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Spectral factorization of bi-infinite multi-index block Toeplitz matrices ☆

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Abstract

In this paper we formulate a theory of LU- and Cholesky factorization of bi-infinite block Toeplitz matrices $A = (A_{i-j})_{i,j\in\mathbb{Z}^d}$ indexed by $i, j \in \mathbb{Z}^d$ and develop two numerical methods to compute such factorizations. © 2002 Elsevier Science Inc. All rights reserved.

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1. Introduction

Given a bi-infinite block Toeplitz matrix $A = (A_{i-j})_{i,j\in\mathbb{Z}}$, indexed by the integers $i, j \in \mathbb{Z}$ and having real $k \times k$ matrices as its entries (called block Laurent operators in some publications; cf., e.g., [15]), it is well known how to factorize it in the form

$$A = LDU, \tag{1.1}$$

where $L = (L_{i-j})_{i,j\in\mathbb{Z}}$ is a lower triangular (i.e., $L_i = 0$ for $i < 0$) block Toeplitz matrix with $L_0 = I_k$ (the $k \times k$ unit matrix) having a block lower triangular inverse, $U = (U_{i-j})_{i,j\in\mathbb{Z}}$ is an upper triangular (i.e., $U_i = 0$ for $i > 0$) block Toeplitz matrix with $U_0 = I_k$ and having a block upper triangular inverse, and D is a nonsingular

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$k \times k$ matrix. Such factorizations are usually studied for A in the Wiener class of block Toeplitz matrices that satisfy

$$\|A\|_{\mathscr{W}} = \sum_{i=-\infty}^{\infty} \|A_i\| < +\infty, \tag{1.2}$$

the norms on the right-hand side being arbitrary $k \times k$ matrix norms, and in that case the factors L and U and their inverses L^{-1} and U^{-1} have finite Wiener norm when the LDU-factorization (1.1) exists. A necessary (but not sufficient) condition for the existence of the factorization (1.1) is that the values of the symbol

$$\hat{A}(z) = \sum_{i=-\infty}^{\infty} z^i A_i, \quad |z| = 1, \tag{1.3}$$

are nonsingular $k \times k$ matrices.

Three special cases should be mentioned. If A is banded (i.e., if $A_i = 0$ for $|i| > m$), then the factors L and U, when they exist, are banded as well (i.e., $L_i = 0$ for $i > m$ and $U_i = 0$ for $i < -m$). If A is positive definite (as a bounded linear operator on the Hilbert space $\ell^2(\mathbb{Z})$) or, equivalently, if $\hat{A}(z)$ is positive definite for every z on the unit circle, the factorization (1.1) exists, $U = L^{\mathrm{T}}$ (the transpose of L) and D is positive definite. In that case, putting $Ł = LD^{1/2}$, we obtain the block Cholesky factorization

$$A = ŁŁ^{\mathrm{T}} \tag{1.4}$$

of A. Finally, if the symbol of A is scalar (i.e., $k = 1$), a necessary and sufficient condition for the existence of the factorization (1.1) is that $z = 0$ has zero winding number with respect to the curve $z \mapsto \hat{A}(z)$. In this case the factorization can be obtained by separating the Fourier expansion of $\log \hat{A}(z)$ in terms analytic inside and outside the unit disks and exponentiating the terms obtained.

The theory of Wiener–Hopf factorization of matrix functions of the form (1.3) with $k \times k$ matrix coefficients A_i satisfying (1.2) is well known from the theoretical point of view. We mention the seminal paper by Gohberg and Krein [18] and several textbooks [8,14,15]. The scalar case goes back to the paper by Krein [24]. In the special case of banded matrices, the symbol A is a trigonometric matrix polynomial and the factorization can be obtained explicitly by applying the theory of matrix polynomials [19,20,28].

Numerical methods for computing the Cholesky factors of a bi-infinite positive definite block Toeplitz matrix have been developed by various authors. In [22] the relative merits of various methods for the scalar case have been discussed in detail. For banded block Toeplitz matrices, a numerical method based on matrix polynomial factorization theory was developed in [25,26] and one based on band extension was given in [27].

In this paper we are primarily interested in multi-index block Toeplitz matrices, i.e., matrices $A = (A_{i-j})_{i,j \in \mathbb{Z}^d}$ which are indexed by $i, j \in \mathbb{Z}^d$ (the lattice points in $\mathbb{R}^d$) and have real $k \times k$ matrices as their entries. Although much of the theory can be developed in analogy with the one-index case of block Toeplitz matrices A indexed by $i, j \in \mathbb{Z}$, much of it has been developed in a lacunary manner and numerical methods are difficult to find.

At first sight, multi-index Toeplitz matrix theory can be developed more or less as in the one-index case. The symbols now are sums of d-variable Fourier series and are continuous $k \times k$ matrix-valued functions on the d-dimensional torus. The usual Banach algebra techniques (see [12,15] for the scalar case and [4,15] for the matrix case) can be applied to study the invertibility of bi-infinite multi-index block Toeplitz matrices. The method of writing the logarithm of the symbol as the sum of two series, either of which is then exponentiated, can be applied, exclusively in the scalar case, to obtain LDU-factorizations [23].

Multi-index block Toeplitz factorization theory has several features that make it more challenging than the corresponding one-index theory, both from the functional analytic and the numerical point of view.

First of all, in order to define LDU-factorizations in a meaningful way, one must introduce a linear order $\preceq$ on $\mathbb{Z}^d$ preserving the displacement structure, using the fact that $\mathbb{Z}^d$ is an ordered group. The net effect is that instead of two such orders as for $\mathbb{Z}$ (the natural and the reversed natural order), there are now infinitely many such orders, leading to very different factorization problems. Using a suitable order, one can now formulate (i) scalar factorizations through separation of logarithms of the symbol and exponentiation [23], (ii) band extension [2,3], and (iii) the projection method of approximating the solutions of the given bi-infinite block Toeplitz systems by the solutions of finite block Toeplitz systems (extending methods given in [5,17,31]). However, the band extension method leads to the approximation of the solution of the original bi-infinite system by the solutions of infinite systems, which makes it as good as useless from the numerical point of view.

Secondly, there is no meaningful multi-variable matrix polynomial theory to assist in the factorization of banded multi-index block Toeplitz matrices. Moreover [32,33], multi-index Toeplitz matrices having a trigonometric polynomial symbol and having an LDU-factorization may not have factors whose symbols are nontrivial trigonometric polynomials. Hence, there is no hope of generalizing the numerical methods developed in [25,26].

Finally, in order to study the algebraic or exponential decay of the coefficients of the factors and their inverses in the case of algebraic or exponential decay of the coefficients of the given matrix, one can either apply Banach algebra techniques with weighted Wiener algebras [12,15] or generalize the so-called exponential equivalence of bi-infinite matrices to the multi-index case [22,23]. In the scalar case ($k = 1$), one does not encounter many problems, as we will show shortly. However, in the block Toeplitz case ($k \geqslant 2$) the nonexistence of a proof of the compactness of semi-infinite multi-index block Hankel matrices (as opposed to the situation in the one-index case, see [14,15,18]) makes it impossible to extend the existing techniques for proving algebraic or exponential decay of the coefficients of the factors and their inverses to the multi-index case.

Hence, as sketched above, we have only two numerical methods in the multi-index case, namely (i) scalar factorizations through separation of logarithms of the symbol and exponentiation [23], and (ii) the projection method. We will discuss both methods in detail.

Let us first discuss the contents of the various sections. Section 2 is of a preliminary nature and contains the definitions (Banach algebras etc.) and main results on the existence of an LDU-factorization in suitable Banach algebras. In Section 3 we deal with the scalar multi-index case and develop factorization theory by Krein's method, taking account also of various decay properties of the coefficients. In Section 4 we develop the band extension method in the multi-index case and discuss the main result of [3]. In Section 5 we discuss the projection method in detail and explain why the band extension method has no multi-index generalization that is meaningful from the numerical point of view. In Section 6 we consider the spectral factorization of a scalar multi-index matrix connected to the numerical solution of the Helmholtz equation in a half-plane.

2. Preliminaries

2.1. Bi-infinite block Toeplitz matrices

Let $\mathbb{Z}^d$ be the set of points in d-dimensional space with integer coordinates. Then by a bi-infinite block Toeplitz matrix, with blocks of order k, we mean a matrix $A = (A_{i-j})_{i,j\in\mathbb{Z}^d}$ whose entries A_{i-j} are complex $k \times k$ matrices. Such a matrix is said to be in the *Wiener class* $\mathscr{W}_k^d$ if

$$\|A\|_{\mathscr{W}_k^d} := \sum_{i\in\mathbb{Z}^d} \|A_i\| < +\infty, \tag{2.1}$$

where $\|\cdot\|$ is an arbitrary $k \times k$ matrix norm. Using multi-index notation, [1] we define its *symbol* by

$$\hat{A}(z) := \sum_{i\in\mathbb{Z}^d} z^i A_i, \quad z \in \mathbb{T}^d, \tag{2.2}$$

where $\mathbb{T} = \{z \in \mathbb{C}\colon |z| = 1\}$. Clearly, the symbol $\hat{A}$ is a continuous complex-valued function on the d-dimensional torus $\mathbb{T}^d$.

Consider a sequence $\boldsymbol{\beta} = (\beta_i)_{i\in\mathbb{Z}^d}$ of weights satisfying the condition $1 \leqslant \beta_{i+j} \leqslant \beta_i\beta_j$ for $i, j \in \mathbb{Z}^d$. Then a bi-infinite block Toeplitz matrix A is said to be in the $\boldsymbol{\beta}$*-weighted Wiener class* $\mathscr{W}_{k,\boldsymbol{\beta}}^d$ if

[1] For $z = (z_1, \ldots, z_d) \in \mathbb{C}^d$ and $i = (i_1, \ldots, i_d) \in \mathbb{Z}^d$ we write $z^i = z_1^{i_1} \cdots z_d^{i_d}$ and $|i| = |i_1| + \cdots + |i_d|$.

$$\|A\|_{\mathscr{W}^d_{k,\boldsymbol{\beta}}} := \sum_{i\in\mathbb{Z}^d} \beta_i \|A_i\| < +\infty. \tag{2.3}$$

Then A is a bounded linear operator on the Banach space $\ell^p_{k,\boldsymbol{\beta}}$ of all sequences $(x_i)_{i\in\mathbb{Z}^d}$ in $\mathbb{C}^k$ which are finite with respect to the norm

$$\|(x_i)_{i\in\mathbb{Z}^d}\|_{p,\boldsymbol{\beta}} = \begin{cases} \left[\sum_{i\in\mathbb{Z}^d} (\beta_i \|x_i\|)^p\right]^{1/p}, & 1 \leqslant p < +\infty, \\ \sup_{i\in\mathbb{Z}^d} \beta_i \|x_i\|, & p = +\infty; \end{cases} \tag{2.4}$$

this can be proved trivially for $p = 1$ and $p = +\infty$ and by interpolation for $p \in (1, \infty)$. We write ℓ^p_k if $\beta_i = 1$ for all $i \in \mathbb{Z}^d$. For $M \subset \mathbb{Z}^d$, we denote by $\ell^p_{k,\boldsymbol{\beta}}(M)$ and $\ell^p_k(M)$ the subspaces of $\ell^p_{k,\boldsymbol{\beta}}$ and ℓ^p_k of those sequences $(x_i)_{i\in\mathbb{Z}^d}$ for which $x_i = 0$ for $i \in \mathbb{Z}^d \setminus M$; their elements are written as sequences indexed by $i \in M$.

The following result is well known for the one-index case [15]; for the multi-index case the proof is similar. In fact, the principal observation in proving this result is that the (continuous) multiplicative linear functionals on $\mathscr{W}^d_{k,\boldsymbol{\beta}}$ are exactly the evaluation maps $A \mapsto \hat{A}(z)$, where $z \in \Omega_{\boldsymbol{\beta}}$, with $\Omega_{\boldsymbol{\beta}}$ as in (2.6).

Proposition 2.1. *The $\boldsymbol{\beta}$-weighted Wiener class $\mathscr{W}^d_{k,\boldsymbol{\beta}}$ is a Banach algebra with respect to the convolution product*

$$(A * B)_i = \sum_{j\in\mathbb{Z}^d} A_j B_{i-j}, \quad i \in \mathbb{Z}^d, \tag{2.5}$$

with involution $A \mapsto A^*$ defined by $(A^*)_i = (A_{-i})^*$, $i \in \mathbb{Z}^d$, where the asterisk superscript denotes the conjugate transpose. Its invertible elements are exactly those $A \in \mathscr{W}^d_{k,\boldsymbol{\beta}}$ for which $\hat{A}(z)$ is a nonsingular $k \times k$ matrix for all $z \in \Omega_{\boldsymbol{\beta}}$, where

$$\Omega_{\boldsymbol{\beta}} := \left\{ z \in \mathbb{C}^d\colon \sup_{i\in\mathbb{Z}^d} \frac{|z^i|}{\beta_i} < +\infty \right\}. \tag{2.6}$$

When the weight sequence $\boldsymbol{\beta} = (\beta_i)_{i\in\mathbb{Z}^d}$ is separated in the sense that $\beta_i = \beta^{(1)}_{i_1} \cdots \beta^{(d)}_{i_d}$, where $1 \leqslant \beta^{(r)}_{i_r+j_r} \leqslant \beta^{(r)}_{i_r}\beta^{(r)}_{j_r}$ for $i_r, j_r \in \mathbb{Z}$ and $r = 1, \ldots, d$, we have

$$\Omega_{\boldsymbol{\beta}} = \prod_{r=1}^{d} \Omega_{\boldsymbol{\beta}^{(r)}} = \prod_{r=1}^{d} \left\{ z \in \mathbb{C}\colon \rho^{(r)}_{-} \leqslant |z| \leqslant \rho^{(r)}_{+} \right\}, \tag{2.7}$$

where, for $r = 1, \ldots, d$, $\boldsymbol{\beta}^{(r)} = (\beta^{(r)}_i)_{i\in\mathbb{Z}}$ and

$$\rho^{(r)}_{-} = \left[\lim \sup_{i\to+\infty} (\beta^{(r)}_{-i})^{1/i}\right]^{-1}, \qquad \rho^{(r)}_{+} = \lim \sup_{i\to+\infty} (\beta^{(r)}_{i})^{1/i}. \tag{2.8}$$

A further generalization consists of considering bi-infinite block Toeplitz matrices $A = (A_{i-j})_{i,j\in\mathbb{Z}^d}$ such that $A_i = 0$ for all $i \notin J$, where J is an additive subgroup of

$\mathbb{Z}^d$. For any weight sequence $\boldsymbol{\beta} = (\beta_i)_{i\in\mathbb{Z}^d}$ one can now define the Banach algebra $\mathscr{W}^{d,J}_{k,\boldsymbol{\beta}}$ (resp., $\mathscr{W}^{d,J}_{k}$) consisting of those $A \in \mathscr{W}^{d}_{k,\boldsymbol{\beta}}$ (resp., $A \in \mathscr{W}^{d}_{k}$) such that $A_i = 0$ for all $i \notin J$. Instead of Proposition 2.1, we then have the following more general result. [2]

Proposition 2.2. *For any additive subgroup J of $\mathbb{Z}^d$, $\mathscr{W}^{d,J}_{k,\boldsymbol{\beta}}$ is a Banach algebra with respect to the convolution product*

$$(A * B)_i = \sum_{j\in\mathbb{Z}^d} A_j B_{i-j} = \sum_{j\in J} A_j B_{i-j}, \quad i \in \mathbb{Z}^d, \tag{2.9}$$

with involution $A \mapsto A^*$ defined by $(A^*)_i = (A_{-i})^*$, $i \in \mathbb{Z}^d$. Its invertible elements are exactly those $A \in \mathscr{W}^{d,J}_{k,\boldsymbol{\beta}}$ for which $\hat{A}(z)$ is a nonsingular $k \times k$ matrix for all $z \in \Omega^J_{\boldsymbol{\beta}}$, where

$$\Omega^J_{\boldsymbol{\beta}} := \left\{ z \in \mathbb{C}^d \colon \sup_{i\in J} \frac{|z^i|}{\beta_i} < +\infty \right\}. \tag{2.10}$$

In the same way one can define bi-infinite block Toeplitz matrices whose elements belong to a Banach algebra $\mathscr{A}$ with unit element (such as the bounded linear operators on some Banach space). Propositions 2.1 and 2.2 are valid in this more general situation which can be proved by using results from [4,21]. Here we use the fact that (1) $\mathscr{W}^{d,J}_{k,\boldsymbol{\beta}} = \mathscr{W}^{d,J}_{1,\boldsymbol{\beta}} \otimes_\pi \mathscr{A}$, $\otimes_\pi$ standing for the projective tensor product, and (2) the multiplicative linear functionals in $\mathscr{W}^{J}_{1,\boldsymbol{\beta}}$ are precisely the evaluation maps $A \mapsto \hat{A}(z)$, where $z \in \Omega^J_{\boldsymbol{\beta}}$.

2.2. Multilevel block Toeplitz matrices

In analogy with [36,37], block Toeplitz matrices with elements indexed by $\mathbb{Z}^d$ can be converted into so-called multi-level block Toeplitz matrices. In fact, given a bi-infinite block Toeplitz matrix $A = (A_{i-j})_{i,j\in\mathbb{Z}^d}$, we can define the block Toeplitz matrix $\mathscr{A} = (\mathscr{A}_{i-j})_{i,j\in\mathbb{Z}}$ indexed by $\mathbb{Z}$ whose entries are in turn block Toeplitz matrices, but this time indexed by $\mathbb{Z}^{d-1}$, by the following conversion rule:

$$\mathscr{A}_{i_1} = (A_{(i_1,i_2,\ldots,i_d)})_{(i_2,\ldots,i_d)\in\mathbb{Z}^{d-1}}, \quad i_1 \in \mathbb{Z}. \tag{2.11}$$

In turn, for $d \geqslant 3$ the block Toeplitz matrices $\mathscr{A}_i$ indexed by $\mathbb{Z}^{d-1}$ can be converted into block Toeplitz matrices indexed by $\mathbb{Z}$ whose elements are block Toeplitz matrices indexed by $\mathbb{Z}^{d-2}$, and so on. For block Toeplitz matrices $A \in \mathscr{W}^d_k$, we easily see that the Wiener norm of $\mathscr{A}$ satisfies

[2] We note that Banach algebras of bi-infinite Toeplitz matrices indexed by arbitrary additive subgroups of $\mathbb{R}$ and $\mathbb{R}^d$ have been studied in [29] and [30,31], respectively.

$$\|\mathscr{A}\|_{\mathscr{W}_k^1} := \sum_{i_1\in\mathbb{Z}} \|\mathscr{A}_{i_1}\|_{\mathscr{W}_k^{d-1}} = \sum_{\substack{i_1\in\mathbb{Z}\\(i_2,\dots,i_d)\in\mathbb{Z}^{d-1}}} \|(\mathscr{A}_{i_1})_{(i_2,\dots,i_d)}\| = \|A\|_{\mathscr{W}_k^d}. \tag{2.12}$$

Analogously, the symbol $\hat{\mathscr{A}}$ of $\mathscr{A}$ is a continuous function on $\mathbb{T}$ with values in the Banach algebra $\mathscr{W}_k^{d-1}$, and for the symbol of each block Toeplitz matrix $\hat{\mathscr{A}}(z_1)$ we have

$$\widehat{\hat{\mathscr{A}}(z_1)}(z_2,\dots,z_d) = \hat{A}(z_1,\dots,z_d), \quad z_1\in\mathbb{T},\ (z_2,\dots,z_d)\in\mathbb{T}^{d-1}. \tag{2.13}$$

Now consider the weight sequence $\boldsymbol{\beta} = (\beta_i)_{i\in\mathbb{Z}^d}$ having the property $\beta_i = \beta_{i_1}^{(1)}\gamma_{(i_2,\dots,i_d)}$ for certain weight sequences $\boldsymbol{\beta}^{(1)} = (\beta_{i_1}^{(1)})_{i_1\in\mathbb{Z}}$ and $\boldsymbol{\gamma} = (\gamma_j)_{j\in\mathbb{Z}^{d-1}}$. Then $\Omega_{\boldsymbol{\beta}} = \Omega_{\boldsymbol{\beta}^{(1)}} \times \Omega_{\boldsymbol{\gamma}}$, while (2.12) is to be replaced by

$$\begin{aligned}\|\mathscr{A}\|_{\mathscr{W}_{\boldsymbol{\beta}^{(1)}}^1} &:= \sum_{i_1\in\mathbb{Z}} \beta_{i_1}^{(1)} \|\mathscr{A}_{i_1}\|_{\mathscr{W}_{\boldsymbol{\gamma}}^{d-1}}\\ &= \sum_{i_1\in\mathbb{Z}} \beta_{i_1}^{(1)} \sum_{(i_2,\dots,i_d)\in\mathbb{Z}^{d-1}} \gamma_{(i_2,\dots,i_d)} \|(\mathscr{A}_{i_1})_{(i_2,\dots,i_d)}\|\\ &= \|A\|_{\mathscr{W}_{k,\boldsymbol{\beta}}^d},\end{aligned} \tag{2.14}$$

where the subscript k indicating the matrix order involved in stating the (weighted) Wiener algebra has been dropped. Using Proposition 2.1, (2.13) and (2.14), we see that A is an invertible element of $\mathscr{W}_{k,\boldsymbol{\beta}}^d$ if and only if $\mathscr{A}$ is an invertible element of $\mathscr{W}_{\boldsymbol{\beta}^{(1)}}^1$, which is the case whenever $\hat{\mathscr{A}}(z_1)$ is an invertible element of $\mathscr{W}_{\boldsymbol{\gamma}}^{d-1}$ for every $z_1 \in \Omega_{\boldsymbol{\beta}^{(1)}}$. But the latter is true if and only if its symbol $\widehat{\mathscr{A}(z_1)}(z_2,\dots,z_d)$ is a nonsingular $k\times k$ matrix for every $z_1\in\Omega_{\boldsymbol{\beta}^{(1)}}$ and every $(z_2,\dots,z_d)\in\Omega_{\boldsymbol{\gamma}}$, in other words, if and only if $\hat{A}(z_1,\dots,z_d)$ is a nonsingular $k\times k$ matrix for every $z=(z_1,\dots,z_d)\in\Omega_{\boldsymbol{\beta}}$.

2.3. *LDU-factorization of bi-infinite block Toeplitz matrices*

Given a block Toeplitz matrix $A = (A_{i-j})_{i,j\in\mathbb{Z}^d}$ of Wiener class, by an *LDU-factorization* of A (with respect to the order $\preceq$) we mean a representation of A in the form

$$A = LDM^*, \tag{2.15}$$

where $L = (L_{i-j})_{i,j\in\mathbb{Z}^d}$, $M = (M_{i-j})_{i,j\in\mathbb{Z}^d}$ and $D = (D_{i-j})_{i,j\in\mathbb{Z}^d}$ are block Toeplitz matrices of Wiener class having the following properties:
(a) $L_0 = M_0 = I_k$ (the $k \times k$ unit matrix),
(b) $D_i = 0$ for $i \neq 0$ and $L_i = M_i = 0$ for $i \prec 0$, and
(c) the inverses L^{-1} and M^{-1} of L and M are block Toeplitz matrices of Wiener class satisfying $[L^{-1}]_i = [M^{-1}]_i = 0$ for $i \prec 0$.
Passing to the respective symbols $\hat{L}$, $\hat{D}(z) \equiv D_0$ and $\hat{M}$, one gets

$$\hat{A}(z) = \hat{L}(z) D_0 \hat{M}(z)^*, \quad z \in \mathbb{T}^d. \tag{2.16}$$

When A is positive definite on the Hilbert space $\ell^2(\mathbb{Z}^d)$ of square integrable sequences on $\mathbb{Z}^d$ (or, equivalently, if $\hat{A}(z)$ is positive definite for all $z \in \mathbb{T}^d$), A always has an LDU-factorization of the form (2.16) with $L = M$ and D_0 a positive definite $k \times k$ matrix. In that case we put $Ł_i = L_i D_0^{1/2}$ and obtain the *block Cholesky factorization*

$$\hat{A}(z) = \hat{Ł}(z)\hat{Ł}(z)^*, \quad z \in \mathbb{T}^d. \tag{2.17}$$

Following the procedure inherent in Theorem XXII 8.2 of [15], the inverses of the factors L and M in (2.15) can be found by solving suitable semi-infinite linear systems. Indeed, writing (2.15) in the form $A(DM^*)^{-1} = L$, restricting oneself to indices $\preceq 0$, applying it to the semi-infinite vector $\boldsymbol{e}_- = (\delta_{0,i} I_k)_{i\preceq 0}$ and changing the sign of all indices, we obtain the linear system

$$\sum_{j\succeq 0} A_{j-i} X_j = (\boldsymbol{e}_+)_i, \quad i \succeq 0, \tag{2.18}$$

where $[(DM^*)^{-1}]_{i,j} = X_{j-i}$ (using the convention that $X_i = 0$ for $i \prec 0$), $\boldsymbol{e}_+ = (\delta_{0,i} I_k)_{i\succeq 0}$ and $\boldsymbol{x}_+ = (X_i)_{i\in\mathbb{Z}^d} \in \ell_k^1$.

Analogously, writing (2.15) as $A^*([LD]^*)^{-1} = M$, restricting oneself to the indices $\succeq 0$ and applying it to $\boldsymbol{e}_+$, we obtain the linear system

$$\sum_{j\succeq 0} A_{j-i}^* Y_j = (\boldsymbol{e}_+)_i, \quad i \succeq 0, \tag{2.19}$$

where $([LD]^*)^{-1} = (Y_{i-j})_{i,j\in\mathbb{Z}^d}$ (using the convention that $Y_i = 0$ for $i \prec 0$) and $\boldsymbol{y}_+ = (Y_i)_{i\in\mathbb{Z}^d} \in \ell_k^1$. Since a solution of either of (2.18) or (2.19) leads to an LDU-factorization of the form (2.15) with the diagonal factor D absorbed in M^* and L, respectively, and such factorizations are unique, those equations are uniquely solvable in ℓ_k^1.

When the weight sequence $\boldsymbol{\beta}$ is to be taken into account, the classical argument of exploiting the compactness of Hankel operators [15] to prove that the vectors $\boldsymbol{x}_+$ and $\boldsymbol{y}_+$ belong to $\ell_{k,\boldsymbol{\beta}}^1$ fails if $k \geqslant 2$. For $k = 1$ one can apply factorization in suitable commutative Banach algebras to establish Theorem 2.3. We therefore postpone its proof until the end of Section 3.

Theorem 2.3. *Let $\boldsymbol{\beta} = (\beta_i)_{i\in\mathbb{Z}^d}$ be a weight sequence and J an additive subgroup of $\mathbb{Z}^d$. Suppose $A \in \mathscr{W}_{k,\boldsymbol{\beta}}^{d,J}$ has an LDU-factorization in $\mathscr{W}_k^d$ of the type* (2.15) *and that $\hat{A}(z)$ is a nonsingular $k \times k$ matrix for $z \in \Omega_{\boldsymbol{\beta}}^J$. Then, in the scalar case $k = 1$, the factors L and M^* and their inverses belong to $\mathscr{W}_{k,\boldsymbol{\beta}}^{d,J}$.*

We now apply Theorem 2.3 to positive definite bi-infinite block Toeplitz matrix $A \in \mathscr{W}_k^{d,J}$, where J is an additive subgroup of $\mathbb{Z}^d$. Then, according to Theorem 14.2 of [9] (applied to the nest algebra generated by (the strong limits of) the orthogonal projections onto $\{(x_j)_{j\in Z} \in \ell_k^{2,J}\colon x_j = 0 \text{ for } j \succ i\}$, $i \in J$), every such matrix has an LDU-factorization of the form (2.15), where L and M^* and their inverses are bounded linear operators on $\ell_k^{2,J}$. In the next theorem we will actually prove that L and M^* and their inverses belong to $\mathscr{W}_k^{d,J}$ if $k = 1$. To do so, we first state the following result on linear orders on $\mathbb{R}^d$ due to Erdös [10] (see [7] for a concise proof).

Lemma 2.4. *Let $\preceq$ be a linear order on $\mathbb{Z}^d$ such that $i + l \preceq j + l$ whenever $i, j, l \in \mathbb{Z}^d$ and $i \preceq j$. Then $\preceq$ can be extended to a so-called term ordering on $\mathbb{R}^d$, i.e., to a linear order $\preceq$ such that $x + z \preceq y + z$ and $cx \preceq cy$ whenever $x, y, z \in \mathbb{R}^d$, $x \preceq y$ and $c \geqslant 0$ in $\mathbb{R}$. Moreover, there exists an orthogonal $d \times d$ matrix Ξ such that the $\preceq$-nonnegative elements of $\mathbb{R}^d$ are exactly the linear combinations of the columns of Ξ with nonnegative coefficients.*

Proof . Following [7,10], there exists a sequence of linear subspaces $\mathbb{R}^d \supset H_{d-1} \supset H_{d-2} \supset \cdots \supset H_1 \supset \{0\}$ and an orthonormal basis $\{\xi_j\}_{j=1}^d$ of $\mathbb{R}^d$ such that $\dim H_j = j$ $(j = 1, \ldots, d-1)$, $\xi_r \in H_{d-r+1}$ $(r = 2, \ldots, d)$, $\xi_s \perp H_{d-s}$ $(s = 1, \ldots d-1)$, and $\xi_t \succ 0$ $(t = 1, \ldots, d)$. We then define Ξ to be the orthogonal $d \times d$ matrix having $\xi_1, \ldots, \xi_d$ as its columns. □

Starting with a linear order $\preceq$ on $\mathbb{Z}^d$ compatible with addition, we first extend it to a term ordering in $\mathbb{R}^d$. This extension is not necessarily unique; see [7]. For example, letting $R(\alpha)$ denote the 2×2 rotation matrix, the linear orders in $\mathbb{R}^2$ described by the orthogonal matrices $R(\alpha)$ and $R(-\alpha)$ lead to the same order on $\mathbb{Z}^2$ if $\tan\alpha \notin \mathbb{Q}$. For a less trivial example, consider $v_1 = (1, 0, 0, 0)^{\mathrm{T}}$, $v_2 = (0, 1, \zeta, 0)$ and $v_3 = (0, 1, 0, \zeta^2)$, where ζ is irrational and does not satisfy a quadratic equation with rational coefficients. Then for any 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with integer entries and determinant 1, the linear orders on $\mathbb{R}^4$ determined by the sequence of linear subspaces $H_3 = \mathrm{span}(v_1, av_2 + bv_3, cv_2 + dv_3)$, $H_2 = \mathrm{span}(v_1, av_2 + bv_3)$ and $H_1 = \mathrm{span}(v_1)$ generate the same linear order on $\mathbb{Z}^4$. Indeed, it is easily seen that $H_3 \cap \mathbb{Z}^4 = H_2 \cap \mathbb{Z}^4 = \{rv_1\colon r \in \mathbb{Z}\}$, no matter the choice of $a, b, c, d \in \mathbb{Z}$ with $ad - bc = 1$.

Now let $\mu = \min\{s\colon H_{d-s} \cap \mathbb{Z}^d = \emptyset\}$ and $\mu = d$ if $H_1 \cap \mathbb{Z}^d \neq \emptyset$. Define the *Besikovich transformation* $\varphi\colon \mathbb{Z}^d \to \mathbb{R}^\mu$ (used in [3] for $d = 2$, but actually already implicit in [10]) by

$$\varphi(i) = (c_1(i), \ldots, c_\mu(i)) = \Xi^{-1} i, \quad i = \sum_{s=1}^{\mu} c_s(i)\xi_s. \tag{2.20}$$

Then $\varphi[\mathbb{Z}^d] = N_1 \times \cdots \times N_\mu$ for certain additive subgroups of $\mathbb{R}$, ordered lexicographically. In particular, if $\mu = 1$, we have $\varphi[\mathbb{Z}^d] = c\mathbb{Z}$ for some $c > 0$, and for $\mu = d$ we have $\varphi[\mathbb{Z}^d] = c\mathbb{Z}^d$ for some $c > 0$. For the above example in $\mathbb{Z}^4$ we have

$$\begin{aligned} &\varphi(i_1, i_2, i_3, i_4) = \left(\frac{i_2\zeta^2 - i_3\zeta - i_4}{\sqrt{\zeta^4 + \zeta^2 + 1}}, i_1 \right), \\ &\varphi[\mathbb{Z}^4] \simeq \operatorname{span}_{\mathbb{Z}}(1, \zeta, \zeta^2) \times \mathbb{Z}. \end{aligned} \tag{2.21}$$

Theorem 2.5. *Let $\boldsymbol{\beta} = (\beta_i)_{i \in \mathbb{Z}^d}$ be a weight sequence and J an additive subgroup of $\mathbb{Z}^d$. Suppose $A \in \mathscr{W}^{d,J}_{k,\boldsymbol{\beta}}$ is positive definite and that $\hat{A}(z)$ is a nonsingular $k \times k$ matrix for $z \in \Omega^J_{\boldsymbol{\beta}}$. Then A has a Cholesky factorization of the type* (2.17), *where the Cholesky factor Ł and its inverse belong to $\mathscr{W}^{d,J}_{k}$. Moreover, in the scalar case $k = 1$, the Cholesky factor Ł and its inverse Ł$^{-1}$ belong to $\mathscr{W}^{d,J}_{k,\boldsymbol{\beta}}$.*

Below we give the proof of Theorem 2.5 with the exception of the proof of the statement that Ł and Ł$^{-1}$ belong to $\mathscr{W}^{d,J}_{k,\boldsymbol{\beta}}$. For $k = 1$, this part will be established at the end of Section 3. We mention that the analogous result for discrete additive subgroups of $\mathbb{R}^d$ (and hence for symbols that are $k \times k$ matrices whose elements are suitable almost periodic functions in d variables) has been established in [30,31], though without accounting for weight sequences $\boldsymbol{\beta}$.

Proof of Theorem 2.5 (*first part*). The proof focuses on the existence of an LDU-factorization; the final part of Theorem 2.5 will be established in Section 3. To establish the existence part, we generalize the proof given in [3] for $d = 2$.

Let $A \in \mathscr{W}^{d}_{k,\boldsymbol{\beta}}$ be positive definite. To prove the existence of an LDU-factorization of A in $\mathscr{W}^{d}_{k}$ (without taking account of the subgroup J and the weight sequence $\boldsymbol{\beta}$), we apply the Besikovich transformation to $\mathbb{Z}^d$ to obtain a positive definite bi-infinite block Toeplitz matrix indexed by $\varphi[\mathbb{Z}^d] = N_1 \times \cdots \times N_\mu$, where $N_1, \ldots, N_\mu$ are additive subgroups of $\mathbb{Z}$. In this way a factorization problem for a matrix indexed by $i \in \mathbb{Z}^d$ (with respect to the order $\preceq$) has been transformed into a factorization problem for a matrix indexed by $r \in \varphi[\mathbb{Z}^d]$ (with respect to the lexicographical order on $\mathbb{R}^\mu$ restricted to $\varphi[\mathbb{Z}^d]$). A subgroup J of $\mathbb{Z}^d$ is converted into a subgroup $\varphi[J]$ of $\varphi[\mathbb{Z}^d]$ and the weight sequence $\boldsymbol{\beta}$ is converted into the weight sequence $\boldsymbol{\gamma} = (\gamma_r)_{r \in \varphi[\mathbb{Z}^d]}$ defined by $\gamma_{\varphi(i)} = \beta_i$ $(i \in \mathbb{Z}^d)$. Now convert the latter μ-index matrix to a multilevel matrix. Then if $\mu = 1$, the factorization result in $\mathscr{W}^{d}_{k}$ follows immediately from Theorem 6.1 of [29]. If $\mu \geqslant 2$, $\varphi[\mathbb{Z}^d]$ is isomorphic to $\mathbb{Z} \otimes N$, where N is an additive subgroup of $\mathbb{R}^{\mu-1}$. Using Lemma 3.1 of [38] (which is a variation of the main result of [21] applied to positive definite finite- or infinite-dimensional block matrices indexed by $\mathbb{Z}$), we obtain the factorization

$$\hat{\mathscr{A}}(z_1) = (I + \hat{U}(z_1))^* D(I + U(z_1)), \quad z_1 \in \mathbb{T}, \tag{2.22}$$

where D is positive definite on $\ell^2(N)$ and U is an element of $\mathscr{W}_k^N$ such that the jth Fourier coefficient of U and $(I + U)^{-1} - I$ vanish for each $j \geqslant 0$. Next, depending on the structure of N as a direct product of $\mu - 1$ additive subgroups of $\mathbb{R}$, we apply the same result again (if $N = \mathbb{Z}$ or if $N = \mathbb{Z} \times M$ for an additive subgroup M of $\mathbb{R}^{\mu-2}$) or the main results of [1,34] (applying to positive definite finite- or infinite-dimensional block matrices indexed by an additive subgroup of $\mathbb{R}$), to factorize the diagonal factor D in (2.22) as a positive definite one-index Toeplitz matrix with $(\mu - 2)$-index matrices as entries. We end up with an LDU-factorization of the form (2.15) for the Besikovich transform of the original bi-infinite matrix A, where $U = L^*$ and D is positive definite. Applying the inverse of φ to the index set, we get an LDU-factorization of A in $\mathscr{W}_k^{d,J}$.

When $A \in \mathscr{W}_k^{d,J}$ for some additive subgroup J of $\mathbb{Z}^d$, we define $\mu = 1$ if $H_{d-1} \cap J = \cdots = H_1 \cap J = \emptyset$ and $\mu = \#\{s\colon H_s \cap J \neq \emptyset\}$. The Besikovich transformation $\varphi_J\colon J \to \mathbb{R}^\mu$ is then defined by

$$\varphi(i) = (c_\nu(i))_{H_{d-i} \cap J \neq \emptyset}, \quad i = \sum_{s=1}^{d} c_s(i)\xi_s, \tag{2.23}$$

where $H_d = \mathbb{R}^d$ and $H_0 = \{0\}$.

The part of the theorem dealing with weight sequences follows directly from Theorem 3.2 [cf. the end of Section 3]. □

A bi-infinite block Toeplitz matrix A is called (*finitely*) *banded* if all but finitely many A_i are equal to the zero matrix. A well-known result (Féjer's theorem if A is positive definite) states that, for $d = 1$, the factors L and M^* (resp., the factor Ł) in an LDU-factorization (resp., Cholesky factorization) of an arbitrary (resp., positive definite) (finitely) banded block Toeplitz matrix of Wiener class are (finitely) banded themselves. This is no longer the case if $d \geqslant 2$ [33]. For instance [6,32], if $d = 2$, $\delta \in (0, \frac{1}{4})$ and

$$\hat{A}(z) = 1 + 2\delta\,[\cos(z_1) + \cos(z_2)]\,, \tag{2.24}$$

then $\hat{A}(z)$ is positive for every $z = (z_1, z_2) \in \mathbb{T}^2$ but cannot be written as the product of two nonconstant trigonometric polynomials in z_1 and z_2. In other words, no matter the choice of the order $\preceq$ in $\mathbb{Z}^2$, the corresponding Toeplitz matrix A has an LDU-factorization (resp., a Cholesky factorization) of the form (2.15) (resp., (2.17)), but its factors L and M^* (resp., the factor Ł) are not (finitely) banded Toeplitz matrices. For $d = 2$, necessary and sufficient conditions to write $\hat{A}(z)$ as the squared absolute value of a stable polynomial in (z_1, z_2) have been given in [13].

3. Factorization of Toeplitz matrices

In this section a well-known method to compute the LDU-factorization of a bi-infinite Toeplitz matrix indexed by $\mathbb{Z}$ is generalized to bi-infinite Toeplitz matrices indexed by $\mathbb{Z}^d$. We need the following result (cf. [11, Theorem I 5.1]).

Theorem 3.1. *Let $A \in \mathscr{W}^{d,J}_{k,\boldsymbol{\beta}}$ for some weight sequence $\boldsymbol{\beta}$ and some additive subgroup J of $\mathbb{Z}^d$, and let φ be an analytic function in a neighborhood of the set $\Sigma(A) = \{\hat{A}(z)\colon z \in \Omega^J_{\boldsymbol{\beta}}\}$. Let Γ be a closed rectifiable Jordan contour in the domain of φ which has winding number 1 with respect to each point of $\Sigma(A)$. Then*

$$\varphi(A) := \frac{1}{2\pi \mathrm{i}} \int_\Gamma \varphi(\lambda)(\lambda I - A)^{-1}\, \mathrm{d}\lambda \tag{3.1}$$

belongs to $\mathscr{W}^{d,J}_{k,\boldsymbol{\beta}}$.

Proof . From the second parts of Propositions 2.1 and 2.2 it follows that $\Sigma(A)$ coincides with the set of all $\lambda \in \mathbb{C}$ for which $\lambda I - A$ does not have an inverse in $\mathscr{W}^{d,J}_{k,\boldsymbol{\beta}}$. As a result, $\lambda I - A$ is an invertible element of $\mathscr{W}^{d,J}_{k,\boldsymbol{\beta}}$ for all $\lambda \in \Gamma$ and hence the right-hand side of (3.1) belongs to $\mathscr{W}^{d,J}_{k,\boldsymbol{\beta}}$, by the usual Riesz functional calculus. □

When $A \in \mathscr{W}^{d,J}_{k,\boldsymbol{\beta}}$, $\exp(A)$ can be defined by choosing $\varphi(\lambda) = \mathrm{e}^{\lambda}$ in (3.1). However, $\log(A)$ can only be defined in this way if $\log(\lambda)$ is an analytic function on a neighborhood of $\Sigma(A)$. This is the case if and only if there exists a (continuous) curve in $\mathbb{C}\backslash\Sigma(A)$ connecting zero to infinity.

The next theorem generalizes a well-known result by Krein [24].

Theorem 3.2. *Let $A \in \mathscr{W}^{d,J}_{1,\boldsymbol{\beta}}$ be a bi-infinite Toeplitz matrix with scalar elements (i.e., with $k = 1$) for some weight sequence $\boldsymbol{\beta}$ and some additive subgroup J of $\mathbb{Z}^d$. Suppose $\log(\lambda)$ is an analytic function in a neighborhood of $\Sigma(A)$. Write $\log(A) = (B_{i-j})_{i,j\in\mathbb{Z}^d}$. Put $L = (L_{i-j})_{i,j\in\mathbb{Z}^d}$, $M = (M_{i-j})_{i,j\in\mathbb{Z}^d}$ and $D = (D_{i-j})_{i,j\in\mathbb{Z}^d}$, where $L_i = B_i$ and $M_i = B_{-i}$ for $i \succ 0$, $L_0 = M_0 = 1$, $L_i = M_i = 0$ for $i \prec 0$, and $D_0 = B_0$ and $D_i = 0$ for $i \neq 0$. Then*

$$A = \exp(L)\exp(D)\exp(M^*) = \exp(L)\exp(D)[\exp(M)]^* \tag{3.2}$$

is an LDU-factorization of A in $\mathscr{W}^{d,J}_{1,\boldsymbol{\beta}}$.

Theorem 3.2 cannot be generalized to block Toeplitz matrices with blocks of order $k \geqslant 2$, because the final part of its proof requires the property $\exp(T + S) = \exp(T)\exp(S)$ for $k \times k$ matrices T and S, which is only true if T and S commute.

We conclude this section by pointing out that the parts of Theorems 2.3 and 2.5 left unproved are immediate from Theorem 3.2, since the factors in (2.15) and their inverses belong to $\mathscr{W}^{d,J}_{1,\boldsymbol{\beta}}$.

4. The band extension method

Let $E \subset \mathbb{Z}^d$. For every bi-infinite block Toeplitz matrix A, we define the bi-infinite block Toeplitz matrix A^E by

$$[A^E]_i = \begin{cases} A_i, & i \in E, \\ 0, & i \notin E. \end{cases} \tag{4.1}$$

Then a bi-infinite block Toeplitz matrix B is called *E-banded* if $B = A^E$ for some bi-infinite block Toeplitz matrix A.

Let $\preceq$ be a given linear order on $\mathbb{Z}^d$ that is compatible with addition, and let $E \subset \mathbb{Z}^d$ be nonempty. Put $E_+ = \{i \in E : i \succ 0\}$, $E_- = \{i \in E : i \prec 0\}$, $E_+^0 = \{i \in E : i \succeq 0\}$, $E_-^0 = \{i \in E : i \preceq 0\}$ and $E_c = E_+ \cup E_- \cup \{0\}$. If E is *$\preceq$-convex* (i.e., if $l \in E$ whenever $i, j \in E$ and $i \preceq l \preceq j$) and *symmetric* (i.e., if $-E = E$), then $E_c = E$ and the following addition table applies:

	$\mathbb{Z}^d_+\backslash E$	E_+	$\{0\}$	E_-	$\mathbb{Z}^d_-\backslash E$
$\mathbb{Z}^d_+\backslash E$	$\mathbb{Z}^d_+\backslash E$	$\mathbb{Z}^d_+\backslash E$	$\mathbb{Z}^d_+\backslash E$	$\mathbb{Z}^d_+\backslash\{0\}$	$\mathbb{Z}^d$
E_+	$\mathbb{Z}^d_+\backslash E$	$\mathbb{Z}^d_+\backslash\{0\}$	E_+	E_c	$\mathbb{Z}^d_-\backslash\{0\}$
$\{0\}$	$\mathbb{Z}^d_+\backslash E$	E_+	$\{0\}$	E_-	$\mathbb{Z}^d_-\backslash E$
E_-	$\mathbb{Z}^d_+\backslash\{0\}$	E_c	E_-	$\mathbb{Z}^d_-\backslash\{0\}$	$\mathbb{Z}^d_-\backslash E$
$\mathbb{Z}^d_-\backslash E$	$\mathbb{Z}^d$	$\mathbb{Z}^d_-\backslash\{0\}$	$\mathbb{Z}^d_-\backslash E$	$\mathbb{Z}^d_-\backslash E$	$\mathbb{Z}^d_-\backslash E$

For every weight sequence $\boldsymbol{\beta} = (\beta_i)_{i\in\mathbb{Z}^d}$, we now introduce the closed linear subspaces $\mathscr{W}^{d,s}_{k,\boldsymbol{\beta}}$ of the Banach algebra $\mathscr{W}^{d}_{k,\boldsymbol{\beta}}$ by

$$\begin{aligned}
\mathscr{W}^{d,1}_{k,\boldsymbol{\beta}} &:= \left\{A \in \mathscr{W}^{d}_{k,\boldsymbol{\beta}} : A_i = 0 \text{ for } i \notin \mathbb{Z}^d_+\backslash E\right\}, \\
\mathscr{W}^{d,2}_{k,\boldsymbol{\beta}} &:= \left\{A \in \mathscr{W}^{d}_{k,\boldsymbol{\beta}} : A_i = 0 \text{ for } i \notin E_+\right\}, \\
\mathscr{W}^{d,3}_{k,\boldsymbol{\beta}} &:= \left\{A \in \mathscr{W}^{d}_{k,\boldsymbol{\beta}} : A_i = 0 \text{ for } i \neq 0\right\}, \\
\mathscr{W}^{d,4}_{k,\boldsymbol{\beta}} &:= \left\{A \in \mathscr{W}^{d}_{k,\boldsymbol{\beta}} : A_i = 0 \text{ for } i \notin E_-\right\}, \\
\mathscr{W}^{d,5}_{k,\boldsymbol{\beta}} &:= \left\{A \in \mathscr{W}^{d}_{k,\boldsymbol{\beta}} : A_i = 0 \text{ for } i \notin \mathbb{Z}^d_-\backslash E\right\}.
\end{aligned} \tag{4.2}$$

Then we have the ℓ^1-direct sum decompositions

$$\mathscr{W}^d_{k,\boldsymbol{\beta}} = \mathscr{W}^{d,1}_{k,\boldsymbol{\beta}} \oplus \mathscr{W}^{d,c}_{k,\boldsymbol{\beta}} \oplus \mathscr{W}^{d,5}_{k,\boldsymbol{\beta}}, \qquad \mathscr{W}^{d,c}_{k,\boldsymbol{\beta}} = \mathscr{W}^{d,2}_{k,\boldsymbol{\beta}} \oplus \mathscr{W}^{d,3}_{k,\boldsymbol{\beta}} \oplus \mathscr{W}^{d,4}_{k,\boldsymbol{\beta}}. \quad (4.3)$$

We also put

$$\mathscr{W}^d_{k,\boldsymbol{\beta},-} = \mathscr{W}^{d,4}_{k,\boldsymbol{\beta}} \oplus \mathscr{W}^{d,5}_{k,\boldsymbol{\beta}}, \qquad \mathscr{W}^d_{k,\boldsymbol{\beta},+} = \mathscr{W}^{d,1}_{k,\boldsymbol{\beta}} \oplus \mathscr{W}^{d,2}_{k,\boldsymbol{\beta}},$$
$$\mathscr{W}^{d,0}_{k,\boldsymbol{\beta},-} = \mathscr{W}^{d,3}_{k,\boldsymbol{\beta}} \oplus \mathscr{W}^{d,4}_{k,\boldsymbol{\beta}} \oplus \mathscr{W}^{d,5}_{k,\boldsymbol{\beta}}$$

and

$$\mathscr{W}^{d,0}_{k,\boldsymbol{\beta},+} = \mathscr{W}^{d,1}_{k,\boldsymbol{\beta}} \oplus \mathscr{W}^{d,2}_{k,\boldsymbol{\beta}} \oplus \mathscr{W}^{d,3}_{k,\boldsymbol{\beta}}.$$

Assuming E to be $\preceq$-convex and symmetric, we get the multiplication table:

	$\mathscr{W}^{d,1}_{k,\boldsymbol{\beta}}$	$\mathscr{W}^{d,2}_{k,\boldsymbol{\beta}}$	$\mathscr{W}^{d,3}_{k,\boldsymbol{\beta}}$	$\mathscr{W}^{d,4}_{k,\boldsymbol{\beta}}$	$\mathscr{W}^{d,5}_{k,\boldsymbol{\beta}}$
$\mathscr{W}^{d,1}_{k,\boldsymbol{\beta}}$	$\mathscr{W}^{d,1}_{k,\boldsymbol{\beta}}$	$\mathscr{W}^{d,1}_{k,\boldsymbol{\beta}}$	$\mathscr{W}^{d,1}_{k,\boldsymbol{\beta}}$	$\mathscr{W}^{d}_{k,\boldsymbol{\beta},+}$	$\mathscr{W}^{d}_{k,\boldsymbol{\beta}}$
$\mathscr{W}^{d,2}_{k,\boldsymbol{\beta}}$	$\mathscr{W}^{d,1}_{k,\boldsymbol{\beta}}$	$\mathscr{W}^{d}_{k,\boldsymbol{\beta},+}$	$\mathscr{W}^{d,2}_{k,\boldsymbol{\beta}}$	$\mathscr{W}^{d,c}_{k,\boldsymbol{\beta}}$	$\mathscr{W}^{d}_{k,\boldsymbol{\beta},-}$
$\mathscr{W}^{d,3}_{k,\boldsymbol{\beta}}$	$\mathscr{W}^{d,1}_{k,\boldsymbol{\beta}}$	$\mathscr{W}^{d,2}_{k,\boldsymbol{\beta}}$	$\mathscr{W}^{d,3}_{k,\boldsymbol{\beta}}$	$\mathscr{W}^{d,4}_{k,\boldsymbol{\beta}}$	$\mathscr{W}^{d,5}_{k,\boldsymbol{\beta}}$
$\mathscr{W}^{d,4}_{k,\boldsymbol{\beta}}$	$\mathscr{W}^{d}_{k,\boldsymbol{\beta},+}$	$\mathscr{W}^{d,c}_{k,\boldsymbol{\beta}}$	$\mathscr{W}^{d,4}_{k,\boldsymbol{\beta}}$	$\mathscr{W}^{d}_{k,\boldsymbol{\beta},-}$	$\mathscr{W}^{d,5}_{k,\boldsymbol{\beta}}$
$\mathscr{W}^{d,5}_{k,\boldsymbol{\beta}}$	$\mathscr{W}^{d}_{k,\boldsymbol{\beta}}$	$\mathscr{W}^{d}_{k,\boldsymbol{\beta},-}$	$\mathscr{W}^{d,5}_{k,\boldsymbol{\beta}}$	$\mathscr{W}^{d,5}_{k,\boldsymbol{\beta}}$	$\mathscr{W}^{d,5}_{k,\boldsymbol{\beta}}$

We also note that the involution $A \mapsto A^*$ defined by $(A^*)_i = (A_{-i})^*$, $i \in \mathbb{Z}^d$, maps $\mathscr{W}^{d,r}_{k,\boldsymbol{\beta}}$ onto $\mathscr{W}^{d,6-r}_{k,\boldsymbol{\beta}}$ ($r = 1, 2, 3, 4, 5$). Hence, in the terminology of Chapter XXXIV of [15], if E is a nonempty, $\preceq$-convex and symmetric subset of $\mathbb{Z}^d$, then $\mathscr{W}^d_{k,\boldsymbol{\beta}}$ is an algebra with band structure (4.3).

Now let $A \in \mathscr{W}^d_{k,\boldsymbol{\beta}}$ be positive definite (as a bounded linear operator on $\ell^2(\mathbb{Z}^d)$) and let its symbol $\hat{A}(z)$ be a nonsingular $k \times k$ matrix for every $z \in \Omega_{\boldsymbol{\beta}}$. Then A is an invertible element of $\mathscr{W}^d_{k,\boldsymbol{\beta}}$. Then A has a block Cholesky factorization with respect to any linear order $\preceq$ on $\mathbb{Z}^d$ that is compatible with addition, and the factors as well as their inverses belong to $\mathscr{W}^d_{k,\boldsymbol{\beta}}$. Moreover, if A is also E-banded, then the LDU-factors and Cholesky factors of A are E-banded as well, as a result of Theorem 1.3 and Lemma 1.4 of Chapter XXXIV of [15].

Let $A \in \mathscr{W}^d_{k,\boldsymbol{\beta}}$ be E-banded, where E is nonempty, $\preceq$-convex and symmetric. Then by a *positive E-band extension* of A in $\mathscr{W}^d_{k,\boldsymbol{\beta}}$ we mean a bi-infinite block Toeplitz matrix $B \in \mathscr{W}^d_{k,\boldsymbol{\beta}}$ that satisfies $B_j = A_j$ for $j \in E_c$, is positive definite (as a bounded linear operator on $\ell^2(\mathbb{Z}^d)$) and has an E-banded inverse in $\mathscr{W}^d_{k,\boldsymbol{\beta}}$.

The following result plus the subsequent remark are immediate from Theorems 1.1, 1.2 and 2.1 in Chapter XXXIV of [15].

Theorem 4.1. *Let $E \subset \mathbb{Z}^d$ be nonempty, $\preceq$-convex and symmetric, and let $A \in \mathscr{W}^d_{k,\boldsymbol{\beta}}$ be E-banded and satisfy $A = A^*$. Then A has a positive E-band extension in $\mathscr{W}^d_{k,\boldsymbol{\beta}}$ if and only if there exists a (unique) vector $(X_i)_{i \in E^0_+}$ of $k \times k$ matrices such that X_0 is positive definite, $\sum_{i \in E^0_+} \beta_i \|X_i\| < +\infty$, $(X_{i-j})_{i,j \in \mathbb{Z}^d}$ has an inverse element in $\mathscr{W}^{d,0}_{k,\boldsymbol{\beta},+}$, and*

$$\sum_{j \in E^0_+} A_{i-j} X_j = \begin{cases} I_k, & i = 0, \\ 0, & i \in E_+. \end{cases} \tag{4.4}$$

Similarly, A has a positive E-band extension in $\mathscr{W}^d_{k,\boldsymbol{\beta}}$ if and only if there exists a (unique) vector $(Y_i)_{i \in E^0_-}$ of $k \times k$ matrices such that Y_0 is positive definite, $\sum_{i \in E^0_-} \beta_i \|Y_i\| < +\infty$, $(Y_{i-j})_{i,j \in \mathbb{Z}^d}$ has an inverse element in $\mathscr{W}^{d,0}_{k,\boldsymbol{\beta},-}$, and

$$\sum_{j \in E^0_-} A_{i-j} Y_j = \begin{cases} I_k, & i = 0, \\ 0, & i \in E_-. \end{cases} \tag{4.5}$$

Here we use the convention that $X_i = 0$ for $i \notin E^0_+$ and $Y_i = 0$ for $i \in E^0_-$. The E-band extension B is then given by either of the expressions

$$B^{-1} = X \mathscr{D}(X_0^{-1}) X^* = Y \mathscr{D}(Y_0^{-1}) Y^*, \tag{4.6}$$

where X is the E^0_+-banded block Toeplitz matrix with coefficients X_i, Y is the E^0_--banded block Toeplitz matrix with coefficients Y_i, and $\mathscr{D}(X_0^{-1})$ and $\mathscr{D}(Y_0^{-1})$ are the block diagonal matrices with diagonal entries X_0^{-1} and Y_0^{-1}, respectively.

Proof . Let us first point out that $\mathscr{W}^d_{k,\boldsymbol{\beta}}$ can be imbedded into the unital C^*-algebra $\mathscr{R}_{\boldsymbol{\beta}}$ of bounded linear operators on $\ell^2(\mathbb{Z}^d)$ of the form

$$(x_i)_{i \in \mathbb{Z}^d} \mapsto \left(\sum_{j \in \mathbb{Z}^d} A_{i-j} x_j \right)_{i \in \mathbb{Z}^d}$$

whose symbol $\hat{A}(z) = \sum_{i \in \mathbb{Z}^d} z^i A_i$ is norm continuous in $z \in \Omega_{\boldsymbol{\beta}}$. Since $\mathscr{W}^d_{k,\boldsymbol{\beta}}$ is an algebra with band structure (4.3) having $\mathscr{R}_{\boldsymbol{\beta}}$ as its "ambient" algebra, Theorem 4.1 follows directly from Theorems 1.1 and 1.2 in Chapter XXXIV of [15]. It follows from Theorem 1.3 of this chapter that the band extensions found using (4.4) and (4.5) are identical. □

Put $U = X\mathscr{D}(X_0^{-1/2})$ and $V = Y\mathscr{D}(Y_0^{-1/2})$. Then every positive definite extension of A in $\mathscr{W}^d_{k,\boldsymbol{\beta}}$ is of the form [3]

$$\mathscr{F}(G) = (G^*V^* + U^*)^{-1}(I - G^*G)(VG + U)^{-1}, \tag{4.7}$$

where G is an arbitrary element of $\mathscr{W}^{d,1}_{k,\boldsymbol{\beta}}$ satisfying $\sup_{z\in\Omega_{\boldsymbol{\beta}}} \|\hat{G}(z)\| < 1$. Similarly, the positive definite extensions of A can be represented in the form

$$\mathscr{F}'(H) = (H^*U^* + V^*)^{-1}(I - H^*H)(UH + V)^{-1}, \tag{4.8}$$

where H is an arbitrary element of $\mathscr{W}^{d,5}_{k,\boldsymbol{\beta}}$ satisfying $\sup_{z\in\Omega_{\boldsymbol{\beta}}} \|\hat{H}(z)\| < 1$. To prove this result, one employs Theorem 2.1 in Chapter XXXIV of [15] and the paragraph following its statement. The only thing to establish is their Axiom (A), which says that $\mathscr{W}^d_{k,\boldsymbol{\beta}}$ can be imbedded in a unital C^*-algebra $\mathscr{R}$ such that $(I - G)^{-1} \in \mathscr{W}^{d,0}_{k,\boldsymbol{\beta},\pm}$ whenever $G \in \mathscr{W}^d_{k,\boldsymbol{\beta},\pm}$ and $\|G\|_{\mathscr{R}} < 1$. As $\mathscr{R}$, we take the algebra $\mathscr{R}_{\boldsymbol{\beta}}$ defined in the first sentence of the proof of Theorem 4.1. With respect to this unital C^*-algebra $\mathscr{R}$, a given $G \in \mathscr{W}^d_{k,\boldsymbol{\beta},\pm}$ such that $\sup_{z\in\Omega_{\boldsymbol{\beta}}} \|\hat{G}(z)\| < 1$, the element $I - G$ is invertible in $\mathscr{W}^d_{k,\boldsymbol{\beta}}$. To prove that in fact $(I - G)^{-1} \in \mathscr{W}^d_{k,\boldsymbol{\beta},\pm}$, we note that $\sup_{z\in\Omega_{\boldsymbol{\beta}}} \|\lambda\hat{G}(z)\| < 1$ for every $\lambda \in \mathbb{C}$ with $|\lambda| \leqslant 1$. As a result, the Neumann series $\sum_{s=0}^{\infty} \lambda^s G^s$ is absolutely convergent in $\mathscr{W}^d_{k,\boldsymbol{\beta},\pm}$ for every $\lambda \in \mathbb{C}$ with $|\lambda| \leqslant 1$ and hence $(I - G)^{-1} \in \mathscr{W}^{d,0}_{k,\boldsymbol{\beta},\pm}$, which establishes Axiom (A).

If $A \in \mathscr{W}^d_{k,\boldsymbol{\beta}}$ is E-banded for some $\preceq$-convex and symmetric index set E and belongs to $\mathscr{W}^{d,J}_{k,\boldsymbol{\beta}}$ for some additive subgroup J of $\mathbb{Z}^d$, then (4.4) and (4.5) can be replaced by

$$\sum_{j\in E_+^0\cap J} A_{i-j}X_j = \begin{cases} I_k, & i = 0, \\ 0, & i \in E_+ \cap J, \end{cases} \tag{4.9}$$

and

$$\sum_{j\in E_-^0\cap J} A_{i-j}Y_j = \begin{cases} I_k, & i = 0, \\ 0, & i \in E_- \cap J, \end{cases} \tag{4.10}$$

respectively. This is easily understood, since $X_j = 0$ for $j \in E_+^0\backslash J$ and $Y_j = 0$ for $j \in E_-^0\backslash J$. Putting $U = X\mathscr{D}(X_0^{-1/2})$ and $V = Y\mathscr{D}(Y_0^{-1/2})$, all positive definite extensions of A in $\mathscr{W}^{d,J}_{k,\boldsymbol{\beta}}$ are given by either (4.7) or (4.8), where G (resp., H) is an arbitrary element of $\mathscr{W}^{d,1}_{k,\boldsymbol{\beta},J}$ (resp., $\mathscr{W}^{d,5}_{k,\boldsymbol{\beta},J}$) satisfying $\sup_{z\in\Omega^J_{\boldsymbol{\beta}}} \|\hat{G}(z)\| < 1$ (resp., $\sup_{z\in\Omega^J_{\boldsymbol{\beta}}} \|\hat{H}(z)\| < 1$).

The band extension method in $\mathbb{Z}^2$ has been developed before by Bakonyi et al. [2,3]. Similar results were obtained for additive subgroups of $\mathbb{R}$ [29] and $\mathbb{R}^d$ [31],

[3] In this paragraph, the symbol $\|\cdot\|$ stands for the spectral $k \times k$ matrix norm.

where the symbol is a univariate or multivariate almost periodic matrix function of Wiener type with spectrum within the subgroup.

5. Projection method for block Toeplitz matrices

The band extension method discussed in Section 4 can in principle be used to compute the inverses of the factors Ł and Ł* in the Cholesky factorization of a positive definite bi-infinite block Toeplitz matrix A (with respect to a linear order $\preceq$ compatible with addition). In fact, replacing A with the E-banded block Toeplitz matrix A^E defined by (4.1) and then replacing A^E in turn by its E-band extension B, we obtain as an approximation

$$(\text{Ł}^*)^{-1} \approx Y\mathscr{D}(Y_0^{-1/2}), \qquad \text{Ł}^{-1} \approx \mathscr{D}(Y_0^{-1/2})Y^*. \tag{5.1}$$

By the same token, if $\tilde{\text{Ł}}$ and $\tilde{\text{Ł}}^*$ denote the Cholesky factors of A with respect to the inverted order $\preceq_0$ (i.e., $i \preceq_0 j$ if and only if $i \succeq j$), then we have as an approximation

$$(\tilde{\text{Ł}}^*)^{-1} \approx X\mathscr{D}(X_0^{-1/2}), \qquad \tilde{\text{Ł}}^{-1} \approx \mathscr{D}(X_0^{-1/2})X^*. \tag{5.2}$$

Eqs. (5.1) and (5.2) represent better approximations of the inverses of the Cholesky factors of A if we choose E to be a member of a sequence of $\preceq$-convex sets $E^{(n)}$ with $E^{(n)} = -E^{(n)}$ and union $\bigcup_{n\in\mathbb{N}} E^{(n)} = \mathbb{Z}^d$, and let n tend to infinity.

Whereas the band extension leads to accurate numerical results [27] if $d = 1$, the problem for $d \geqslant 2$ is to convert a theoretically valid approximation method into a sequence of operations involving only finite matrices, because for $d \geqslant 2$ one cannot write $\mathbb{Z}^d$ as the union of countably many finite $\preceq$-convex sets (as is possible for $d = 1$). As a result, for $d \geqslant 2$ any method to compute the inverses of the Cholesky factors of A based on the band extension method involves operations with infinite matrices.

We now follow the procedure of the band extension method but choose a countable sequence of finite symmetric sets $E^{(n)}$ with union $\mathbb{Z}^d$, dropping the assumption that each $E^{(n)}$ is $\preceq$-convex. This leads to the projection method, which was first formulated for $d = k = 1$ in [14]. Here we draw on results of [17], in particular the paragraph following the statement of Theorem 4.1. Similar results appeared in [5]. We put $(E^{(n)})^0_+ = \{i \in E^{(n)} : i \succeq 0\}$ and $(E^{(n)})^0_- = \{i \in E^{(n)} : i \preceq 0\}$.

Theorem 5.1. *Let A be a positive definite bi-infinite block Toeplitz matrix in $\mathscr{W}^d_{k,\boldsymbol{\beta}}$, where $\boldsymbol{\beta}$ is a given weight sequence, and let $\hat{A}(z)$ be a nonsingular $k \times k$ matrix for every $z \in \Omega_{\boldsymbol{\beta}}$. Suppose $(E^{(n)})_{n\in\mathbb{N}}$ is a sequence of symmetric sets with union $\bigcup_{n\in\mathbb{N}} E^{(n)} = \mathbb{Z}^d$. Then for sufficiently large n there exist unique vectors $(X_i^{(n)})_{i\in(E^{(n)})^0_+}$ and $(Y_i^{(n)})_{i\in(E^{(n)})^0_-}$ of $k \times k$ matrices such that $X_0^{(n)}$ and $Y_0^{(n)}$ are positive definite, $\sum_{i\in(E^{(n)})^0_+} \beta_i \|X_i^{(n)}\|$ and $\sum_{i\in(E^{(n)})^0_-} \beta_i \|Y_i^{(n)}\|$ are finite,*

$$\sum_{j\in(E^{(n)})^0_+} A_{i-j}X_j^{(n)} = \begin{cases} I_k, & i=0,\\ 0, & i\in(E^{(n)})_+,\end{cases} \tag{5.3}$$

and

$$\sum_{j\in(E^{(n)})^0_-} A_{i-j}Y_j^{(n)} = \begin{cases} I_k, & i=0,\\ 0, & i\in(E^{(n)})_-.\end{cases} \tag{5.4}$$

Moreover, there exist vectors $(X_i)_{i\in\mathbb{Z}^d_+}$ *and* $(Y_i)_{i\in\mathbb{Z}^d_-}$ *of* $k\times k$ *matrices such that*

$$\begin{aligned}\lim_{n\to\infty}&\sum_{i\in(E^{(n)})^0_+}\left(\beta_i\|X_i-X_i^{(n)}\|\right)^2\\ &=\lim_{n\to\infty}\sum_{i\in(E^{(n)})^0_-}\left(\beta_i\|Y_i-Y_i^{(n)}\|\right)^2=0.\end{aligned} \tag{5.5}$$

Further,

$$A^{-1}=X\mathscr{D}(X_0^{-1})X^*=Y\mathscr{D}(Y_0^{-1})Y^*, \tag{5.6}$$

where the triangular matrices $X=(X_{i-j})_{i,j\in\mathbb{Z}^d}$ *and* $Y=(Y_{j-i})_{i,j\in\mathbb{Z}^d}$ *belong to* $\mathscr{W}^d_{k,\boldsymbol{\beta}}$. [4]

Proof . Let us apply the projection method to either of the linear systems

$$\sum_{j\in\mathbb{Z}^d_+} A_{i-j}X_j = \begin{cases} I_k, & i=0,\\ 0, & i\succ 0,\end{cases} \tag{5.7}$$

defined on the Hilbert space $\ell^2_{k,\boldsymbol{\beta}}(\mathbb{Z}^d_+)$, and

$$\sum_{j\in\mathbb{Z}^d_-} A_{i-j}Y_j = \begin{cases} I_k, & i=0,\\ 0, & i\prec 0,\end{cases} \tag{5.7$'$}$$

defined on the Hilbert space $\ell^2_{k,\boldsymbol{\beta}}(\mathbb{Z}^d_-)$. Let us define the projections $P^{(n)}_\pm$ on $\ell^2_{k,\boldsymbol{\beta}}(\mathbb{Z}^d_\pm)$ as follows:

$$\left(P^{(n)}_\pm(x_j)_{j\in\mathbb{Z}^d_\pm}\right)_i = \begin{cases} x_i, & i\in(E^{(n)})_\pm,\\ 0, & i\in\mathbb{Z}^d_\pm\setminus(E^{(n)})_\pm.\end{cases}$$

Then $P^{(n)}_\pm$ converges strongly to the identity operator on $\ell^2_{k,\boldsymbol{\beta}}(\mathbb{Z}^d_\pm)$ as $n\to\infty$.

If all the weights $\beta_i\equiv 1$, the infinite system matrices in (5.7) and (5.7′) are self-adjoint and the projections $P^{(n)}_\pm$ are orthogonal. Then the projection method can be applied (see the sufficient condition following Theorem II 2.1 of [14]) and the

[4] We used the convention $X_i=0$ for $i\prec 0$ and $Y_i=0$ for $i\succ 0$.

conclusions of Theorem 5.1 are immediate. For more general weights, consider the commutative diagram

$$\begin{array}{ccc} \ell^2_{k,\boldsymbol{\beta}}(\mathbb{Z}^d_\pm) & \stackrel{D_\pm}{\longrightarrow} & \ell^2_k(\mathbb{Z}^d_\pm) \\ \Big\downarrow A_\pm & & \Big\downarrow T_\pm \\ \ell^2_{k,\boldsymbol{\beta}}(\mathbb{Z}^d_\pm) & \stackrel{D_\pm}{\longrightarrow} & \ell^2_k(\mathbb{Z}^d_\pm) \end{array}$$

where $A_\pm$ is the compression of A to $\ell^2_{k,\boldsymbol{\beta}}(\mathbb{Z}^d_\pm)$ and $D_\pm : \ell^2_{k,\boldsymbol{\beta}}(\mathbb{Z}^d_\pm) \to \ell^2_k(\mathbb{Z}^d_\pm)$ is the unitary operator defined by $(D_\pm(x_j)_{j\in\mathbb{Z}^d})_i = \beta_i x_i$. Clearly, since $D(\mathbb{Z}^d_\pm)$ commutes with each $P^{(n)}_\pm$, the projections $P^{(n)}_\pm$ are also selfadjoint on $\ell^2_{k,\boldsymbol{\beta}}(\mathbb{Z}^d_\pm)$ and converge to the identity operator on $\ell^2_{k,\boldsymbol{\beta}}(\mathbb{Z}^d_\pm)$. Now note that the real part of $T_\pm$, given by

$$\frac{1}{2}(T_{i,j} + T^*_{j,i}) = \frac{1}{2}\left(\frac{\beta_i}{\beta_j} + \frac{\beta_j}{\beta_i}\right) A_{i-j},$$

is positive selfadjoint. We may then employ the same result of [14], followed by an application of Theorem II 2.2 of [14], to prove the applicability of the projection method and hence the validity of Theorem 5.1 in the case of general weights. □

When applying Theorem 5.1 to a sequence of $\preceq$-convex symmetric sets $E^{(n)}$, one finds a justification of the band extension method as described in Section 4 and in the first paragraph of Section 5. However, as explained above, for $d \geqslant 2$ the sets $E^{(n)}$ are infinite for sufficiently large n.

When applying Theorem 5.1 to finite sets $E^{(n)}$, one obtains a numerically implementable method for computing the inverses of the Cholesky factors of a positive definite bi-infinite block Toeplitz matrix. However, for $d \geqslant 2$ and sufficiently large n the sets $E^{(n)}$ are not $\preceq$-convex, and hence the finite linear systems (5.3) and (5.4) are not finite multi-index Toeplitz systems.

6. Example

Let us consider the bi-infinite Toeplitz matrix A with entries $A_{(0,0)} = 2\zeta$, $A_{(-1,0)} = A_{(1,0)} = A_{(0,-1)} = A_{(0,1)} = -1$, and $A_i = 0$ otherwise, where $\zeta > 2$. Then its symbol is positive

$$\hat{A}(z) = 2[\zeta - \cos(\theta_1) - \cos(\theta_2)] > 0, \quad z = (\mathrm{e}^{\mathrm{i}\theta_1}, \mathrm{e}^{\mathrm{i}\theta_2}) \in \mathbb{T}^2. \tag{6.1}$$

Let $E^N = \{i = (i_1, i_2) \in \mathbb{Z}^2 : |i_1| \leqslant N,\ |i_2| \leqslant N\}$. The set E^N is symmetric and, with respect to the lexicographical order $\preceq$,

$$E^N_+ = \left\{i \in \mathbb{Z}^2 : (i_1 = 0 \text{ and } 0 \leqslant i_2 \leqslant N) \text{ or } (1 \leqslant i_1 \leqslant N \text{ and } |i_2| \leqslant N)\right\}. \tag{6.2}$$

In this case the linear system (5.3) takes the form

$$B\boldsymbol{x} = \boldsymbol{e}_0, \tag{6.3}$$

where

$$B = \begin{bmatrix} H_{N+1} & \begin{bmatrix} 0 & -I_{N+1} \end{bmatrix} & & & & \\ \begin{bmatrix} 0 \\ -I_{N+1} \end{bmatrix} & H_{2N+1} & -I_{2N+1} & & & \\ & -I_{2N+1} & H_{2N+1} & -I_{2N+1} & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & -I_{2N+1} \\ & & & & -I_{2N+1} & H_{2N+1} \end{bmatrix},$$

$\boldsymbol{x} = (x_0, x_1, \ldots, x_m)^{\mathrm{T}} \in \mathbb{R}^{m+1}$ and $\boldsymbol{e}_0 = (1, 0, \ldots, 0)^{\mathrm{T}} \in \mathbb{R}^{m+1}$, with $m = 2N(N+1)$. Here I_j stands for the identity matrix of order j, the dimensions of the various zero matrices are not indicated and $H_j = H_j(\zeta)$ is the tridiagonal matrix of order j defined by

$$H_j(\zeta) = \begin{bmatrix} 2\zeta & -1 & & & & \\ -1 & 2\zeta & -1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & -1 \\ & & & & -1 & 2\zeta \end{bmatrix}. \tag{6.4}$$

Recalling the definition of the Chebyshev polynomials of the second kind $U_j(\zeta) = \sin((j+1)t)/\sin t$ for $\zeta = \cos t \in [-1, 1]$ and $U_j(\zeta) = \sinh((j+1)t)/\sinh t$ for $\zeta = \cosh t \in (1, \infty)$, we easily derive that $\det H_j(\zeta) = U_j(\zeta) \neq 0$.

Let us write the system (6.3) in the form

$$\begin{bmatrix} H_{N+1} & C_N \\ C_N^{\mathrm{T}} & S_N \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_0 \\ \boldsymbol{x}_+ \end{bmatrix} = \begin{bmatrix} \boldsymbol{e}_0 \\ 0 \end{bmatrix}, \tag{6.5}$$

where $\boldsymbol{x}_0 = \operatorname{col}[x_j]_{j=0}^N$, $\boldsymbol{x}_+ = \operatorname{col}[x_j]_{j=N+1}^m$, $\boldsymbol{e}_0 = \operatorname{col}[\delta_{j,0}]_{j=0}^N$,

$$C_N = \begin{bmatrix} 0_{(N+1),N} & -I_{N+1} & 0_{(N+1),(N-1)(2N+1)} \end{bmatrix},$$

where the dimensions of the zero matrices have been indicated, and S_N is the block tridiagonal $N \times N$ matrix with diagonal blocks H_{2N+1} and off-diagonal blocks $-I_{N+1}$. Using the recurrence relation for the Chebyshev polynomials of the second kind, $-U_{j-1}(\zeta) + 2\zeta U_j(\zeta) - U_{j+1}(\zeta) = 0$ with $U_{-1}(\zeta) \equiv 0$ and $U_0(\zeta) = 1$, we obtain

$$S_N \operatorname{col}[U_{N-j}(\tilde{H}_{2N+1})]_{j=1}^N = \operatorname{col}[\delta_{j,1} I_{2N+1}]_{j=1}^N U_N(\tilde{H}_{2N+1}), \tag{6.6}$$

where the matrix $\tilde{H}_{2N+1} = \frac{1}{2} H_{2N+1}$ has been substituted into various scalar polynomials. Since the set of eigenvalues of $\tilde{H}_{2N+1}$ is given by

$$\sigma(\tilde{H}_{2N+1}) = \left\{\zeta - \cos\left(\frac{j\pi}{2N+2}\right) : j = 1, 2, \ldots, 2N+1\right\}, \tag{6.7}$$

$\zeta > 2$ and $U_N(z) \neq 0$ for $z \geqslant 1$, the matrix $U_N(\tilde{H}_{2N+1})$ is nonsingular. [5] Hence, S_N is nonsingular [16] and the first column (of the $N \times N$ matrix having entries of order $2N+1$) of its inverse is given by

$$S_N^{-1}\,\mathrm{col}\,[\delta_{j,1} I_{2N+1}]_{j=1}^{N} = \mathrm{col}\,[U_{N-j}(\tilde{H}_{2N+1})]_{j=1}^{N} U_N(\tilde{H}_{2N+1})^{-1}. \tag{6.8}$$

It now follows from straightforward calculation that the 2×2 block matrix in (6.5) is nonsingular if and only if the $(N+1) \times (N+1)$ matrix

$$\begin{aligned} M &= H_{N+1} - C_N S_N^{-1} C_N^{\mathrm{T}} \\ &= \begin{bmatrix} 0_{N+1,N} & I_{N+1} \end{bmatrix} \left(H_{2N+1} - U_{N-1}(\tilde{H}_{2N+1}) U_N(\tilde{H}_{2N+1})^{-1}\right) \begin{bmatrix} 0_{N,N+1} \\ I_{N+1} \end{bmatrix} \end{aligned} \tag{6.9}$$

is nonsingular. In that case the inverse is given by

$$\begin{bmatrix} M^{-1} & -M^{-1} C_N S_N^{-1} \\ -S_N^{-1} C_N^{\mathrm{T}} M^{-1} & S_N^{-1} + S_N^{-1} C_N^{\mathrm{T}} M^{-1} C_N S_N^{-1} \end{bmatrix}. \tag{6.10}$$

The solution of (6.5) is the first column of the $(m+1) \times (m+1)$ matrix in (6.10), where $m = N + N(2N+1)$. Since the eigenvalues of $\tilde{H}_{2N+1}$ all belong to $(1, +\infty)$ (because $\zeta > 2$), we can employ the relation (based on the monotonicity of $\sinh\tau$ for $\tau > 0$)

$$\begin{aligned} f_N(z) &= z - \frac{U_{N-1}(z/2)}{U_N(z/2)} \\ &= 2\cosh t - \frac{\sinh(Nt)}{\sinh((N+1)t)} > 0, \quad z = 2\cosh t > 1, \end{aligned} \tag{6.11}$$

to prove that the matrix M in (6.9) is positive definite and hence nonsingular. Note that $f_N(1^+) = (N+2)/(N+1)$, $f_N(z) < z$ and $f_N(z) \sim z$ as $z \to +\infty$, while

$$\begin{aligned} f_N'(z) = 1 &+ \frac{\cosh(Nt)\cosh((N+1)t)}{2\sinh t \sinh^2((N+1)t)} N(N+1) \\ &\times \left[\frac{\tanh(Nt)}{N} - \frac{\tanh((N+1)t)}{N+1}\right] \end{aligned}$$

is positive. We also remark that the matrix $-S_N^{-1} C_N^{\mathrm{T}}$ (of dimension $(m+1) \times (N+1)$) in the left lower corner of (6.10) consists exactly of the $(N+1)$th up to the $(2N+1)$th column of the matrix given by (6.8). Moreover, the condition number

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[5] Its eigenvalues are the numbers $U_N\left(\zeta - \cos\left(\frac{j\pi}{2N+2}\right)\right)$, $j = 1, 2, \ldots, 2N+1$.

of M (with respect to the spectral norm) is bounded above by that of $H_{2N+1} - U_{N-1}(\tilde{H}_{2N+1})U_N(\tilde{H}_{2N+1})^{-1}$, which equals

$$\frac{f_N\left(U_N\left(2\zeta+2\cos[\pi/(2N+2)]\right)\right)}{f_N\left(U_N\left(2\zeta-2\cos[\pi/(2N+2)]\right)\right)}, \tag{6.12}$$

which behaves as $C(\zeta)[(\zeta+1)/(\zeta-1)]^N$ as $N \to \infty$ for fixed $\zeta > 2$ ([35], 8.21.9 for $\alpha = \beta = \frac{1}{2}$, in combination with $f_N(z) \sim z$ as $z \to \infty$).

Finally, since $A_{-i} = A_i$ for $i \in \mathbb{Z}^d$, the solution of the system (5.4) is given by the vector $\operatorname{col}[x_{-j}]_{j\in(E^{(n)})^0_-}$, where $\operatorname{col}[x_j]_{j\in(E^{(n)})^0_+}$ is the solution of (5.3).

When taking the limit as $N \to \infty$, one gets the lower triangular Toeplitz matrix $X = (x_{i-j})_{i,j\in\mathbb{Z}^d}$, with $x_k = 0$ whenever $k \prec 0$, and the Cholesky factorization

$$A^{-1} = ŁŁ^{\mathrm{T}} = Ł^{\mathrm{T}}Ł, \tag{6.13}$$

where Ł is the lower triangular matrix $(x_{i-j}x_0^{-1/2})_{i,j\in\mathbb{Z}^d}$.

The computation of the solution of system (6.3) can be greatly improved, both in terms of stability and of computational complexity, by resorting to the factorization

$$\tilde{H}_{2N+1} = VDV^{\mathrm{T}}, \tag{6.14}$$

where D is the diagonal matrix of the eigenvalues of $\tilde{H}_{2N+1}$, given by (6.7), and V is the orthogonal matrix formed by its eigenvectors, whose entries are

$$V_{ij} = \frac{1}{\sqrt{N+1}}\sin\frac{ij\pi}{2N+2}, \quad i, j = 1, \ldots, 2N+1.$$

Substituting (6.14) in (6.9), we get

$$M = \begin{bmatrix} 0_{N+1,N} & I_{N+1} \end{bmatrix} V \left(2D - U_{N-1}(D)U_N(D)^{-1}\right) V^{\mathrm{T}} \begin{bmatrix} 0_{N,N+1} \\ I_{N+1} \end{bmatrix},$$

so that the matrix M can be computed by evaluating the scalar Chebyshev polynomials on the eigenvalues of $\tilde{H}_{2N+1}$ and by applying two sine transforms.

Finally, the matrix $-S_N^{-1}C_N$, necessary to obtain the lower part $\boldsymbol{x}_+$ of vector $\boldsymbol{x}$, is given by the last $N+1$ columns of the matrix

$$\operatorname{diag}(V, V, \ldots, V)\operatorname{col}[U_{N-j}(D)]_{j=1}^N U_N(D)^{-1}V^{\mathrm{T}}.$$

In order to give an idea of the numerical performance of the method just illustrated, in Fig. 1 we depicted $\log_{10}|x_{(i_1,i_2)}|$ for $(i_1, i_2) \in E_+^N$ and $N = 40$, where $x_k = x_{(i_1,i_2)}$, for $k = (2N+1)i_1 + i_2$ and $k = 0, 1, \ldots, m$, are the solutions of system (6.3) obtained by the above algorithm. Fig. 1 clearly shows that only a small number (890) of the components of $\boldsymbol{x}$ have an absolute value exceeding computer precision and also that they decay exponentially with respect to $|i| = |i_1| + |i_2|$.

As $k = 1$, the Cholesky factorization of A^{-1} can also be obtained by the method due to Krein, illustrated in Theorem 3.2. For a comparison of the computational effectiveness of the two numerical methods, in Fig. 2 we reported the values of $\log_{10}|x_{(i_1,i_2)}|$, for $(i_1, i_2) \in E_+^{40}$, obtained by Krein's method.

Even though the numerical results are both acceptable, the algorithm based on the projection method leads to more accurate results. In fact, as is evident in Fig. 2, our implementation of Krein's algorithm raises to a value close to the machine precision *eps* all the components of $\boldsymbol{x}$ whose absolute values are smaller than or equal to *eps* (roughly 10^{-16} in double precision). The reason for this is that the *fast Fourier transform* (FFT), on which our implementation of Krein's algorithm heavily depends, does not distinguish between quantities whose difference does not exceed the computer floating point precision in absolute value.

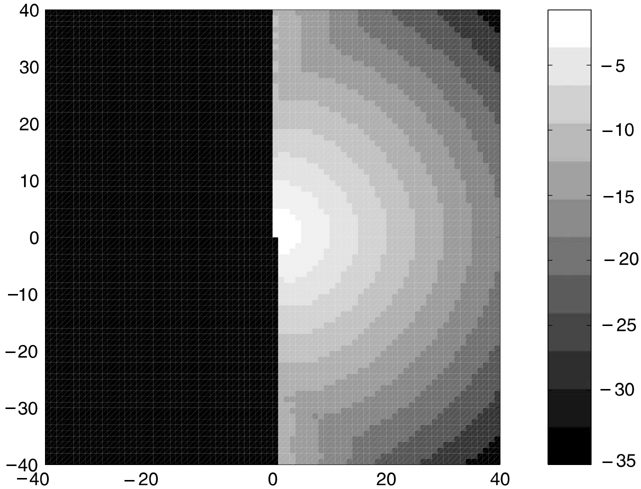

Fig. 1. $\log_{10}|x_{(i_1,i_2)}|$, projection method.

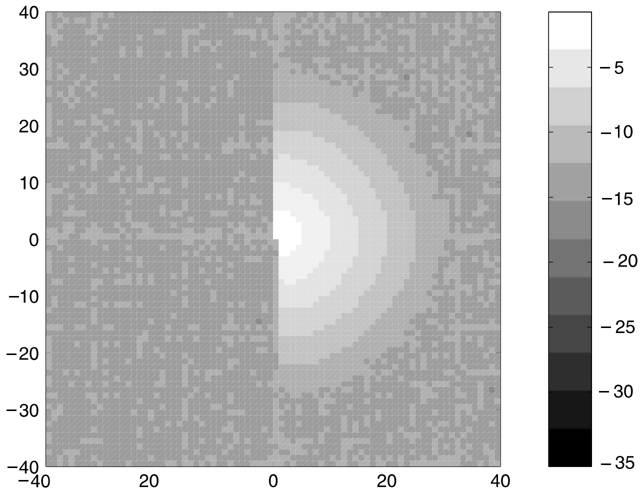

Fig. 2. $\log_{10}|x_{(i_1,i_2)}|$, Krein's method.

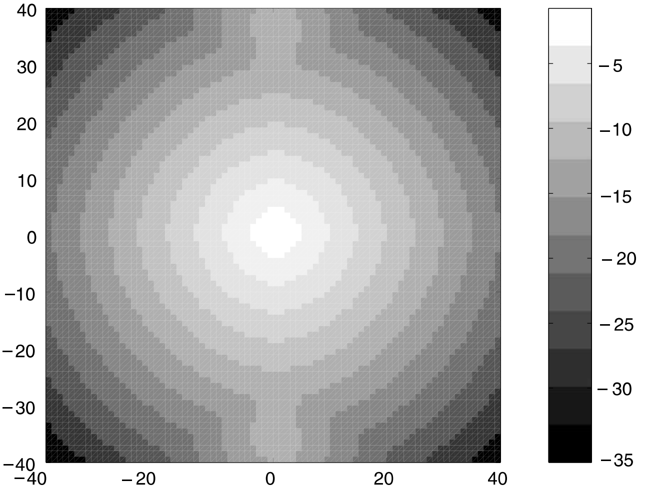

Fig. 3. $\log_{10}|(A^{-1})_{(i_1,i_2)}|$, projection method.

In general it may be impossible to simplify system (6.3) in order to reduce complexity and memory storage, as in this example. So, even though less accurate, Krein's method might be the most convenient algorithm to deal with in a scalar multi-index factorization problem.

Finally, Fig. 3, reporting the values of $\log_{10}|(A^{-1})_{(i_1,i_2)}|$ for $(i_1, i_2) \in E_+^{40}$, shows that, as should be expected, A^{-1} decays exponentially. In this figure, the deviation from radial symmetry along the vertical axis is a numerical effect, due to the columnwise lexicographical ordering chosen.

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