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DEAD CORES FOR PARABOLIC REACTION-DIFFUSION PROBLEMS

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Abstract. In this article sufficient conditions are derived for the existence and the nonexistence of a dead core for at least one of the solutions (u, v) of a nonlinear parabolic reaction-diffusion system with Robin boundary conditions. Simplifying assumptions on the diffusion coefficients allow one to decouple the system and to use super and subsolution methods to guarantee or to exclude the existence of a dead core.
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1 Introduction

In chemical engineering, one studies reaction-diffusion equations which have the form of the coupled system

$$\frac{\partial u}{\partial t} - \mu \Delta u = -u^m \, e^{v/(1+\varepsilon v)}, \tag{1.1}$$

$$\nu \frac{\partial v}{\partial t} - \Delta v = \beta u^m \, e^{v/(1+\varepsilon v)}, \tag{1.2}$$

where $\mu, \nu, \varepsilon > 0$ are physical constants, $m = 1, 2, 3, \cdots$, and β is positive, zero or negative depending on whether the reaction is exothermic, isothermic or endothermic [11]. Depending on the specific chemical applications, various boundary conditions are imposed. The function $e^{v/(1+\varepsilon v)}$ can be replaced by e.g. $e^{-\gamma/v}$ for some $\gamma > 0$ [10].

In this article we consider the following initial-boundary value problem for $u(x,t)$ and $v(x,t)$:

$$u_t - Lu = -\lambda f(u)g(v) \qquad \text{in } Q = \Omega \times \mathbb{R}^+, \tag{1.3}$$

$$v_t - Lv = -k\lambda f(u)g(v) \qquad \text{in } Q = \Omega \times \mathbb{R}^+, \tag{1.4}$$

$$\frac{\partial u}{\partial n} + \sigma u(x,t) = \chi(x) \qquad \text{in } \Gamma = \partial\Omega \times \mathbb{R}^+, \tag{1.5}$$

$$\frac{\partial v}{\partial n} + \sigma v(x,t) = \eta(x) \qquad \text{in } \Gamma = \partial\Omega \times \mathbb{R}^+, \tag{1.6}$$

$$u(x,0) = u_0(x) \qquad \text{in } \Omega, \tag{1.7}$$

$$v(x,0) = v_0(x) \qquad \text{in } \Omega, \tag{1.8}$$

where the domain Ω is a bounded open connected set in $\mathbb{R}^N$ $(N \geq 1)$ whose boundary is a surface of class C^3, n is the unit outer normal to $\partial\Omega$, k, λ are positive constants, the Robin boundary conditions (1.5) and (1.6) with the same $\sigma > 0$ are imposed, the differential operator L defined by

$$Lu = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(A_{ij}(x,t)\frac{\partial u}{\partial x_j}\right) \tag{1.9}$$

has coefficients $A_{ij}(x,t)$ which are bounded and continuous on Q, are continuously differentiable in x for every t, and satisfy the uniform ellipticity condition

$$\Lambda_{\min}|\xi|^2 \leq \sum_{i,j=1}^{n} A_{ij}(x,t)\xi_i\xi_j \leq \Lambda_{\max}|\xi|^2 \tag{1.10}$$

for certain $\Lambda_{\min}, \Lambda_{\max} > 0$, $\chi(x)$ and $\eta(x)$ are continuous and nonnegative on $\partial\Omega$, $u_0(x)$ and $v_0(x)$ can be extended to nonnegative continuous functions on $\overline{\Omega}$, and

$$\begin{cases} f, g \in C[0,\infty) \cap C^2(0,\infty), \\ f(0) = g(0) = O(s), \ s \to 0^+, \\ f'(s) > 0, \ g'(s) > 0 \ (s > 0). \end{cases}$$

In other words, we study a class of initial-boundary value problems modelling reaction-diffusion where the chemical reaction is endothermic and the diffusion coefficients of the two types of reactants are equal.

The existence and nonexistence of a dead core (i.e., of a region where an otherwise nontrivial solution vanishes identically) have been studied in many papers. To mention just a few key results, Bandle and Stakgold [3] have studied such a problem for a single parabolic equation of the type $u_t - Lu = -\lambda f(u)$ in Q with initial condition (1.7) and a Dirichlet condition instead of (1.5), whereas Bobisud and Stakgold [4] have tackled the stationary problem corresponding to the system of equations (1.3)-(1.8), where (1.5) and (1.6) are replaced by a Dirichlet condition and L is the Laplacian. In [12], the methods developed in [3, 4] have been applied to the parabolic counterpart of the problem studied in [4]. In particular, in [12] the differential operator L

is the Laplacian and Dirichlet conditions are imposed. In this article we seek sufficient conditions on $f, g, k, \lambda, L, \sigma$ in order that the system of equations (1.3)-(1.8) has or does not have a dead core. In contrast to the previous papers, we impose Robin boundary conditions.

The basic technique underlying the analysis of (1.3)-(1.8) is to introduce the auxiliary function $w = -v + ku$, to subtract (1.4), (1.6) and (1.8) from the respective equations (1.3), (1.5) and (1.7), and to arrive at a *linear* parabolic system for w and *uncoupled* systems of equations for u and v. Once the relevant properties of w are known, super and subsolution methods are applied to the resulting system for u if $w \leq 0$ and to the resulting system for v if $w \geq 0$. The necessary upper and lower solutions are to be generated as solutions of either the stationary problem (where t-dependence is absent) or the lumped problem (where x-dependence is absent). We will employ this technique in the present situation, relying on estimates by Díaz [5] to deal with an arbitrary uniformly elliptic operator L and existence results for parabolic systems with Robin boundary conditions [6, 8].

2 Uncoupling the Parabolic System

Following [12], we introduce

$$w = -v + ku. \tag{2.1}$$

From (1.3)-(1.8) we then obtain the initial-boundary value problem

$$\begin{cases} w_t - Lw = 0 & \text{in } Q, \\ \dfrac{\partial w}{\partial n} + \sigma w(x,t) = -\eta(x) + k\chi(x) & \text{in } \Gamma, \\ w(x,0) = -v_0(x) + ku_0(x) & \text{in } \Omega. \end{cases} \tag{2.2}$$

Problem (2.2) has a unique classical solution and has been studied in [6, 8, 9].

Eliminating either v or u, we obtain the initial-boundary value problems

$$\begin{cases} u_t - Lu = -\lambda f(u) g(ku - w) & \text{in } Q, \\ \dfrac{\partial u}{\partial n} + \sigma u(x,t) = \chi(x) & \text{in } \Gamma, \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{2.3}$$

and

$$\begin{cases} v_t - Lv = -k\lambda f\left(\frac{v+w}{k}\right) g(v) & \text{in } Q, \\ \dfrac{\partial v}{\partial n} + \sigma v(x,t) = \eta(x) & \text{in } \Gamma, \\ v(x,0) = v_0(x) & \text{in } \Omega. \end{cases} \tag{2.4}$$

Let us define a *supersolution* $(\overline{u}, \overline{v})$ (resp. *subsolution* $(\underline{u}, \underline{v})$) of (1.3)-(1.8) to be a pair of functions (u, v) such that (1.3)-(1.8) hold with the inequality sign $\geq$ (resp. $\leq$) instead of the equality sign.

3 Comparison theorems

3.1. Following [2, 12], we obtain the following comparison results for the solutions of problem (1.3)-(1.8):

(a) Let (u_1, v_1) and (u_2, v_2) be the solutions corresponding to λ_1 and λ_2, respectively, with $\lambda_1 \leq \lambda_2$; then $u_2 \leq u_1$ and $v_2 \leq v_1$ in Q.

(b) If $f_1 \leq f_2$, $g_1 \leq g_2$ and $w_2 \leq w_1 \leq 0$, then $u_2 \leq u_1$ in Q. Analogously, if $f_1 \leq f_2$, $g_1 \leq g_2$ and $0 \leq w_1 \leq w_2$, then $v_2 \leq v_1$ in Q.

(c) If any of the initial and/or boundary data is decreased while w remains the same, either of the solutions u and v is decreased.

The comparison results (a)-(c) can easily be proved by standard super and subsolution techniques (cf. e.g. [1, 12]).

3.2. Prior to analyzing the existence of dead cores of (u, v), we make the following observations. Consider the unique solution (z, y) of the so-called lumped problem

$$\begin{cases} z_t = -\lambda f(z)g(y) & \text{in } \mathbb{R}^+, \\ y_t = -k\lambda f(z)g(y) & \text{in } \mathbb{R}^+, \\ z(0) = z_0, \\ y(0) = y_0, \end{cases} \tag{3.1}$$

which is obtained from (1.3)-(1.8) by dropping the x-dependent diffusion term. Then it is easily seen that $(\tilde{u}, \tilde{v})$ with

$$\tilde{u}(x,t) = z(t), \qquad \tilde{v}(x,t) = y(t),$$

is the unique solution of (1.3)-(1.8), where $\tilde{u}|_{\partial\Omega} \equiv z_0$, $\tilde{u}|_{t=0} \equiv z_0$, $\tilde{v}|_{\partial\Omega} \equiv y_0$, and $\tilde{v}|_{t=0} \equiv y_0$. In other words, if the initial and boundary values of u (resp. v) are equal to the same positive constant z_0 (resp. y_0), then u and v do not depend on x.

Since the lumped problem does not depend on the diffusion term, we may employ one of its solutions as either a supersolution or a subsolution of the system of equations (1.3)-(1.8). Therefore, the first two parts of Theorem 3.1 of [12] still hold.

Theorem 3.1 *Let $v_0 = ku_0$ in $\overline{\Omega}$ be nonnegative. Put $H(s) = \int_0^s f(t)g(kt)dt$. Then the following statements are true:*

1. *If $\chi = \eta = 0$ in $\partial\Omega$ and $\int_0^1 \frac{ds}{f(s)g(ks)} < \infty$ (strong absorption), then there is simultaneous extinction of both species in finite time, i.e., $u(x,t) = v(x,t) = 0$ for $t \geq t_*$ and $x \in \overline{\Omega}$, where $t_* = \frac{1}{\lambda}\int_0^{z_0} \frac{ds}{f(s)g(ks)}$ and $z_0 = \max_{\overline{\Omega}} u_0$.*
2. *If $\min_{\overline{\Omega}} v_0 = k \min_{\overline{\Omega}} u_0 > 0$ and $\int_0^1 \frac{ds}{f(s)g(ks)} = \infty$ (weak absorption), then $v(x,t) = ku(x,t) > 0$ for all $(x,t) \in Q$.*

4 Dead core for the associated steady-state problem

Let us associate to (2.3) the corresponding steady-state problem

$$\begin{cases} -Lu = -\lambda h(u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} + \sigma u(x) = \chi(x) & \text{in } \partial\Omega, \end{cases} \tag{4.1}$$

where $h(u) = f(u)g(ku - w)$ whenever $w \le 0$ is fixed, or let us associate to (2.4) a steady-state problem of the same form for v, where $h(v) = kf(\frac{v+w}{k})g(v)$ whenever $w \ge 0$ is fixed. Here $\chi \in C(\partial\Omega)$ and is nonnegative. In particular, $h \in C[0,\infty) \cap C^1(0,\infty)$, $h(0) = 0$, and $h'(s) > 0$ for $s > 0$. Moreover, recall that $\overline{u}$ is defined to be a supersolution of (4.1) if

$$-L\overline{u} \ge -\lambda h(\overline{u}), \qquad \left(\frac{\partial \overline{u}}{\partial n} + \sigma\overline{u}\right)\bigg|_{\partial\Omega} \ge \chi.$$

Similarly, if we reverse the inequalities, $\underline{u} \ge 0$ is a subsolution of (4.1). In order to find the existence of a subregion of Ω where the solution of the associated steady-state problem vanishes identically, we prove the following

Theorem 4.1 *Let $v_0 = ku_0$ in $\overline{\Omega}$ be nonnegative. Put $H(s) = \int_0^s h(t)\,dt$ such that the integral $\int_0^1 H(t)^{-1/2}\,dt < \infty$. Then every classical solution of the steady-state problem (4.1) has a dead core, i.e., there exists a nonempty open subset $D \subset \Omega$ on which it vanishes identically.*

Proof. Let u be a weak solution of (4.1). Then it is classical when restricted to any open connected subregion Ω_0 with C^∞ boundary such that $\overline{\Omega_0} \subset \Omega$ (cf. [7]).

Let us now consider

$$\begin{cases} -Lu = -\lambda h(u) & \text{in } \Omega_0, \\ u|_{\partial\Omega_0} = u(x) & \text{in } \partial\Omega_0, \end{cases} \tag{4.2}$$

and construct a supersolution of (4.2) with a dead core.

With no loss of generality, we consider the following ball:

$$B_R(0) = \{x \in R^n : |x| < R\} \subset \Omega_0.$$

Further, we extend h to an odd function defined on R. For any radially symmetric function $\eta \in C[0,R] \cap C^2(0,R)$, we compute $(L\eta)(x)$ as in Appendix A and obtain

$$-L\eta(r) \ge -\Lambda_{\max}\eta''(r) - M_0\frac{\eta'(r)}{r}, \qquad r \in (0,R),$$

where M_0 is some constant. Put $M_1 = M_0/\Lambda_{max}$. Then, if $\eta(r)$ satisfies $\eta(r) = 0$ for $0 < r < R/2$, then

$$\begin{cases} \eta''(r) + M_1 \dfrac{\eta'(r)}{r} = \dfrac{\lambda}{\Lambda_{max}} h(\eta(r)), & r \in (R/2, R), \\ \eta(R/2) = \eta'(R/2) = 0, \end{cases} \tag{4.3}$$

Then $\overline{\phi}(x) = \eta(|x|)$ is a supersolution of (4.1) with $\chi(x) = 0$. Indeed,

$$-L(\overline{\phi}) \geq -\Lambda_{max}\eta'' - M_0 \frac{\eta'}{r} = -\lambda h(\eta) = -\lambda h(\overline{\phi}).$$

According to [5], the condition $\int_0^1 H(t)^{-1/2}\,dt < \infty$ implies the existence of a dead core of the solution η of (4.3) which is nonnegative. Since $\eta(r) = 0$ for $0 \leq r \leq R/2$ and $0 \leq u \leq \eta$, we have $u(|x|) = 0$ for $0 \leq r \leq R/2$.

As a result, the solution u of (4.1) on Ω has a dead core. □

5 Dead core for the solution (u(x,t), v(x,t))

In this section we study the behavior of the solution $(u(x,t), v(x,t))$ in comparison with the dead cores of the associated lumped and steady-state problems. We derive two results, one where $v_0 \equiv ku_0$ and $\eta \equiv k\chi$ are nonnegative, so that the auxiliary function w solving (2.2) vanishes identically, and one where u_0, v_0, χ and η are nonnegative but $v_0 \not\equiv ku_0$ and $\eta \not\equiv k\chi$.

Theorem 5.1 *Let (u, v) be the solution of (1.3)-(1.8) with $-v_0 + ku_0 \equiv 0$ in $\overline{\Omega}$ and $-\eta + k\chi \equiv 0$ in $\partial\Omega$, so that $v \equiv ku$. Put $h(u) = f(u)g(ku)$. Then we have the following:*

1. *If $\int_0^1 \frac{ds}{\sqrt{H(s)}} < \infty$ (dead core for steady-state), $I = \int_0^1 \frac{ds}{f(s)g(ks)}\,ds < \infty$ (dead core for the lumped problem), and $\lambda_0 = \inf_\lambda\{\phi(x_0, \lambda) = 0\}$ for a fixed $x_0 \in \Omega$, then we have a dead core for both $u(x_0, t)$ and $v(x_0, t)$ for $t \geq I/(\lambda - \lambda_0)$.*
2. *If $\int_0^1 \frac{ds}{\sqrt{H(s)}} < \infty$, and $I = \infty$, and $\min_{\overline{\Omega}} v_0(x) = \min_{\overline{\Omega}} ku_0(x) > 0$, then $v(x,t) = ku(x,t) > 0$ for all $(x,t) \in Q$.*

Proof. To prove the first part of the theorem, under the above hypotheses we will introduce a supersolution $\overline{u}(x,t)$ of (1.3)-(1.8) such that $\overline{u}(x_0, t) = 0$ for $t > I/(\lambda - \lambda_0)$. Following [1], let $\alpha(x,t) = z(t) + \phi(x)$, where $z(t)$ is the solution of the lumped problem

$$\begin{cases} z_t = -\gamma h(z), & t > 0, \\ z(0) = z_0 = \max_{\overline{\Omega}} u_0(x), \end{cases}$$

for some $\gamma > 0$, and $\phi(x)$ is the solution of the steady-state problem

$$\begin{cases} -L(\phi(x)) = -\lambda_0 h(\phi(x)), & x \in \Omega, \\ \dfrac{\partial \phi}{\partial n}(x) + \sigma\phi(x) = \chi(x), & x \in \partial\Omega. \end{cases}$$

We then have

$$\alpha(x_0, t) = z(t) + \phi(x_0) = 0, \qquad t > (I/\gamma).$$

Moreover, $\alpha(x,t) \geq \max(z(t), \phi(x))$. Now $\alpha(x,0) \geq u_0(x)$ and

$$\frac{\partial \phi}{\partial n}(x) + \sigma\phi(x) \geq \chi(x), \qquad x \in \partial\Omega.$$

Now we determine the value of γ such that α is a supersolution. To this end, let us recall that

$$\alpha_t = z_t = -\gamma h(z),$$
$$\alpha_t - L\alpha + \lambda h(\alpha) = -\gamma h(z) - L\phi + \lambda h(\alpha) \geq (-\gamma - \lambda_0 + \lambda) h(\alpha) = 0,$$

if we choose $\gamma = \lambda - \lambda_0$. Thus, if $\gamma = \lambda - \lambda_0$, we can write

$$\begin{cases} \alpha_t - L\alpha \geq -\lambda h(\alpha) & \text{in } Q, \\ \dfrac{\partial \alpha}{\partial n} + \sigma\alpha(x) \geq \chi(x) & \text{on } \Gamma, \\ \alpha(x,0) \geq u_0(x) & \text{in } \Omega. \end{cases}$$

Since α is a supersolution of (1.3)-(1.8) which has a dead core, also $v = ku$ has a dead core.

To prove the second part, we construct positive subsolutions, following the second part of Theorem 3.1. □

The final result of this paper can be proved as Theorem 3.2 of [12].

Theorem 5.2 *Let $-v_0 + ku_0 \not\equiv 0$ in $\overline{\Omega}$ and be nonnegative there. Put $G(s) = \int_0^s g(t)\,dt$, let $I_G = \int_0^1 \frac{ds}{\sqrt{G(s)}} < \infty$, and let w be the solution of the boundary value problem*

$$\begin{cases} Lw = 0 & \text{in } \Omega, \\ \dfrac{\partial w}{\partial n} + \sigma w = -\eta + k\chi & \text{in } \partial\Omega. \end{cases} \tag{5.1}$$

Then the following statements are true:

1. *If $I_g = \int_0^1 \frac{ds}{g(s)} < \infty$, then for all $x_0 \in \Omega$ there exists λ_0 such that $v(x_0, t) = 0$ whenever $(\lambda - \lambda_0)t \geq I_g$.*
2. *If $I_g = \infty$, then $u(x,t) > 0$ and $v(x,t) > 0$ for all $(x,t) \in Q$, where $u \to (w/k)$ and $v \to 0$ as $t \to \infty$.*

Analogously, let $-v_0 + ku_0 \not\equiv 0$ *in* $\overline{\Omega}$ *and be nonpositive there. Put* $F(s) = \int_0^s f(t)\,dt$, *let* $I_F = \int_0^1 \frac{ds}{\sqrt{F(s)}} < \infty$, *and let* w *be as in* (5.1). *Then the following statements are true:*

1′. *If* $I_f = \int_0^1 \frac{ds}{f(s)} < \infty$, *then for all* $x_0 \in \Omega$ *there exists* λ_0 *such that* $u(x_0, t) = 0$ *whenever* $(\lambda - \lambda_0)t \geq I_f$.

2′. *If* $I_f = \infty$, *then* $u(x,t) > 0$ *and* $v(x,t) > 0$ *for all* $(x,t) \in Q$, *where* $u \to 0$ *and* $v \to -w$ *as* $t \to \infty$.

A Differentiating Spherically Symmetric Functions

Let us compute $(Lu)(x)$ where $u = \eta(r)$ is spherically symmetry. Using that $(\partial\eta/\partial x_j) = \eta'(r)(x_j/r)$, we obtain

$$\begin{aligned}(L\eta)(x) &= \sum_{i,j} A_{ij}(x,t)\frac{\partial^2\eta}{\partial x_i \partial x_j} + \sum_{i,j}\left(\frac{\partial\eta}{\partial x_j}\frac{\partial}{\partial x_i}A_{ij}(x,t)\right)\\
&= \sum_{i\neq j} A_{ij}(x,t)\frac{x_i x_j}{r^2}\left(\eta''(r) - \frac{\eta'(r)}{r}\right)\\
&\quad + \sum_i A_{ii}(x,t)\left[\frac{x_i^2}{r^2}\eta''(r) + \left(1 - \frac{x_i^2}{r^2}\right)\frac{\eta'(r)}{r}\right]\\
&\quad + \frac{\eta'(r)}{r}\sum_{i,j} x_j \frac{\partial}{\partial x_i}A_{ij}(x,t)\\
&= \eta''(r)\sum_{i,j} A_{ij}(x,t)\frac{x_i x_j}{r^2}\\
&\quad + \frac{\eta'(r)}{r}\left[-\sum_{i,j} A_{ij}(x,t)\frac{x_i x_j}{r^2} + \sum_i A_{ii}(x,t) + \sum_{i,j} x_j \frac{\partial}{\partial x_i}A_{ij}(x,t)\right].\end{aligned}$$

Now note that, due to (1.10),

$$\Lambda_{\min} \leq \sum_{i,j} A_{ij}(x,t)\frac{x_i x_j}{r^2} \leq \Lambda_{\max}, \qquad \left|\sum_{i,j} x_j \frac{\partial}{\partial x_i}A_{ij}(x,t)\right| \leq nMr,$$

where

$$\left[\sum_{j=1}^n \left|\frac{\partial}{\partial x_i}A_{ij}(x,t)\right|^2\right]^{1/2} \leq M, \qquad (x,t) \in Q.$$

Thus for any nondecreasing convex function $\eta \in C[0,R] \cap C^2(0,R)$ we have

$$-(L\eta)(x) \geq -\Lambda_{\max}\eta''(r) - M_0\frac{\eta'(r)}{r},$$

where $M_0 = nMR + \Lambda_{\max} + \sup |\sum_i A_{ii}(x,t)|$.

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