ELECTRONIC DELIVERY COVER SHEET

Warning Concerning Copyright Restrictions

The copyright law of the United States (Title 17, United States Code) governs the making of photocopies or other reproductions of copyrighted material. Under certain conditions specified in the law, libraries and archives are authorized to furnish a photocopy or other reproduction. One of these specific conditions is that the photocopy or reproduction is not to be “used for any purpose other than private study, scholarship, or research.” If a user makes a request for, or later uses, a photocopy or reproduction for purposes in excess of “fair use,” that user may be liable for copyright infringement. This institution reserves the right to refuse to accept a copying order if, in its judgment, fulfillment of the order would involve violation of copyright law.
ABSTRACT BOUNDARY VALUE PROBLEMS
FROM KINETIC THEORY

by

William Greenberg and Cor van der Mee

Laboratory for Transport Theory and Mathematical Physics
Virginia Polytechnic Institute and State University
Blacksburg, Virginia 24061

ABSTRACT

The abstract differential equation $(Tf)' = -Af$ with "partial range" boundary conditions is solved on a Hilbert space. $T$ and $A$ are (possibly unbounded) self-adjoint operators, $A \geq 0$ and semi-Fredholm. Examples from kinetic theory are given.

I. INTRODUCTION

In a recent paper\(^1\) the partial range boundary value problem associated with the abstract kinetic equation

\[
\frac{3}{3x} (Tf) = -Af \\
(Q_Af)(0) = \phi_0 \\
\lim_{x \to \infty} \|f(x)\| < =
\]

was studied on an abstract Hilbert space, under the assumption that $A$ is positive (possibly unbounded) self-adjoint and Fredholm, and $T$ is bounded self-adjoint and one-one. Here $Q_A$ is the maximal positive projection associated with $T$. Such an abstract equation includes as special
cases a number of one-dimensional problems in neutron transport, electron transport, and radiative transfer. However, the boundedness restriction on \( T \) explicitly excludes those one-dimensional problems in gas kinetics for which explicit representations of solutions are already known.

The present paper removes the boundedness restriction on \( T \), and existence and uniqueness of solutions to (1) will be proved. The approach we utilize, from the theory of strongly continuous linear semigroups, provides a rigorous framework for the method of singular eigenfunctions introduced by K. M. Case to construct solutions of transport equations. This approach was introduced into neutron transport theory by R. Hangelbroeck in 1973, and has been extended to a wide range of abstract problems by R. Beals.

The solution of the boundary value problem (1) we give here follows closely the development in Ref. 1, and in some cases, the reader is referred to that source for the proofs of preliminary propositions which are quite similar. The notation in this paper and the previous one is consistent. For completeness, we give the principle result of the earlier paper.

**THEOREM 1.**

If \( A \) is positive, self-adjoint, and Fredholm, and \( T \) is bounded self-adjoint and one-one, then for \( f \in \mathcal{D}(T) \), the boundary value problem (1) has a (differentiable) solution which is unique if and only if \( \ker A \) is positive definite with respect to the \( T \)-indefinite inner product (defined in Eq. 12).

In Section II, we obtain a reduction of the operator \( T^{-1} A \) which enables us to restate the half-space problem. The existence of a suitable signature operator provides \( T^{-1} A \) with self-adjoint extensions. In the following section, these extensions are used to construct the Larsen-Habetler albedo operator and obtain half-range expansions. The existence and uniqueness theorems are stated in Section IV. Finally, in the last section, applications from kinetic theory are presented.

**II. DEFINITIONS AND DECOMPOSITIONS**

Throughout we consider a (possibly unbounded) self-adjoint operator \( T \) with trivial kernel \( \ker T = \{0\} \) and a (possibly unbounded) positive self-adjoint operator \( A \) with closed range \( \text{Ran} A \) in an abstract Hilbert space \( H \) and with kernel \( \ker A \) of finite dimension:

\[
\dim \ker A = n < \infty .
\] (2)

Such a pair \((T, A)\) is called a symmetric pair on \( H \) if the following conditions are satisfied:

(i) \( D(T) \cap D(A) \) is dense in \( H \). This assumption implies that \( T^{-1} A \) is a closable operator. Indeed, \( AT^{-1} \) densely defined and extension \((AT^{-1})^* \supset T^{-1} A \) imply that \((AT^{-1})^* \) is closed and \( T^{-1} A \) is closable.

(ii) Let \( K_0 = T^{-1} A \) be the closure of \( T^{-1} A \) on \( H \). Define the zero root linear manifold \( Z_0(K_0) \) of \( K_0 \) by

\[
Z_0(K_0) = \{ f \in D(K_0) | f_0 \in D(K_0^0) \text{ and } K_0 f_0 = 0 \text{ for some } n \in \mathbb{Z}_+ \} .
\]
and similarly the zero root linear manifold \( Z_0(B) \) for any linear operator \( B \). We assume that \( Z_0(K_0) \subset D(T) \) and if \( x \in \text{Ker} \, A \) with \( x \notin \text{Ran} \, K_0^* \), then there exists \( y \in \text{Ker} \, A \) such that \( (x, Ty) \neq 0 \). This is precisely the requirement that a maximal subspace of eigenvectors which are not the image of generalized eigenvectors be non-degenerate with respect to the indefinite inner product of Eq. (12).

**Lemma 1:** \( T^{-1}A \) is densely defined.

**Proof** The proof depends only upon the closeness of \( R = \text{Ran} \, A \) and its finite co-dimension. Write \( D = D(T^{-1}) \). If \( D \cap R \) is not dense in \( R \), then there exists \( q \in R \setminus \text{Ran} \, A \) and a functional \( \ell \in X^* \) such that \( \ell(q) = 1 \), \( \ell(R \cap D) = \{0\} \). But codim \( R \setminus D \) dense imply that \( H = R \oplus N \), \( N \subset D \). Extending \( \ell \) to \( H \) by setting \( \ell(N) = \{0\} \) gives \( \ell \in H^* \) and \( \ell(D) = \{0\} \). Thus \( \ell = 0 \), which is a contradiction.

**Lemma 2:** \( \text{Ker} \, K_0 = \text{Ker} \, A \) and \( Z_0(K_0) = Z_0(T^{-1}A) \).

**Proof** Assume \( x \in D(T^{-1}A) \cap (\text{Ker} \, A)^* \rightarrow x \) and \( T^{-1}A \, x = y \). Then \( y \in D(T), y = 0 \) and \( A^{-1}T \neq 0 \), where \( A^{-1} \) is the bounded operator from \( \text{ran} \, A \) into \( (\text{Ker} \, A)^* \) defined by \( A^{-1}(Ax) = x \) for \( x \in (\text{Ker} \, A)^* \). But \( A^{-1}T \) is closable, because \( A^{-1}T \subset (TA)^{-1} \) and \( TA^{-1} \) is a densely defined operator from \( \text{ran} \, A \) into \( H \) (cf. Lemma 1). Thus \( x = 0 \).

Similarly, if \( T_0 = A \) for some \( x \in D(T^{-1}A) \), then \( x \in D(T^{-1}A) \) and similarly the zero root linear manifold \( Z_0(B) \) for any linear operator \( B \). We assume that \( Z_0(K_0) \subset D(T) \) and \( \ell \in \text{Ker} \, A \) with \( \ell \notin \text{Ran} \, K_0^* \), then there exists \( y \in \text{Ker} \, A \) such that \( (x, Ty) \neq 0 \). This is precisely the requirement that a maximal subspace of eigenvectors which are not the image of generalized eigenvectors be non-degenerate with respect to the indefinite inner product of Eq. (12).

**Lemma 3:** If \( f_0 \in Z_0(K_0) \), then there exists \( f_1 \in Z_0(K_0) \) such that \( K_0 f_1 = f_1^\prime, K_0^* f_1 = 0 \).

**Proof** We remark that, by virtue of Lemma 2, \( Z_0(K_0) = Z_0(T^{-1}A) \subset D(A) \). By Condition (ii) in the definition of a symmetric pair, \( Z_0(K_0) \subset D(T) \). Now the lemma can be proved in the same way as Lemma 1 of Ref. 1.

Lemma 3 implies that the length of a zero Jordan chain of \( K_0 \) cannot exceed 2. For special cases similar results were found in Refs. 1 and 3. The next proposition relates \( Z_0(K_0) \) and \( Z_0(K_0^*) \) and yields two useful decompositions of \( H \).

**Proposition 1.** One has
TZ₀(K₀) = Z₀(K₀), \quad A(Z₀(K₀) ⊆ D(A))

= T(Z₀(K₀) ⊆ D(T)) = Z₀(TK₀),

(3)

and the following decompositions hold true:

Z₀(K₀) ⊗ Z₀(K₀) ⊆ H ;

(4a)

Z₀(K₀) ⊗ Z₀(K₀) ⊆ H .

(4b)

**Proof** Let us first prove the identity

TZ₀(K₀) = Z₀(K₀∗).

(5)

If α ⊆ Ker A, then αD(T), TαD(T⁻¹) and T⁻¹αT₀ = K₀T₀ = 0.

If T⁻¹Ag = α ⊆ Ker A for some g ∈ Z₀(K₀) = Z₀(T⁻¹)A, then αD(T),

Ag = T₀ + α⁻¹, where A⁻¹f ∈ (K₀)∗ and α⁻¹K₀α.

But g ∈ D(T), so T₀ = T⁻¹αT₀ + T₀ and T⁻¹T₀ = K₀T₀T₀ = T₀. Therefore,

TZ₀(K₀) = Z₀(K₀∗).

Next assume f ∈ Ker K₀∗. Then (f, T⁻¹Ag) = 0 for all g ∈ D(T⁻¹)A,
or (f, T⁻¹α + α⁻¹) = 0 for all α ⊆ Ker A and D(T⁻¹) = (K₀)∗ ⊆ D(T⁻¹).

Then h ∈ (f, T⁻¹h) is bounded on a manifold of finite co-dimension,
and thus on all of D(T⁻¹). Since T⁻¹ is self-adjoint, f ∈ D(T⁻¹) and
(T⁻¹f, h) = 0. Therefore T⁻¹f = α ⊆ Ker A, or f = T₀. We have shown that Ker K₀∗ = T₀K₀.

Assume K₀ = T₀f ∈ Ker K₀∗ ⊆ D(K₀∗) and α ⊆ Ker A. Then

T₀f ∈ Ker A and f ∈ D(A⁻¹), so A⁻¹K₀f = A⁻¹T₀f ∈ Ker A. We claim T⁻¹Ag = α. For if this is not the case,

ABSTRACT BOUNDARY VALUE PROBLEMS

T₁ ∈ Ker A = (K₀)∗. Thus TA⁻¹K₀f = T₀, since Z₀(K₀) ⊆ D(T).

But TA⁻¹K₀ = I on (K₀)∗ and (f ∈ D(K₀∗) ∩ f ∈ D(TA⁻¹)). We

have shown K₀f ∈ D(TA⁻¹). Therefore, f = T₀. Repeating this

argument one shows that Ker(K₀∗) = T Ker(T⁻¹) for all α ∈ Z₀,

and thus Ker(K₀∗) = Z₀(K₀∗) = T₀K₀, which establishes (5).

Next take x ∈ Z₀(K₀) ∩ Z₀(K₀)∗. Then x ∈ D(T⁻¹) and x ∈ Ker K₀∗.

and thus x ∈ T₀K₀. But x ∈ Z₀(K₀∗). So (Ax, x) = 0, which,

by the positivity of A, implies x ∈ Ker A. However, we

also have x ∈ Z₀(K₀∗) ∩ Z₀(K₀∗). Condition (ii) in

the definition of a symmetric pair yields x ∈ Ker K₀∗, and Lemma 2

yields x = T⁻¹Ay for some y ∈ D(T⁻¹). Because Ty ∈ Z₀(K₀∗), one gets

(Ay, y) = (Ty, y) = (x, Ty) = 0,

implying Tx = Ay = 0. Thus x = 0 and Z₀(K₀) ∩ Z₀(K₀)∗ = {0}.

Take y ∈ Z₀(K₀) ∩ Z₀(K₀)∗ and z ∈ Z₀(K₀). Then (5) implies

y = Tx with x ∈ Z₀(K₀) and z = T₀y with y ∈ Z₀(K₀). So (x, z) = (x, Tu) = (Tx, u) = (y, u) = 0. Thus x ∈ Z₀(K₀) ∩ Z₀(K₀) = {0} and y = Tx

= 0. Hence,

Z₀(K₀) ∩ Z₀(K₀)∗ = Z₀(K₀) ∩ Z₀(K₀)∗ = {0}.

(6)

The remaining part of the proof is the same as the corresponding
part of Ref.1. The decompositions (4a)-(4b) follows from (5), (6) and
a simple dimension argument.

Now, modifying an idea originally introduced by Hangelbroek, let

H = D(A⁻¹) ∩ Z₀(K₀)∗ be the Hilbert space with inner product

(x, y) = (A⁻¹x, A⁻¹y).

(7)
Note that \((x, y) \in (Ax, y)\) for \(x, y \in D(A) \cap Z_0(K_0^*) \subseteq H_A\), and that \(H_A\) is continuously and densely embedded in \(Z_0(K_0^*)^\perp\). We define

\[
K_1 = T^{-1}A = T^{-1}A|_{Z_0(K_0^*)^\perp}
\]

to be the direct sum of two operators: (i) the restriction of \(T^{-1}A\) to \(Z_0(K_0^*)\), which is bounded; (ii) the closure in \((, )_A\)-topology of the restriction of \(T^{-1}A\) to \(Z_0(K_0^*)^\perp\). Note that

\[
T^{-1}A \subseteq K_1 \subseteq K_0^*,
\]

with \(Z_0(K_0^*)\) in the domain of all three operators. Then Lemma 2 and 3 and Proposition 1 are also valid for \(K_1\). Henceforth we write \(T\) and \(A\) also for their restrictions on \(H_A\).

Obviously the operator \(T^{-1}A|_{Z_0(K_0^*)^\perp}\) is symmetric with respect to the inner product (7) and \(K_1\) is its second adjoint with respect to (7). In general, it is quite difficult to find out whether \(K_1|_{Z_0(K_0^*)^\perp}\) is \((, )_A\)-self-adjoint or even if it has a self-adjoint extension. However, if either \(T\) or \(A\) is bounded, then \(K_1 = T^{-1}A\) is closed and its restriction to \(Z_0(K_0^*)^\perp\) is \((, )_A\)-self-adjoint. The following lemma gives the existence of self-adjoint extensions for the most interesting kinetic models.

**Lemma 4:** Let \(F\) be a signature operator on \(H\) (i.e., \(F = F^*, F^2 = I\)) such that \(FD(T) \subseteq D(T), F(D(A) \subseteq D(A)\) and

\[
F^*x = -T^*F^*x, F^*y = AFy, x \in D(T), y \in D(A).
\]

**Abstract Boundary Value Problems**

Then \(K_1|_{Z_0(K_0^*)^\perp}\) has a \((, )_A\)-self-adjoint extension \(K\) which satisfies

\[
FD(K) \subseteq D(K); FK = -KF^*, x \in D(K) \subseteq H_A.
\]

**Proof** Certainly \(F \in H_A\) and \(F\) induces a \((, )_A\)-signature operator on \(H_A\), which has the following properties:

\[
FD(T^{-1}A) \subseteq D(T^{-1}A); FT^{-1}Az = -T^{-1}AFz, z \in D(T^{-1}A).
\]

Hence, a similar property holds for the second \((, )_A\)-adjoint \(K_1|_{Z_0(K_0^*)^\perp}\) of \(T^{-1}A|_{Z_0(K_0^*)^\perp}\). Let us denote the \((, )_A\)-adjoint of \(B\) by \(B^\dagger\). Then the relations

\[
FD(K_1^+) \subseteq D(K_1^+); FK_1^+z = -K_1^+Fz, z \in D(K_1^+),
\]

imply

\[
F \text{ Ker}(K_1^+ - i) = \text{ Ker}(K_1^+ + i),
\]

and thus \(K_1\) has equal deficiency indices. Using the procedure of Theorem X.2 of Ref. 6, one defines

\[
D(K) = \{x + x_+ + Fx_+ | x \in D(K_1^+), x_+ \in \text{ Ker}(K_1^+ - i)\};
\]

\[
K(x + x_+ + Fx_+) = K_1x + i(x_+ - Fx_+).
\]

Then \(K|_{Z_0(K_0^*)^\perp}\) is a self-adjoint extension of \(K_1|_{Z_0(K_0^*)^\perp}\).
A signature operator $F$ on $H$ that leaves invariant $D(T)$ and $D(A)$, and anticommutates with $T$ and commutes with $A$, will be called an
inversion symmetry for the symmetric pair $(T,A)$. It is easy to
prove that in this case $\dim Z_0(K) = 2m = \text{even}$ and the spectra of
$T$ and $K$ are real and symmetric with respect to $\lambda = 0$ (cf. Refs.
3 and 7; the dimensional statement will be proved at the end of
this section.).

As in Ref. 1 define a matrix operator $\beta$ on $Z_0(K_0)$ and a
positive operator $A_\beta$ to reduce the half-space problem to one
where $\ker A = \{0\}$.

**PROPOSITION 2.** Let $(T,A)$ be a symmetric pair and let $P$
denote the projection of $H$ onto $Z_0(K_0^*)^\perp$ along $Z_0(K_0)$. For
some invertible operator $\beta$ on the finite-dimensional space
$Z_0(K_0)$ put

$$A_\beta = A + T^\beta(I - P). \quad (10)$$

Then the operator $A_\beta$ is densely defined with bounded inverse and

$$A_\beta^{-1} T = \beta \oplus (\beta^{-1}|_{Z_0(K_0^*)}^{-1}). \quad (11)$$

One may choose $\beta$ in such a way that $(T^\beta - x, x) \geq 0$ for all $x \in Z_0(K_0)$, in which case $A_\beta$ will be a positive operator.

The proof of this proposition is the same as the one of Proposition
2 of Ref. 1. Whenever $(T^\beta - x, x) \geq 0$ for all $x \in Z_0(K_0)$, we
define a Hilbert space $H_{A_\beta}$ and its inner product $(\cdot, \cdot)_{A_\beta}$
$= (A_\beta', \cdot)$ in analogy with $H_A$ and $(\cdot, \cdot)_A$. Because $(T,A)$ is a

**ABSTRACT BOUNDARY VALUE PROBLEMS**

symmetric pair on $H$ with $\ker A = \{0\}$ (which one easily checks),
the operator $T^{-1}_A A_\beta$ is closable in $H_{A_\beta}$. However,

$$D(A_\beta) = D(A) = (D(A) \cap Z_0(K_0^*)^\perp) \oplus Z_0(K_0)$$

and $\dim Z_0(K_0) < \infty$. Thus $H_{A_\beta}$ does not depend on the particular
choice of $\beta$ and so we suppress $\beta$ and write $H_{A_\perp}$. (Note that the
new $H_{A_\perp}$ equals $H_A \oplus Z_0(K_0)$). The minimal closure of $A_\beta^{-1} T$
in $H_A$ is $K_1^{-1} \oplus \beta$. If on $Z_0(K_0^*)^\perp$ the operator $T^{-1}_A$
has a $(\cdot, \cdot)_A^{-1}$-self-adjoint extension $K$, then $K^{-1} \oplus \beta$ will be a self-adjoint
extension of $A_\beta^{-1} T$.

**LEMMA 4:** The subspace $Z_0(K_0)$ is a Pontryagin space with respect
to the indefinite inner product

$$[u,v] = (Tu, v). \quad (12)$$

If $M$ is a complement of $K_0[Z_0(K_0)]$ in $Z_0(K_0)$ and $N_\pm$ is a maximal
positive/negative subspace of $M$ with respect to (12), then

$$K_0[Z_0(K_0)] \oplus N_\pm$$
is a maximal positive/negative subspace of $Z_0(K_0)$
and there exists a maximal negative/positive subspace $M_\mp$ of
$Z_0(K_0)$ orthogonal to $K_0[Z_0(K_0)] \oplus N_\pm$ such that

$$K_0[Z_0(K_0)] \oplus N_\pm \oplus M_\mp = Z_0(K_0).$$

This lemma can be proved in the same way as Lemma 2 of Ref. 1.
As in Ref. 1 we may derive the following: in order that there
exists a unique maximal positive subspace $M_+$ such that $K_0[Z_0(K_0)]$
\( \subseteq \mathcal{M} \subseteq \ker K_0 = \ker A \), it is necessary and sufficient that \( \ker A \) is definite (i.e., either positive or negative) with respect to (12). If all zero eigenvectors of \( K_0 \) (or \( T^{-1}A \); see Lemma 2) are the image under \( K_0 \) (or \( T^{-1}A \)) of a generalized eigenvector, then \( \ker K_0 = \ker A \) is neutral (i.e., consists of zero norm vectors only) and \( \mathcal{M} \) is uniquely specified. In fact, in this case \( \mathcal{M} = \ker K_0 = K_0(\mathcal{Z}_0(K_0)) \).

In case the symmetric pair (\( T, A \)) has an inversion symmetry \( F \), one obviously has

\[
F \ker A = \ker A, \quad F \mathcal{Z}_0(K_0) = \mathcal{Z}_0(K_0). \tag{13}
\]

Then

\[
(u, v)_F = (Fu, v) \tag{14}
\]

is another indefinite inner product on \( \mathcal{Z}_0(K_0) \) in which 
\( T^{-1}A \mid \mathcal{Z}_0(K_0) \) is self-adjoint. (Note that \( FT^{-1}Ax = T^{-1}AFx \) for \( x \in \mathcal{Z}_0(K_0) \)). The subspace \( \mathcal{Z}_0 = \{x \in \mathcal{Z}_0(K_0) \mid Fx = xx \} \) of even/odd vectors in \( \mathcal{Z}_0(K_0) \) is strictly positive/negative with respect to (14) and for \( x, x \in \mathcal{Z}_0 \) one has

\[
(x, x)_F = (Fx, x) = (x, Fx). \tag{15}
\]

As \( Fx = x \), one gets \( (x, x)_F = 0 \), and thus \( \mathcal{Z}_0 \) are orthogonal in (14) with \( \mathcal{Z}_0^+ \oplus \mathcal{Z}_0^- = \mathcal{Z}_0(K_0) \). (For the latter we note that
\[
\frac{1}{2}(I + F) + \frac{1}{2}(I - F) = I.
\]
Let us make the connection of (12) and (14). As \( T \) and \( F \) anticommute, the subspaces \( \mathcal{Z}_0 \) satisfy

\[
[x, x] = 0, \quad x \in \mathcal{Z}_0 \,
\]

Hence, the dimension \( m_+ \) of a maximal \([ , ]\)-positive/negative subspace of \( \mathcal{Z}_0(K_0) \) equals or exceeds \( \max(\dim \mathcal{Z}_0^+, \dim \mathcal{Z}_0^-) \). But

\[
m_+ + m_- = \dim \mathcal{Z}_0^+ + \dim \mathcal{Z}_0^- = \dim \mathcal{Z}_0(K_0) = m.
\]

Hence, \( m = m_+ = m_- = \dim \mathcal{Z}_0^+ = \dim \mathcal{Z}_0^- \) and \( \dim \mathcal{Z}_0(K_0) = 2m \) even.

For symmetric pairs (\( T, A \)) for which \( T \) is bounded and \( I - A \) is compact this has been observed before in Ref. 3. The presence of the inversion symmetry \( F \) implies that \( \ker A \) is definite in (12) if and only if every zero eigenvector of \( T^{-1}A \) is the image under \( T^{-1}A \) of a generalized eigenvector.

### III. Half-Range Expansions

Throughout Sections III to V we assume that \( T^{-1}A \mid \mathcal{Z}_0(K_0) \) has a \([ , ]_A \)-self-adjoint extension \( K \) which we extend linearly to \( H_A \) by putting \( Kx = T^{-1}Ax \) for \( x \in \mathcal{Z}_0(K_0) \). This assumption is satisfied if (\( T, A \)) has an inversion symmetry \( F \) (cf. Lemma 4) or if either \( T \) or \( A \) is bounded. In the latter case one simply takes \( K = T^{-1}A \). We define \( Q_x \) to be the \( H \)-orthogonal projections of \( H \) onto the maximal \( T \)-invariant subspace on which \( T \) is positive/negative. We already defined the \([ , ]_A \)-inner product on \( H_A \) and took the decision to suppress \( \beta \) in \( H_A \) (because this space does not depend on \( \beta \)); then \( H_A \subseteq H \) densely. In analogy with \( Q_x \), we define \( P_x \) to be the \([ , ]_A \)-orthogonal projections of \( H_A \) onto the maximal \( \mathcal{K}_x \)-invariant subspace on which

\[
\mathcal{K}_x = \beta \oplus (K \mid \mathcal{Z}_0(K_0^+))^\perp
\]

(15)
is \((A, ) -\text{selfadjoint.}\)

Let us introduce two additional inner products; namely

\[
(x, y)_T = \left( |T|x, y \right), \quad (x, yD(T))
\]

with the completion of \(D(T)\) denoted \(H_T\), and

\[
(x, y)_{K_B} = \left( |K_B|x, y \right)_{A_B}, \quad (x, yD(K_B))
\]

with the completion of \(D(K_B)\) denoted \(H_{K_B}\). We remark that

\[
D(K_B) = Z_0(K_0)^\perp \cap D(A) \ni Z_0(K_0)^\perp.
\]

(c.f. (15)), and thus \(D(K_B)\) does not depend on \(\beta\). Also \((x, y)_{A_B} = (Ax, y)\) for \(x, yD(A) \ni Z_0(K_0)^\perp\). Therefore, all norms (17) are equivalent on the set (18) and henceforth we shall suppress \(\beta\) in \(H_{K_B}\) and write \(H_k\).

One of the main differences with the case when \(T\) is bounded is that \(H\) is not "naturally" embedded in \(H_T\). However, the set \(D(|T|^{1/2})\) is complete with respect to the graph inner product

\[
(x, y)_{GT} = (x, y) + \left( |T|^{1/2}x, |T|^{1/2}y \right), \quad (x, yD(|T|^{1/2}))
\]

(note that \(|T|^{1/2}\) is closed) and densely embedded in both \(H\) and \(H_T\).

The domains \(D(T)\) and \(D(|T|^{1/2})\) are invariant under \(Q_+\) and \(Q_-\) is orthogonal with respect to \((, )_{GT}\). In a straightforward way one shows that \(Q_\perp\) extends to an orthogonal projection with respect to (16) also. In an analogous way one shows that the set \(D(|K_B|^{1/2})\), which is a complete Hilbert space with respect to the graph inner product

\[
(x, y)_{K_B} = (x, y)_{A_B} + \left( |K_B|^{1/2}x, |K_B|^{1/2}y \right)_{A_B}, \quad (x, yD(|K_B|^{1/2}))
\]

is densely embedded in both \(H_A\) and \(H_K\), while \(P_\perp\) extends to a \((, )_{K_B}\)-orthogonal projection on \(H_A\). Finally, the identity \(\text{Ker } P = Z_0(K_0)^\perp \subseteq D(A \cap D(T)\) allows us to extend continuously the projection \(P\) of \(H\) onto \(Z_0(K_0)^\perp\) along \(Z_0(K_0)^\perp\) to bounded projections on the spaces \(H_A\) and \(H_K\). In both cases, \(\text{Ker } P = Z_0(K_0)^\perp\).

Let us introduce the Larsen-Habetsch albedo operator \(E\). This operator is defined by the conditions that, for all \(f \in D(T)\),

\[
\begin{align*}
Q_\perp E f &= Q_\perp Ef, \\
P_\perp E f &= 0.
\end{align*}
\]

In transport theory terminology, these conditions imply that if \(f \in \text{Ran } Q_+\) is an incoming flux for a right half-space problem, then \(Ef\) will be the corresponding total (incoming plus reflected) flux, and if \(f \in \text{Ran } Q_-\) is an incoming flux for a left half-space problem, then \(Ef\) will be the corresponding total flux. In this way \(E\) will depend on the particular self-adjoint extension \(K\), contrary to the case when \(T\) is bounded or \(A\) is bounded, because in these cases \(K = T^{-1}A\) is uniquely specified by \(T\) and \(A\).

To derive an explicit representation for \(E: H_T \rightarrow H_K\), we establish first the intertwining relation

\[
P_\perp = E Q_\perp = Q_\perp E = P_\perp E
\]

on \(H_T\). We have

\[
P_\perp = P_\perp E (Q_+ + Q_-) = P_\perp E Q_\perp = Q_\perp E = P_\perp E
\]
where we have used Eqs. (19). Now by (19) again,

$$Q_\pm \pm Q_\pm = Q_\pm$$

whence, by adding the $\pm$ equations,

$$Q_+Q_+ + Q_-Q_- = (Q_+ + Q_-)I = I.$$ (20)

**PROPOSITION 3.** There exists a unique albedo operator $E: H_T \to H_K$ that is bounded, injective and satisfies the conditions (19). Further, $E$ acts as a bounded operator from $H_T$ into $H_K$.

**PROOF.** On $H_A$ we define the Hangelbroek operators $5$

$$V = Q_+ + Q_+: H_A \to H, \quad W = Q_+ + Q_-: H_A \to H.$$ (21)

Following an argument of Beals $11$ we compute that, for $f \in H_A$, if the terms on the left hand side are both finite, then

$$\|Vf\|_T^2 - \|Wf\|_T^2 = \{(TP_+f, Q_+f) - (TP_-f, Q_-f)\}$$

$$- \{(TP_-f, Q_+f) - (TP_+f, Q_-f)\}$$ (22)

$$= (TP_+f, P_+f) - (TP_-f, P_-f) = (\|K_B^{-1}f\|_A, \|f\|_K^2)$$

Let us prove that $E$ extends to a bounded operator from $H_T$ into $H_K$. A straightforward calculation shows that, for $f \in H_A$, $12$

$$(Q_+ - Q_-)(P_+ - P_-)f = Vf - Wf = (2V - 1)f,$$ (23)

and therefore, for $f \in H_A \cap D(T)$,

$$((2V - 1)f, f)_T = (\|f\|_T^2 - f, f)_A$$

$$= (K_B^{-1}(P_+ - P_-)f, f)_A = (K_B^{-1}f, f)_A$$

$$= \|f\|_K^2.$$ (24)

This implies the following identity:

$$2(Vf, f)_T = \|f\|_T^2 + \|f\|_K^2, \quad f \in H_A \cap D(T).$$ (25)

Introduce the semi-bounded quadratic form

$$q(f, g) = 2(Vf, g)_T, \quad f, g \in H_A \cap D(T)$$ (26)

on the Hilbert space $H_T$. Note that $q$ can be extended to a closed form with domain $D(q) = H_A \cap H_K$, and $H_A \cap D(T)$ is a form core for $q$. Now $q$ is the quadratic form of a unique self-adjoint operator whose domain $D$ satisfies $6$

$$H_A \cap D(T) \subset D \subset H_A \cap H_K \subset H_T.$$ 

Hence $V$ extends to a unique self-adjoint operator on $H_T$ (with domain $D$), and moreover,

$$2(Vf, f)_T \geq \|f\|_T^2, \quad f \in D.$$ (27)

From this we find $V$ to have trivial kernel and dense range in $H_T$.

Putting $E$ on the manifold $D_0(E)$ as

$$D_0(E) = VD \in H_T, \quad E(Vf) = f \in D,$$
E extends to a bounded operator on $H_T$.

Since $V(H_A \cap D(T))$ is dense in $H_T$, we may consider $E$ as a densely defined operator on $D(E)$:

$$D(E) = \{ Vf : f \in H_A \cap D(T) \}, \quad E(Vf) = f \in H_A \cap D(T).$$

From Eq. (22) it follows that

$$\| E_g \|_{K_B}^2 \leq \| g \|_T^2, \quad g \in D(E) \subseteq H_T^1.$$ (28)

which establishes the existence of $E$ as a bounded operator from $H_T$ into $H_K$.

For arbitrary $T$ and bounded injective $A$ the invertibility of the Hangelbroek operator $V$ from $H_K$ into $H_T$ and the equivalence of the $(\cdot, \cdot)_A^T$ and $(\cdot, \cdot)_T$ inner products on $D(A) \cap D(T)$ were proved by Beals, after which he could simply put $E = V^{-1}$. Subsequently these results were generalized to the case when $T$ is bounded and $A$ may be unbounded with non-trivial kernel (see Refs. 1 and 11), but as in the present article the proof of the boundedness of $V$ (rather than $E = V^{-1}$) can not be obtained. For a discussion of the implications of the boundedness of $V$ we refer to Lemma 3 in Ref. 1.

Earlier, Hangelbroek proved the invertibility of $V$ as an operator from $H$ into $H$ for neutron transport with isotropic (and later also anisotropic) scattering kernels. In that work $T-A$ was assumed compact. Under the conditions that $T-A$ is compact and $\text{Ran } (T-A) \subseteq \text{Ran } |T|$ for $0 < \alpha < 1$, van der Mee proved the invertibility of $V$ and of $TVT^{-1}$ on $H$, which in that case implies Beals' result on $H_T$.

IV. EXISTENCE AND UNIQUENESS THEORY FOR HALF SPACE PROBLEMS

To solve the half-space problem, one seeks a solution of Eq. (1), $f : [0, \infty) \to H_K$, subject to

$$Q_+ f(0) = f_+ \in \text{Ran } Q_+ \quad (29a)$$

$$\lim_{x \to \infty} \| f(x) \| \text{ finite}.$$ (29b)

Because the albedo operator $E$ acts from $H_T$ into $H_K$, a statement of this type is required. Below, we give a more precise statement of the problem.

The decomposition of $H$ into reducing subspaces of $K$, Proposition 1, decouples the half-space problem, into a half-space problem on $PH$ (with a different $f_+$) and a finite-dimensional first order system on $(I - P)H$. However, the use of a suitable operator $A_8$ makes it possible to extend the half-space problem on $PH$ to one on $H$ of a simpler structure than the original problem, the simplicity stemming from the injectivity of $A_8$. The main difficulty of the newly obtained half-space problem is that the albedo operator $E$ acts from $H_T$ into $H_K$ and might not act from $H$ into $H$. For this reason we state the following weakened version of the half-space problem:

Given $f_+ \in \text{Ran } Q_+$, construct a continuous function $\phi : [0, \infty) \to H_K$, with both $K\phi$ and $(I - P)\phi$ differentiable on $(0, \infty)$, such that

$$\frac{d}{dx} K\phi = -P\phi \quad \text{on } P H_K \quad (30a)$$

$$\frac{d}{dx} (I - P)\phi = -T^{-1}A_8 \phi \quad \text{on } Z_0(K_0) \quad (30b)$$
\[ \phi(0) \in \mathcal{H}_T \quad \text{and} \quad Q_+ \phi(0) = f_+ \]  

(30c)

\[ \| P_+ (x) \|_K = o(1), \quad \| (I-P) \phi(x) \|_K = o(1)(x \to \infty). \]  

(30d)

We did not make use of 6 in this statement of the half-space problem.

In (30d) it is immaterial which \( \beta \) one applies in the \( K_6 \)-norm.

The decompositions of \( \mathcal{H}_K \) into reducing subspaces of \( K \), Proposition 1 extended to \( \mathcal{H}_K \), decouples the weak half-space problem (30) into an infinite dimensional evolution equation on \( \mathcal{H}_K \) (namely, (30a)) with initial value \( P_+ \phi \) and a finite-dimensional first order system on

\[ (I-P)H_K = Z_0(K_0). \]

On \( \mathcal{H}_K \), the weak half-space problem is equivalent to the semigroup problem

\[ \frac{\partial \phi}{\partial x} + T \phi = -A \phi; \]

\[ \phi(0) = P_+ \phi; \]

\[ \| \phi(x) \|_K = o(1)(x \to \infty), \]

which has a unique solution once \( \phi(0) = P_+ \phi \) is specified uniquely.

The albedo operator \( E \) satisfies conditions (19). On \( (I-P)H_K = Z_0(K_0) \), boundedness at infinity requires that \( (I-P)E_+ \in \text{Ker} \ A \), after which the solution on \( Z_0(K_0) \) can be written as a constant; more precisely,

\[ (I-P) \phi(x) = e^{-xtA} (I-P)E_+ \cong (I-P)E_+ . \]

**THEOREM 2.** For every \( f_+ \in Q_+ (H_T) \), the half-space problem has a unique (differentiable) solution if and only if \( \text{Ker} \ A \) is positive definite with respect to the indefinite inner product (12).

This will be the case if each \( \lambda = 0 \) eigenvector of \( K \) has a corresponding generalized eigenvector. If \( \text{Ker} \ A \) is not positive definite, there exist non-trivial solutions with incoming flux \( f_+ = 0 \) (non-uniqueness). On \( \mathcal{H}_K \),

\[ \lim_{x \to \infty} \| P(x) \|_K = 0. \]

The theorem follows immediately from standard semigroup theory, assuming the construction of \( E \) (which depends on \( \beta \)) gives a unique albedo operator \( E \). We observe that \( PE_+ H_K = \mathcal{H}_K \), independent of the choice of \( \beta \). The reasoning involving the uniqueness or non-uniqueness of the construction of \( E \) is precisely the same as in Ref. 1. As in Ref. 1 we may derive the following measure of non-uniqueness:

\[ \delta = \dim \left( \text{Ran} \ P_+ \oplus \text{Ran} \ Q_- \right) \cap \text{Ker} \ A, \]

which is the dimension of the maximal strictly negative subspace of \( \text{Ker} \ A \) with respect to the indefinite inner product (12).

**V. APPLICATIONS**

This section contains several physical models leading to an equation of the form (1), which were not contained in Ref. 1. However, all models in Ref. 1 could be added here as applications.

All of them (the present ones and those in Ref. 1) involve a time-independent one-dimensional transport problem in a semi-infinite medium with spatial variable \( x(0, \infty) \). For all these models we shall specify the Hilbert space \( H \), the operators \( T \) and \( A \), whether or not \( (T,A) \) is a symmetric pair on \( H \), and the structure of the zero root linear manifold \( Z_0(K_0) \). All models will involve an unbounded operator \( T \).
1. SCALAR BGK EQUATION

\[ \nu \frac{\partial f}{\partial x}(x,\nu) = -f(x,\nu) + \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(x,\nu)e^{-\nu^2} d\nu, \,-\infty < \nu < \infty. \]

In this case we take \( H = L^2_{\delta}(-\infty,\infty) \) with \( \delta \) the measure on \((\infty,\infty)\) with Radon-Nikodym derivative \( d\delta/d\nu = e^{-\nu^2}. \) We define \( A \) and \( T \) by

\[ (Af)(\nu) = f(\nu) - \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(\nu)e^{-\nu^2} d\nu, \quad (Tf)(\nu) = \nu f(\nu). \]

Then \( T \) is unbounded self-adjoint, \( A \) bounded positive and \( I - A \) compact. Further,

\[ \text{Ker} A = \text{span}(1), \quad Z_0(K_0) = \text{span}(1, \nu) \subseteq D(T) \quad (31) \]

The map \((Ff)(\nu) = f(-\nu)\) is an inversion symmetry of the symmetric pair \((T,A)\). As the assumption (2) is fulfilled, the half-space problem can be solved and has a unique solution (cf. (31)).

2. BGK EQUATION FOR HEAT TRANSFER

\[ \nu \frac{\partial}{\partial x} \begin{bmatrix} f_1(x,\nu) \\ f_2(x,\nu) \end{bmatrix} = -\begin{bmatrix} f_1(x,\nu) \\ f_2(x,\nu) \end{bmatrix} + \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \begin{bmatrix} 1 + \frac{2}{3}(\nu^2 - \frac{1}{2}) \nu \frac{2}{3}(\nu^2 - \frac{1}{2}) \\ \frac{2}{3}(\nu^2 - \frac{1}{2}) \end{bmatrix} f_1(x,\nu) e^{-\nu^2} d\nu. \]

\[ (\infty < \nu < \infty) \]

We take \( H = L^2_{\delta}(\infty,\infty) \oplus L^2_{\delta}(\infty,\infty) \) with \( d\delta/d\nu = e^{-\nu^2} \) and define \( A \) and \( T \) by

\[ \begin{bmatrix} (Af_1)(\nu) \\ (Af_2)(\nu) \end{bmatrix} = \begin{bmatrix} f_1(\nu) \\ f_2(\nu) \end{bmatrix} - \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \begin{bmatrix} 1 + \frac{2}{3}(\nu^2 - \frac{1}{2}) \nu \frac{2}{3}(\nu^2 - \frac{1}{2}) \\ \frac{2}{3}(\nu^2 - \frac{1}{2}) \end{bmatrix} f_1(\nu) e^{-\nu^2} d\nu; \]

\[ \begin{bmatrix} (Tf_1)(\nu) \\ (Tf_2)(\nu) \end{bmatrix} = \nu f_1(\nu) f_2(\nu). \]

Then \( T \) is self-adjoint, \( A \) is self-adjoint with closed range.

Abstract Boundary Value Problems

\[ (Tf_i)(\nu) = \mu f_i(\nu) \quad (i = 1, 2), \]

where \( f \) is the column vector with entries \( f_1 \) and \( f_2 \). Then \( T \) is unbounded, self-adjoint, \( A \) bounded positive, \( I - A \) compact and

\[ \text{Ker} A = \text{span}(1), \quad Z_0(K_0) = \text{span}(1, \nu) \subseteq D(T) \quad (32) \]

The map \((Ff)(\nu) = f_i(-\nu) \quad (i = 1, 2; f = (f_1, f_2))\) is an inversion symmetry of the inversion symmetric pair \((T,A)\). As (2) is fulfilled, the half-space problem can be solved and has a unique solution (cf. (32)).

3. NEUTRON TRANSPORT WITH ANGULARLY DEPENDENT CROSS-SECTIONS

\[ \nu \frac{\partial f(x,\nu)}{\partial x} = -\Sigma(\nu)f(x,\nu) + \frac{1}{2} \int_{-1}^{+1} \Sigma_S(\nu') f(x,\nu') d\nu', \quad -1 \leq \nu \leq 1. \quad (33) \]

We assume that \( \Sigma \) and \( \Sigma_S \) are measurable and \( \Sigma \geq \Sigma_S \geq \Sigma_0 > 0. \) Now premultiply Eq. (33) by \( \Sigma_S(\nu) \) and consider the new equation on \( H = L^2_{\delta}(-1,1) \). Assume that \( \int_{-1}^{1} \Sigma_S(\nu)^2 d\nu < \infty. \) Put

\[ (Af)(\nu) = \Sigma_S(\nu) (\Sigma(\nu)f(\nu) - \frac{1}{2} \int_{-1}^{+1} \Sigma_S(\nu') f(\nu') d\nu') \]

\[ (Tf)(\nu) = \mu \Sigma_S(\nu) f(\nu). \]

Then \( T \) is self-adjoint, \( A \) is self-adjoint with closed range.
Schwarz's inequality implies that
\[ \frac{1}{2} \int_{-1}^{1} |E_s f| du \leq \frac{1}{2} \int_{-1}^{1} |E_s f|^2 du \]
and therefore \( A \) is positive. Note that \( T \) (resp. \( A \)) is bounded if and only if \( E_s \) (resp. \( E \)) is bounded. As \( E_s \in L_2([-1,1]) \), it is clear that
\[ D(A) = \{ f|E_s f \in L_2([-1,1]) \}, \quad D(T) = \{ f|\mu E_s f \in L_2([-1,1]) \}. \]

Thus \( D(A) \cap D(T) \) is dense in \( L_2([-1,1]) \).

In the same way as in Ref. 1 (Ch. VI, Appl. 6) we compute the zero root linear manifold. We find that \( \text{Ker} \ A \neq \{0\} \) if and only if \( E(\mu) = E_s(\mu) \) almost everywhere, in which case
\[ Z_0(K_0) = \left\{ \begin{array}{l}
\text{span}(E^{-1}, \mu E_s(\mu)^{-2}) \text{ if } \int_{-1}^{1} \mu E_s(\mu)^{-1} du = 0 ; \\
\text{span}(E^{-1}) = \text{Ker} \ A \text{ if } \int_{-1}^{1} \mu E_s(\mu)^{-1} du \neq 0 .
\end{array} \right. \]

We observe that \( Z_0(K_0) \subseteq D(T) \) and Eq. (2) is satisfied. Note that \( [E^{-1}, E^{-1}] = \int_{-1}^{1} \mu E_s(\mu)^{-1} du \). Thus \( \text{Ker} \ A \) is positive definite if the integral is non-negative, and strictly negative otherwise.

Now let us calculate the deficiency subspaces of \( T^{-1}A + i \). If \( 0 \neq f \in \text{Ker} \ (T^{-1}A + i) \), then
\[ (E_s(\mu) \pm i \mu)f(\mu) = \frac{1}{2} \int_{-1}^{1} E_s(\mu)f(\mu')d\mu' , \]
and thus \( \int_{-1}^{1} E_s(\mu)f(\mu')d\mu' \neq 0 \). Thus \( f(\mu) = a(E_s(\mu) \pm i \mu)^{-1} \), \( a \neq 0 \),

ABSTRACT BOUNDARY VALUE PROBLEMS

which certainly is an \( L_2 \)-function. By substitution into (34) one gets
\[ 0 = 1 - \frac{1}{2} \int_{-1}^{1} E_s(\mu) d\mu = \frac{1}{2} \int_{-1}^{1} (E_s(\mu) + \mu)^{-2} \]
\[ \pm i \mu \int_{-1}^{1} \frac{\mu E_s(\mu)}{\mu E_s(\mu) + \mu^2} d\mu , \]
which cannot possibly hold true. (Note that \( E(\mu) + \mu^2 \gtrless 0 \). Thus \( \text{Ker}(T^{-1}A + i) = \{0\} \). An easy calculation gives the invertibility of \( T^{-1}A + i \). In fact,
\[(T^{-1}A + i)^{-1}g(\mu) = \frac{\mu g(\mu) + \phi(g)}{E_s(\mu) + i \mu} , \]
where
\[ \phi(g) = \frac{1}{2} \int_{-1}^{1} \frac{E_s(\mu)\mu g(\mu')d\mu'}{E_s(\mu') + i \mu'} \left\{ 1 - \frac{1}{2} \int_{-1}^{1} \frac{E_s(\mu')d\mu'}{E_s(\mu') + i \mu'} \right\}^{-1} . \]
(Note that \( \phi \) is a bounded functional, because \( \int_{-1}^{1} (E_s(\mu) + \mu^2)^{-1} \mu^2 E_s d\mu < \infty \).) Hence, \( T^{-1}A \) is essentially self-adjoint in \( L_2 \).

All information considered, we may conclude that \( (T, A) \) is a symmetric pair on \( L_2([-1,1]) \) and the half-space problem has a unique solution if and only if \( \int_{-1}^{1} \mu E_s(\mu)^{-1} du \geq 0 \). Otherwise the measure of non-uniqueness \( \delta = 1 \).

ACKNOWLEDGEMENT

This work was supported in part by the Department of Energy under grant number DE-AS05-80ER10711. The article was written while
the second author was visiting Blacksburg. The authors are indebted to Prof. P. F. Zweifel for helpful discussions and to Prof. R. J. Hangelbroek for a useful communication.

Permanent Address:

Cor van der Mee
Dept. of Physics & Astronomy
Vrije Universiteit
De Boelelaan 1081
1081 HV Amsterdam
THE NETHERLANDS

REFERENCES


18. R. L. Bowden and W. L. Cameron, TTSP 8, 45 (1979).

