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Integral Equations and Operator Theory

Polar Decompositions and Related Classes of Operators in Spaces Π_κ

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Polar decompositions with respect to an indefinite inner product are studied for bounded linear operators acting on a Π_κ space. Criteria are given for existence of various forms of the polar decompositions, under the conditions that the range of a given operator X is closed and that zero is not an irregular critical point of the selfadjoint operator $X^{[*]}X$. Both real and complex spaces Π_κ are considered. Relevant classes of operators having a selfadjoint (in the sense of the indefinite inner product) square root, or a selfadjoint logarithm, are characterized.

1 Introduction

In this work we study polar decompositions of bounded linear operators in infinite dimensional linear spaces with respect to an indefinite inner product, and related classes of operators, such as operators having selfadjoint square roots or selfadjoint logarithms. It will be assumed throughout that the underlying linear spaces are separable Hilbert spaces, and that the indefinite inner product under consideration is non-degenerate (i.e., only the zero vector is orthogonal to the whole space) and has a finite upper bound for the dimensions of its negative (with respect to the indefinite inner product) subspaces. Such indefinite inner product spaces are commonly known as Pontryagin or Π_κ spaces. Many results in the present paper represent extensions to Pontryagin spaces of earlier results obtained in [BR, BMR31, BMR32, BMR33] for finite dimensional indefinite inner product spaces.

We fix an infinite dimensional separable Hilbert space $\mathcal{G}$ over F, where F is the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$, and an invertible selfadjoint operator $H \in \mathcal{L}(\mathcal{G})$ (here and elsewhere $\mathcal{L}(\mathcal{G})$ stands for the Banach algebra of bounded linear operators on $\mathcal{G}$) such that the H-invariant subspace corresponding to the negative part of the spectrum of H is finite dimensional; let us denote by κ the (finite) dimension of this subspace.

Consider the inner product induced by H by the formula $[x, y] = \langle Hx, y\rangle$, $x, y \in \mathcal{G}$. Here $\langle \cdot\,, \cdot \rangle$ stands for the Hilbert space inner product in $\mathcal{G}$. The space $\mathcal{G}$ equipped with the indefinite inner product $[\,\cdot\,, \cdot\,]$ will be denoted by Π_κ. The vector $x \in \mathcal{G}$ is called *H-positive* if $[x, x] > 0$, *H-neutral* if $[x, x] = 0$, and *H-negative* if $[x, x] < 0$. A subspace (always understood as a closed linear set) $\mathcal{L}$ of $\mathcal{G}$ is called *H-neutral* if all elements of $\mathcal{L}$ are H-neutral vectors. A subspace is

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called *strictly H-positive* (resp., *strictly H-negative*) if all its nonzero elements are H-positive (resp., H-negative). We denote by $\mathcal{L}^{[\perp]}$ the *H-orthogonal companion* of a subspace $\mathcal{L}$:

$$\mathcal{L}^{[\perp]} = \{x \in \mathcal{G} : [x, y] = 0 \text{ for every } y \in \mathcal{L}\}.$$

All operators are assumed to be bounded and linear (with respect to the Hilbert space norm $\|\cdot\|$ in $\mathcal{G}$), unless explicitly stated otherwise.

Well-known concepts related to inner products are defined in an obvious way. Thus, given an operator A on $\mathcal{G}$, the H-adjoint $A^{[*]}$ is defined by $[Ax, y] = [x, A^{[*]}y]$ for all $x, y \in \mathcal{G}$. In that case $A^{[*]} = H^{-1}A^*H$, where A^* denotes the Hilbert space adjoint of A. An operator A is called *H-selfadjoint* if $A^{[*]} = A$ (or equivalently, if HA is Hilbert space selfadjoint). An operator U is called *H-isometry* if $[Ux, Uy] = [x, y]$ for all $x, y \in \mathcal{G}$ (or equivalently, if $U^*HU = H$). Note that every H-isometry has zero kernel; moreover, every linear transformation (not assumed bounded a priori) U on $\mathcal{G}$ such that $[Ux, Uy] = [x, y]$ for all $x, y \in \mathcal{G}$ is in fact bounded (see Corollary 2, Section 6 of [IKL]). An operator is called *H-unitary* if it is an H-isometry defined on $\mathcal{G}$ and maps onto $\mathcal{G}$. Clearly, H-unitary operators have a (bounded) inverse which is also H-unitary.

We now introduce the notion of polar decomposition with respect to indefinite inner products in Π_κ spaces. A representation $X = UA$, where A is H-selfadjoint and U is an H-isometry defined on a linear set containing the range of A, will be called an *H-polar decomposition* of X. Note that in contrast with the standard polar decompositions for Hilbert space operators, we allow the factor A to be merely H-selfadjoint, not necessarily H-semidefinite. In finite dimensional spaces, H-polar decompositions have been studied recently in [BR, BMR31, BMR32, BMR33, LMMR, MRR1, MRR2, MRR3].

Besides the introduction, the paper consists of 4 sections. In Section 2, operators that admit H-polar decompositions with various additional properties are described using a suitable version of Witt's theorem. This theorem provides extensions of isometries under appropriate conditions. Spectral properties of operators of the form $X^{[*]}X$ and those operators that admit a selfadjoint square root are described in Section 3. In Section 3 we present also a result concerning existence of H-selfadjoint logaritms of invertible H-selfadjoint operators. The main results describing H-polar decompositions in terms of a canonical form of $X^{[*]}X$, under the hypotheses that X has a closed range and the H-selfadjoint operator $X^{[*]}X$ has no irregular singular point at zero, are stated and proved in Section 4: Theorem 4.1 concerns general H-polar decompositions, and Theorem 4.2 concerns H-polar decompositions with the H-unitary factor U. We also indicate in Section 4 a result concerning existence of H-polar decompositions of H-normal operators. Both Sections 3 and 4 concern the complex case only. Several corresponding results for the real case are stated in Section 5.

The results on H-polar decompositions are closely related to results on certain decompositions arising from the theory of an H-modulus of an operator as developed by Potapov [P1, P2]. An operator R is called an *H-modulus* of a (bounded) operator X in a Krein space if $R^2 = X^{[*]}X$, $\operatorname{Ker} R = \operatorname{Ker} X^{[*]}X$ and finally, the spectrum of R is nonnegative. In this setting the concept of H-polar decomposition requires the H-selfadjoint factor to be an H-modulus of X and allows the H-unitary factor to be a partial H-isometry. In [P1, P2] it was shown that any H-nonexpansive matrix allows such a restricted H-polar decomposition (and in fact the H-partial isometry can then be taken to be H-unitary). These concepts were extended to the infinite dimensional case by Ginzburg [Gi1, Gi2], and finally, in the paper by Krein and Shmul'jan [KS2] (see also [KS1]), it was shown that a strict H-plus operator X admits such a restricted H-polar decomposition if and only if the spectrum of $X^{[*]}X$ is nonnegative and the image of X is an H-nondegenerate subspace. If in addition to this, $\operatorname{Ker} X^{[*]}$ is uniformly H-negative and $\operatorname{Ker} X^{[*]}$ has the same dimension as $\operatorname{Ker} X$, then the H-partially isometric factor in the H-polar decomposition can be chosen to be H-unitary.

2 Polar decompositions and extensions of isometries

Let $\mathcal{G}$ be a Π_κ space. We recall the definition of an H-polar decomposition $X = UA$ of an operator $X \in \mathcal{L}(\mathcal{G})$ given in the introduction. An H-polar decomposition $X = UA$ is called *H-unitary* if the operator U appearing in it is H-unitary. Classes of operators that are closely related to H-polar decompositions, include H-selfadjoint operators of the form $X^{[*]}X$, and H-selfadjoint operators having an H-selfadjoint square root. The existence of a close connection between these classes of operators, which is sufficiently well understood in the finite dimensional case, is apparent from the following three results.

Theorem 2.1 *An operator X admits an H-polar decomposition if and only if there exists an H-selfadjoint operator A with the following properties:*

(a) $X^{[*]}X = A^2$;

(b) $\operatorname{Ker} A = \operatorname{Ker} X$.

Moreover, only such operators A appear in H-polar decompositions $X = UA$ of X.

Proof. Conditions (a) and (b) are immediate if there exists an H-isometry U defined on $\operatorname{Im} A$ such that $X = UA$. Conversely, if conditions (a) and (b) are satisfied, then the map $U : \operatorname{Im} A \to \operatorname{Im} X$ defined by $U(Ax) = Xx$, is well-defined, linear, and an H-isometry. Hence $X = UA$ is an H-polar decomposition. □

For a (closed) subspace $\mathcal{V}$ of $\mathcal{G}$, we denote by $\operatorname{codim} \mathcal{V}$ the (finite or infinite) dimension of a direct complement of $\mathcal{V}$ in $\mathcal{G}$.

Theorem 2.2 *An operator X admits an H-polar decomposition $X = UA$ satisfying the condition*

$$\operatorname{codim} \overline{\operatorname{Im} A} \leq \operatorname{codim} \overline{\operatorname{Im} X} \tag{2.1}$$

and with U defined on all of $\mathcal{G}$ and is bounded below, i. e., there exists a positive constant C such that $C\|x\| \leq \|Ux\|$ for every $x \in \mathcal{G}$, if and only if there exists an H-selfadjoint operator A with the following properties:

(a) $X^{[*]}X = A^2$;

(b) $\operatorname{Ker} A = \operatorname{Ker} X$;

(c) there exist positive constants C_1, C_2 such that $C_1\|Ax\| \leq \|Xx\| \leq C_2\|Ax\|$ for every $x \in \mathcal{G}$;

(d) $\operatorname{codim} \overline{\operatorname{Im} A} \leq \operatorname{codim} \overline{\operatorname{Im} X}$.

Moreover, every such A, and only such A, appears as the H-selfadjoint factor in the H-polar decomposition $X = UA$ *with the property* (2.1).

Theorem 2.3 *An operator X admits an H-unitary H-polar decomposition if and only if there exists an H-selfadjoint operator A with the following properties:*

(a) $X^{[*]}X = A^2$;

(b) $\operatorname{Ker} A = \operatorname{Ker} X$;

(c) there exist positive constants C_1, C_2 *such that* $C_1\|Ax\| \leq \|Xx\| \leq C_2\|Ax\|$ *for every* $x \in \mathcal{G}$;

(d) $\operatorname{codim} \overline{\operatorname{Im} A} = \operatorname{codim} \overline{\operatorname{Im} X}$.

Moreover, every such A, and only such A, appears as the H-selfadjoint factor in the H-unitary H-polar decomposition $X = UA$.

As in the finite dimensional case, the proofs of Theorems 2.2 and 2.3 are based on the extension of isometries described by Witt's theorem. Before proving this theorem in Pontryagin spaces, we discuss the concept of "skewly linked" H-neutral subspaces (cf. [IKL], Lemma I 3.1). Indeed, if $\mathcal{E}$ is an H-neutral subspace of Π_κ, there exists a subspace $\mathcal{F}$ of the same finite dimension as $\mathcal{E}$ with the following properties:

(i) $\mathcal{F}$ is an H-neutral subspace;

(ii) no nonzero vector of $\mathcal{E}$ is H-orthogonal to $\mathcal{F}$;

(iii) no nonzero vector of $\mathcal{F}$ is H-orthogonal to $\mathcal{E}$;

(iv) the H-inner product does not degenerate on the direct sum $\mathcal{F}\dot{+}\mathcal{E}$. (Here $\mathcal{F}\dot{+}\mathcal{E}$ is the (algebraic) direct sum and $\mathcal{F} \oplus \mathcal{G}$ the direct sum of $\mathcal{F}$ and $\mathcal{E}$ with H-orthogonal components.)

Note that (ii) and (iii) follow from (iv). Two H-neutral subspaces $\mathcal{F}$ and $\mathcal{E}$ are called *skewly linked* if conditions (i)-(iv) are satisfied. Corollary 1 to Theorem I 3.4 of [IKL] shows that the orthogonal companion $\mathcal{E}^{[\perp]}$ of $\mathcal{E}$ is a direct complement of $\mathcal{F}$ if $\mathcal{F}$ is skewly linked with $\mathcal{E}$. Now Theorem I 3.4 of [IKL] (originally proved in [IK, Bo]) is as follows:

Theorem 2.4 *Let* $\mathcal{L}$ *be an arbitrary subspace of* Π_κ, *let* $\mathcal{M} = \mathcal{L}^{[\perp]}$, *and denote by* $\mathcal{E}$ *the isotropic part* $\mathcal{L} \cap \mathcal{M}$ *of* $\mathcal{L}$.

(i) *If we choose decompositions* $\mathcal{L} = \mathcal{L}_1 \oplus \mathcal{E}$ *and* $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{E}$, *then there exists at least one subspace* $\mathcal{F} \subset \Pi_\kappa$ *which is skewly linked with* $\mathcal{E}$ *and such that*

$$\Pi_\kappa = \mathcal{L}_1 \oplus (\mathcal{E}\dot{+}\mathcal{F}) \oplus \mathcal{M}_1. \tag{2.2}$$

(ii) *If* $\mathcal{F}$ *is an arbitrary subspace of* Π_κ *which is skewly linked with* $\mathcal{E}$, *then there exist unique decompositions* $\mathcal{L} = \mathcal{L}_1 \oplus \mathcal{E}$ *and* $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{E}$ *such that* (2.2) *holds.*

Notice that Theorem 2.4 is valid in the real case as well, with essentially the same proof (it is stated and proved in [IKL] for the complex case only).

Let us now apply these results to obtain Witt's theorem for Pontryagin spaces. In the finite dimensional case, this theorem and various refinements can be found in [BMR33], although the basic result itself has been known for decades.

Theorem 2.5 *Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be closed subspaces of Π_κ and let U_0 be a continuous H-isometric operator mapping $\mathcal{V}_1$ onto $\mathcal{V}_2$. Assume that* $\operatorname{codim}\mathcal{V}_1 \geq \operatorname{codim}\mathcal{V}_2$. *Then there exists a linear operator $U : \Pi_\kappa \to \Pi_\kappa$ such that*

$$[Ux, Uy] = [x, y], \quad x, y \in \Pi_\kappa, \qquad \text{and} \qquad Ux = U_0 x, \quad x \in \mathcal{V}_1.$$

Moreover, if $\operatorname{codim}\mathcal{V}_1 = \operatorname{codim}\mathcal{V}_2$, *then U_0 can be extended to an H-unitary operator on Π_κ.*

Proof. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be the isotropic parts of $\mathcal{V}_1$ and $\mathcal{V}_2$, respectively, so that U_0 maps $\mathcal{E}_1$ isometrically onto $\mathcal{E}_2$. Let $\mathcal{L}_1^1$ and $\mathcal{M}_1^1$ be a closed strictly positive and a strictly negative subspace, respectively, of $\mathcal{V}_1$ such that $\mathcal{V}_1 = \mathcal{L}_1^1 \oplus \mathcal{E}_1 \oplus \mathcal{M}_1^1$. Now let $\mathcal{L}_1^2 = U_0[\mathcal{L}_1^1]$ and $\mathcal{M}_1^2 = U_0[\mathcal{L}_1^2]$. Then $\mathcal{L}_1^2$ and $\mathcal{M}_1^2$ are a closed strictly positive and a strictly negative subspace, respectively, of $\mathcal{V}_2$ such that $\mathcal{V}_2 = \mathcal{L}_1^2 \oplus \mathcal{E}_2 \oplus \mathcal{M}_1^2$. We now choose the subspace $\mathcal{F}_1$ skewly linked with $\mathcal{E}_1$ and the subspace $\mathcal{F}_2$ skewly linked with $\mathcal{E}_2$, and define

$$\Pi_{\kappa 1} = \mathcal{L}_1^1 \oplus (\mathcal{E}_1 \dot{+} \mathcal{F}_1) \oplus \mathcal{M}_1^1; \qquad \Pi_{\kappa 2} = \mathcal{L}_1^2 \oplus (\mathcal{E}_2 \dot{+} \mathcal{F}_2) \oplus \mathcal{M}_1^2.$$

Then choosing a basis $e_1, \cdots, e_{m_0}$ in $\mathcal{E}_1$ and a basis $f_1, \cdots, f_{m_0}$ in $\mathcal{E}_2$ with

$$f_1 = U_0 e_1, \cdots, f_{m_0} = U_0 e_{m_0},$$

we find vectors $\tilde{e}_1, \cdots, \tilde{e}_{m_0}$ to form a basis of $\mathcal{F}_1$ such that $[\tilde{e}_i, e_j] = \delta_{i,j}$ for $i, j = 1, \cdots, m_0$, and vectors $\tilde{f}_1, \cdots, \tilde{f}_{m_0}$ to form a basis of $\mathcal{F}_2$ such that $[\tilde{f}_i, f_j] = \delta_{i,j}$ for $i, j = 1, \cdots, m_0$. Now readjust the vectors $\tilde{e}_1, \cdots, \tilde{e}_{m_0}$ and $\tilde{f}_1, \cdots, \tilde{f}_{m_0}$ by defining

$$\hat{e}_i = \tilde{e}_i - \frac{1}{2}[\tilde{e}_i, \tilde{e}_i] e_i, \qquad \hat{f}_i = \tilde{f}_i - \frac{1}{2}[\tilde{f}_i, \tilde{f}_i] f_i.$$

Now defining $U_1 x = U_0 x$ on $\mathcal{V}_1$ and $U_1(\sum_{j=1}^{m_0} \xi_j \hat{e}_j) = \sum_{j=1}^{m_0} \xi_j \hat{f}_j$, we extend U_0 to an isometry U_1 from $\Pi_{\kappa 1}$ onto $\Pi_{\kappa 2}$. Indeed, U_1 maps $\mathcal{F}_1 \dot{+} \mathcal{E}_1$ isometrically onto $\mathcal{F}_2 \dot{+} \mathcal{E}_2$, as shown by the two identities

$$\left[\sum_{i=1}^{m_0} (\xi_i \hat{e}_i + \eta_i e_i), \sum_{j=1}^{m_0} (\mu_j \hat{e}_j + \nu_j e_j)\right] = \sum_{i=1}^{m_0} \left(\xi_i \overline{\nu_i} + \eta_i \overline{\mu_i}\right);$$

$$\left[\sum_{i=1}^{m_0} (\xi_i \hat{f}_i + \eta_i f_i), \sum_{j=1}^{m_0} (\mu_j \hat{f}_j + \nu_j f_j)\right] = \sum_{i=1}^{m_0} \left(\xi_i \overline{\nu_i} + \eta_i \overline{\mu_i}\right),$$

while the three spaces in the decompositions $\Pi_{\kappa j} = \mathcal{L}_1^j \oplus (\mathcal{F}_j \dot{+} \mathcal{E}_j) \oplus \mathcal{M}_j$, $j = 1, 2$, are orthogonal. (In the real case, the complex conjugation is omitted in the above formulas.) Since U_1 is now an H-isometry from the nondegenerate subspace $\Pi_{\kappa 1}$ onto the nondegenerate subspace $\Pi_{\kappa 2}$, and $\operatorname{codim}\Pi_{\kappa 1} \geq \operatorname{codim}\Pi_{\kappa 2}$, it is now simple to extend it to an H-isometry defined on Π_κ (cf. Section 9.1 of [IKL]).

If U_0 maps $\mathcal{V}_1$ onto $\mathcal{V}_2$ and $\operatorname{codim}\mathcal{V}_1 = \operatorname{codim}\mathcal{V}_2$, then U_0 is first extended to an H-isometry U_1 mapping the nondegenerate closed subspace $\Pi_{\kappa 1}$ onto the nondegenerate closed subspace $\Pi_{\kappa 2}$, as explained above. However, the codimension condition implies that $\Pi_{\kappa 1}$ and $\Pi_{\kappa 2}$ have the same codimension. It now follows from the last paragraph of Section 9.2 of [IKL] that U_1 can in turn be extended to an H-unitary operator on Π_κ, which completes the proof. □

In the above proof we have stipulated bases on the isotropic parts of the closed subspaces $\mathcal{V}_1$ and $\mathcal{V}_2$ between which the given isometry is acting. However, since the full space is a Π_κ-space, the finite dimensionality of these H-neutral subspaces allows one to choose these bases. Note that in the space Π_κ, the dimensions of H-negative and H-neutral linear sets are bounded above by κ; in particular, they are finite dimensional, and therefore automatically closed.

Proof of Theorem 2.2. We mimic the argument given in [BR]. The existence of an H-polar decomposition $X = UA$, where U is defined on all of $\mathcal{G}$ and is bounded below, easily implies that A satisfies conditions (a) - (c). Condition (d) need not be proved, since it is included in the type of H-polar decomposition described in the statement of Theorem 2.2.

Conversely, if there exists an H-selfadjoint A satisfying conditions (a) - (d), then define U_0 from $\operatorname{Im} A$ onto $\operatorname{Im} X$ by $U_0(Az) = Xz$ for every $z \in \Pi_\kappa$. Clearly, U_0 is a well-defined and bounded H-isometry defined on $\operatorname{Im} A$ which is bounded above and below (because of (c)). Therefore, U_0 can be continuously extended to $\overline{\operatorname{Im} A}$ and U_0^{-1} can be continuously extended to $\overline{\operatorname{Im} X}$. Writing U_1 for the former extension, one easily sees that U_1 maps $\overline{\operatorname{Im} A}$ onto $\overline{\operatorname{Im} X}$. Next, condition (d) allows us to apply Theorem 2.5 to extend U_1 to an H-isometry U on Π_κ. Since one obviously has $X = UA$, this completes the proof. □

The proof of Theorem 2.3 is similar to that of Theorem 2.2 and will therefore be omitted.

3 Operators of the form $X^{[*]}X$ and selfadjoint square roots

In connection with Theorems 2.1, 2.2 and 2.3, it is of interest to study operators of the form $X^{[*]}X$. It will be convenient to consider in this section the complex case only (except for Theorem 3.10), and postpone the discussion of the real case to Section 5.

The following result is a particular case of Theorem VII.2.1 of [Bo]. A transparent proof for the finite-dimensional case [BR] can be generalized to the present context.

Proposition 3.1 *In an infinite dimensional Π_κ-space, an H-selfadjoint operator Z is of the form $X^{[*]}X$ for some $X \in \mathcal{L}(\mathcal{G})$ if and only if the spectral subspace of HZ corresponding to the negative part of $\sigma(HZ)$ has dimension at most κ.*

It is a natural question to ask whether a restriction of $X^{[*]}X$ to an H-orthogonally reducing invariant subspace is again of the from $Y^{[*]}Y$, where the operator Y, as well as the corresponding indefinite inner product, act in the invariant subspace. This turns out to be false in general, as the next example shows.

Example 3.2 Consider on a finite dimensional space $H = I_\nu \oplus -I_\kappa$, $Z = Z_1 \oplus Z_2$, where Z_1 and Z_2 are negative. Clearly, if $\nu \leq \kappa$ the condition of the above proposition is satisfied. However, $\kappa(I) = 0$, and Z_1 is negative. So, unless $\nu = 0$, the condition is not satisfied for the pair (Z_1, I).

Corollary 3.3 *The set of all operators of the form $X^{[*]}X$ is closed in the strong operator topology.*

Theorem 3.4 $\sigma(X^{[*]}X)\setminus[0,\infty)$ *is a finite set consisting only of eigenvalues λ_0 of finite algebraic multiplicities $m(\lambda_0)$.*

Proof. First observe that $X^{[*]}X$ is a finite rank perturbation of X^*X. Therefore [Go, GoKr], $\sigma(X^{[*]}X)\setminus[0,\infty)$ is a discrete set in $\mathbb{C}\setminus[0,\infty)$ with possible accumulation points only in $[0,\infty)$. Moreover, each of the points in $\sigma(X^{[*]}X)\setminus[0,\infty)$ is an eigenvalue of finite algebraic multiplicity.

Since $X^{[*]}X$ is H-selfadjoint, the number of non real points in the spectrum of $X^{[*]}X$ is finite (see, e.g., [Bo]). Next we show that there are only a finite number of negative eigenvalues of $X^{[*]}X$. Let $\lambda<0$ and $x=x(\lambda)\neq 0$ be such that $X^{[*]}Xx=\lambda x$. Then

$$\lambda[x,x]=[X^{[*]}Xx,x]=[Xx,Xx].$$

Since $\lambda<0$, there are three possibilities: (a) $[x,x]=[Xx,Xx]=0$; (b) $[x,x]<0$; (c) $[Xx,Xx]<0$. Denote

$$\Sigma_1=\{\lambda\in\sigma(X^{[*]}X)\cap(-\infty,0)\ :\ [x,x]\leq 0\},$$

i.e., (a) or (b) holds for $\lambda\in\Sigma_1$. Also, denote

$$\Sigma_2=\{\lambda\in\sigma(X^{[*]}X)\cap(-\infty,0)\ :\ [Xx,Xx]\leq 0\},$$

so that (a) or (c) holds for $\lambda\in\Sigma_2$. Then $\sigma(X^{[*]}X)\cap(-\infty,0)=\Sigma_1\cup\Sigma_2$ by the preceding paragraph. Next, we show that both Σ_1 and Σ_2 are finite sets containing at most κ points each.

It is well-known (and easy to check) that the vectors $x(\lambda_1),\cdots,x(\lambda_k)$ are linearly independent and H-orthogonal to each other, assuming that $\lambda_1,\cdots,\lambda_k$ are distinct negative eigenvalues of $X^{[*]}X$. Thus, if either (a) or (b) holds for every $x(\lambda_j)$, the subspace

$$\text{span}\,\{x(\lambda_1),\cdots,x(\lambda_k)\}$$

is a k-dimensional H-nonnegative subspace, and therefore $k\leq\kappa$. On the other hand, the vectors

$$Xx(\lambda_1),\cdots,Xx(\lambda_k)$$

are also linearly independent and H-orthogonal (we continue to assume that $\lambda_1,\cdots,\lambda_k$ are distinct). Indeed,

$$[Xx(\lambda_i),Xx(\lambda_j)]=\lambda_i[x(\lambda_i),x(\lambda_j)]=0,\quad (i\neq j)$$

and if

$$\sum_{j=1}^{k}\alpha_j Xx(\lambda_j)=0 \tag{3.1}$$

for some $\alpha_j\in\mathbb{C}$, then multiplying (3.1) by $X^{[*]}$ yields $\sum_{j=1}^{k}\alpha_j\lambda_j x(\lambda_j)=0$, and therefore $\alpha_j=0$ in view of the linear independence of the vectors $x(\lambda_1),\cdots,x(\lambda_k)$. Thus, if (c) holds for every $x(\lambda_j)$ we obtain a k-dimensional H-nonnegative subspace

$$\text{span}\,\{Xx(\lambda_1),\cdots,Xx(\lambda_k)\},$$

and hence again $k\leq\kappa$. It follows that the number of negative eigenvalues of $X^{[*]}X$ cannot exceed 2κ. □

A more careful analysis of the proof of Theorem 3.4 shows that the number of distinct eigenvalues of $X^{[*]}X$ in $\mathbb{C} \setminus [0, \infty)$ does not exceed 2κ. In fact, the following better estimate holds:

$$\sum_{\lambda_0 \in \sigma(X^{[*]}X) \setminus [0,\infty)} m(\lambda_0) \leq 2\kappa. \tag{3.2}$$

Further analysis of operators of the form $X^{[*]}X$ depends on the properties of the point 0 as a critical point of $X^{[*]}X$, namely, whether it is a regular or a singular critical point. Let us recall from [Lan] the definition and properties of singular points of the operator A. Any selfadjoint operator on a Pontryagin space is definitizable, that is, there exists a polynomial $p(z) \not\equiv 0$ such that $p(A)$ is a nonnegative operator in the indefinite inner product. The set of critical points is then defined as the intersection of the set of real zeros of p with the spectrum of A. Denote by R_A the semiring of all bounded real intervals with endpoints not in the set of critical points, together with their complements. Then there is a map $\Delta \mapsto E_\Delta$ from R_A into the set of selfadjoint projections in the Pontryagin space inner product such that all the usual properties of a spectral function for A are satisfied (see Section II.3 in [Lan]).

Critical points come in two kinds. Let α be a critical point, and let $\lambda_0 < \alpha$ and $\lambda_1 > \alpha$ be real numbers that are not critical points. If for any choice of λ_0, λ_1 we have that the limits $\lim_{\lambda \uparrow \alpha} E([\lambda_0, \lambda])$ and $\lim_{\lambda \downarrow \alpha} E([\lambda, \lambda_1])$ exist in the strong operator topology, then we say that α is a *regular* critical point. Otherwise α is called a *singular* critical point. These concepts are introduced and discussed in Section II.5 in [Lan]. It turns out that α is a regular critical point if and only if there is a neighbourhood $\mathcal{U}$ of α, with α the only critical point in $\mathcal{U}$, such that the spectral projections $E(\Delta)$, with $\Delta \subset \mathcal{U}$, are uniformly bounded in norm. In particular, this means that we can define the spectral projection of A corresponding to $\{\alpha\}$. Its image, called the *spectral subspace* of A associated with α and denoted $\mathcal{P}_A(\alpha)$, is the intersection of the ranges of $E(\Delta)$ with $\alpha \in \Delta$ and $\Delta \in R_A$. Note also that, for a regular critical point α, the subspace $\mathcal{P}_A(\alpha)$ coincides with Ker $(\alpha I - A)^k$ for a suitable integer k, and the H-inner product does not degenerate on $\mathcal{P}_A(\alpha)$.

In the next proposition a well-known canonical form for H-selfadjoint operators in finite dimensional spaces will be used. For the reader's convenience, we recall this form. Denote by $J_p(\lambda)$ the $p \times p$ upper triangular Jordan block with eigenvalue λ, and by $Z_p = [\delta_{i+j,p+1}]_{i,j=1}^p$ the $p \times p$ matrix with ones on the southwest-northeast diagonal and zeros elsewhere. The notation diag $(X_1, \cdots, X_p)$ stands for the block diagonal matrix with the diagonal blocks $X_1, \cdots, X_p$.

Proposition 3.5 *Let $A \in \mathbb{C}^n$ be G-selfadjoint, where G is an invertible Hermitian $n \times n$ matrix. Then there exists a nonsingular matrix $S \in \mathbb{C}^n$, such that*

$$S^{-1}AS = \text{diag}(A_1, \dots, A_k), \quad S^*GS = \text{diag}\,(G_1, \dots, G_k), \tag{3.3}$$

where A_j, G_j are of the same size and the pair (A_j, G_j) has one and only one of the following forms:

1. *Blocks associated with real eigenvalues:*

$$A_j = J_p(\lambda), \quad \text{and} \quad G_j = \varepsilon Z_p, \tag{3.4}$$

where $\lambda \in \mathbb{R}$, $\varepsilon \in \{1, -1\}$, and p depends on j.

2. *Blocks associated with a pair of nonreal eigenvalues:*

$$A_j = \begin{bmatrix} J_p(\lambda) & 0 \\ 0 & J_p(\lambda^*) \end{bmatrix}, \quad \text{and} \quad G_j = \begin{bmatrix} 0 & Z_p \\ Z_p & 0 \end{bmatrix} \tag{3.5}$$

where $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and p depends on j.

Moreover, the form $(S^{-1}AS, S^*GS)$ of (A, G) is uniquely determined up to a simultaneous permutation of blocks, and is called the canonical form of (A, G).

We are now in a position to state and prove the main result regarding existence of H-selfadjoint square roots of operators of the form $X^{[*]}X$.

Theorem 3.6 *Let $Z = X^{[*]}X$ be an H-selfadjoint operator on a Π_κ-space $\mathcal{G}$ that does not have a singular critical point at $\lambda = 0$. Then $\mathcal{G}$ can be decomposed H-orthogonally as*

$$\mathcal{G} = \mathcal{M}_{00} \oplus \mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \mathcal{M}_3, \tag{3.6}$$

where the subspaces $\mathcal{M}_{00}$, $\mathcal{M}_1$, $\mathcal{M}_2$, and $\mathcal{M}_3$ are Z-invariant and have the following properties: $\mathcal{M}_{00}$ is finite dimensional and the restriction Z_{00} of Z to $\mathcal{M}_{00}$ is nilpotent, $\mathcal{M}_1$ is strictly H-positive and contained in the kernel of Z, the restriction of Z to $\mathcal{M}_2$ has its spectrum in $[0, \infty)$ and has trivial kernel, and $\mathcal{M}_3$ is finite dimensional and the restriction of Z to $\mathcal{M}_3$ does not have spectrum in $[0, \infty)$. Write $[x, y] = \langle H_{00}x, y\rangle$ for all $x, y \in \mathcal{M}_{00}$. Then there exists an H-selfadjoint operator A such that $Z = A^2$, if and only if the conditions (i) *and* (ii) *below are satisfied:*

(i) *For each negative eigenvalue λ of Z the part of the canonical form of (Z, H) corresponding to λ can be presented in the form*

$$(\operatorname{diag}(A_1, \ldots, A_m), \ \operatorname{diag}(H_1, \ldots, H_m)), \tag{3.7}$$

where for $i = 1, \ldots, m$

$$A_i = \begin{bmatrix} J_{k_i}(\lambda) & 0 \\ 0 & J_{k_i}(\lambda) \end{bmatrix}, \qquad H_i = \begin{bmatrix} Q_{k_i} & 0 \\ 0 & -Q_{k_i} \end{bmatrix};$$

(ii) *the canonical form of (Z_{00}, H_{00}) can be presented in the form*

$$(\operatorname{diag}(B_0, B_1, \ldots, B_l), \ \operatorname{diag}(H_0, H_1, \ldots, H_l)), \tag{3.8}$$

where $B_0 = 0_{k_0 \times k_0}$, $H_0 = I_{p_0} \oplus -I_{n_0}$, $p_0 + n_0 = k_0$, and for each $i = 1, \ldots, l$ the pair (B_i, H_i) has one of the following two forms:

$$B_i = \begin{bmatrix} J_{k_i}(0) & 0 \\ 0 & J_{k_i}(0) \end{bmatrix}, \qquad H_i = \begin{bmatrix} Q_{k_i} & 0 \\ 0 & -Q_{k_i} \end{bmatrix}, \qquad k_i \geq 1;$$

or

$$B_i = \begin{bmatrix} J_{k_i}(0) & 0 \\ 0 & J_{k_i - 1}(0) \end{bmatrix}, \qquad H_i = \varepsilon_i \begin{bmatrix} Q_{k_i} & 0 \\ 0 & Q_{k_i - 1} \end{bmatrix},$$

with $\varepsilon_i = \pm 1$ and $k_i > 1$.

Before proving Theorem 3.6 we need the following lemma.

Lemma 3.7 *Let Y be H-selfadjoint in Π_κ, and let $\sigma(Y) = \{0\}$ (i.e., Y is quasinilpotent). Then Y is nilpotent and of finite rank. Moreover, the space $\mathcal{H}$ admits an H-orthogonal decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ such that $\mathcal{H}_1$ is in the kernel of Y, and $\mathcal{H}_0$ is Y-invariant and finite dimensional.*

Proof. Let κ be the number of negative squares of the Pontryagin space. We proceed by induction on κ. For $\kappa = 0$ there is nothing to prove, as $Y = 0$ in the Hilbert space case.

Let $\kappa > 0$. Then there is a maximal Y-invariant H-nonpositive subspace. This subspace is finite dimensional, and non-zero. Thus there is an eigenvector of Y: $Yx = 0$ for some non-zero vector x. Now either $\langle Hx, x\rangle < 0$ or $\langle Hx, x\rangle = 0$.

In the former case we assume without loss of generality that $\langle Hx, x\rangle = -1$. Put $\mathcal{M}_0 = \operatorname{span}\{x\}$. As $\mathcal{M}_0$ is H-nondegenerate we can decompose the space $\mathcal{G}$ as $\mathcal{G} = \mathcal{M}_0 \oplus \mathcal{M}_0^{[\perp]}$, and with respect to this decomposition we have

$$Y = \begin{bmatrix} 0 & a \\ 0 & Y_2 \end{bmatrix}, \qquad H = \begin{bmatrix} -1 & 0 \\ 0 & H_2 \end{bmatrix},$$

where H_2 is selfadjoint and invertible. Using the fact that Y is H-selfadjoint we see that $a = 0$, and Y_2 is H_2-selfadjoint. Moreover, $\sigma(Y_2) = \{0\}$, and $\kappa(H_2) = \kappa(H) - 1$. So, by induction we get the conclusion of the lemma in this case.

Now assume that $\langle Hx, x\rangle = 0$. As $\mathcal{G}$ is H-nondegenerate there is a y such that $\langle Hx, y\rangle = 1$. Moreover, considering $y + \alpha x$ in place of y, by a suitable choice of α, we get that we can assume $\langle Hy, y\rangle = 0$ without loss of generality. Put $\mathcal{M}_0 = \operatorname{span}\{x, y\}$. Then $\mathcal{M}_0$ is H-nondegenerate. Again, decompose the space $\mathcal{G}$ as $\mathcal{G} = \mathcal{M}_0 \oplus \mathcal{M}_0^{[\perp]}$. With respect to this decomposition and the basis $\{x, y\}$ in $\mathcal{M}_0$ we write Y and H as

$$Y = \begin{bmatrix} 0 & \alpha & a_1 \\ 0 & \beta & a_2 \\ 0 & \mu & Y_2 \end{bmatrix}, \qquad H = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & H_2 \end{bmatrix},$$

where $\mu \in \mathcal{M}_0^{[\perp]}$ and H_2 is selfadjoint and invertible. Using the fact that Y is H-selfadjoint we see that $\alpha \in \mathbb{R}$, $\beta = 0$, $a_2 = 0$, $a_1 = (H_2\mu)^*$ and Y_2 is H_2-selfadjoint. Thus

$$Y = \begin{bmatrix} 0 & \alpha & (H_2\mu)^* \\ 0 & 0 & 0 \\ 0 & \mu & Y_2 \end{bmatrix}.$$

Next, we show that Y_2 is quasinilpotent. Take $\lambda \neq 0$. Then $\lambda I - Y$ is invertible as well as $\lambda I_2 - \begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix}$. It follows by a Schur complement argument that

$$\lambda I - Y_2 - \begin{bmatrix} 0 & \mu \end{bmatrix} \left(\lambda - \begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix}\right)^{-1} \begin{bmatrix} (H_2\mu)^* \\ 0 \end{bmatrix}$$

is invertible. However, the latter is just $\lambda I - Y_2$. So Y_2 is quasinilpotent and H_2-selfadjoint. Also $\kappa(H_2) = \kappa(H) - 1$. Using the induction hypothesis we see that Y_2 is nilpotent and of finite rank. So Y is of finite rank as well. As Y is quasinilpotent and of finite rank it must be nilpotent.

It remains to prove the last part of the lemma. To see that this is true, first consider the image of Y. It is in general not H-nondegenerate, but we can extend it to a finite dimensional subspace $\mathcal{M}_0$ that is H-nondegenerate. It is obviously Y-invariant as it contains the image of Y. With respect to the decomposition $\mathcal{M}_0 \oplus \mathcal{M}_0^{[\perp]}$ write the operators Y and H as

$$Y = \begin{bmatrix} Y_{11} & Y_{12} \\ 0 & 0 \end{bmatrix}, \qquad H = \begin{bmatrix} H_{11} & 0 \\ 0 & H_{22} \end{bmatrix}.$$

As $\mathcal{M}_0$ is H-nondegenerate H_{11} is invertible. We have only to show that $Y_{12} = 0$. To see this note that for $x \in \mathcal{M}_0^{[\perp]}$ and for any z we have

$$\langle HYx, z\rangle = \langle Hx, Yz\rangle = 0,$$

as Yz is in the image of Y and hence in $\mathcal{M}_0$. This shows that $Y_{12} = 0$. This proves the lemma. □

Proof of Theorem 3.6. Let Z be an H-selfadjoint operator on Π_κ satisfying the conditions of Theorem 3.6. Then there exists an H-orthogonal decomposition of Π_κ reducing Z such that

$$Z = Z_0 \oplus Z_2 \oplus Z_3, \tag{3.9}$$

where Z_0 is quasinilpotent and defined on the spectral subspace $\mathcal{M}_0$ of Z corresponding to the spectral point at $\lambda = 0$, $\sigma(Z_2) \subset [0, \infty)$ and $\operatorname{Ker} Z_2 = \{0\}$, and $\sigma(Z_3) \cap [0, \infty) = \emptyset$ (see Theorem 5.7 in [Lan]).

As the decomposition reducing Z in (3.9) is H-orthogonal, with respect to the same decomposition we can write $H = H_0 \oplus H_2 \oplus H_3$. Then Z_0 is H_0-selfadjoint and quasinilpotent. Thus, according to Lemma 3.7 we may write $\mathcal{M}_0 = \widetilde{\mathcal{M}}_{00} \oplus \widetilde{\mathcal{M}}_1$, where $\widetilde{\mathcal{M}}_1 \subseteq \operatorname{Ker} Z_0$ and $\widetilde{\mathcal{M}}_{00}$ is Z_0-invariant and finite dimensional. Then $\widetilde{\mathcal{M}}_1$ is H_0-nondegenerate, and we can split it as $\widetilde{\mathcal{M}}_1 = \mathcal{M}_{01} \oplus \mathcal{M}_1$, where $\mathcal{M}_{01}$ is strictly H_0-negative and $\mathcal{M}_1$ is strictly H_0-positive. Obviously, $\mathcal{M}_{01}$ is finite dimensional. Now put $\mathcal{M}_{00} = \widetilde{\mathcal{M}}_{00} \oplus \mathcal{M}_{01}$. Then $\mathcal{M}_{00}$ is Z_0-invariant (as $\mathcal{M}_{01} \subseteq \operatorname{Ker} Z_0$) and finite dimensional, and $\mathcal{M}_1$ is H_0-positive and is contained in $\operatorname{Ker} Z_0$. Clearly, representing the indefinite inner product on $\mathcal{M}_{00}$ by $[x, y] = \langle H_{00}x, y\rangle$ for $x, y \in \mathcal{M}_{00}$, we have the decomposition as described in the first paragraph of the theorem. Further, Theorem 3.4 implies that Z_3 is defined on a finite dimensional subspace $\mathcal{M}_3$ whose inner product is represented by $[x, y] = \langle H_3x, y\rangle$ for $x, y \in \mathcal{M}_3$.

We verify next that if $Z = A^2$ for some H-selfadjoint operator A, then A also does not have a singular critical point at $\lambda = 0$. Indeed, otherwise $\|E_A[-\alpha, \alpha]x\| \to \infty$ as $\alpha \to 0$, where E_A is the spectral function of A. Since by the functional calculus we have $E_Z[0, \alpha^2] = E_A[-\alpha, \alpha]$ for $\alpha > 0$, it follows that $\|E_Z[0, \alpha^2]\| \to \infty$ as $\alpha \to 0$, which is a contradiction with Z not having a singular critical point at $\lambda = 0$. Now by repeating the above decomposition for A rather than for Z, we conclude, as in the finite dimensional case (see [BMR31]), that if $Z = A^2$ for some H-selfadjoint A, then conditions (i) and (ii) hold.

Conversely, suppose that (i) and (ii) hold. Then the canonical form of (Z_{00}, H_{00}) is as in condition (ii) of Theorem 3.6, while at the negative eigenvalues the canonical form of (Z_3, H_3) is as in condition (i) of Theorem 3.6. Now write $Z_3 = Z_- \oplus Z_{nr}$ with Z_- defined on $\mathcal{M}_-$ where $\sigma(Z_-) \subset (-\infty, 0)$ and $\sigma(Z_{nr}) \cap \mathbb{R} = \emptyset$, and represent the inner product induced by H on $\mathcal{M}_-$ by $[x, y] = \langle H_-x, y\rangle$ for $x, y \in \mathcal{M}_-$.

The proof of the existence of an H_{00}-selfadjoint square root of Z_{00} and an H_{-}-selfadjoint square root of Z_{-} now is identical to the proof of Theorem 4.4 of [BMR31] with the following modifications: one replaces $(X^{[*]}X, H)$ by $(Z_{00} \oplus Z_3, H_{00} \oplus H_3)$ (the special form $X^{[*]}X$ does not play any role), and one disregards the part of this proof involving condition (iii) of Theorem 4.4 of [BMR31]. Obviously, once Z_{00} has an H-selfadjoint square root, so has Z_0, by just extending it with 0.

Denoting the closed subspace on which Z_2 is defined by $\mathcal{M}_2$ and representing the indefinite inner product on $\mathcal{M}_2$ by $[x, y] = \langle H_2 x, y\rangle$ for $x, y \in \mathcal{M}_2$, it remains to prove the existence of an H_2-selfadjoint square root of Z_2. If Z does not have any nonzero singular critical points, one has the spectral representation

$$Z_2 = \int_{(0,\infty)} t\, E(dt)$$

for some bounded H_2-selfadjoint spectral measure $E(\cdot)$ supported on $[0, \infty)$ [Bo, IKL]. In that case, an H_2-selfadjoint square root of Z_2 is given by

$$A_2 = \int_{(0,\infty)} \sqrt{t}\, E(dt).$$

On the other hand, if Z_2 is boundedly invertible (which means that $\sigma(Z) \cap (0, \delta) = \emptyset$ for some $\delta > 0$), an H_2-selfadjoint square root of Z_2 is given by

$$A_2 = \frac{1}{2\pi i} \int_{\Gamma} \sqrt{\lambda}\, (\lambda I - Z)^{-1}\, d\lambda, \tag{3.10}$$

where Γ is a simple closed rectifiable Jordan contour having winding number one with respect to any point of $\sigma(Z) \cap (0, \infty)$ and winding number zero with respect to any of the finitely many points of $\sigma(Z) \setminus (0, \infty)$. Equation (3.10) is also valid if Z has nonzero singular critical points and $\sigma(Z) \cap (0, \delta) = \emptyset$ for some $\delta > 0$. Finally, if neither situation occurs, there exists $\delta > 0$ such all nonzero singular critical points of Z are contained in $(2\delta, \infty)$. Using the spectral measure $E(\cdot)$ of Z on $(0, 2\delta]$ and a closed rectifiable Jordan contour having winding number one with respect to any point of $\sigma(Z) \cap (2\delta, \infty)$ and winding number zero with respect to any of the finitely many points of $\sigma(Z) \setminus (0, \infty)$, one finds the following expression for an H_2-selfadjoint square root of Z_2:

$$A_2 = \int_{(0,2\delta]} \sqrt{t}\, E(dt) + \frac{1}{2\pi i} \int_{\Gamma} \sqrt{\lambda}\, (\lambda I - Z_+)^{-1}\, d\lambda.$$

This completes the proof. □

If $\lambda = 0$ is a singular critical point of Z, the proof of Theorem 3.6 breaks down, because one can no longer single out a spectral subspace of Z corresponding to the spectrum at $\lambda = 0$. The next example due to H. Langer [Lan], however, shows that many H-selfadjoint operators on a Pontryagin space with a singular critical point at $\lambda = 0$ still have an H-selfadjoint square root.

Example 3.8 Let $\mathcal{G} = L_2([0, 1]) \oplus \mathbb{C}^2$, $H = I \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and consider the operator

$$Z = \begin{bmatrix} M_t & 0 & z\sqrt{t} \\ \bar{z}\langle \cdot, \sqrt{t}\rangle & 0 & |z|^2 \\ 0 & 0 & 0 \end{bmatrix} \in \mathcal{L}(\mathcal{G}),$$

where $z \in \mathbb{C} \setminus \{0\}$ and M_t is the operator of multiplication by the function $f(t) = t$. The operator Z is H-selfadjoint, $\sigma(Z) = [0, 1]$, and $Z = A^2$, where

$$A = \begin{bmatrix} M_{\sqrt{t}} & 0 & z \cdot 1 \\ \bar{z}\langle \cdot, 1 \rangle & 0 & a \\ 0 & 0 & 0 \end{bmatrix}$$

is H-selfadjoint with $\sigma(A) = [0, 1]$ for any $a \in \mathbb{R}$. Zero is a singular critical point of both Z and A.

The spectral properties of operators of the form $X^{[*]}X$ described in Theorem 3.4 are not sufficient to guarantee the existence of an H for which the operator is of this form, as the following example shows.

Example 3.9 Let V be the Volterra operator

$$(Vf)(t) = \int_0^t f(u)du, \qquad f \in L_2([0, 1]).$$

Consider the operator

$$Z = \alpha I + iV = \alpha I + \frac{i}{2}(V - V^*) + \frac{i}{2}(V + V^*),$$

where $\alpha > 0$ is chosen so that $\alpha I + \frac{i}{2}(V - V^*)$ is positive definite (any $\alpha > 1$ will do). It is well-known that $V + V^*$ has rank one. Thus, Z is a rank one perturbation of a positive definite operator. Nevertheless, Z is not H-selfadjoint for any H (selfadjoint, invertible, and with finite dimensional spectral subspace corresponding to the negative part of $\sigma(H)$). Indeed, V, and therefore also Z, has no (nonzero) finite dimensional invariant subspaces (see [GoKr]). However, by Pontryagin's theorem (see [Bo, IKL]) every H-selfadjoint operator has a κ-dimensional invariant subspace. Also, Z is not H-selfadjoint for any positive definite H, because Z is unicellular, and on the other hand, by the spectral theorem no selfadjoint operator is unicellular (unless it acts on the one-dimensional space).

We conclude this section with a side result concerning logarithms of H-selfadjoint operators. It was proved in [LMMR] that in finite dimensions an invertible H-selfadjoint operator has an H-selfadjoint square root if and only if it has an H-selfadjoint logarithm. The same result is true in infinite dimensions, in both the complex and the real case:

Theorem 3.10 *Let Z be an invertible H-selfadjoint operator. Then $Z = A^2$ for some H-selfadjoint A if and only if there exists an H-selfadjoint B such that $Z = e^B$.*

Proof. By Theorem 3.4, $\sigma(Z) \setminus [0, \infty)$ is a finite set consisting only of eigenvalues with a finite algebraic multiplicity. The spectral subspace $\mathcal{G}_-$ of Z corresponding to the negative eigenvalues coincides with the spectral subspace of A corresponding to its purely imaginary eigenvalues, which is finite dimensional. By the finite dimensional analysis in [LMMR], there exists an $H|_{\mathcal{G}_-}$-selfadjoint logarithm of $Z|_{\mathcal{G}_-}$. On the H-orthogonal companion of $\mathcal{G}_-$, which is the spectral subspace of Z corresponding to $\sigma(Z) \setminus (-\infty, 0]$, the $H|_{\mathcal{G}_-^{[\perp]}}$-selfadjoint logarithm exists by the functional calculus. Putting the two logarithms together, we obtain an H-selfadjoint algorithm of Z. □

4 Polar decompositions revisited

In this section we present our main results concerning polar decompositions. As in the preceding section, we consider here the complex case only.

Theorem 4.1 *Let X be such that the range of X is closed, and the operator $Z = X^{[*]}X$ does not have a singular critical point at $\lambda = 0$. Then there exists an H-polar decomposition $X = UA$ of X if and only if the conditions* (i) *and* (ii) *in Theorem* 3.6 *are satisfied and in addition the following condition holds:*

(iii) *there is a decomposition as in condition* (ii) *and there is a choice of basis $\{e_{i,j}\}_{i=0}^{l}{}_{j=1}^{s_i}$, where $s_0 = k_0$ and s_i is the order of B_i for $i > 0$, in $\mathcal{M}_{00}$ with respect to which* (ii) *holds and for which we have*

$$\operatorname{Ker} X = \mathcal{M}_1 \oplus \operatorname{span}\{e_{i,1} + e_{i,\,k_i+1} |\ s_i = 2k_i,\ i = 1, \dots, m\} \oplus$$
$$\oplus\ \operatorname{span}\{e_{i,1} |\ s_i = 2k_i - 1,\ i = m+1, \dots, l\} \oplus\ \operatorname{span}\ \{e_{0,j}\}_{j=1}^{k_0},$$

where $\mathcal{M}_{00}$ and $\mathcal{M}_1$ are defined in Theorem 3.6.

Proof. The "if" part. Assume conditions (i) - (iii). With no loss of generality, we assume that the space $\mathcal{M}_1$ is srtrictly H-positive and the vectors $e_{0,j}$ ($j = 1, \dots, k_0$) are H-negative. Now, we reduce Z:

$$Z = Z_{00} \oplus 0 \oplus Z_2 \oplus Z_3$$

with respect to the H-orthogonal decomposition (3.6), where Z_{00} is nilpotent and defined on the finite-dimensional subspace $\mathcal{M}_{00}$, $\mathcal{M}_1 \subseteq \operatorname{Ker} X$, $\sigma(Z_2) \subset [0, +\infty)$ and $\operatorname{Ker} Z_2 = \{0\}$, and $\sigma(Z_3) \cap [0, \infty) = \emptyset$. Note that $\dim \mathcal{M}_3 < \infty$.

We now refine the decomposition (3.6), by obtaining a further decomposition of $\mathcal{M}_2$. Let H_2 be the unique invertible selfadjoint operator on $\mathcal{M}_2$ such that $[x, y] = \langle H_2 x, y\rangle$ for all $x, y \in \mathcal{M}_2$. We denote by $\widetilde{E}$ the spectral function of Z_2 as an H_2-selfadjoint operator. There are two possibilities: either 0 is not a critical point of Z_2, or 0 is a critical point of Z_2 (in the latter case the critical point 0 must be regular, as follows from the construction of the spectral function given in [Lan]). If 0 is not a critical point of Z_2, then, by definition, $\operatorname{Im} \widetilde{E}[0, \delta)$ is either strictly H_2-positive, or strictly H_2-negative, for all $\delta > 0$ small enough. If 0 is a critical point of Z_2, then by Proposition 5.3 in [Lan] there exists a definitizing polynomial for Z_2 that has a simple zero at 0. By the properties of the spectral function (Theorem 3.1(4) and the remark after Theorem 5.7 in [Lan]) we have again that $\operatorname{Im} \widetilde{E}[0, \delta)$ is either strictly H_2-positive, or strictly H_2-negative, for all $\delta > 0$ small enough. (We also use here the fact that $\widetilde{E}(0) = 0$, as $\operatorname{Ker} Z_2 = \{0\}$.)

Next, suppose $\operatorname{Im} \widetilde{E}[0, \delta)$ is strictly H_2-negative for all $\delta > 0$ small enough. Then $\operatorname{Im} \widetilde{E}[0, \delta)$ is finite dimensional, and since $\operatorname{Ker} Z_2 = \{0\}$, it follows that $\operatorname{Im} \widetilde{E}[0, \delta) = \{0\}$ for all $\delta > 0$ small enough. In this case, we put $\mathcal{M}_{2s} = \{0\}$. Otherwise, if $\operatorname{Im} \widetilde{E}[0, \delta)$ is strictly H_2-positive, we put $\mathcal{M}_{2s} = \operatorname{Im} \widetilde{E}[0, \delta)$ for $\delta > 0$ sufficiently small, and let $\mathcal{M}_{2c} = \operatorname{Im} \widetilde{E}[\delta, \infty)$. Write Z_2 as a direct sum $Z_2 = Z_{2c} \oplus Z_{2s}$ corresponding to the H-orthogonal decomposition

$$\mathcal{M}_2 = \mathcal{M}_{2c} \oplus \mathcal{M}_{2s}, \tag{4.1}$$

where, by construction of $\mathcal{M}_{2s}$ and $\mathcal{M}_{2c}$, Z_{2c} has its spectrum bounded away from zero, and Z_{2s} is defined on a strictly H_2-positive subspace $\mathcal{M}_{2s}$. Note that every strictly positive subspace in a Pontryagin space is uniformly positive (see [IKL] for a proof): If $[x, x] > 0$ for every non-zero x in a subspace $\mathcal{Q}$ of a Pontryagin space, then there exists $\varepsilon > 0$ such that $[x, x] \geq \varepsilon\|x\|^2$ for every $x \in \mathcal{Q}$. Thus, $[x, x] \geq \varepsilon\|x\|^2$ for every $x \in \mathcal{M}_{2s}$, where $\varepsilon > 0$ is independent of x.

Put $\mathcal{N}_2 = X[\mathcal{M}_2]$, and $\mathcal{N}_3 = X[\mathcal{M}_3]$. Then $\mathcal{N}_3$ has the same dimension as $\mathcal{M}_3$, and is H-nondegenerate. Indeed, suppose that $\mathcal{N}_3$ is H-degenerate. Then there is a nonzero $x \in \mathcal{M}_3$ such that $[Xx, Xy] = 0$ for all $y \in \mathcal{M}_3$. So $0 = [Xx, Xy] = [X^{[*]}Xx, y] = [Zx, y]$ for all $y \in \mathcal{M}_3$. As $Zx \in \mathcal{M}_3$ and $\mathcal{M}_3$ is H-nondegenerate, it follows that $Zx = 0$. However, as $\sigma(Z_3) \cap [0, \infty) = \emptyset$ we would get $x = 0$, which is a contradiction. The subspace $\mathcal{N}_2$ is closed (because X is assumed to have closed range), and is H-nondegenerate (the proof is analogous to the above proof of H-nondegeneracy of $\mathcal{N}_3$). Introduce also the subspaces $\mathcal{N}_{2c} = X[\mathcal{M}_{2c}]$ and $\mathcal{N}_{2s} = X[\mathcal{M}_{2s}]$.

Notice that each of the three subspaces $\mathcal{M}_{2c}$, $\mathcal{M}_{2s}$, and $\mathcal{M}_3$ has the same number of negative squares (with respect to the indefinite inner product induced by H) as the corresponding subspace $\mathcal{N}_{2c}$, $\mathcal{N}_{2s}$, or $\mathcal{N}_3$. Let us verify this. For $x, y \in \mathcal{M}_3$ we have $[Xx, Xy] = [Z_3x, y]$, where $Z_3 = Z|\mathcal{M}_3$, and therefore the number of negative squares of $\mathcal{M}_3$ with respect to H is equal to that of $\mathcal{M}_3$ with respect to HZ_3. Now by condition (i) of Theorem 3.6, the number of negative squares of H on $\mathcal{N}_3$ is equal to the number of negative squares of H on $\mathcal{M}_3$, and the same thing holds for HZ_3. As for $\mathcal{M}_{2s}$, notice that $\mathcal{M}_{2s}$ is a Hilbert space (with respect to the H-indefinite inner product). The equality

$$[Xx, Xy] = [Z_{2s}x, y], \qquad x, y \in \mathcal{M}_{2s}$$

shows that the number of negative squares of H on $\mathcal{N}_{2s}$ is equal to the number of negative squares of HZ_{2s} on $\mathcal{M}_{2s}$. But since $\mathcal{M}_{2s}$ is a Hilbert space and $\sigma(Z_{2s}) \subset [0, \infty)$, the number of negative squares of HZ_{2s} on $\mathcal{M}_{2s}$ is zero, which is equal to the number of negative squares of H on $\mathcal{M}_{2s}$. For $\mathcal{M}_{2c}$, the verification is analogous.

Recall that (Z_{00}, H_{00}) has the representation (3.8), where $H_{00} = -I_{k_0}$; the latter is because the vectors $e_{0,j}$, $(j = 1, \ldots, k_0)$ in the description of $\operatorname{Ker} X$ are all H-negative. Then we easily see that the vectors

$$Xe_{i,2}, \ldots, Xe_{i,k_i}; Xe_{i,k_i+2}, \ldots, Xe_{i,2k_i}; X(e_{i,1} - e_{i,k_i+1}) \quad \text{for} \quad i = 1, \ldots, m \tag{4.2}$$

together with the vectors

$$Xe_{i,2}, \ldots, Xe_{i,2k_i-1} \quad \text{for} \quad i = m+1, \ldots, l \tag{4.3}$$

form a basis of $X[\mathcal{M}_{00}]$. The Gram matrix of the set (4.2) (with respect to the indefinite inner product induced by H) has the form $Q_{k_i-1} \oplus -Q_{k_i-1} \oplus 0_{1\times1}$, while the Gram matrix of the set (4.3) has the form $\varepsilon_i(Q_{k_i-1} \oplus 0_{1\times1} \oplus Q_{k_i-2})$. As a result, $X[\mathcal{M}_{00}]$ has an isotropic part of dimension l, has l positive squares less than $\mathcal{M}_{00}$, and has $k_0 + l$ negative squares less than $\mathcal{M}_{00}$. Now let $\mathcal{F}$ be a subspace of $\mathcal{G}$ (necessarily of dimension l) that is skewly linked to the isotropic part of $X[\mathcal{M}_{00}]$ and is contained in the H-orthogonal companion of $\mathcal{N}_2 \oplus \mathcal{N}_3$. Define

$$\mathcal{N}_{00} = X[\mathcal{M}_{00}] \dot{+} \mathcal{F}.$$

Then $\mathcal{M}_{00}$ and $\mathcal{N}_{00}$ are both H-nondegenerate subspaces and have the same number of positive squares (with respect to the H-indefinite inner product); however, $\mathcal{M}_{00}$ has k_0 more negative squares than $\mathcal{N}_{00}$. We denote by $\widetilde{\mathcal{M}}_{00}$ the linear span of the vectors $e_{i,j}$ ($j = 1, \ldots, s_i; i = 1, \ldots, l$).

Let us now define an H-isometry U as follows. Clearly, there exist H-isometries

$$V_{00} : \mathcal{N}_{00} \to \widetilde{\mathcal{M}}_{00}, \quad V_{2c} : \mathcal{N}_{2c} \to \mathcal{M}_{2c}, \quad V_{2s} : \mathcal{N}_{2s} \to \mathcal{M}_{2s}, \quad \text{and} \quad V_3 : \mathcal{N}_3 \to \mathcal{M}_3,$$

all of which are onto. We then define the surjective H-isometry

$$V = V_{00} \oplus V_{2c} \oplus V_{2s} \oplus V_3 : \mathcal{N}_{00} \oplus \mathcal{N}_{2c} \oplus \mathcal{N}_{2s} \oplus \mathcal{N}_3 \to \widetilde{\mathcal{M}}_{00} \oplus \mathcal{M}_{2c} \oplus \mathcal{M}_{2s} \oplus \mathcal{M}_3.$$

The H-orthogonal companions of the domain of V and of the subspace span $\{e_{0,j}\}_{j=1}^{k_0} \oplus \mathcal{M}_1$ are again Π_{κ_0} spaces with $\kappa_0 \le \kappa$. We have then defined an H-isometry on a closed H-nondegenerate subspace containing $\operatorname{Im} X$.

We shall construct an H-polar decomposition of VX by giving H-polar decompositions of the restrictions of VX to its invariant subspaces $\mathcal{M}_{00}$, $\mathcal{M}_{2c}$, $\mathcal{M}_{2s}$, $\mathcal{M}_3$, and $\mathcal{M}_1$. Combining these H-polar decompositions in a direct sum manner, we will arrive at an H-polar decomposition of VX.

Let $(VX)_{00}$ be the restriction of VX to its invariant subspace $\mathcal{M}_{00}$. Then conditions (i) - (iii) and a finite dimensional result in [BMR31] imply the existence of an H-isometry W_{00} and an H-selfadjoint A_{00} on $\mathcal{M}_{00}$ such that $(VX)_{00} = W_{00}A_{00}$.

Decompose $(VX)_2$ (the restriction of VX to $\mathcal{M}_2$) as $(VX)_2 = (VX)_{2c} \oplus (VX)_{2s}$, according to (4.1). Then $(VX)_2$ has the H_2-polar decomposition

$$(VX)_2 = W_{2c}A_{2c} \oplus W_{2s}A_{2s},$$

where $W_{2c} = (VX)_{2c}A_{2c}^{-1}$, and where $(VX)_{2s} = W_{2s}A_{2s}$ is the usual polar decomposition on the Hilbert space $\mathcal{M}_{2s}$ (endowed with the inner product $\langle H_2\cdot, \cdot\rangle$).

On $\mathcal{M}_3$, we obtain by conditions (i) - (ii) and the finite dimensional result in [BMR31] that $(VX)_3 = W_3A_3$, where W_3 is an H_3-isometry, and A_3 is H_3-selfadjoint.

We now put together the various polar decompositions above to arrive at the representation

$$(VX)_{00} \oplus (VX)_2 \oplus (VX)_3 = [W_{00} \oplus W_2 \oplus W_3]\ [A_{00} \oplus A_2 \oplus A_3]$$

as an operator on $\mathcal{M}_{00} \oplus \mathcal{M}_2 \oplus \mathcal{M}_3$. Now let U be the H-isometry defined on $\overline{\operatorname{Im} A}$ by

$$U = V^{-1}[W_{00} \oplus W_2 \oplus W_3] : \widetilde{\mathcal{M}}_{00} \oplus \mathcal{M}_2 \oplus \mathcal{M}_3 \to \mathcal{N}_{00} \oplus \mathcal{N}_2 \oplus \mathcal{N}_3.$$

Then $X = UA$ is an H-polar decomposition of X.

The converse part. Let $X = UA$ be an H-polar decomposition of X. Then there exist H-orthogonal H-nondegenerate subspaces $\mathcal{M}_{00}$, $\mathcal{M}_1$, $\mathcal{M}_{2s}^+$, $\mathcal{M}_{2c}^+$, $\mathcal{M}_2^-$, and $\mathcal{M}_{nr}$, with respect to which A decomposes as follows:

$$A = A_{00} \oplus 0 \oplus A_{2s}^+ \oplus A_{2c}^+ \oplus A_2^- \oplus A_{nr},$$

and the subspaces and operators involved have the following properties:

(a) $\dim \mathcal{M}_{00} < \infty$ and A_{00} is nilpotent;

(b) $\mathcal{M}_1$ is strictly H-positive;

(c) $\mathcal{M}_{2s}^+$ is strictly H-positive, $\sigma(A_{2s}^+) \subset [0, \infty)$, and $\operatorname{Ker} A_{2s}^+ = \{0\}$;

(d) $\dim \mathcal{M}_{2c}^+ < \infty$ and $\sigma(A_{2c}^+) \subset (0, \infty)$;

(e) $\dim \mathcal{M}_2^- < \infty$ and $\sigma(A_2^-) \subset (-\infty, 0)$;

(f) $\dim \mathcal{M}_{nr} < \infty$ and $\sigma(A_{nr}) \cap \mathbb{R} = \emptyset$.

Here we have used the fact the zero is not an irregular critical point of A (it follows from the construction of the spectral function in [Lan] that if zero is not an irregular singular point of an H-selfadjoint operator Z which is the square of an H-selfadjoint operator A, then the operator A also doesn't have an irregular critical point at zero). Next, put $\mathcal{N}_{00} = U[\mathcal{M}_{00}]$; then $\mathcal{N}_{00}$ and $\mathcal{M}_{00}$ have the same numbers of positive and negative squares (with respect to H), due to the H-isometric property of U on $\operatorname{Im} A$. So there exists an H-isometry mapping $\mathcal{N}_{00}$ onto $\mathcal{M}_{00}$.

Applying the finite dimensional H-polar decomposition result [BMR31] to $V_{00}X|_{\mathcal{M}_{00}}$, there exists a basis $\{e_{i,j} : i = 0, \dots, l; j = 0, \dots, s_i\}$ of $\mathcal{M}_{00}$ with respect to which $\operatorname{Ker} A$ has the form

$$\begin{aligned}\operatorname{Ker} A = \operatorname{Ker}(V_{00}X|_{\mathcal{M}_{00}}) &= \operatorname{span}\{e_{i,1} + e_{i,\,k_i+1} \mid s_i = 2k_i,\ i = 1, \dots, m\} \oplus \\ &\oplus \operatorname{span}\{e_{i,1} \mid s_i = 2k_i - 1,\ i = m+1, \dots, l\} \oplus \operatorname{span}\{e_{0,j}\}_{j=1}^{k_0},\end{aligned}$$

whereas the canonical form of (A^2, H_{00}) is of the type (3.8) (here H_{00} is the Hermitian matrix that produces the H-indefinite inner product on $\mathcal{M}_{00}$). Hence, since $\operatorname{Ker} X = \operatorname{Ker} A$, the subspace $\operatorname{Ker} X$ satisfies the condition (iii) of Theorem 4.1. Conditions (i) and (ii) follow directly from the existence of an H-selfadjoint operator A such that $Z = X^{[*]}X = A^2$, upon applying Theorem 3.6. □

Easy examples (for example, the backward shift in a Hilbert space) show that under the hypotheses of Theorem 4.1 the operator X that admits an H-polar decomposition need not admit an H-unitary H-polar decomposition. An additional hypothesis is used in the next theorem to guarantee an H-unitary H-polar decomposition (provided the criterion for the existence of an H-polar decomposition is fulfilled):

Theorem 4.2 *Let X be as in Theorem 4.1, and assume in addition that the dimension of $\operatorname{Ker} X$ is equal to the codimension of $\operatorname{Im} X$ (in particular, this condition is satisfied if X is Fredholm with index zero). Then X admits an H-unitary H-polar decomposition if and only if the conditions (i) and (ii) of Theorem 3.6 and the condition (iii) if Theorem 4.1 are fulfilled.*

Thus, Theorem 4.2 represents an immediate generalization of the finite dimensional results on existence of H-polar decompositions obtained in [BMR31].

Proof. If $X = UA$ for an H-unitary operator U and an H-selfadjoint operator A, and X has closed range, then

$$\operatorname{codim} \operatorname{Im} X = \operatorname{codim} \operatorname{Im} A = \dim \operatorname{Ker} A = \dim \operatorname{Ker} X.$$

Conversely, if the conditions (i) and (ii) of Theorem 3.6 and the condition (iii) of Theorem 4.1 are satisfied and X has closed range, there exists an H-isometry U on $\mathrm{Im}\, A$ such that $X = UA$. Actually, following the proof of the "if" part of Theorem 4.1, we construct an H-isometry U on $\widetilde{\mathcal{M}}_{00} \oplus \mathcal{M}_2 \oplus \mathcal{M}_3$ which has $\mathcal{N}_{00} \oplus \mathcal{N}_2 \oplus \mathcal{N}_3$ as its range. Now note that the H-orthogonal companion of the former subspace is span $\{e_{0,j}\}_{j=1}^{k_0} \oplus \mathcal{M}_1$, which is a closed subspace of $\mathrm{Ker}\, X$ of codimension l. On the other hand, $\mathrm{Im}\, X$ is a closed subspace of $\mathcal{N}_{00} \oplus \mathcal{N}_2 \oplus \mathcal{N}_3$ of codimension l. Since the dimension of $\mathrm{Ker}\, X$ equals the codimension of $\mathrm{Im}\, X$, we have

$$\dim \left[\mathrm{span}\, \{e_{0,j}\}_{j=1}^{k_0} \oplus \mathcal{M}_1\right] = \mathrm{codim}\, (\mathcal{N}_{00} \oplus \mathcal{N}_2 \oplus \mathcal{N}_3).$$

Further, span $\{e_{0,j}\}_{j=1}^{k_0} \oplus \mathcal{M}_1$ is H-nondegenerate with k_0 negative squares, whereas $\mathcal{N}_{00} \oplus \mathcal{N}_2 \oplus \mathcal{N}_3$ is H-nondegenerate with $\kappa - k_0$ negative squares. The latter is immediate, since $\mathcal{N}_{00}$, $\mathcal{N}_2$ and $\mathcal{N}_3$ have the same number of negative squares as $\widetilde{\mathcal{M}}_{00}$, $\mathcal{M}_2$ and $\mathcal{M}_3$, respectively, while $\widetilde{\mathcal{M}}_{00} \oplus \mathcal{M}_2 \oplus \mathcal{M}_3$ is the H-nondegenerate H-orthogonal companion of span $\{e_{0,j}\}_{j=1}^{k_0} \oplus \mathcal{M}_1$. As a result, there exists an H-isometry mapping span $\{e_{0,j}\}_{j=1}^{k_0} \oplus \mathcal{M}_1$ onto the H-orthogonal companion of $\mathcal{N}_{00} \oplus \mathcal{N}_2 \oplus \mathcal{N}_3$. This H-isometry then allows us to extend U to an H-unitary operator on all of $\mathcal{G}$. Consequently, for this U, $X = UA$ is an H-unitary H-polar decomposition of X. □

For some particular classes of operators, existence of H-polar decompositions can be obtained by entirely different means than the general criteria of Theorems 4.1 and 4.2. We present one such result concerning H-normal operators. An operator X is called *H-normal* if $XX^{[*]} = X^{[*]}X$.

Theorem 4.3 *If X is an invertible H-normal operator such that the spectrum of X does not surround zero, then X has an H-normal logarithm, and therefore admits an H-unitary H-polar decomposition $X = UA$ with commuting factors U and A.*

The proof is essentially the same as in the finite dimensional case (see [LMMR]), and is therefore omitted.

Theorem 4.3 is valid for the complex case only. A general description of all H-normal operators that have H-normal logarithms is an open problem in the real case even in finite dimensions. Partial results in this direction are found in [LMMR]. In connection with Theorem 4.3, note that there exist H-normal operators that admit an H-unitary H-polar decomposition but do not admit an H-polar decomposition with commuting factors (see [LMMR] for a finite dimensional example).

5 The real case

It is convenient to recast the real case, i.e., the case of linear operators in a real Π_κ space, using the notion of a semilinear involution on a complex Hilbert space. We say that a function K that maps a complex Hilbert space $\mathcal{G}$ into itself, is a *semilinear involution* if $K^2 = I$ and

$$K(x+y) = K(x) + K(y), \quad K(\alpha x) = \overline{\alpha} x, \quad \langle Kx, Ky \rangle = \overline{\langle x, y \rangle}$$

for all $x, y \in \mathcal{G}$ and all $\alpha \in \mathbb{C}$. We assume that a fixed semilinear involution K is given on $\mathcal{G}$.

A linear operator A in the complex space is called real (more precisely, K-real) if $AK = KA$. It is easy to see that this definition is equivalent to the standard definition of operators on a real Pontryagin space, after a complexification. It follows that if A is K-real then so is A^*.

Now let H be an invertible selfadjoint operator which is also K-real. Then $A^{[*]} = H^{-1}A^*H$ is K-real whenever both A and H are K-real. Further, if H is K-real and A is H-selfadjoint with spectral function $\Delta \mapsto E_\Delta$, then KAK is H-selfadjoint with spectral function $\Delta \mapsto KE_\Delta K$. In particular, it follows from the uniqueness of the spectral function of an H-selfadjoint operator that, if H is K-real, the spectral function of a K-real H-selfadjoint operator is K-real.

A subspace $\mathcal{M} \subseteq \mathcal{G}$ will be called *K-real* if $Kx \in \mathcal{M}$ for every $x \in \mathcal{M}$.

With these definitions, it is now a routine matter to extend the results of this paper concerning operators of the form $X^{[*]}X$, selfadjoint square roots, polar decompositions, and their proofs, to the real case (with the exception of Theorem 4.3). As an example, we formulate the real version of Theorem 3.6.

Theorem 5.1 *Let K be a semilinear involution on a Π_κ-space $\mathcal{G}$ defined by a K-real invertible selfadjoint operator H. Assume that X is K-real, and assume that $Z = X^{[*]}X$ does not have a singular critical point at $\lambda = 0$. Then $\mathcal{G}$ can be decomposed H-orthogonally as*

$$\mathcal{G} = \mathcal{M}_{00} \oplus \mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \mathcal{M}_3, \tag{5.1}$$

where the subspaces $\mathcal{M}_{00}$, $\mathcal{M}_1$, $\mathcal{M}_2$, and $\mathcal{M}_3$ are Z-invariant, K-real and have the properties described in Theorem 3.6. *Furthermore, there exists an H-selfadjoint K-real operator A such that $Z = A^2$, if and only if the conditions* (i) *and* (ii) *of Theorem* 3.6 *are satisfied.*

We also formulate the real versions of Theorems 4.1 and 4.2.

Theorem 5.2 *Let H, X and $Z = X^{[*]}X$ be as in Theorem* 5.1. *Assume also that the range of X is closed. Then X admits a real H-polar decomposition if and only if the conditions* (i) *and* (ii) *of Theorem* 3.6 *and the condition* (iii) *of Theorem* 4.1 *are satisfied.*

Theorem 5.3 *Let H, X and $Z = X^{[*]}X$ be as in Theorem* 5.1. *Assume also that the range of X is closed. Then X admits a real H-unitary H-polar decomposition if and only if the conditions* (i) *and* (ii) *of Theorem* 3.6 *and the condition* (iii) *of Theorem* 4.1 *are satisfied, and the dimension of* $\operatorname{Ker} X$ *is equal to the codimension of* $\operatorname{Im} X$.

For the proofs observe that by Lemma 4.2 of [BMR31], in the finite dimensional case with H real, a real matrix X has a complex H-polar decomposition if and only if X has a real H-polar decomposition. Using this observation, and the fact that the decompositions (5.1) and (4.1) can be implemented using K-real subspaces only, the proofs of Theorems 4.1 and 4.2 carry through under the hypotheses of Theorems 5.2 and 5.3.

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