infty < p\}$$

and partition the system (1.3) as in equation (3.6). In particular, the numerical solution $\hat{\mathbf{x}}$ of the semi-infinite linear system is written in the form

$$\hat{\mathbf{x}} = \begin{bmatrix} \hat{\mathbf{x}}_E \\ \tilde{\mathbf{x}}_F \end{bmatrix},$$

where $F = \mathbb{Z}^2_+ \setminus E$. The infinite part $\tilde{\mathbf{x}}_F$ of this vector is extracted from the solution of the Toeplitz system $T\tilde{\mathbf{x}} = \mathbf{b}$ computed by equation (3.3), where the spectral factorization (3.1) is obtained by means of the additive decomposition of the logarithm of the symbol [19]. In solving this semi-infinite Toeplitz system, we need to approximate the series (3.3) by a finite summation, so that, if we introduce a truncation parameter N, this formula takes the form

$$x_i = \sum_{\substack{j \succeq 0 \\ \|j\|_\infty < N}} \Gamma_{ij} b_j, \quad i \succeq 0,\ \|i\|_\infty < N.$$

The finite part has been computed by solving the finite linear system

$$A_{EE}\hat{\mathbf{x}}_E = \mathbf{b}_E - A_{EF}\,\tilde{\mathbf{x}}_F \tag{4.4}$$

by Gaussian elimination, after lexicographical reordering of the unknowns. The choice for a direct *general purpose* algorithm is due to our wish to initially ascertain the accuracy of the method. We plan to study in a future paper the application of a preconditioned iterative method to system (4.4) which takes advantage of the particular structure of matrix A_{EE}. This step is essential, since the number of unknowns in system (4.4) is huge even for moderate values of p (for $p = 50$ it is over 5000).

As a measure of the accuracy of the algorithm we take the relative error

$$\epsilon_i = \frac{|x_i - \hat{x}_i|}{|x_i|}, \quad i \in \mathbb{Z}_+^2,$$

which is well defined because in our experiments we considered only solutions with components different from zero.

Fig. 1 reports the logarithm of the relative error corresponding to the perturbation of the Gram matrix of box-splines, with parameters $N = 64$ and $p = 10$. The increase in the error occurring for $\|i\|_\infty \sim N$ is an obvious consequence of the truncation of infinite series; to improve the precision for these components of the solution it is sufficient to increase the value of N. What is clear, is that the value of the projection parameter p is definitely inadequate, as testified by the sharp peak in the error near the center of the square. In Fig. 2 the value of p has been taken to 20, showing a clear improvement in the solution. By taking $p = 30$ the relative error

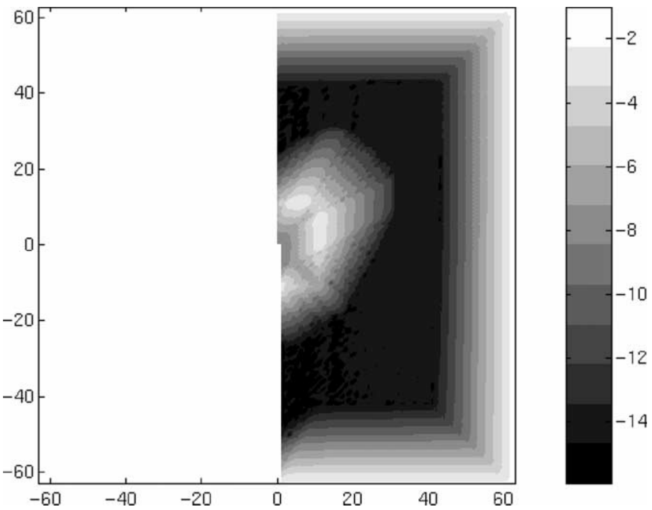

Fig. 1. Relative error in $\log_{10}$-scale (box-splines, $N = 64$, $p = 10$).

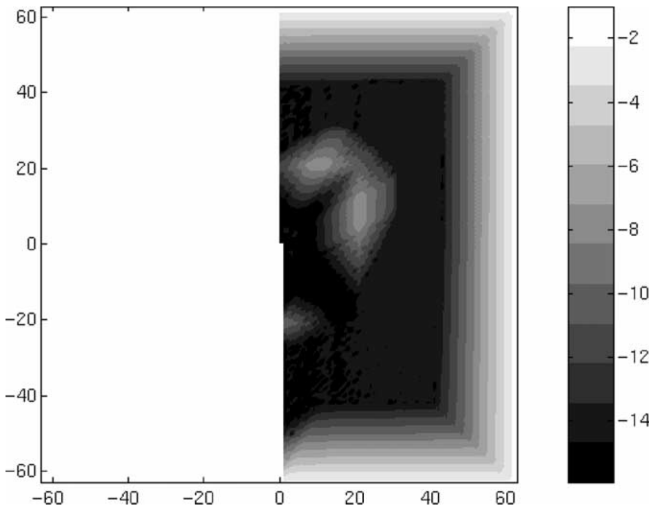

Fig. 2. Relative error in $\log_{10}$-scale (box-splines, $N = 64$, $p = 20$).

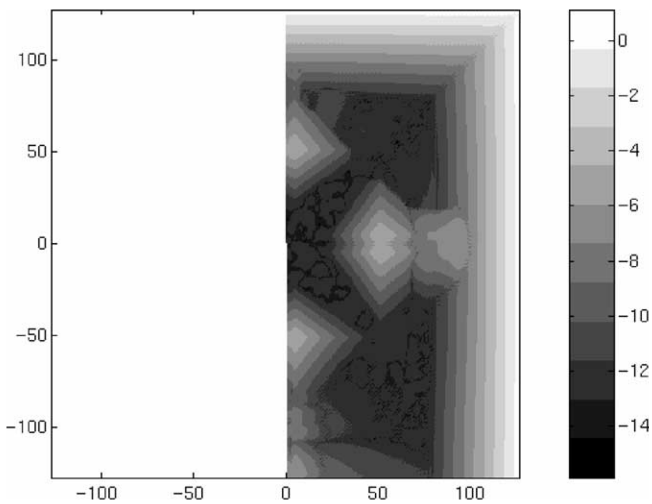

Fig. 3. Relative error in $\log_{10}$-scale (Gaussians, $N = 128$, $p = 50$).

reaches machine precision for all the components located far enough from the edges of the square.

Fig. 3 shows the result obtained with the perturbed Gram matrix of the Gaussians, with $N = 128$ and $p = 50$. Due to the different decay properties of this matrix, this value of p allows only five exact digits on the mantissa of the solution, but while there is no particular difficulty in increasing (even doubling) the value of N, the value of p cannot be increased indefinitely, without resorting to a rapidly converging iterative algorithm for the solution of system (4.4).

5. Gram matrices on the half-plane

We now turn our attention to the Gram matrix associated to the restriction to the half-plane $\mathbb{R}^2_+ = \{\mathbf{x} \in \mathbb{R}^2 \colon x_1 \geqslant 0\}$ of a set of functions $\{\phi_i\}_{i\in\mathbb{Z}^2_+}$, which are integer translates of a fixed function $\phi(x)$, that is to the semi-infinite matrix

$$G_{ij} = \int_{\mathbb{R}^2_+} \phi_i(\mathbf{x})\phi_j(\mathbf{x})\,\mathrm{d}\mathbf{x}, \quad i, j \in \mathbb{Z}^2_+. \tag{5.1}$$

A linear system characterized by this matrix may occur, for example, in the computation of a least squares approximation to a function defined on the half-plane. We will analyze the situations in which ϕ is one of the functions considered in the preceding section.

In the first case, that is when $\phi(x)$ is the box spline (4.1), the Gram matrix (5.1) takes the form $G_{ij} = T_{i-j} + S_{ij}$, with

$$S_{ij} = \begin{cases} -\frac{1}{4} & \text{if } i_1 = j_1 = 0 \text{ and } i_2 = j_2, \\ -\frac{1}{24} & \text{if } i_1 = j_1 = 0 \text{ and } i_2 = j_2 \pm 1, \\ 0 & \text{otherwise.} \end{cases}$$

Since the box-splines form an unconditionally stable basis [7], the matrix G is positive definite. It is immediate to observe that G cannot be considered $\boldsymbol{\beta}$-equivalent to the Toeplitz matrix T in the sense of formula (2.10). In fact we have, for example, $|S_{\ell\ell}| = \frac{1}{4}$ for $\ell = (0, r)$ and for any $r \in \mathbb{Z}_+$, and these values cannot be bounded by any $\boldsymbol{\beta}$-summable sequence $\{h_{2\ell}\}$.

To study the second case, that is when $\phi(x)$ is a bivariate Gaussian distribution, we need some preliminary results. It is easily verified that for $\mathscr{R} \in \mathrm{SO}(d)$ and $\phi_{\mathscr{R}}(\mathbf{x}) = \phi(\mathscr{R}^{-1}\mathbf{x})$ we have

$$\kappa_{\phi_{\mathscr{R}}}(\mathbf{t}) = \kappa_\phi(\mathscr{R}^{-1}\mathbf{t}), \quad \mathbf{t} \in \mathbb{R}^d,$$

where κ_ϕ is the function introduced in (4.2). In the following, let

$$F(z) = \frac{1}{\sqrt{\pi}} \int_z^\infty \mathrm{e}^{-y^2}\,\mathrm{d}y$$

stand for half the complementary error function [1].

Proposition 5.1. *Let* $\mathbb{R}^d_+ = \{\mathbf{x} \in \mathbb{R}^d \colon x_1 \geqslant 0\}$. *Then for* $\mathbf{t}, \mathbf{s} \in \mathbb{R}^d_+$ *we have*

$$\int_{\mathbb{R}^d_+} \phi(\mathbf{x}-\mathbf{t})\phi(\mathbf{x}-\mathbf{s})\,\mathrm{d}\mathbf{x} = (2\pi)^{-d/2}(\det\Sigma)^{1/2}\mathrm{e}^{-\frac{1}{2}(\Sigma[\mathbf{t}-\mathbf{s}],\mathbf{t}-\mathbf{s})}\left[1 - F(\delta_1)\right],$$

where

$$\delta_1 = \frac{(t_1+s_1)/\sqrt{2}}{\sqrt{\tau_{11}^2+\tau_{12}^2+\cdots+\tau_{1d}^2}} \tag{5.2}$$

and $\Sigma^{-1/2} = (\tau_{ij})_{i,j=1}^d$.

Proof. For $\mathbf{t},\mathbf{s}\in\mathbb{R}_+^d$ and $\mathbf{u}=\frac{1}{2}(\mathbf{t}+\mathbf{s})$ we have

$$\begin{aligned}
&\textstyle\int_{\mathbb{R}_+^d} \phi(\mathbf{x}-\mathbf{t})\phi(\mathbf{x}-\mathbf{s})\,\mathrm{d}\mathbf{x}\\
&\quad = (2\pi)^{-d/2}(\det\Sigma)^{1/2}\mathrm{e}^{-\frac{1}{2}(\Sigma[\mathbf{t}-\mathbf{s}],\mathbf{t}-\mathbf{s})}\textstyle\int_{\mathbb{R}_+^d} \mathrm{e}^{-2(\Sigma[\mathbf{x}-\mathbf{u}],\mathbf{x}-\mathbf{u})}\,\mathrm{d}\mathbf{x}\\
&\quad = 2^{-d/2}\pi^{-d}(\det\Sigma)^{1/2}\mathrm{e}^{-\frac{1}{2}(\Sigma[\mathbf{t}-\mathbf{s}],\mathbf{t}-\mathbf{s})}\textstyle\int_{\sqrt{2}\,\Sigma^{1/2}(\mathbb{R}_+^d-\mathbf{u})} \mathrm{e}^{-\|\mathbf{y}\|^2}\,\mathrm{d}\mathbf{y},
\end{aligned}$$

where

$$\sqrt{2}\,\Sigma^{1/2}(\mathbb{R}_\pm^d-\mathbf{u}) = \left\{\Sigma^{1/2}(y_1-u_1\sqrt{2},y_2,\ldots,y_d):(\pm y_1)\geqslant 0\right\}.$$

Now note that the orthogonal projection of the origin onto the hyperplane $\mathscr{L} = \{\Sigma^{1/2}(-u_1\sqrt{2},x_2,\ldots,x_d):x_2,\ldots,x_d\in\mathbb{R}\}$ is the point $(-\delta_1,\ldots,-\delta_d)$, where δ_1 is given by (5.2). Since

$$\begin{aligned}
\pi^{-d/2}\int_{\sqrt{2}\,\Sigma^{1/2}(\mathbb{R}_+^d-\mathbf{u})} \mathrm{e}^{-\|\mathbf{y}\|^2}\,\mathrm{d}\mathbf{y} &= \pi^{-d/2}\int_{-\delta_1}^{\infty}\mathrm{e}^{-y_1^2}\,\mathrm{d}y_1\prod_{j=2}^{d}\int_{-\infty}^{\infty}\mathrm{e}^{-y_j^2}\,\mathrm{d}y_j\\
&= \frac{1}{\pi}\int_{-\delta_1}^{\infty}\mathrm{e}^{-y_1^2}\,\mathrm{d}y_1 = 1-F(\delta_1),
\end{aligned}$$

the result of the proposition is clear. □

We now prove the most general result.

Theorem 5.2. *Let $\preceq$ be a linear order on $\mathbb{R}^d$ such that $i+l\preceq j+l$ and $ci\preceq cj$ whenever $i\preceq j$ and $c\geqslant 0$, and let $\mathscr{R}\in\mathrm{SO}(d)$ be such a rotation that*

$$\mathbb{R}_+^d(\preceq) = \{\mathbf{x}\in\mathbb{R}^d:\mathbf{x}\succeq 0\} = \mathscr{R}\{(x_1,\ldots,x_d):x_1\geqslant 0\}.$$

Then for $\mathbf{t},\mathbf{s}\in\mathbb{R}_+^d$ we have

$$\begin{aligned}
\int_{\mathbb{R}_+^d(\preceq)}\phi(\mathbf{x}-\mathbf{t})\phi(\mathbf{x}-\mathbf{s})\,\mathrm{d}\mathbf{x} &= (2\pi)^{-d/2}(\det\Sigma)^{1/2}\mathrm{e}^{-\frac{1}{2}(\Sigma[\mathbf{t}-\mathbf{s}],\mathbf{t}-\mathbf{s})}\\
&\quad\times\left[1-F\left(\frac{[\mathscr{R}(\mathbf{t}+\mathbf{s})]_1/\sqrt{2}}{\sqrt{\tau_{11}^2+\cdots+\tau_{1d}^2}}\right)\right],
\end{aligned}$$

where $\Sigma^{1/2} = (\tau_{ij})_{i,j=1}^d$.

Proof. The proof requires a simple change of variables, where it is to be observed that $\det(\mathscr{R}\Sigma\mathscr{R}^{-1}) = \det\Sigma$ and the orthogonality of $\mathscr{R}$ assures that the rows of $(\mathscr{R}\Sigma\mathscr{R}^{-1})^{1/2}$ have the same lengths as those of $\Sigma^{1/2}$. □

Corollary 5.3. *We have the estimate*

$$0 \leqslant k_\phi(\mathbf{t},\mathbf{s}) - \int_{\mathbb{R}^d_+(\preceq)} \phi(\mathbf{x}-\mathbf{t})\phi(\mathbf{x}-\mathbf{s})\,\mathrm{d}\mathbf{x} \leqslant (2\pi)^{-d/2}(\det\Sigma)^{1/2}\mathrm{e}^{-\frac{1}{2}(\Sigma[\mathbf{t}-\mathbf{s}],\mathbf{t}-\mathbf{s})}. \tag{5.3}$$

Using the asymptotic expression ([1], 7.1.23)

$$F(z) \sim \frac{1}{2z\sqrt{\pi}}\mathrm{e}^{-z^2}, \quad z \to +\infty,$$

it is clear that the middle member of (5.3) has arbitrarily fast exponential decay in the direction of positive increasing x, orthogonal to the $\preceq$-separating hyperplane $\mathscr{R}(\{x_1 = 0\})$.

As in Section 4, we take $d = 2$, $\Sigma = I$ and let $\mathscr{R}$ be the identity of SO(2), so that the order $\preceq$ concides with the lexicographical order on $\mathbb{Z}^2$, already introduced in the previous example. Then, the Gram matrix we seek is given by

$$G_{ij} = \int_{\mathbb{R}^2_+} \phi(\mathbf{x}-i)\phi(\mathbf{x}-j)\,\mathrm{d}\mathbf{x} = T_{i-j} - T_{i-j}H_{i+j}, \quad i,j \in \mathbb{Z}^2_+,$$

where T is given in (4.3) and

$$H_{i+j} = F\left(\frac{i_1+j_1}{\sqrt{2}}\right).$$

The remark following Corollary 5.3 shows that the perturbation matrix $S = (-T_{i-j}H_{i+j})_{i,j\in\mathbb{Z}^2_+}$ decays exponentially in the direction orthogonal to the line $i_1 = 0$ of $\mathbb{Z}^2$, but **not** along the same line. This is true, in general, of any Gram matrix defined on the half-space, since the perturbation of the entries with indexes along the separating hyperplane does not decay on the points of the hyperplane approaching infinity.

Even if this general situation does not fall within the limits of definition (2.10), we applied the projection algorithm of Section 3 as well, to analyze its numerical performance on these more difficult problems.

Fig. 4 reports the logarithm of the relative error for a semi-infinite linear system characterized by the Gram matrix of the box-splines on the half-plane. The solution of the system is the same one used in the other numerical tests, and the parameters, introduced in the preceding section, are fixed at $N = 128$ and $p = 30$. Again, the

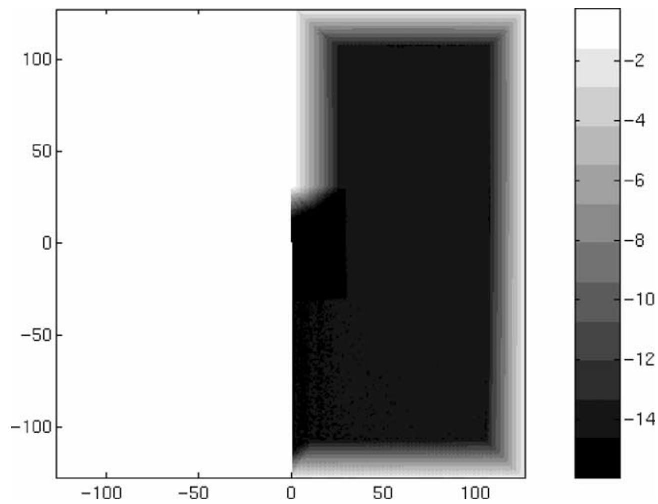

Fig. 4. Relative error in $\log_{10}$-scale (box-splines, $N = 128$, $p = 30$).

relative error is high for those components $\hat{x}_i$ of the solution with $\|i\|_\infty \sim N$, but, as already pointed out, this effect can be circumvented by increasing the value of N, without a significant increase in the computational effort.

On the contrary, the high relative error found for $i_1 \sim 0$ and $i_2 > 0$ is the effect of the perturbation induced by the border of the half-plane, and cannot be avoided while applying the projection algorithm, i.e., partitioning the infinite system into a finite unstructured system and a semi-infinite Toeplitz system. This error is negligible only for that part $\hat{\mathbf{x}}_E$ of the solution which is computed by solving a finite linear system, and this suggests a possible workaround. In fact, it would suffice to modify the definition of the finite set E, extending it along the vertical axis so that it includes the indexes of all the components of the solution near the border and inside the region of interest. This approach obviously makes it necessary to solve a large finite system, and again asks for the use of an iterative algorithm.

The same effect can be observed in Fig. 5, reporting the relative error for the Gram matrix of Gaussians on the half-plane. In this case the effect of the perturbation is much more evident since the perturbation is not localized only on some columns of the matrix, but decays exponentially in the direction orthogonal to the border of the half plane. A deeper look into the behavior of the solution is given by Fig. 6, reporting the absolute error

$$e_i = |x_i - \hat{x}_i|, \quad i \in \mathbb{Z}_+^2,$$

for the same example, and clearly showing that this kind of error is decreasing also in the vertical direction, but with a slower rate compared to the other directions.

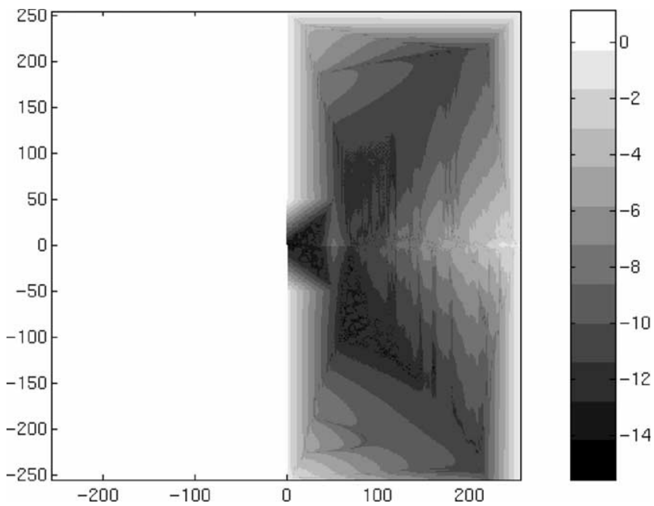

Fig. 5. Relative error in $\log_{10}$-scale (Gaussians, $N = 256$, $p = 50$).

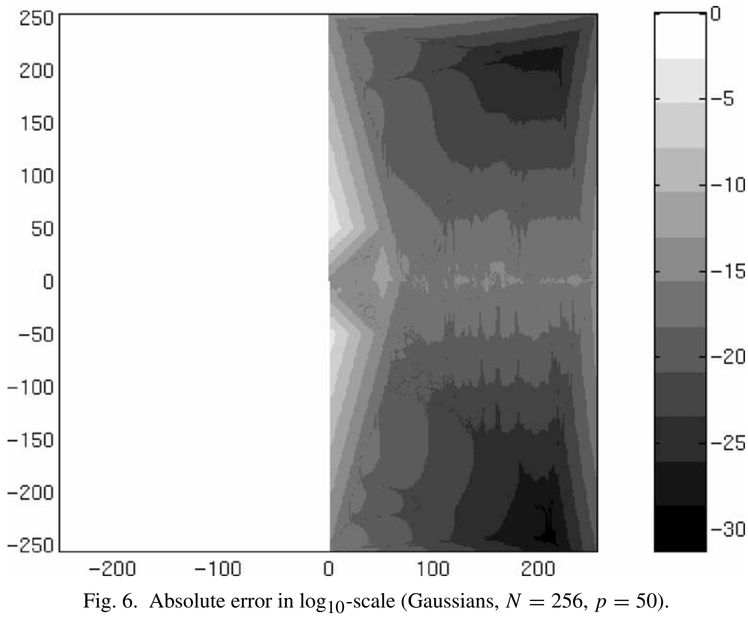

Fig. 6. Absolute error in $\log_{10}$-scale (Gaussians, $N = 256$, $p = 50$).

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