INSTITUTE OF PHYSICS PUBLISHING INVERSE PROBLEMS

Inverse Problems **17** (2001) 1831–1845 PII: S0266-5611(01)27405-X

The inverse generalized Regge problem

Vjacheslav Pivovarchik[1] **and Cornelis van der Mee**[2]

[1] Odessa State University of Civil Engineering and Architecture, Didrichson Ul. 4, 65029 Odessa, Ukraine
[2] Dipartimento di Matematica, Università di Cagliari, Via Ospedale 72, 09124 Cagliari, Italy

E-mail: v.pivovarchik@paco.net, vnp.@dtp.odessa.ua and cornelis@krein.sc.unica.it

Received 30 July 2001
Published 15 November 2001
Online at stacks.iop.org/IP/17/1831

Abstract
In this paper the inverse eigenvalue problem of recovering the real coefficients in a Sturm–Liouville problem with nonselfadjoint boundary conditions depending on the spectral parameter from the eigenvalues is solved using entire-function theory and the solution of a Marchenko integral equation.

1. Introduction

Small transversal vibrations of a nonhomogeneous string are described by the boundary value problem

$$\frac{\partial^2 u}{\partial s^2}(s,t) - \rho(s)\frac{\partial^2 u}{\partial t^2}(s,t) = 0, \tag{1.1}$$

$$u(0,t) = 0, \tag{1.2}$$

$$\left.\frac{\partial u}{\partial s}\right|_{s=l} + \nu \left.\frac{\partial u}{\partial t}\right|_{s=l} = 0. \tag{1.3}$$

Here $u(s,t)$ stands for the transversal displacement at the position s and time t, $\rho(s)$ is the density of the string, l is its length and $\nu > 0$ is the damping coefficient at the right-hand endpoint of the string. The left-hand endpoint is fixed. Substituting $u(s,t) = v(s)\mathrm{e}^{\mathrm{i}\lambda t}$ into (1.1)–(1.3), we obtain

$$v_{ss} + \lambda^2\rho(s)v = 0, \tag{1.4}$$

$$v(\lambda,0) = 0, \tag{1.5}$$

$$v_s(\lambda,s)|_{s=l} + \mathrm{i}\lambda\nu v(\lambda,l) = 0. \tag{1.6}$$

If the density $\rho(s)$ satisfies $\rho \in W_2^2(0,l)$ and $\rho(s) > 0$ for $s \in [0,l]$, then we write $\rho[x] := \rho(s(x))$, apply the Liouville transformation [3]

$$x(s) = \int_0^s \rho(s')^{1/2}\,\mathrm{d}s', \tag{1.7}$$

$$y(\lambda,x) = \rho[x]^{1/2}v(\lambda,s(x)), \tag{1.8}$$

and obtain

$$y''(\lambda, x) + \lambda^2 y(\lambda, x) - q(x)y(\lambda, x) = 0, \tag{1.9}$$

$$y(\lambda, 0) = 0, \tag{1.10}$$

$$y'(\lambda, a) + (\mathrm{i}\alpha\lambda + \beta)y(\lambda, a) = 0, \tag{1.11}$$

where primes denote x-differentiation and

$$q(x) = \rho[x]^{-1/4}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\rho[x]^{1/4}\right), \tag{1.12}$$

$$a = \int_0^l \rho(s)^{1/2}\,\mathrm{d}s, \tag{1.13}$$

$$\alpha = \rho[a]^{-1/2}\nu, \tag{1.14}$$

$$\beta = -\frac{1}{4}\rho[a]^{-1}\left.\frac{\mathrm{d}\rho[x]}{\mathrm{d}x}\right|_{x=a}. \tag{1.15}$$

We remark that $\alpha > 0$, $\beta \in \boldsymbol{R}$ and $q \in L_2(0, a)$ is real valued. It follows from (1.12), (1.15) and the inequality $\rho[x] \geqslant \varepsilon > 0$ that the operator A defined by

$$Af = -f'' + qf,$$
$$D(A) = \left\{f \in W_2^2(0, a) : f'(a) + \beta f(a) = 0,\ f(0) = 0\right\}$$

is strictly positive selfadjoint. We shall consider the problem (1.9)–(1.11) under the general conditions that $q \in L_2(0, a)$ and A is selfadjoint and bounded below, but not necessarily strictly positive. For $\alpha = 1$ and $\beta = 0$ we obtain the so-called Regge problem [11, 19, 20]. When $\alpha > 0$, $\alpha \neq 1$ and $\beta \in \boldsymbol{R}$, the Riesz basis properties of the root functions of problem (1.9)–(1.11) were studied in [4, 10, 21].

Problem (1.9)–(1.11) can be treated as a spectral problem for a nonmonic quadratic operator polynomial, i.e. a quadratic operator polynomial where the leading coefficient is not the identity operator. Indeed, denote by A_1 the following operator acting on the Hilbert space $L_2(0, a) \oplus \boldsymbol{C}$:

$$A_1\begin{pmatrix} y \\ c \end{pmatrix} = \begin{pmatrix} -y'' + qy \\ y'(a) + \beta c \end{pmatrix},$$
$$D(A_1) = \left\{\begin{pmatrix} y \\ c \end{pmatrix} : y \in W_2^2(0, a),\ y(0) = 0,\ y(a) = c\right\}.$$

This operator with dense domain is selfadjoint and bounded below and has the same spectrum as A. Its spectrum consists of normal eigenvalues accumulating at infinity, because it has a compact resolvent. Put

$$K = \alpha K_1 = \begin{pmatrix} 0 & 0 \\ 0 & \alpha I \end{pmatrix}, \qquad M = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \tag{1.16}$$

where I denotes the identity operator. It is clear that $M \geqslant 0$ and $K \geqslant 0$ (i.e. that M and K are bounded non-negative selfadjoint). Problem (1.9)–(1.11) is nothing but the spectral problem for the operator polynomial

$$L(\lambda) := \lambda^2 M - \mathrm{i}\lambda K - A_1, \tag{1.17}$$

where $D(L(\lambda)) = D(A_1)$ is independent of λ.

In this paper we primarily study the inverse problem of constructing the real potential q in the problem (1.9)–(1.11) from its eigenvalues. First we prove that the eigenvalues of this problem coincide with the zeros of the entire function

$$\varphi(\lambda) = s'(\lambda, a) + (\mathrm{i}\lambda\alpha + \beta)s(\lambda, a),$$

where $s(\lambda, x)$ is the solution of (1.9) with initial conditions $\varphi(\lambda, 0) = 0$ and $\varphi'(\lambda, 0) = 1$. If $\alpha > 0$ and $\alpha \neq 1$, the asymptotics of the eigenvalues can easily be established. In this case well known properties of so-called sine-type entire functions can be employed to solve the inverse spectral problem, leading to two variants, one for $0 < \alpha < 1$ and one for $\alpha > 1$. When eigenvalues in the closed lower half-plane are absent, $\varphi(\lambda)$ belongs to the so-called Hermite–Biehler class [12] and hence the solution of the inverse eigenvalue problem simplifies considerably. For $\alpha = 1$ it is generally very difficult to solve the inverse problem, because of anomalies in the asymptotic behaviour of the eigenvalues. For instance, if $\alpha = 1$, $\beta = 0$ and $q = 0$, we have $\varphi(\lambda) = \mathrm{e}^{\mathrm{i}\lambda a}$, and hence there are no eigenvalues at all [19]. Further, as one easily verifies, if $\alpha = 1$ and $\beta \neq 0$, the eigenvalues are not confined to a horizontal strip (as when $\alpha > 0$ and $\alpha \neq 1$), but instead tend to infinity in absolute value with a logarithmic growth of the imaginary part. For these reasons we have decided to study the inverse eigenvalue problem only for $0 < \alpha < 1$ and $\alpha > 1$.

The present inverse eigenvalue study fits into a long tradition of reconstructions of Schrödinger equation potentials from spectral data, ever since the seminal papers by Borg [1, 2] on the necessity of using the spectra for two different boundary conditions for the same equation to reconstruct the potential. A more profound study of the properties of pairs of sequences leading to a unique determination of the potential was made by Levitan and Gasymov [15]. Expositions and reviews of the existing literature on the subject have been given by Levitan [14]. More recent results were obtained by Gesztesy and Simon [6]. Generalizations for some first-order systems and, by consequence, for higher-order equations were obtained by Malamud [16].

Let us describe the organization of this paper. In section 2 we give definitions and mention certain properties of operator polynomials. In section 3 we study the direct problem and in section 4 the inverse problem.

2. Basic properties

In this section we give some definitions and basic properties involving isolated eigenvalues of operator polynomials.

Definition 2.1. *Let $L(\lambda)$ be an operator polynomial defined on a complex Hilbert space $\mathcal{H}$. The set of values $\lambda \in \boldsymbol{C}$ such that $L(\lambda)^{-1}$ exists as a bounded linear operator on $\mathcal{H}$ is called the resolvent set $\varrho(L)$ of the operator polynomial $L(\lambda)$. We denote by $\sigma(L)$ the spectrum of $L(\lambda)$, i.e. the set $\sigma(L) = \boldsymbol{C}\backslash\varrho(L)$. The number $\lambda_0 \in \boldsymbol{C}$ is said to be an eigenvalue of $L(\lambda)$ if there exists a nonzero vector y_0 (called an eigenvector) such that $L(\lambda_0)y_0 = 0$. The vectors $y_1, y_2, \ldots, y_{r-1}$ are called corresponding associated eigenvectors if*

$$\sum_{s=0}^{n} \frac{1}{s!} \frac{\mathrm{d}^s}{\mathrm{d}\lambda^s} L(\lambda) \big|_{\lambda=\lambda_0} y_{n-s} = 0, \qquad n = 1, \ldots, r-1. \tag{2.1}$$

The number r is called the length of the chain composed of the eigenvector and its associated eigenvectors. The geometric multiplicity of an eigenvalue is defined to be the maximal number of corresponding linearly independent eigenvectors. Its algebraic multiplicity is defined as the maximal value of the sum of the lengths of chains corresponding to linearly independent eigenvectors. An eigenvalue is said to be isolated if it has a deleted neighbourhood contained in the resolvent set. An isolated eigenvalue λ_0 of finite algebraic multiplicity is said to be normal if the image $\operatorname{Im} L(\lambda_0)$ *is closed. We denote by $\sigma_0(L)$ the set of normal eigenvalues of $L(\lambda)$.*

Lemma 2.2. *The operator polynomial $L(\lambda)$ has the following properties.*

(1) The spectrum of $L(\lambda)$ consists only of normal eigenvalues.

(2) *All eigenvalues of $L(\lambda)$ have unit geometric multiplicity.*

(3) *The eigenvalues of $L(\lambda)$ located in the closed lower half-plane are purely imaginary, are algebraically and geometrically simple and are finite in number, and their number coincides with the number of nonpositive eigenvalues of the operator A_1 (or, what is the same thing, of A).*

The first part follows from the fact that the operator polynomial $L(\lambda)$ is compactly invertible for $\lambda = -i\gamma$ with $\gamma > 0$ large enough. The second statement follows from the existence of only one linearly independent solution of (1.9) which vanishes at $x = 0$. The proof of the third part is almost the same as the proofs of corollary 2.1, lemma 2.4 and theorem 2.1 in [18], where the operator $M \gg 0$ (i.e. M is strictly positive). However, if $M \geqslant 0$, $K \geqslant 0$ and $M + K \gg 0$, which is true in our case, the proof given in [18] remains valid.

3. Main results on the direct problem

To state these results, let $\{-i\gamma_k\}_{k=1}^{\kappa}$ be the eigenvalues of $L(\lambda)$ located in the closed lower half-plane and hence on the nonpositive imaginary axis, numbered such that $0 \leqslant \gamma_1 < \cdots < \gamma_{\kappa-1} < \gamma_\kappa$.

Theorem 3.1. *We have the following.*

(1) $i\gamma_k \notin \sigma(L) \setminus \{0\}$ *for* $k = 1, \ldots, \kappa$.

(2) *If* $\kappa > 1$, *the number of purely imaginary eigenvalues in each of the intervals* $(i\gamma_k, i\gamma_{k+1})$ *for* $k = 1, \ldots, \kappa - 1$ *is odd.*

(3) *If* $\gamma_1 > 0$, *then the number of eigenvalues in the interval* $(0, i\gamma_1)$ *is even or zero.*

(4) *If* $\kappa > 0$ *and* $0 < \alpha < 1$, *the number of eigenvalues in the interval* $(i\gamma_\kappa, i\infty)$ *is odd; if* $\kappa > 0$ *and* $\alpha > 1$, *this number is even and positive.*

(5) *If* $\kappa = 0$, *the number of positive imaginary eigenvalues is odd if* $\alpha > 1$ *and even or zero if* $0 < \alpha < 1$.

In parts (2)–(5), algebraic multiplicities are taken into account when counting eigenvalues.

Proof. The spectrum of problem (1.9)–(1.11) coincides with the set of zeros of the entire function

$$\varphi(\lambda) = s'(\lambda, a) + (i\lambda\alpha + \beta)s(\lambda, a), \tag{3.1}$$

where the prime denotes the derivative with respect to x. Here $s(\lambda, x)$ is the solution of (1.9) satisfying the initial conditions

$$s(\lambda, 0) = 0, \qquad s'(\lambda, 0) = 1.$$

□

We need the following proposition.

Proposition 3.2. *If* $\varphi(-i\tau) = 0$ *for some* $\tau > 0$, *then* $\varphi(i\tau) \neq 0$.

Proof of proposition 3.2. The functions $s'(\lambda, a)$ and $s(\lambda, a)$ are even functions of λ and therefore

$$s'(-i\gamma, a) = s'(i\gamma, a), \tag{3.2}$$

$$s(-i\gamma, a) = s(i\gamma, a). \tag{3.3}$$

Let $\varphi(-i\gamma) = 0$ for some $\gamma > 0$. Then due to (3.1) we obtain

$$s'(-i\gamma, a) + (\gamma\alpha + \beta)s(-i\gamma, a) = 0. \tag{3.4}$$

Substituting (3.2) and (3.3) into (3.4), we obtain

$$s'(\mathrm{i}\gamma, a) + (\gamma\alpha + \beta)s(\mathrm{i}\gamma, a) = 0. \tag{3.5}$$

Suppose now that $\varphi(\mathrm{i}\gamma) = 0$. Then

$$s'(\mathrm{i}\gamma, a) + (-\gamma\alpha + \beta)s(\mathrm{i}\gamma, a) = 0. \tag{3.6}$$

Combining (3.5) and (3.6) we obtain

$$s(\mathrm{i}\gamma, a) = 0. \tag{3.7}$$

Substituting (3.7) into (3.5) we obtain

$$s(\mathrm{i}\gamma, a) = s'(\mathrm{i}\gamma, a) = 0,$$

which implies that $s(\mathrm{i}\gamma, x) \equiv 0$, a contradiction. □

To continue the proof, let us consider the operator polynomial $L(\lambda)$ in which α occurs as a parameter, as an operator polynomial $L(\lambda, \alpha)$, keeping λ as the spectral parameter. The spectrum of $L(\lambda, 0) = \lambda^2 I - A_1$ is symmetric with respect to both the real and the imaginary axis. The eigenvalues of $L(\lambda, \alpha)$ are piecewise analytic functions of α, which may lose their analyticity only when they collide [5, 7].

In the case of the nonmonic operator polynomial (1.17), differentiation of the identity $(L(\lambda_j, \alpha)y_j(\alpha), y_j(\alpha)) = 0$ with respect to α, where $\lambda_j(\alpha)$ is a purely imaginary eigenvalue of $L(\lambda, \alpha)$ and $y_j(\alpha)$ is the corresponding eigenvector, leads to the following identity (cf, e.g., [18]):

$$\lambda_j'(\alpha) = \frac{\mathrm{i}\,\lambda_j(\alpha)(K_1 y_j(\alpha), y_j(\alpha))}{2\lambda_j(\alpha)(My_j(\alpha), y_j(\alpha)) - \mathrm{i}\alpha(K_1 y_j(\alpha), y_j(\alpha))}, \tag{3.8}$$

where K_1 is defined by (1.16) and the prime indicates the derivative with respect to α. Consequently,

$$\lambda_j'(0) = \frac{\mathrm{i}\,(K_1 y_j(0), y_j(0))}{2\,(My_j(0), y_j(0))}. \tag{3.9}$$

In the next proposition it is proved that the numerator and denominator of (3.9) are positive numbers.

Proposition 3.3. *We have* $(My_j(0), y_j(0)) > 0$ *and* $(K_1 y_j(0), y_j(0)) > 0$. *Hence,*

$$\operatorname{Re} \lambda_j'(0) = 0, \qquad \operatorname{Im} \lambda_j'(0) > 0.$$

Proof. Since $M \geqslant 0$ and $K \geqslant 0$, the identities $(My_j(0), y_j(0)) = 0$ and $(K_1 y_j(0), y_j(0)) = 0$ imply that $My_j(0) = 0$ and $K_1 y_j(0) = 0$, respectively. The former yields $y_j(\alpha = 0, x) \equiv 0$, implying $y_j(\alpha = 0, a) = 0$ and hence $y_j(\alpha = 0) = 0$, which is a contradiction. The latter yields $y_j(\alpha = 0, a) = 0$. On the other hand,

$$\lambda_j(0)^2 My_j(0) - A_1 y_j(0) = 0,$$

which means that

$$\lambda_j(\alpha = 0)^2 y_j(\alpha = 0, x) + \frac{\mathrm{d}^2}{\mathrm{d}x^2} y_j(\alpha = 0, x) - q y_j(\alpha = 0, x) = 0,$$

$$y_j(\alpha = 0, a) = \left.\frac{\mathrm{d}}{\mathrm{d}x} y_j(\alpha = 0, x)\right|_{x=a} = 0.$$

Hence $y_j(\alpha = 0, x) \equiv 0$, which is a contradiction. □

Let us continue the proof of theorem 3.1. Taking into account the symmetry of the problem on reflection with respect to the imaginary line, we have $\lambda_{-k}(\alpha) = -\overline{\lambda_k(\alpha)}$ for all not purely imaginary $\lambda_{-k}(\alpha)$ with $\alpha \geqslant 0$, and hence new eigenvalues can appear on the imaginary axis only in pairs, which implies statements (1)–(3) of the theorem.

Let us consider the asymptotics of $\varphi(\mathrm{i}\tau, \alpha)$ for $\alpha \in (0, 1)$ and $\tau \to +\infty$. We use the well known formulae [17]

$$s(\lambda, a) = \frac{\sin \lambda a}{\lambda} - K(a, a)\frac{\cos \lambda a}{\lambda^2} + \int_0^a K_t(a, t)\frac{\cos \lambda t}{\lambda^2}\,\mathrm{d}t, \tag{3.10}$$

$$s'(\lambda, a) = \cos \lambda a + K(a, a)\frac{\sin \lambda a}{\lambda} + \int_0^a K_x(a, t)\frac{\sin \lambda t}{\lambda}\,\mathrm{d}t, \tag{3.11}$$

where the function $K(x, t)$ has partial derivatives of first order such that $K_x(a, t)$ and $K_t(a, t)$ belong to $L_2(0, a)$ as functions of t, and

$$K(x, x) = \tfrac{1}{2}\int_0^x q(t)\,\mathrm{d}t.$$

Substituting (3.10)–(3.11) into (3.1) and setting $\lambda = \mathrm{i}\tau$ we obtain

$$\varphi(\mathrm{i}\tau; \alpha) = \frac{\mathrm{e}^{\tau a}(1-\alpha)}{2} + (1+\alpha)\mathrm{O}\left(\frac{\mathrm{e}^{\tau a}}{\tau}\right), \qquad \tau \to +\infty, \tag{3.12}$$

where the α-dependence has been expressed explicitly and the order constant does not depend on α. This means that no eigenvalues arrive from or arrive at $+\mathrm{i}\infty$ along the imaginary axis as α increases from zero to any value in (0, 1). As a result, the statements (4) and (5) regarding the case $\alpha \in (0, 1)$ are true.

Now let $\alpha > 1$ be fixed and replace $q(x)$ by $\eta q(x)$. Then the eigenvalues of problem (1.9)–(1.11) are piecewise analytic functions of η [5, 7]. For $\eta = 0$ the problem (1.9)–(1.11) can be easily solved explicitly and hence it is clear that the number of purely imaginary eigenvalues is equal to unity. Equation (3.12) shows that $|\varphi(\mathrm{i}\tau)| \to +\infty$ as $\tau \to +\infty$ uniformly in $\eta \in [0, 1]$ for any $q \in L_2(0, a)$. This means that there are no eigenvalues arriving from or arriving at infinity along the imaginary axis as η increases from zero to unity. By the same token, since

$$\varphi(-\mathrm{i}\tau; \alpha) = \frac{\mathrm{e}^{\tau a}(1+\alpha)}{2} + (1+\alpha)\mathrm{O}\left(\frac{\mathrm{e}^{\tau a}}{\tau}\right), \qquad \tau \to +\infty, \tag{3.13}$$

there are also no eigenvalues arriving from or arriving at $-\mathrm{i}\infty$ along the imaginary axis as η increases from zero to unity.

Due to the symmetry of the problem on reflection with respect to the imaginary line, new eigenvalues can appear on the imaginary axis only in pairs. This means that for $\alpha > 1$ the number of purely imaginary eigenvalues (with multiplicities taken into account) is odd. Then assertion (5) follows. Assertion (4) now follows if we take into account assertion (3).

4. Inverse problem

In [9] the following theorem was stated without proof.

Theorem 4.1. *Let the sequence of complex numbers* $\{\lambda_k\}_{0\neq k\in Z}$ *satisfy the following conditions.*

(1) $\lambda_{-k} = -\overline{\lambda_k}$ *for all* λ_k *not purely imaginary.*
(2) $\operatorname{Im} \lambda_k > 0$ *for all k.*
(3) The number of purely imaginary terms of $\{\lambda_k\}_{0\neq k\in Z}$ *is even.*

(4) We have the asymptotic relation

$$\lambda_k = \frac{(k-\frac{1}{2})\pi}{a} + \mathrm{i}g + \frac{h}{k} + \frac{\mu_k}{k}, \qquad k \to +\infty, \tag{4.1}$$

where $g > 0$, $h \in \boldsymbol{R}$ *and* $\sum_{0\neq k\in \boldsymbol{Z}} |\mu_k|^2 < +\infty$.

Then there exist $\alpha \in (0,1)$, $\beta \in \boldsymbol{R}$ *and real* $q \in L_2(0,a)$ *which generate the problem (1.9)–(1.11) with spectrum* $\{\lambda_k\}_{0\neq k\in \boldsymbol{Z}}$.

The proof of this theorem was given in [8]. In the proof, it was used that, under the conditions of theorem 4.1, the function $\varphi(\lambda)$ in (3.1) can be represented in the form

$$\varphi(\lambda) = C \lim_{n\to\infty} \prod_{0<|k|\leqslant n} \left(1 - \frac{\lambda}{\lambda_k}\right) = C \prod_{0\neq k\in \boldsymbol{Z}} \left(1 - \frac{\lambda}{\lambda_k}\right)$$

and belongs to the so-called Hermite–Biehler class (see below for the definition of this class).

In [8] the inverse eigenvalue problem of constructing the potential $q \in L_2(0,a)$ in (1.9)–(1.11) from its eigenvalue spectrum was solved under the condition that there are no eigenvalues in the closed lower half-plane. In this paper we consider the same inverse problem, but without restricting the eigenvalues to the upper half-plane. In other words, we aim at a generalization of theorem 4.1 in which condition (2) no longer appears. In this case the function $\varphi(\lambda)$ in (3.1) does not belong to the Hermite–Biehler class.

Definition 4.2. *A sequence of complex numbers* $\{\lambda_k\}_{k\in \boldsymbol{Z}}$ *or* $\{\lambda_k\}_{0\neq k\in \boldsymbol{Z}}$ *is said to be properly enumerated if*

(1) $\mathrm{Re}\,\lambda_k \geqslant \mathrm{Re}\,\lambda_p$ *for all* $k > p$.
(2) $\lambda_{-k} = -\overline{\lambda_k}$ *for all* λ_k *not purely imaginary.*
(3) A certain complex number appears in the sequence at most finitely many times.

All sequences of complex numbers used in this paper are properly enumerated.

Definition 4.3. *An entire function* $\omega(\lambda)$ *is said to belong to the Hermite–Biehler class* $\mathcal{HB}$ *[12] if it has no zeros in the closed lower half-plane and*

$$\left|\omega(\lambda)/\overline{\omega(\overline{\lambda})}\right| < 1, \qquad \mathrm{Im}\,\lambda > 0. \tag{4.2}$$

Definition 4.4. *The function* $\omega \in \mathcal{HB}$ *is said to belong to the symmetric Hermite–Biehler class* $\mathcal{SHB}$ *if*

$$\omega(-\lambda) = \overline{\omega(\overline{\lambda})}, \qquad \lambda \in \boldsymbol{C}. \tag{4.3}$$

Any function $\omega \in \mathcal{HB}$ can be presented in the form [12]

$$\omega(\lambda) = P(\lambda) + \mathrm{i}Q(\lambda),$$

where $P(\lambda)$ and $Q(\lambda)$ are real entire functions (i.e., they are real on the real line). Moreover, if $\omega \in \mathcal{SHB}$, then (4.3) implies that

$$P(-\lambda) + \mathrm{i}Q(-\lambda) = P(\lambda) - \mathrm{i}Q(\lambda), \qquad \lambda \in \boldsymbol{R}.$$

This means that for $\omega \in \mathcal{SHB}$ the functions $P(\lambda)$ and $\hat{Q}(\lambda) = \lambda^{-1}Q(\lambda)$ are even entire functions satisfying

$$P(\lambda) = \frac{\omega(\lambda) + \omega(-\lambda)}{2}, \tag{4.4}$$

$$\hat{Q}(\lambda) = \frac{\omega(\lambda) - \omega(-\lambda)}{2\mathrm{i}\lambda}. \tag{4.5}$$

Definition 4.5. *Let κ be a non-negative integer. Then the properly enumerated sequence $\{\lambda_k\}_{k\in \mathbf{Z}}$ or $\{\lambda_k\}_{0\neq k\in \mathbf{Z}}$ is said to have the $\mathcal{SHB}_\kappa^+$ (respectively, $\mathcal{SHB}_\kappa^-$) property if we have the following.*

(1) All but κ terms of the sequence lie in the open upper half-plane.
(2) All terms in the closed lower half-plane are purely imaginary and occur only once. If $\kappa \geqslant 1$, we denote them as $\lambda_{-j} = -\mathrm{i}|\lambda_{-j}|(j = 1, \ldots, \kappa)$. We assume that $|\lambda_{-j}| < |\lambda_{-(j+1)}|(j = 1, \ldots, \kappa - 1)$.
(3) If $\kappa \geqslant 1$, the numbers $\mathrm{i}|\lambda_{-j}|(j = 1, \ldots, \kappa)$ (with the exception of λ_{-1} if it is equal to zero) are not terms of the sequence.
(4) If $\kappa \geqslant 2$, then the number of terms in the intervals $(\mathrm{i}|\lambda_{-j}|, \mathrm{i}|\lambda_{-(j+1)}|)(j = 1, \ldots, \kappa - 1)$ is odd.
(5) If $|\lambda_{-1}| > 0$, then the interval $(0, \mathrm{i}|\lambda_{-1}|)$ contains no terms at all or an even number of terms.
(6) If $\kappa \geqslant 1$, then the interval $(\mathrm{i}|\lambda_{-\kappa}|, \mathrm{i}\infty)$ contains a nonzero even (in the case of $\mathcal{SHB}_\kappa^+$) or odd (in the case of $\mathcal{SHB}_\kappa^-$) number of terms.
(7) If $\kappa = 0$, then the sequence has an odd (in the case of $\mathcal{SHB}_\kappa^+$) or even or zero (in the case of $\mathcal{SHB}_\kappa^-$) number of positive imaginary terms.

Notice that the set of zeros of a function $\omega \in \mathcal{SHB}$ belongs to one of the two disjoint sets of properly enumerated sequences $\mathcal{SHB}_0^+$ and $\mathcal{SHB}_0^-$.

Theorem 3.1 states, among other things, that the spectrum of problem (1.9)–(1.11) belongs to $\mathcal{SHB}_\kappa^+$ if $\alpha > 1$ and to $\mathcal{SHB}_\kappa^-$ if $0 < \alpha < 1$, where κ is the number of nonpositive imaginary eigenvalues.

Definition 4.6. *Let us denote by $\mathcal{B}^+$ (respectively, $\mathcal{B}^-$) the class of four-tuples $\{a, q, \beta, \alpha\}$ such that $a > 0$, $q \in L_2(0, a)$ is real valued, $\beta \in \mathbf{R}$ and $\alpha > 1$ (respectively, $0 < \alpha < 1$).*

Theorem 4.7. *Let*

(1) $\{\lambda_k\}_{0\neq k\in \mathbf{Z}} \in \mathcal{SHB}_\kappa^-$, and
(2) let the asymptotic relation (4.1) be true for some $g > 0$, $h \in \mathbf{R}$, and $\sum_{0\neq k\in \mathbf{Z}} |\mu_k|^2 < +\infty$.

Then there exists a unique four-tuple $\{a, q, \beta, \alpha\} \in \mathcal{B}^-$ such that $\{\lambda_k\}_{0\neq k\in \mathbf{Z}}$ is the spectrum of problem (1.9)–(1.11) generated by the four-tuple.

Proof. Put

$$a = \lim_{k\to\infty} (\pi k/\lambda_k). \tag{4.6}$$

By (4.1), this limit exists. As will be shown shortly, this limit a will turn out to be the length of the interval on which (1.9) is valid.

Consider the auxiliary function

$$\varphi_0(\lambda) = \cos((\lambda - \mathrm{i}\varepsilon)a), \qquad \varepsilon > 0.$$

Then $\varphi_0 \in \mathcal{SHB}$. Let us denote its properly enumerated sequence of zeros by $\{\lambda_k^0\}_{0\neq k\in \mathbf{Z}}$, where $\lambda_k^0 = \mathrm{i}\varepsilon + \mathrm{sgn}(k)(|k| - \frac{1}{2})(\pi/a)$.

Put

$$P_0(\lambda) = \frac{\varphi_0(\lambda) + \varphi_0(-\lambda)}{2} = \cos(\lambda a)\cosh(\varepsilon a),$$
$$\hat{Q}_0(\lambda) = \frac{\varphi_0(\lambda) - \varphi_0(-\lambda)}{2\mathrm{i}\lambda} = \frac{\sin(\lambda a)}{\lambda}\sinh(\varepsilon a).$$

Then the functions $P_0(\lambda)$ and $\hat{Q}_0(\lambda)$ are both even functions. Let us introduce the real entire functions

$$\tilde{P}_0(\lambda) = P_0(\sqrt{\lambda}) = \cos(a\sqrt{\lambda})\cosh(\varepsilon a),$$
$$\tilde{\hat{Q}}_0(\lambda) = \hat{Q}_0(\sqrt{\lambda}) = \frac{\sin(a\sqrt{\lambda})}{\sqrt{\lambda}}\sinh(\varepsilon a).$$

Since the zeros $\{\zeta_k^{(0)}\}_{0\neq k\in Z}$ of $P_0(\lambda)$ and the zeros $\{\xi_k^{(0)}\}_{0\neq k\in Z}$ of $\hat{Q}_0(\lambda)$ interlace, we have

$$\cdots < \xi_{-2}^{(0)} < \zeta_{-2}^{(0)} < \xi_{-1}^{(0)} < \zeta_{-1}^{(0)} < 0 < \zeta_1^{(0)} < \xi_1^{(0)} < \zeta_2^{(0)} < \xi_2^{(0)} < \cdots. \quad (4.7)$$

Proposition 4.8. *There exists a sequence of continuous and piecewise analytic functions* $\{\lambda_k(t)\}_{0\neq k\in Z}$ *such that* $\lambda_k(0) = \lambda_k^{(0)}$ *and* $\lambda_k(1) = \lambda_k (0 \neq k \in \boldsymbol{Z})$ *and*

$$\{\lambda_k(t)\}_{0\neq k\in Z} \in \mathcal{SHB}^-_{\kappa(t)}$$

for any fixed $t \in [0, 1]$.

Proof of proposition 4.8. There exists $\ell \in \boldsymbol{N}$ such that Re $\lambda_s > 0$ for $s \geqslant \ell+1$ and Re $\lambda_s < 0$ for $s \leqslant -(\ell+1)$. For such λ_s we define

$$\lambda_s(t) = \lambda_s^{(0)} + (\lambda_s - \lambda_s^{(0)})t.$$

So it remains to construct the piecewise analytic functions involving the purely imaginary eigenvalues λ_s $(0 < |s| \leqslant \ell)$.

Now let λ_{-j} be a (purely imaginary) term in the closed lower half-plane. Consider the interval $(\mathrm{i}|\lambda_{-j}|, \mathrm{i}|\lambda_{-(j+1)}|)$. Then the number of terms on this interval is odd. Let us choose one of them and denote it by λ_{+j}, thus resulting in the positive imaginary, so-called principal, eigenvalues $\lambda_{+1}, \ldots, \lambda_{+\kappa}$. The remaining positive imaginary eigenvalues, which are even in number when multiplicities are taken into account, are grouped into pairs $\{\lambda_{+s}, \lambda_{-s}\}$ $(s = \kappa+1, \ldots, \ell)$ such that each of the intervals $(\mathrm{i}|\lambda_{-j}|, \mathrm{i}|\lambda_{-(j+1)}|)$ contains an integer number of such pairs or no such pairs at all.

Let us move the points $\lambda_s^{(0)}$ $(0 < |s| \leqslant \ell)$ in various steps. First, we define for $0 \leqslant t \leqslant (1/(\kappa+2))$

$$\lambda_j(t) = \begin{cases} \lambda_j^{(0)}, & 0 < |j| < |\kappa|, \\ \lambda_{\pm\kappa}^{(0)} + (\kappa+2)t\left(\dfrac{\lambda_{-\kappa}+\lambda_{+\kappa}}{2} - \lambda_{\pm\kappa}^{(0)}\right), & j = \pm\kappa, \\ \lambda_j^{(0)}, & |\kappa| < |j| \leqslant \ell. \end{cases}$$

Next, for $s = 0, \ldots, \kappa-2$ and $t \in \left[\frac{1+s}{\kappa+2}, \frac{2+s}{\kappa+2}\right]$ we define

$$\lambda_j(t) = \begin{cases} \dfrac{\lambda_j+\lambda_{-j}}{2} + ((\kappa+2)t - s - 1)\dfrac{\lambda_j-\lambda_{-j}}{2}, & j = \pm(\kappa - s), \\ \lambda_j^{(0)} + ((\kappa+2)t - s - 1)\left(\dfrac{\lambda_j+\lambda_{-j}}{2} - \lambda_j^{(0)}\right), & j = \pm(\kappa - s - 1), \\ \lambda_j^{(0)}, & 0 < |j| < \kappa - s - 1, \\ \lambda_j, & \kappa - s < |j| \leqslant \ell. \end{cases}$$

Next, for $t \in \left[\frac{\kappa}{\kappa+2}, \frac{\kappa+1}{\kappa+2}\right]$ we define

$$\lambda_j(t) = \begin{cases} \dfrac{\lambda_1+\lambda_{-1}}{2} + ((\kappa+2)t - \kappa)\dfrac{\lambda_{\pm1}-\lambda_{\mp1}}{2}, & j = \pm1, \\ \lambda_j, & 1 < |j| \leqslant \kappa, \\ \lambda_j^{(0)} + ((\kappa+2)t - \kappa)\left(\dfrac{\lambda_j+\lambda_{-j}}{2} - \lambda_j^{(0)}\right), & \kappa < |j| \leqslant \ell. \end{cases}$$

Finally, for $t \in \left[\frac{\kappa+1}{\kappa+2}, 1\right]$ we define

$$\lambda_j(t) = \begin{cases} \lambda_j, & 0 < |j| \leqslant \kappa, \\ \dfrac{\lambda_j + \lambda_{-j}}{2} + ((\kappa+2)t - \kappa - 1)\dfrac{\lambda_j - \lambda_{-j}}{2}, & \kappa < |j| \leqslant \ell, \end{cases}$$

which completes the proof. □

Continuing the proof of theorem 4.7, let us construct the function

$$\varphi(\lambda, t) = \prod_{0 \neq k \in Z} \frac{(\lambda - \lambda_k(t))a}{(|k| - \frac{1}{2})\pi}. \tag{4.8}$$

Then

$$\varphi(\lambda, 0) = C\cos((\lambda - i\varepsilon)a), \qquad C \neq 0,$$
$$\varphi(\lambda) := \varphi(\lambda, 1) = \prod_{0 \neq k \in Z} \frac{(\lambda - \lambda_k)a}{(|k| - \frac{1}{2})\pi}. \tag{4.9}$$

Put

$$P(\lambda, t) = \frac{\varphi(\lambda, t) + \varphi(-\lambda, t)}{2}, \tag{4.10}$$

$$\hat{Q}(\lambda, t) = \frac{\varphi(\lambda, t) - \varphi(-\lambda, t)}{2i\lambda}, \tag{4.11}$$

and then define $P(\lambda) = P(\lambda, 1)$ and $\hat{Q}(\lambda) = \hat{Q}(\lambda, 1)$. Next, put

$$\tilde{P}(\lambda, t) = P(\sqrt{\lambda}, t),$$
$$\tilde{\hat{Q}}(\lambda, t) = \hat{Q}(\sqrt{\lambda}, t),$$

and then define $\tilde{P}(\lambda) = \tilde{P}(\lambda, 1)$ and $\tilde{\hat{Q}}(\lambda) = \tilde{\hat{Q}}(\lambda, 1)$.

Denote by $\{\zeta_k(t)\}_{0 \neq k \in Z}$ the set of zeros of $P(\lambda, t)$ and by $\{\xi_k(t)\}_{0 \neq k \in Z}$ the set of zeros of $\hat{Q}(\lambda, t)$. Then $\{\zeta_k(t)^2\}_{k=1}^{\infty}$ are the zeros of $\tilde{P}(\lambda, t)$ and $\{\xi_k(t)^2\}_{k=1}^{\infty}$ are the zeros of $\tilde{\hat{Q}}(\lambda, t)$.

Proposition 4.9. *For any fixed $t \in [0, 1]$, the sets of zeros $\{\zeta_k(t)^2\}_{k=1}^{\infty}$ and $\{\xi_k(t)^2\}_{k=1}^{\infty}$ interlace; i.e.,*

$$-\infty < \zeta_1(t)^2 < \xi_1(t)^2 < \zeta_2(t)^2 < \xi_2(t)^2 < \cdots. \tag{4.12}$$

Proof of proposition 4.9. Due to (4.7), this statement is true for $t = 0$. The functions $\tilde{P}(\lambda, t)$ and $\tilde{\hat{Q}}(\lambda, t)$ are entire functions of λ for every $t \in [0, 1]$ and continuous functions of $t \in [0, 1]$ for every $\lambda \in \boldsymbol{C}$. This means that the inequalities (4.12) can only be violated if for some $t_1 \in [0, 1]$ we have $\xi_k(t_1)^2 = \zeta_k(t_1)^2$ or $\xi_k(t_1)^2 = \zeta_{k+1}(t_1)^2$. However, either identity implies

$$\tilde{P}(\xi_k(t_1)^2, t_1) = \tilde{\hat{Q}}(\xi_k(t_1)^2, t_1) = 0,$$

and, consequently,

$$\varphi(\xi_k(t_1), t_1) = \varphi(-\xi_k(t_1), t_1) = 0,$$

where $\xi_k(t_1)$ is real or purely imaginary. Since proposition 4.8 implies that $\varphi(\lambda, t)$ can have a real zero only at the origin for any $t \in [0, 1]$, we conclude that $\xi_k(t_1)$ is purely imaginary or zero. Since $P(\lambda, t)$ and $\hat{Q}(\lambda, t)$ are even functions of λ, we have

$$P(\xi_k(t_1), t_1) = P(-\xi_k(t_1), t_1) = 0$$

and

$$\hat{Q}(\xi_k(t_1), t_1) = \hat{Q}(-\xi_k(t_1), t_1) = 0,$$

and consequently $\varphi(\xi_k(t_1), t_1) = \varphi(-\xi_k(t_1), t_1) = 0$, which contradicts proposition 4.8 if $\xi_k(t_1) \neq 0$. On the other hand, if $\xi_k(t_1) = 0$, then it is a double zero, because it is a zero of the even functions $\tilde{P}(\lambda, t_1)$ and $\hat{\tilde{Q}}(\lambda, t_1)$ and consequently of $\varphi(\lambda, t_1)$, which contradicts proposition 4.8.

We now need the following definition.

Definition 4.10. *An entire function $\omega(\lambda)$ of exponential type $\sigma > 0$ is said to be a function of sine type [13] if we have the following.*

(1) There exists $h > 0$ such that the zeros of $\omega(\lambda)$ are lying in the strip $|\mathrm{Im}\ \lambda| < h$.
(2) There exists $h_1 \in \boldsymbol{R}$ such that $0 < m \leqslant |\omega(\lambda)| \leqslant M < \infty$ for all λ with $\mathrm{Im}\ \lambda = h_1$.
(3) The exponential type of $\omega(\lambda)$ in the lower half-plane coincides with its exponential type in the upper half-plane.

Let us consider the auxiliary function

$$\varphi_{00}(\lambda) = \cos\left(a\sqrt{\lambda^2 - 2\pi h a^{-1}}\right) + \mathrm{i}\lambda \tanh(ga) \frac{\sin\left(a\sqrt{\lambda^2 - 2\pi h a^{-1}}\right)}{\sqrt{\lambda^2 - 2\pi h a^{-1}}}. \tag{4.13}$$

Then it is clear that $\varphi_{00}(\lambda)$ is a sine-type function. Direct computation shows that the zeros of $\varphi_{00}(\lambda)$ behave asymptotically as follows:

$$\lambda_k^{(00)} = \frac{\pi}{a}\left(k - \frac{1}{2}\right) + \mathrm{i}g + \frac{h}{k} + \frac{d_k}{k}, \tag{4.14}$$

where $\sum_{0\neq k\in \boldsymbol{Z}} |d_k|^2 < \infty$. Comparing (4.14) with (4.1) we obtain

$$\lambda_k = \lambda_k^{(00)} + \frac{b_k}{\lambda_k^{(00)}},$$

where $\sum_{0\neq k\in \boldsymbol{Z}} |b_k|^2 < \infty$. Using lemma 5 of [13] , we obtain

$$\varphi(\lambda) = c_0\left(1 + \frac{\mathrm{i}T}{\lambda}\right)\varphi_{00}(\lambda) + \frac{\psi(\lambda)}{\lambda}, \tag{4.15}$$

where $c_0 \in \boldsymbol{C} \setminus \{0\}$ and $T \in \boldsymbol{C}$ are constants and $\psi \in \mathcal{L}^a$, the class of entire functions of exponential type $\leqslant a$ whose restrictions to the real line belong to $L_2(\boldsymbol{R})$. Due to the symmetry of our problem with respect to the imaginary axis, we have $\{c_0, T\} \subset \boldsymbol{R}$. Substituting (4.13) into (4.15) we obtain

$$\varphi(\lambda) = c_0\bigg(\cos(\lambda a) + \mathrm{i}\tanh(ag)\sin(\lambda a) + \frac{\pi h - T\tanh(ag)}{\lambda}\sin(\lambda a) + \mathrm{i}\frac{T - \pi h\tanh(ag)}{\lambda}\cos(\lambda a)\bigg) + \frac{\psi_1(\lambda)}{\lambda}, \tag{4.16}$$

where $\psi_1 \in \mathcal{L}^a$.

The constants c_0, T, g and h can be found as follows:

$$c_0 = \lim_{k\to\infty} \varphi\left(\frac{2\pi k}{a}\right), \tag{4.17}$$

$$g = -\mathrm{i}\lim_{k\to\infty}\left(\lambda_k - \frac{\pi(k - \frac{1}{2})}{a}\right), \tag{4.18}$$

$$h = \lim_{k\to\infty} k\left(\lambda_k - \frac{\pi(k - \frac{1}{2})}{a} - \mathrm{i}g\right), \tag{4.19}$$

$$T = -\frac{1}{c_0 \tanh(ag)} \lim_{k\to\infty} \left(\left(\varphi\left(\frac{\pi(2k+\frac{1}{2})}{a}\right) + \varphi\left(-\frac{\pi(2k+\frac{1}{2})}{a}\right)\right) \frac{\pi(2k+\frac{1}{2})}{a} - \pi h \right), \tag{4.20}$$

where (4.17) and (4.20) follow from (4.16) and (4.18) and (4.19) from (4.1). Substituting (4.16) into (4.10) and (4.11) with $t = 1$, we obtain

$$P(\lambda) = c_0 \left[\cos(\lambda a) + \frac{\pi h - T \tanh(ag)}{\lambda} \sin(\lambda a) \right] + \frac{\tilde{\psi}_1(\lambda)}{\lambda}, \tag{4.21}$$

$$\hat{Q}(\lambda) = c_0 \left[\tanh(ag) \frac{\sin(\lambda a)}{\lambda} + \frac{T - \pi h \tanh(ag)}{\lambda} \cos(\lambda a) \right] + \frac{\tilde{\psi}_2(\lambda)}{\lambda}, \tag{4.22}$$

where $\tilde{\psi}_j \in \mathcal{L}^a$ $(j = 1, 2)$.

Put

$$\alpha = \tanh(ag), \tag{4.23}$$

$$\beta = \frac{(1-\alpha^2)T}{\alpha}, \tag{4.24}$$

where, clearly, $\alpha > 0$. We will prove below that these α and β are the numbers α and β we are looking for.

Put

$$\hat{P}(\lambda) := P(\lambda) - \frac{\beta}{\alpha} \hat{Q}(\lambda), \tag{4.25}$$

where α and β follow from (4.23) and (4.24). The function $\hat{P}(\lambda)$ is even. Thus we define the entire function

$$\tilde{\hat{P}}(\lambda) = \hat{P}(\sqrt{\lambda}) = \tilde{P}(\lambda) - \frac{\beta}{\alpha} \tilde{\hat{Q}}(\lambda). \tag{4.26}$$

Proposition 4.11. *The sets of zeros $\{\tilde{\zeta}_k\}_{k=1}^{\infty}$ of $\tilde{\hat{P}}(\lambda)$ and $\{\tilde{\xi}_k\}_{k=1}^{\infty}$ of $\tilde{\hat{Q}}(\lambda)$ interlace; i.e.,*

$$-\infty < \tilde{\zeta}_1 < \tilde{\xi}_1 < \tilde{\zeta}_2 < \tilde{\xi}_2 < \cdots.$$

Proof of proposition 4.11. Consider the function

$$\tilde{\hat{P}}(\lambda, \beta) = \tilde{P}(\lambda) - \frac{\beta}{\alpha} \tilde{\hat{Q}}(\lambda),$$

where $\alpha > 0$ is fixed. Then the zeros $\{\zeta_k(1)^2\}_{k=1}^{\infty}$ of $\tilde{\hat{P}}(\lambda, 0)$ interlace with the zeros $\{\tilde{\xi}_k\}_{k=1}^{\infty} = \{\xi_k(1)^2\}_{k=1}^{\infty}$ of $\tilde{\hat{Q}}(\lambda)$ (cf, proposition 4.9). Being piecewise analytic functions of β, the zeros $\{\tilde{\xi}_k(\beta)\}_{k=1}^{\infty}$ of $\tilde{\hat{P}}(\lambda, \beta)$ can violate interlacing only at collisions with $\tilde{\xi}_k$. This is impossible, because if $\tilde{\hat{P}}(\lambda, \beta) = \tilde{P}(\lambda) - (\beta/\alpha)\tilde{\hat{Q}}(\lambda)$ for some real λ, then proposition 4.9 is contradicted. □

Let us now pass to the final part of the proof of theorem 4.7. Let us consider the asymptotic properties of $\{\tilde{\zeta}_k\}_{k=1}^{\infty}$ and $\{\tilde{\xi}_k\}_{k=1}^{\infty}$. Substituting (4.21) and (4.22) into (4.25) and using (4.26), (4.23) and (4.24), we obtain

$$\tilde{\hat{P}}(\lambda) = c_0 \left(\cos(a\sqrt{\lambda}) + h \frac{\sin(a\sqrt{\lambda})}{\sqrt{\lambda}} + \frac{\psi_3(\sqrt{\lambda})}{\sqrt{\lambda}} \right),$$

$$\tilde{\hat{Q}}(\lambda) = \alpha c_0 \left(\frac{\sin(a\sqrt{\lambda})}{\sqrt{\lambda}} - h \frac{\cos(a\sqrt{\lambda})}{\lambda} + \frac{\psi_4(\sqrt{\lambda})}{\lambda} \right),$$

where $\psi_j \in \mathcal{L}^a$ $(j = 3, 4)$ are even functions. Making use of lemma 3.4.2 of [17], we obtain

$$\tilde{\zeta}_k = \frac{\pi^2(k - \frac{1}{2})^2}{a^2} + \frac{2\pi h}{a} + \beta_k^{(1)}, \tag{4.27}$$

$$\tilde{\xi}_k = \frac{\pi^2 k^2}{a^2} + \frac{2\pi h}{a} + \beta_k^{(2)}, \tag{4.28}$$

where $\sum_{k=1}^{\infty} |\beta_k^{(j)}|^2 < \infty$ $(j = 1, 2)$. As a result of (4.27), (4.28) and proposition 4.11, the sequences $\{\tilde{\zeta}_k\}_{k=1}^{\infty}$ and $\{\tilde{\xi}_k\}_{k=1}^{\infty}$ satisfy the requirements of theorem 3.4.3 of [17] when restated on an interval of length a. As a consequence, there exists a unique real $q \in L_2(0, a)$ such that $\{\tilde{\zeta}_k\}_{k=1}^{\infty}$ are the eigenvalues of the Dirichlet–Neumann problem

$$y'' + \lambda y - qy = 0,$$
$$y(0) = y'(a) = 0,$$

and $\{\tilde{\xi}_k\}_{k=1}^{\infty}$ are the eigenvalues of the Dirichlet problem

$$y'' + \lambda y - qy = 0,$$
$$y(0) = y(a) = 0.$$

Let us prove that q is the potential we are looking for and that it is unique. Indeed, we have

$$s(\lambda, a) = \frac{1}{c_0\alpha}\hat{Q}(\lambda),$$
$$s'(\lambda, a) = \frac{1}{c_0}\hat{P}(\lambda),$$

where $s(\lambda, x)$ is the solution of (1.9) with initial conditions $s(\lambda, 0) = 0$ and $s'(\lambda, 0) = 1$. Then

$$\frac{\varphi(\lambda)}{c_0} = \frac{\hat{P}(\lambda)}{c_0} + \frac{\mathrm{i}\alpha\lambda + \beta}{c_0}\hat{Q}(\lambda) = s'(\lambda, a) + (\mathrm{i}\alpha\lambda + \beta)s(\lambda, a),$$

and hence the set of zeros of $\varphi(\lambda)$ coincides with the spectrum of problem (1.9)–(1.11). Since $\alpha \in (0, 1)$ as a result of (4.23), the four-tuple $\{a, q, \beta, \alpha\} \in \mathcal{B}^-$. This completes the proof of theorem 4.7.

Let us give more details on the construction of the real potential q of the problem (1.9)–(1.11) from the properly enumerated sequence $\{\lambda_k\}_{0\neq k\in Z}$ such that this sequence becomes the spectrum of the problem. Let us first indicate the quantities to be computed from $\{\lambda_k\}_{0\neq Z}$ using previous equations. First we evaluate the interval length a with the help of (4.6). We also evaluate g and h from (4.18) and (4.19) and then α from (4.23). Note that α thus obtained belongs to (0,1). Next we compute the function $\varphi(\lambda)$ with the help of (4.9), and the functions $P(\lambda)$ and $\hat{Q}(\lambda)$ using (4.10) and (4.11) with $t = 1$. Next, we evaluate c_0 and T from (4.17) and (4.20) and then β from (4.24). Finally, we compute $\hat{P}(\lambda)$ using (4.25).

Let us now continue the inversion algorithm. Let us define the S-function

$$S(\lambda) = \frac{\alpha\hat{P}(\lambda) - \mathrm{i}\lambda\hat{Q}(\lambda)}{\alpha\hat{P}(\lambda) + \mathrm{i}\lambda\hat{Q}(\lambda)}\,\mathrm{e}^{2\mathrm{i}\lambda a},$$

and then evaluate the Marchenko kernel function

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}(1 - S(\lambda))\mathrm{e}^{-\mathrm{i}\lambda x}\,\mathrm{d}\lambda + \sum_{\substack{j=1,\ldots,\kappa \\ \lambda_{-1}\neq 0}} \mathrm{e}^{-|\lambda_{-j}|x}\mathrm{Res}\ (1 - S(\lambda))|_{\lambda=\lambda_{-j}}\,.$$

We then solve the Marchenko integral equation

$$F(x+t)+K(x,t)+\int_x^{\infty} K(x,z)F(z+t)\,\mathrm{d}z=0, \qquad t \geqslant x,$$

which has a unique solution [17]. With the help of the expression [17]

$$K(x,x)=\tfrac{1}{2}\int_x^{\infty} q(t)\,\mathrm{d}t,$$

we obtain the potential q in the form

$$q(x)=-2\frac{\mathrm{d}}{\mathrm{d}x}K(x,x).$$

The potential q found in this way has its support on $[0,a]$ and its restriction to $[0,a]$ belongs to $L_2(0,a)$.

Let us now derive the analogue of theorem 4.7 for $\alpha > 1$.

Theorem 4.12. *Let*

(1) $\{\lambda_k\}_{k\in Z} \in \mathcal{SHB}_{\kappa}^{+}$, *and*
(2) let the asymptotic relation

$$\lambda_k=\frac{k\pi}{a}+\mathrm{i}g+\frac{h}{k}+\frac{\mu_k}{k}, \qquad k\to+\infty, \tag{4.29}$$

be true for some $g>0$, $h\in \boldsymbol{R}$ *and* $\sum_{k\in Z}|\mu_k|^2<+\infty$.

Then there exists a unique four-tuple $\{a,q,\beta,\alpha\}\in\mathcal{B}^{+}$ *such that* $\{\lambda_k\}_{k\in Z}$ *is the spectrum of problem (1.9)–(1.11) generated by the four-tuple.*

The proof of theorem 4.12 is similar to the proof of theorem 4.7. Equation (4.23) for constructing α from the spectral data is now as follows:

$$\alpha=\operatorname{cotanh}(ag). \tag{4.30}$$

Instead of sequences indexed by $0\neq k\in \boldsymbol{Z}$ we now have sequences indexed by $k\in\boldsymbol{Z}$. In particular, in (4.18) and (4.19), $k-\frac{1}{2}$ should be replaced by k.

Acknowledgments

One of the authors (VP) expresses his gratitude to the Department of Mathematics of the University of Cagliari for its hospitality during a visit in which a considerable part of this paper was written. VP's research was supported in part by the Autonomous Region of Sardinia. CvdM's research was supported in part by INdAM-GNCS and MURST.

References

[1] Borg G 1946 Eine Umkehrung der Sturm–Liouvilleschen Eigenwertaufgabe. Bestimmung der Differentialgleichung durch die Eigenwerte *Acta Math.* **78** 1–96
[2] Borg G 1952 Uniqueness theorems in the spectral theory of $y''+(\lambda-q(x))y=0$ *Proc. 11th Scand. Congr. Mathematicians* (Oslo: Johan Grundt Tanums) pp 276–87
[3] Courant R and Hilbert D 1953 *Methods of Mathematical Physics* vol 1 (New York: Wiley-Interscience)
[4] Cox S and Zuazua E 1995 The rate at which energy decays in a string damped at one end *Indiana Univ. Math. J.* **44** 545–73
[5] Eni V M 1967 Stability of the root-number of an analytic operator-function and perturbations of its characteristic numbers and eigenvectors *Dokl. Akad. Nauk SSSR* **173** 1251–4 (in Russian) (Engl. transl. 1967 *Sov. Math. Dokl.* **8** 542–5)

[6] Gesztesy F and Simon B 1999 Inverse spectral analysis with partial information on the potential II: the case of discrete spectrum *Trans. Am. Math. Soc.* **352** 2765–87

[7] Gohberg I C and Sigal E I 1971 An operator generalization of the logarithmic residue theorem and the theorem of Rouché *Mat. Sbornik* **84** 607–29 (in Russian) (Engl. transl. 1971 *Math. USSR Sbornik* **13** 603–25)

[8] Gomilko A M and Pivovarchik V N On bases of root functions of a boundary value associated with small vibrations of a damped nonsmooth inhomogeneous string *Preprint*

[9] Gubreev G M and Pivovarchik V N 1997 Spectral analysis of the parametrized Regge problem *Funkt. Anal. Pril.* **31** 70–4 (in Russian) (Engl. transl. 1997 *Funct. Anal. Appl.* **1** 54–7)

[10] Hruščëv S V, Nikolskiĭ N K and Pavlov B S 1981 Unconditional bases of exponentials and of reproducing kernels *Complex Analysis and Spectral Theory (Lecture Notes in Mathematics vol 864)* ed V P Havin and N K Nikolskiĭ (Berlin: Springer) pp 214–335

[11] Kravickiĭ A O 1968 The two-fold expansion of a certain non-selfadjoint boundary value problem in series of eigenfunctions *Differ. Uravn.* **4** 165–77 (in Russian)

[12] Levin B Ja 1956 *Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow* (in Russian) (Engl. transl. 1980 *Distribution of Zeros of Entire Functions* (*Transl. Math. Monographs) vol 5* (Providence, RI: American Mathematical Society))

[13] Levin B Ja and Ostrovskii I V 1979 *Izv. Akad. Nauk SSSR, Ser. Mat.* **43** 87–110, 238 (in Russian) (Engl. transl. 1980 On small perturbations of sets of root of sinus-type functions *Math. USSR–Izv.* **14** 79–101)

[14] Levitan B M 1984 *Inverse Sturm–Liouville Problems* (Zeist: VSP) (Moscow: Nauka (in Russian))

[15] Levitan B M and Gasymov M G 1964 Determination of a differential equation by two spectra *Usp. Mat. Nauk* **19** 3–63 (in Russian)

[16] Malamud M M 1999 Questions of uniqueness in inverse problems for systems of differential equations on a finite interval *Tr. Mosk. Mat. Obs.* **60** 199–258 (in Russian) (Engl. transl. 1999 *Trans. Moscow Math. Soc.* 173–224)

[17] Marchenko V A 1986 *Sturm–Liouville Operators and Applications* (Basel: Birkhäuser)

[18] Pivovarchik V 1994 On positive spectra of one class of polynomial operator pencils *Int. Eq. Operator Theory* **19** 314–26

[19] Regge T 1958 Construction of potential from resonances *Nuovo Cimento* **9** 491–503
Regge T 1958 Construction of potential from resonances *Nuovo Cimento* **9** 671–9

[20] Sergeev A G 1972 The asymptotic behavior of the Jost function and of the eigenvalues of the Regge problem *Diff. Uravn.* **8** 925–7 (in Russian)

[21] Shubov M A 1996 Basis property of eigenfunctions of nonselfadjoint operator pencils generated by the equation of nonhomogeneous damped string *Int. Eq. Operator Theory* **25** 289–328