TRANSPORT THEORY AND STATISTICAL PHYSICS, 30(1), 63–90 (2001)

TIME-DEPENDENT KINETIC EQUATIONS WITH COLLISION TERMS RELATIVELY BOUNDED WITH RESPECT TO THE COLLISION FREQUENCY

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ABSTRACT

In this article the existence and uniqueness theory of time dependent kinetic equations is developed for collision terms dominated in the norm by the collision frequency, thus generalizing prior work by Beals and Protopopescu.

1 INTRODUCTION

In this article we study initial-boundary value problems of the type

$$\frac{\partial u}{\partial t}(x, v, t) + v\frac{\partial u}{\partial x} + a(x, v, t)\frac{\partial u}{\partial v} + h(x, v, t)u(x, v, t)$$
$$= (Ju)(x, v, t) + f(x, v, t), \quad (x, v, t) \in \Omega \times \mathcal{V} \times \mathbb{R}_+; \tag{1.1}$$

$$u(x, v, 0) = g_0(x, v), \quad (x, v) \in \Omega \times \mathcal{V}; \tag{1.2}$$

$$u_-(x, v, t) = (Ku_+)(x, v, t) + g_-(x, v, t), \quad (x, v, t) \in \Sigma_- \times \mathbb{R}_+; \tag{1.3}$$

where the position $x \in \Omega$ (Ω a region in $\mathbb{R}^n$), the velocity $v \in \mathcal{V}$ ($\mathcal{V}$ a region in $\mathbb{R}^n$), the time $t \geq 0$, and $\Sigma_\pm$ are the parts of the boundary Σ of the phase space $\Omega \times \mathcal{V}$ where the integral curves of the vector field $a(x, v, t)$ enter and leave. A comprehensive theory of the existence and uniqueness of solutions of Eqs. (1.1)–(1.3) in an L_p-space setting has been developed by Beals and Protopopescu [3]. This theory can also be found in Chapter XI of [16]. In this theory the major assumptions on Eqs. (1.1)–(1.3) are as follows:

1. The vector field $a(x, v, t)$ is Lipschitz continuous and divergence free;
2. The function $h(x, v, t)$ is nonnegative and locally integrable, the operator J is bounded, and the operator K has norm strictly less than 1. If J and K are positive operators (in lattice sense), K is allowed to have unit norm;
3. The operators J and K are real (i.e., for all u, $J\overline{u} = \overline{Ju}$ and $K\overline{u} = \overline{Ku}$) and local in time (i.e., they commute with the multiplication by any bounded measurable function of t only);
4. The integral curves of the vector field $a(x, v, t)$ do not reach infinity in finite time. This condition is always satisfied if $|a(x, v, t)| \leq$ const. $(1 + |x| + |v|)$ for $(x, v, t) \in \Omega \times \mathcal{V} \times \mathbb{R}_+$.

Assuming a phase space $\Omega \times \mathcal{V}$ equipped with a Borel measure μ and a vector field independent of t and writing

$$X = v \cdot \frac{\partial}{\partial x} + a(x, v) \cdot \frac{\partial}{\partial v}, \tag{1.4}$$

the fact that the vector field is divergence free may be expressed through the Green's identity

$$\int_{\Omega \times \mathcal{V}} X\phi \, d\mu = \int_{\Sigma_+} \phi \, d\nu_+ - \int_{\Sigma_-} \phi \, d\nu_- \tag{1.5}$$

for ϕ in a suitable test function space, where $\nu_\pm$ are suitable measures on $\Sigma_\pm$. After constructing the boundary measures $\nu^\pm$ and the test function space pertaining to the vector field $Y = (\partial/\partial t) + X$, Eqs. (1.1)–(1.3) with $J = 0$ and $K = 0$ reduce to ordinary first order differential equations along the integral curves of Y which can be solved trivially. Two perturbation arguments then allow one to incorporate a bounded J and K with $\|K\| < 1$ into the theory. If J and K are positive operators, a monotonicity argument allows one to extend the existence and uniqueness result to operators K of unit norm. We mention that important earlier work on Eqs. (1.1)–(1.3) was done by Voigt [31] for the case where $a \equiv 0$ and $J = 0$, and Ukai [30] for $J = 0$, while

Cessenat [8, 9] derived trace theorems for $a = 0$. The non divergence free case was pioneered, for $h = 0$ and $K = 0$, by Bardos [2], and the results of [3, 16] were extended to the non divergence free case by van der Mee [21]. In addition to these papers, there exist hundreds of papers dealing with particular examples of Eqs. (1.1)–(1.3), but discussing them is beyond the scope of this article.

Let us devote a few words to the physical meaning of Eqs. (1.1)–(1.3). Equations (1.1)–(1.3) describe the time evolution of the distribution function $u(x, v, t)$ of particles in a phase space of functions of position and velocity (x, v), or the time evolution of the specific intensity $u(x, v, t)$ of unpolarized light in a phase space of functions of position and frequency (x, v), given the initial distribution $u_0(x, v)$ and subject to the reflection law (1.3). The function $f(x, v, t)$ describes internal sources and the function $g_0(x, v, t)$ describes particles or radiation entering the medium at the boundary. The term containing $a(x, v, t)$ accounts for the effect of external forces.

Let us now discuss some of the open problems regarding Eqs. (1.1)–(1.3). First, there are a number of problems in which it is useful to enlarge the scope of [3, 16]. In problems where the natural functional space is $L_1(\Omega \times \mathcal{V}; d\mu)$ and some equilibrium condition demands that

$$\int_{\Omega\times\mathcal{V}} \{hu - Ju\}\, d\mu = 0, \quad u \in L_1(\Omega \times \mathcal{V}; d\mu). \tag{1.6}$$

the theory in an L_1-setting should be extended to deal with J for which $\|hu\|_1 = \|Ju\|_1$. Further, in various applications with unbounded $h \geq 0$ one deals with positive J satisfying $\|Ju\|_1 \leq \delta\|hu\|_1$ for some $\delta \in [0, 1]$. To mention a few, in neutron transport the collision frequency dominates the collision kernel integrated over outgoing velocities if the medium is nonmultiplying [5]. In radiative transfer, the phase function integrated over outgoing directions is dominated by the extinction coefficient [7, 29]. In cell growth modeling [28], electron transport in weakly ionized gases [15], rarefied gas dynamics [6], and modeling of electron-phonon interaction in semiconductors [18, 19], the integrated (nonnegative) collision kernel is exactly equal to the collision frequency. In fact, Eq. (1.6) is the linear counterpart of the balance condition involved in the nonlinear Boltzmann equation.

Secondly, once an existence and uniqueness theory of Eqs. (1.1)–(1.3) has been put in place, the consideration of various moments of the solutions $u(x, v, t)$ makes it mandatory to also study Eqs. (1.1)–(1.3) in suitably weighted L_p-spaces. However, on adding a weight to the phase space measure μ one usually turns a divergence free vector field X for which the Green's identity (1.5) is valid into a vector field for which (1.5) cannot be formulated any more. To overcome this difficulty, one may either use the far more complicated theory involving non divergence free vector fields or employ the solutions of Eqs. (1.1)–(1.3) already found in the weighted L_p-setting. The latter requires an enhancement of the theory with estimates of solutions in suitably weighted L_p-norms.

In this article we will not study all of these questions in full generality. We will restrict ourselves to the following situation:

1. The vector field $a(x, v, t)$ is Lipschitz continuous and divergence free and the phase space $\Omega \times \mathcal{V}$ has piecewise C^1 boundary;
2. The function $h(x, v, t)$ is nonnegative and locally integrable, $J = J_1 + J_2$, $\|J_1 u\|_1 \leq \delta \|hu\|_1$ for some $\delta \in [0, 1]$, $\|J_2 u\|_1 \leq j\|u\|_1$ for some $j \geq 0$, and K is a contraction from $L_1(\Sigma_+; d\nu_+)$ into $L_1(\Sigma_-; d\nu_-)$;
3. The operators J_1, J_2 and K are real (i.e., for all u, $J_1\overline{u} = \overline{J_1 u}$, $J_2\overline{u} = \overline{J_2 u}$ and $K\overline{u} = \overline{Ku}$) and local in time (i.e., they commute with the multiplication by any bounded measurable function of t only);
4. The integral curves of the vector field $a(x, v, t)$ do not reach infinity in finite time.

Alongside we study the analogous L_p-problem where $1 < p < +\infty$. Here the condition on J_1 is that $\|J_1 u\|_p \leq \delta \|h|u|^p\|_1^{1/p}$ for some $\delta \geq 0$. Here there is no restriction on the size of δ.

If h is unbounded, the operator J_1 is unbounded and has the L_p-functions u for which $h^{1/p}u$ is also an L_p-function, as its domain. Hence, when h is bounded, the theory developed in this article reduces to the theory expounded in [3].

We now briefly describe the organization of the paper. In Section 2 we develop the vector field formalism, in analogy with Sections XI.2 and XI.3 of [16]. As far as Eqs. (1.1)–(1.3) are concerned, this formalism can be copied verbatim from [3, 16]. In Section 3 we prove the existence and uniqueness of the solution of Eqs. (1.1)–(1.3). The incorporation of K proceeds exactly as in [3, 16], but with J one has to deal differently. To maintain a clear exposition of the theory, we only discuss the L_1-case in the main text and defer the discussion of the general L_p-case to Appendix A. In Section 4 we prove that the solution is generated by an evolution family in the sense of [23] (or by a C_0-semigroup if a, h, J and K are time independent). In Section 5 the existence and uniqueness theory of Eqs. (1.1)–(1.3) is adapted to deal with suitably weighted L_p-norms. Our insights on Eqs. (1.1)–(1.3) in weighted L_p-spaces then allow us to derive certain results under multiplying boundary conditions (i.e., for $\|K\| > 1$), using a weight function proposed by Boulanouar [4].

Section 6, divided into two subsections, is devoted entirely to applications. Subsection 6.1 deals with an equation describing the time evolution of the electron distribution in a weakly ionized gas, studied before by Frosali et al. [15], Poupaud [25, 26], and Arlotti et al. [1]. In Subsection 6.2 we briefly discuss the linearized Boltzmann equation studied by Pettersson [24], Chvála [11] and by Chvála et al. [12] on general spatial domains, incorporating boundary conditions.

Two appendices have been included. In Appendix A the L_p-counterpart of the contents of Section 3 is presented. In Appendix B an auxiliary result on strong limits of evolution families on $L_p(E, d\mu)$ is established.

2 PRELIMINARIES

Let Ω be an open subset of $\mathbb{R}^n$, let $\mathcal{V}$ be a subset of $\mathbb{R}^n$ equipped with a positive Borel measure μ_0 such that all bounded Borel sets in $\mathbb{R}^n$ have finite μ_0-measure, and denote by μ and μ_T the product measures $d\mu(x, v) = dx d\mu_0(v)$ on $\Omega \times \mathcal{V}$ and $d\mu_T(x, v, t) = dx\, d\mu_0(v)\, dt$ on $\Omega \times \mathcal{V} \times [0, T]$, respectively. When Ω and $\mathcal{V}$ have boundaries, they are assumed to be piecewise C^1. In typical applications $\mathcal{V} = \mathcal{V}_0 \times S^{n-1}$ and $d\mu_0(v) = d\mu_{00}(|v|)d\theta$, where μ_{00} is a measure on a subset $\mathcal{V}_0$ of $\mathbb{R}_+$ and $d\theta$ is the surface Lebesgue measure on the unit sphere S^{n-1} of $\mathbb{R}^n$. We assume that for each $T > 0$, h is μ_T-integrable on every bounded μ_T-measurable subset of $\Lambda_T = \Omega \times \mathcal{V} \times (0, T)$.

Let us assume that $a(x, v, t)$ is continuous in (x, v, t) and locally Lipschitz continuous in x and v on the closure of $\Omega \times \mathcal{V} \times \mathbb{R}_+$, and let us define the vector field Y by

$$Y = \frac{\partial}{\partial t} + v \cdot \frac{\partial}{\partial x} + a(x, v, t) \cdot \frac{\partial}{\partial v},$$

and suppose that for each $T > 0$ and each C^1-function ϕ of compact support in Λ_T

$$\int_{\Lambda_T} Y\phi\, d\mu_T = 0,$$

meaning that Y is divergence free. Then through every point of Λ_T there passes exactly one integral curve of Y, and this curve has length $\leq T$ and has left and right limits on the boundary of Λ_T.[1] The left endpoints form the

[1]In this article, as in [3, 16], "length" stands for travel time and does not necessarily have the same meaning as the usual arclength.

incoming boundary Σ_T^- and the right endpoints the outgoing boundary Σ_T^+. Every point of $x \in \Lambda_T$ may thus be parametrized as (z, s) where $z \in \Sigma_T^-$ is the left endpoint of the integral curve passing through x and s is the time needed to travel from the left endpoint to x along the integral curve. We have for the measure space (Λ_T, μ_T)

$$\Lambda_T \simeq \{(z, s)\colon z \in \Sigma_T^-, 0 < s < \ell(z)\}, \quad d\mu_T = d\nu^-(z)\, ds,$$

where ds is the travel time measure, $\ell(z)$ $(\leq T)$ is the total length of the integral curve starting at z and ν^- is a Borel measure on Σ_T^-. Using the integral curves to map $(\Sigma_T^-, d\nu^-)$ in a measure preserving fashion onto $(\Sigma_T^+, d\nu^+)$ we obtain the Green's identity

$$\int_{\Lambda_T} Yu\, d\mu_T = \int_{\Sigma_T^+} u\, d\nu^+ - \int_{\Sigma_T^-} u\, d\nu^-, \quad u \in \Phi_T,$$

where Φ_T is the test function space of all Borel functions u on Λ_T such that (i) u is continuously differentiable on each integral curve, (ii) u and Yu are bounded, and (iii) the support of u is bounded and the length of the integral curves meeting the support of u is bounded away from zero.

Let us define $L_{p,\mathrm{loc}}(\Lambda_T; d\mu_T)$ as the linear space of all μ_T-measurable functions u on Λ_T such that $|u|^p$ is μ_T-integrable on every bounded μ_T-measurable subset of Λ_T on which $\ell(z, s) \equiv \ell(z)$ is bounded away from zero. Then if $u, Yu \in L_p(\Lambda_T; d\mu_T)$, we define a trace for u as a pair of functions $u^\pm \in L_{p,\mathrm{loc}}(\Sigma_T^\pm; d\nu^\pm)$ such that for each $\phi \in \Phi_T$

$$< Yu, \phi > + < u, Y\phi > = \int_{\Sigma_T^+} u^+ \phi\, d\nu^+ - \int_{\Sigma_T^-} u^- \phi\, d\nu^-.$$

Then if $1 \leq p < \infty$ and $\{u, (Y + h)u\} \subset L_p(\Lambda_T, d\mu_T)$, u has a unique trace $u^\pm$. Moreover, if $u^- \in L_p(\Sigma_T^-; d\nu^-)$, then $u^+ \in L_p(\Sigma_T^+; d\nu^+)$, $h|u|^p$ and $|u|^{p-1} Yu$ are μ_T-integrable and

$$\begin{aligned} &\int_{\Sigma_T^+} |u^+|^p d\nu^+ + p \int_{\Lambda_T} h|u|^p d\mu_T \\ &\quad = \int_{\Sigma_T^-} |u^-|^p d\nu^- + p \int_{\Lambda_T} \mathrm{sgn}(u)|u|^{p-1}(Y + h)u\, du_T. \end{aligned} \tag{2.1}$$

The present formalism may be unfamiliar to whomever pursues kinetic theory applications, where usually the time variable on the one hand and the spatial and velocity variables on the other hand are treated as mutually alien species. However, when examined more closely, one sees that, up to sets of measure zero,

$$\Sigma^- = \{(x, v, 0)\colon (x, v) \in \Omega \times \mathcal{V}\} \cup \{(x, v, t)\colon (x, v) \in \Sigma_{-,t}, t \in (0, T)\};$$
$$\Sigma^+ = \{(x, v, T)\colon (x, v) \in \Omega \times \mathcal{V}\} \cup \{(x, v, t)\colon (x, v) \in \Sigma_{+,t}, t \in (0, T)\},$$

where $X_t = v(\partial/\partial x) + a(x, v, t)(\partial/\partial v)$, and

$$\int_{\Omega\times\mathcal{V}} X_t \phi \, d\mu = \int_{\Sigma_{+,t}} \phi \, d\nu_{+,t} - \int_{\Sigma_{-,t}} \phi \, d\nu_{-,t}.$$

If a does not depend on t, the subscripts t on X_t, $\Sigma_{\pm,t}$ and $\nu_{\pm,t}$ may be dropped, while $d\mu_T = d\mu \, dt$ and $d\nu^\pm = (d\mu, d\nu_\pm dt)$.

3 USING THE METHOD OF CHARACTERISTICS

Let us suppose that J and K are real (i.e., $J\overline{u} = \overline{Ju}$ and $K\overline{u} = \overline{Ku}$, $u \in L_p(\Lambda_T; d\mu_T)$) and local in time (i.e., for all $u \in L_p(\Lambda_T; d\mu_T)$, $J(\varphi u) = \varphi Ju$ and $K(\varphi u) = \varphi Ku$ for any bounded measurable function $\varphi(t)$ of t only). In analogy with [3, 16], we set $u_\lambda(x, v, t) = e^{-\lambda t}u(x, v, t)$, $f_\lambda(x, v, t) = e^{-\lambda t}f(x, v, t)$, $g_{\lambda,0}(x, v, t) = e^{-\lambda t}g_0(x, v, t)$ and $g_{\lambda,-}(x, v, t) = e^{-\lambda t}g_-(x, v, t)$, where $\lambda > 0$. Since J and K are local in time, Eqs. (1.1)–(1.3) are transformed to the initial-boundary value problem

$$\frac{\partial u_\lambda}{\partial t}(x, v, t) + v \cdot \frac{\partial u_\lambda}{\partial x} + a(x, v, t)\frac{\partial u_\lambda}{\partial v} + \{h(x, v, t) + \lambda\}u_\lambda(x, v, t)$$
$$= (Ju_\lambda)(x, v, t) + f_\lambda(x, v, t), \quad (x, v, t) \in \Omega \times \mathcal{V} \times \mathbb{R}_+; \tag{3.1}$$

$$u_\lambda(x, v, 0) = g_{0,\lambda}(x, v), \quad (x, v) \in \Omega \times \mathcal{V}; \tag{3.2}$$

$$u_{-,\lambda}(x, v, t) = (Ku_{+,\lambda})(x, v, t) + g_{-,\lambda}(x, v, t), \quad (x, v, t) \in \Sigma_- \times \mathbb{R}_+. \tag{3.3}$$

To treat the temporal, spatial and velocity variables on an equal footing as far as the initial and boundary conditions are concerned, we introduce $\mathcal{K} = (0, K)$, allowing one to write (3.2) and (3.3) together as $u_\lambda^- = \mathcal{K}u_\lambda^+ + g_\lambda^-$.

Theorem 3.1. *The initial-boundary value problem (3.1)–(3.3) has a unique solution* $u_\lambda \in L_1(\Lambda_T; d\mu_T)$ *for every* $f \in L_1(\Lambda_T; d\mu_T)$ *and* $(g_0, g_-) \in L_1(\Sigma_T^-; d\nu^-)$*, provided* $J = J_1 + J_2$ *with*

$$\|J_1 u_\lambda\|_1 \le \delta\|hu_\lambda\|_1, \quad \|J_2 u_\lambda\|_1 \le j\|u_\lambda\|_1, \quad \|\mathcal{K}u_\lambda^+\|_1 \le \kappa\|u_\lambda^+\|_1,$$

for certain $\delta, \kappa \in [0,1)$ and $\lambda > j \geq 0$. Then the solution u_λ satisfies the conditions $hu_\lambda \in L_1(\Lambda_T; d\mu_T)$ and $u_\lambda^\pm \in L_1(\Sigma_T^\pm; d\nu^\pm)$, and is nonnegative if J_1, J_2, $\mathcal{K}$, f and (g_0, g_-) are nonnegative.

Proof. Integrating Eqs. (3.1)–(3.3) with $J = 0$ and $\mathcal{K} = 0$ along integral curves, we find a unique solution $u_\lambda = S_\lambda(f_\lambda, g_\lambda^-)$, where $g_\lambda^- = (g_{0,\lambda}, g_{-,\lambda})$ and

$$\begin{aligned} \|S_\lambda(f_\lambda, g_\lambda^-)\|_1 &\leq \frac{1}{\lambda}(\|f_\lambda\|_1 + \|g_\lambda^-\|_1); \\ \|S_\lambda(f_\lambda, g_\lambda^-)^+\|_1 &\leq \|f_\lambda\|_1 + \|g_\lambda^-\|_1; \\ \|hS_\lambda(f_\lambda, g_\lambda^-)\|_1 &\leq \|f_\lambda\|_1 + \|g_\lambda^-\|_1. \end{aligned}$$

Now suppose $J = 0$ and $\kappa \in [0,1)$. Then any solution of Eqs. (3.1)–(3.3) satisfies $u_\lambda = S_\lambda(f_\lambda, \mathcal{K}u_\lambda^+ + g_\lambda^-)$, where

$$u_\lambda^+ = S_\lambda(0, \mathcal{K}u_\lambda^+)^+ + S_\lambda(f_\lambda, g_\lambda^-)^+.$$

Since $\|S_\lambda(0, \mathcal{K}u_\lambda^+)^+\|_1 \leq \|\mathcal{K}u_\lambda^+\|_1 \leq \kappa\|u_\lambda^+\|_1$, a contraction mapping argument yields $u_\lambda^+ \in L_1(\Sigma_T^+; d\nu^+)$ uniquely. We denote the so-obtained solution by $u_\lambda = Z_\lambda(f_\lambda, g_\lambda^-)$. We then have

$$\begin{aligned} \|hZ_\lambda(f_\lambda, g_\lambda^-)\|_1 &\leq \|f_\lambda\|_1 + \|g_\lambda^-\|_1 + \kappa\|u_\lambda^+\|_1 \leq \frac{1}{1-\kappa}(\|f_\lambda\|_1 + \|g_\lambda^-\|_1); \\ \|Z_\lambda(f_\lambda, g_\lambda^-)^+\|_1 &\leq \|f_\lambda\|_1 + \|g_\lambda^-\|_1 + \kappa\|u_\lambda^+\|_1 \leq \frac{1}{1-\kappa}(\|f_\lambda\|_1 + \|g_\lambda^-\|_1); \\ \|Z_\lambda(f_\lambda, g_\lambda^-)\|_1 &\leq \lambda^{-1}(\|f_\lambda\|_1 + \|g_\lambda^-\|_1 + \kappa\|u_\lambda^+\|_1) \leq \frac{1}{\lambda(1-\kappa)}(\|f_\lambda\|_1 + \|g_\lambda^-\|_1). \end{aligned}$$

Let us now consider Eqs. (3.1)–(3.3) for $J = J_1$ and $\mathcal{K}$ with $\delta + \kappa < 1$. Then any solution u_λ satisfies

$$u_\lambda = Z_\lambda(J_1 u_\lambda + f_\lambda, g_\lambda^-) = Z_\lambda(J_1 u_\lambda, 0) + Z_\lambda(f_\lambda, g_\lambda^-).$$

Moreover, since

$$\|hZ_\lambda(J_1 u_\lambda, 0)\|_1 \leq \frac{1}{1-\kappa}\|J_1 u_\lambda\|_1 \leq \frac{\delta}{1-\kappa}\|hu_\lambda\|_1,$$

a contraction mapping argument yields the existence of u_λ if $\delta + \kappa < 1$. We denote the so-obtained solution by $u_\lambda = W_\lambda(f_\lambda, g_\lambda^-)$. Then

$$\|hW_\lambda(f_\lambda, g_\lambda^-)\|_1 \leq \frac{\delta\|hu_\lambda\|_1 + \|f_\lambda\|_1 + \|g_\lambda^-\|_1}{1-\kappa} \leq \frac{\|f_\lambda\|_1 + \|g_\lambda^-\|_1}{1-\delta-\kappa}; \quad (3.4)$$

$$\|W_\lambda(f_\lambda, g_\lambda^-)^+\|_1 \leq \frac{\delta\|hu_\lambda\|_1 + \|f_\lambda\|_1 + \|g_\lambda^-\|_1}{1-\kappa} \leq \frac{\|f_\lambda\|_1 + \|g_\lambda^-\|_1}{1-\delta-\kappa}; \quad (3.5)$$

$$\|W_\lambda(f_\lambda, g_\lambda^-)\|_1 \le \frac{\delta\|hu_\lambda\|_1 + \|f_\lambda\|_1 + \|g_\lambda^-\|_1}{\lambda(1-\kappa)} \le \frac{\|f_\lambda\|_1 + \|g_\lambda^-\|_1}{\lambda(1-\delta-\kappa)}, \quad (3.6)$$

where (3.4) has been used to derive Eqs. (3.5) and (3.6).

Applying (2.1) to Eqs. (3.1–3.3) with $J = J_1$ while changing $h \mapsto h + \lambda$, we find

$$\|u_\lambda^+\|_1 + \|hu_\lambda\|_1 + \lambda\|u_\lambda\|_1 \le \|\mathcal{K}u_\lambda^+\|_1 + \|g_\lambda^-\|_1 + \|J_1 u_\lambda\|_1 + \|f_\lambda\|_1, \quad (3.7)$$

where the equality sign occurs if J_1 and $\mathcal{K}$ are positive operators, $f_\lambda \ge 0$ and $g_\lambda^- \ge 0$. Hence

$$(1-\kappa)\|u_\lambda^+\|_1 + (1-\delta)\|hu_\lambda\|_1 + \lambda\|u_\lambda\|_1 \le \|f_\lambda\|_1 + \|g_\lambda^-\|_1, \quad (3.8)$$

which suggests that the restriction to $\delta, \kappa \in [0, 1)$ with $\delta + \kappa < 1$ is not necessary.

Let us now extend the above estimates for $W_\lambda(f_\lambda, g_\lambda^-)$ to the case where $\delta < 1$ and $\kappa < 1$, without assuming that $\delta + \kappa < 1$. Now choose $\kappa_0, \kappa_1 \in [0, 1)$ such that $\kappa = \kappa_0 + \kappa_1$ and $\delta + \kappa_0 < 1$, and let $u_\lambda = V_\lambda(f_\lambda, g_\lambda^-)$ denote the solution of Eqs. (3.1)–(3.3) with $\mathcal{K}$ replaced by $(\kappa_0/\kappa)\mathcal{K}$ and $J = J_1$. Replacing $\mathcal{K}$ by $(\kappa_0/\kappa)\mathcal{K}$ and observing that the latter boundary operator has norm κ_0 and that $\delta + \kappa_0 < 1$, we obtain from Eqs. (3.4)–(3.6) the bounds

$$\|hV_\lambda(f_\lambda, g_\lambda^-)\|_1 \le \frac{\|f_\lambda\|_1 + \|g_\lambda^-\|_1}{1-\delta-\kappa_0};$$

$$\|V_\lambda(f_\lambda, g_\lambda^-)^+\|_1 \le \frac{\|f_\lambda\|_1 + \|g_\lambda^-\|_1}{1-\delta-\kappa_0};$$

$$\|V_\lambda(f_\lambda, g_\lambda^-)\|_1 \le \frac{\|f_\lambda\|_1 + \|g_\lambda^-\|_1}{\lambda(1-\delta-\kappa_0)}.$$

Now observe that the solution of Eqs. (3.1)–(3.3) with $J = J_1$ has the form

$$u_\lambda = V_\lambda(f_\lambda, (\kappa_1/\kappa)\mathcal{K}u_\lambda^+ + g_\lambda^-) = V_\lambda(0, (\kappa_1/\kappa)\mathcal{K}u_\lambda^+) + V_\lambda(f_\lambda, g_\lambda^-).$$

Since Eq. (3.8) (applied for $(\kappa_0/\kappa)\mathcal{K}$ instead of $\mathcal{K}$, and hence with κ_0 taking the place of κ) implies that

$$\left\| V_\lambda\left(0, \frac{\kappa_1}{\kappa}\mathcal{K}u_\lambda^+\right)^+ \right\|_1 \le \frac{(\kappa_1/\kappa)\|\mathcal{K}u_\lambda^+\|_1}{1-\kappa_0} \le \frac{\kappa_1}{1-\kappa_0}\|u_\lambda^+\|_1,$$

and since $(\kappa_1/(1-\kappa_0)) < 1$, a contraction argument yields the existence of the solution u_λ of Eqs. (3.1)–(3.3) for $J = J_1$. As a result of Eq. (3.8), we now obtain the three estimates

$$\|hW_\lambda(f_\lambda, g_\lambda^-)\|_1 \leq \frac{\|f_\lambda\|_1 + \|g_\lambda^-\|_1}{1-\delta}; \tag{3.9}$$

$$\|W_\lambda(f_\lambda, g_\lambda^-)^+\|_1 \leq \frac{\|f_\lambda\|_1 + \|g_\lambda^-\|_1}{1-\kappa}; \tag{3.10}$$

$$\|W_\lambda(f_\lambda, g_\lambda^-)\|_1 \leq \frac{\|f_\lambda\|_1 + \|g_\lambda^-\|_1}{\lambda}, \tag{3.11}$$

valid under the hypothesis that $\delta, \kappa \in [0, 1)$.

Considering the full problem (3.1)–(3.3), we must solve u_λ from the equation

$$u_\lambda = W_\lambda(J_2 u_\lambda, 0) + W_\lambda(f_\lambda, g_\lambda^-).$$

Since Eq. (3.11) implies that $\|W_\lambda(J_2 u_\lambda, 0)\|_1 \leq (1/\lambda)\|J_2 u_\lambda\|_1 \leq (j/\lambda)\|u_\lambda\|_1$, we apply a contraction mapping argument for $\lambda > j$ to obtain the solution. As a result, for $\lambda > j$ we have

$$(1-\kappa)\|u_\lambda^+\|_1 + (1-\delta)\|hu_\lambda\|_1 + (\lambda - j)\|u_\lambda\|_1 \leq \|f_\lambda\|_1 + \|g_\lambda^-\|_1. \tag{3.12}$$

The general solution u_λ satisfies the estimates (3.9) and (3.10) (which actually apply to the case $J = J_1$), but Eq. (3.11) is to be modified by replacing the denominator λ by $\lambda - j$. This completes the proof.

Using the monotonicity argument of [3] and Section XI.5 of [16], we obtain

Theorem 3.2. *The initial-boundary value problem (3.1)–(3.3) has a unique solution* $u_\lambda \in L_1(\Lambda_T; d\mu_T)$ *for every* $f \in L_1(\Lambda_T; d\mu_T)$ *and* $(g_0, g_-) \in L_1(\Sigma_T^-; d\nu^-)$, *provided* $J = J_1 + J_2$, J_1, J_2 *and* $\mathcal{K}$ *are positive operators and*

$$\|J_1 u_\lambda\|_1 \leq \delta\|hu_\lambda\|_1, \quad \|\mathcal{K}u_\lambda^+\|_1 \leq \kappa\|u_\lambda^+\|_1, \quad \|J_2 u_\lambda\|_1 \leq j\|u_\lambda\|_1,$$

for certain $\delta, \kappa \in [0, 1]$ *and* $j \geq 0$.

In general, under the conditions of Theorem 3.2 the solution u of Eqs. (3.1)–(3.3) need not satisfy $hu \in L_1(\Lambda_T; d\mu_T)$ if $\delta = 1$. Moreover, if $\kappa = 1$, the trace $u^\pm$ of the solution may no longer belong to $L_1(\Sigma_T^\pm; d\nu^\pm)$.

Decomposing $u_\lambda^\pm$ in temporal and spatial-velocity sections before making the conversion from (3.7) to (3.12) and recalling that $\mathcal{K} = (0, K)$, we get

$$\|u_\lambda(t = T)\|_1 + (1-k)\|u_{\lambda,+}\|_1 + (1-\delta)\|hu_\lambda\|_1 + (\lambda - j)\|u_\lambda\|_1$$

$$\leq \|f_\lambda\|_1 + \|g_0, \lambda\|_1 + \|g_{-,\lambda}\|_1$$

Thus if $J = J_1$ (and hence $j = 0$) and $\lambda = 0$, we obtain the contractivity property

$$\|u(t = T)\|_1 \leq \|f\|_1 + \|g_0\|_1 + \|g_-\|_1.$$

An interesting special case occurs if $\lambda = 0$, $J = J_1 \geq 0$ and $\mathcal{K} \geq 0$, while

$$\|Ju\|_1 = \|hu\|_1, \quad u \geq 0 \text{ in } L_1(\Gamma_T; d\mu_T); \tag{3.13}$$

$$\|\mathcal{K}u^+\|_1 = \|u^+\|_1, \quad u^+ \geq 0 \text{ in } L_1\left(\sum_T^+; d\nu^+\right). \tag{3.14}$$

When premultiplying J by $\delta \in (0, 1)$ and $\mathcal{K}$ by $\kappa \in (0, 1)$, we find the equality

$$\|u^{(\delta,\kappa)}(t = T)\|_1 + (1 - \kappa)\|u_+^{(\delta,\kappa)}\|_1 + (1 - \delta)\|hu^{(\delta,\kappa)}\|_1 = \|f\|_1 + \|g_0\|_1 + \|g_-\|_1,$$

where f, g_0 and g_- are assumed to be nonnegative. Here the solution $u^{(\delta,\kappa)}$ is monotonically nondecreasing in δ and κ. So if we let $\delta \uparrow 1$ and $\kappa \uparrow 1$, we find the isometry condition

$$\|u^{(\delta=1,\kappa=1)}(t = T)\|_1 = \|f\|_1 + \|g_0\|_1 + \|g_-\|_1, \tag{3.15}$$

provided the terms having the factors $1 - \kappa$ and $1 - \delta$ vanish in the limit. This is automatic if Eqs. (1.1)–(1.3) with $\delta = \kappa = 1$ have a solution $u \in L_1(\Lambda_T; d\mu_T)$ satisfying $u^+ \in L_1(\Sigma_T^+; d\nu^+)$ and $hu \in L_1(\Lambda_T; d\mu_T)$. We may then simply pass to the limit as $\delta \uparrow 1$ and $\kappa \uparrow 1$, using that $\|hu^{(\delta,\kappa)}\|_1 \leq \|hu\|_1 < \infty$ and $\|u^{(\delta,\kappa)^+}\|_1 \leq \|u^+\|_1 < \infty$.

The situation as described in the preceding paragraph occurs in various applications, such as electron transport in weakly ionized gases [15], cell growth modeling [20], and modeling of electron-phonon interactions in semiconductors [19]. In these papers, sufficient conditions for having the isometry condition (3.15) are derived and the possibility of not having (3.15) is discussed, though without providing explicit models in which the isometry relation (3.15) is not satisfied.

We have

Theorem 3.3. *Suppose J and $\mathcal{K}$ are positive operators satisfying (3.13) for every $u \geq 0$ in $L_1(\Lambda_T; d\mu_T)$ and (3.14) for every $u^+ \geq 0$ in $L_1(\Sigma_T^+; d\nu^+)$. Then for nonnegative f, g_0 and g_- the solution of Eqs. (1.1)–(1.3) satisfies the isometry condition (3.15) if the following conditions are fulfilled:*

1. *$hu \in L_1(\Lambda_T; d\mu_T)$;*
2. *$u^+ \in L_1(\Sigma_T^+; d\nu^+)$.*

This occurs if

a. *the integral curves do not ever meet the boundary of the phase space Σ, and*

b. *for some finite constant C independent of the choice of* $z \in \Sigma_T^-$, *we have*

$$\int_0^{\ell(z)} h(z,s)\,ds \leq C. \tag{3.16}$$

Proof. The first part of the theorem has actually been proven in the few lines following (3.15). Let us therefore focus on the second part of the theorem in which the sufficient conditions (a) and (b) are assumed.

As explained in the first few lines of Section 3, the solution $u_{\lambda=0}$ of Eqs. (3.1)–(3.3) with $\lambda = 0$ exists. Unfortunately, the estimates derived in the proof of Theorem 3.1 do not apply if $\lambda = 0$. In order to derive such estimates if (3.16) holds, we follow the various steps of the proof of Theorem 3.1 for $\lambda = 0$. To simplify notation, we drop the subscript $\lambda = 0$ throughout the present proof. Because of condition (a), we only have to deal with temporal boundaries. We therefore have $\mathcal{K} = 0$ (and hence $\kappa = 0$), $g^- = g_0$, $u^- = u(t=0) = u_0$, and $u^+ = u(t=T) = u_T$.

Let us first devote one paragraph to derive two estimates that also hold if condition (a) is not assumed. Let $u = S(f, g^-)$ be the solution of Eqs. (3.1)–(3.3) for $\lambda = 0$, $\mathcal{K} = 0$ and $J = 0$. Then the explicit expression

$$u(z,s) = e^{-\int_0^s h(z,t)\,dt} g^- + \int_0^s e^{-\int_\tau^s h(z,t)\,dt} f(z,\tau)\,d\tau$$

immediately gives the estimates

$$\|S(f,g^-)^+\|_1 \leq \|f\|_1 + \|g^-\|_1; \tag{3.17}$$

$$\|hS(f,g^-)\|_1 \leq (1 - e^{-C})(\|f\|_1 + \|g^-\|_1). \tag{3.18}$$

Let us again assume both of the conditions (a) and (b). Consider Eq. (3.1) with initial condition (3.2) for $\lambda = 0$ and $J = J_1$ with $\delta(1 - e^{-C}) < 1$. Then any solution u satisfies

$$u = S(J_1 u + f, g_0) = S(J_1 u, 0) + S(f, g_0).$$

Since (3.17) implies that

$$\|hS(J_1 u, 0)\|_1 \leq (1 - e^{-C})\|J_1 u\|_1 \leq \delta(1 - e^{-C})\|hu\|_1,$$

a contraction mapping argument yields the existence of $u = W(f, g_0)$. Using (3.17) and (3.18), we then get the estimates

$$\|hW(f,g_0)\|_1 \leq \frac{1 - e^{-C}}{1 - \delta(1 - e^{-C})}(\|f\|_1 + \|g_0\|_1);$$

$$\|W(f,g_0)_T\|_1 \leq \frac{1}{1 - \delta(1 - e^{-C})}(\|f\|_1 + \|g_0\|_1).$$

Finally, premultiplying $J = J_1$ by δ and denoting the so-obtained solution by $u^{(\delta)}$, we obtain

$$\|u^{(\delta)}(t = T)\|_1 + (1 - \delta)\|hu^{(\delta)}\|_1 = \|f\|_1 + \|g_0\|_1,$$

implying that $hu^{(\delta)} \in L_1(\Lambda_T; d\mu_T)$ if $\delta(1 - e^{-C}) < 1$. Since $\|hu^{(\delta)}\|_1 \leq \|hu^{(1)}\|_1 < \infty$, we find the isometry condition 3.15.

The L_p-case with $p \in (1, +\infty)$ can be treated similarly, assuming that $\mathcal{K}$ is a bounded operator from $L_p(\Sigma_T^+; d\nu^+)$ into $L_p(\Sigma_T^-; d\nu^-)$ with norm $\kappa \in [0, 1)$ and $J = J_1 + J_2$ with J_2 bounded on $L_p(\Lambda_T; d\mu_T)$ and J_1 satisfying the norm estimate $\|J_1 u\|_p \leq \delta\|h|u|^p\|_1^{1/p}$ for some $\delta \geq 0$. Taking sufficiently large λ the existence of a unique solution u_λ of Eqs. (3.1)–(3.3) can be proved and this solution satisfies $h|u_\lambda|^p \in L_1(\Lambda_T; d\mu_T)$ and $u_\lambda^\pm \in L_p(\Sigma_T^\pm; d\nu^\pm)$. We remark that for $p > 1$ there is no restriction on the size of $\delta \in \mathbb{R}^+$. This is due to the presence of the factor $\lambda^{-1+1/p}$ in the right-hand side of Eq. (XI 4.9) of [16], which leads to the norm estimates involving strict contractions for sufficiently large λ required in the proof. The details will be worked out in Appendix A.

4 USING AN EVOLUTION SYSTEM

In this section we prove that the solution of Eqs. (3.1)–(3.3) is representable by means of an evolution family. By an *evolution family* on a Banach space $\mathcal{X}$ we mean a family of boundary linear operators $U(t, t_0)$, $t \geq t_0$, on $\mathcal{X}$ such that

1. for $t_0 \in \mathbb{R}$ we have $U(t_0, t_0) = I$, the identity operator;
2. for $t \geq r \geq t_0$ we have the product rule $U(t, r)U(r, t_0) = U(t, t_0)$;
3. for every $\xi \in \mathcal{X}$, $U(t, t_0)\xi$ is a continuous function of $(t, t_0) \in \Delta$;
4. there exist constants M, ω such that $\|U(t, t_0)\| \leq Me^{\omega(t-t_0)}$ for $(t, t_0) \in \Delta$.

Here we have defined $\Delta = \{(t_1, t_2) \in \mathbb{R}^2\colon t_1 \geq t_2\}$. Evolution families are studied in detail in [23, 10].

Consider Eqs. (1.1)–(1.3) with $J = 0$ and $K = 0$, where $f \in L_1(\Lambda_T; d\mu_T)$ and $(g_0, g_-) \in L_1(\Sigma_T^-; d\nu^-)$. Then the unique solution is given by

$$u(z, s) = e^{-\int_0^s h(z,t)\,dt} u(z, 0) + \int_0^s e^{-\int_\tau^s h(z,t)\,dt} f(z, \tau)\, d\tau,$$

where the phase space Λ_T has been decomposed along integral curves and $u(z,0) = (g_0, g_-)$. Let us choose s, s_1 such that $0 < s < s_1 < \ell(z)$. Then

$$\begin{aligned} u(z,s) - u(z,s_1) &= e^{-\int_0^s h(z,t)\,dt}\Big[1 - e^{-\int_s^{s_1} h(z,t)\,dt}\Big]u(z,0) \\ &\quad + \int_0^s e^{-\int_\tau^s h(z,t)\,dt}\Big[1 - e^{-\int_s^{s_1} h(z,t)\,dt}\Big] f(z,\tau)\,d\tau \\ &\quad + \int_s^{s_1} e^{-\int_\tau^{s_1} h(z,t)\,dt} f(z,\tau)\,d\tau, \end{aligned}$$

where $h \in L_{1,\mathrm{loc}}(\Lambda_T; d\mu_T)$ for every $T > 0$. Hence a dominated convergence argument shows that

$$\int_{\Sigma_T^-} |u(z,s) - u(z,s_1)|\,d\nu^-(z) \tag{4.1}$$

tends to zero as $(s_1 - s) \downarrow 0$. The integrand in (4.1) is set equal to zero on that part of the incoming boundary Σ_T^- where $\ell(z) < s_1$.

When dealing with Eqs. (3.1)–(3.3) with general J and $\mathcal{K}$, the above argument remains the same, because f and g^- are replaced by $Ju_\lambda + f$ and $\mathcal{K}u_\lambda^+ + g^-$, respectively, and h is replaced by $h + \lambda$, provided $Ju_\lambda + f \in L_1(\Lambda_T; d\mu_T)$, and $\mathcal{K}u_\lambda^+ + g^- \in L_1(\Sigma_T^-; d\nu^-)$. This is the case if $\{hu_\lambda, u_\lambda\} \subset L_1(\Lambda_T; d\mu_T)$ and $u_\lambda^+ \in L_1(\Sigma_T^+; d\nu^+)$. Thus under the conditions of either Theorem 3.1 or Theorem 3.3, the solution of Eqs. (3.1)–(3.3) is strongly continuous in $L_1(\Lambda_T; d\mu_T)$.

We have

Proposition 4.1. *Under the conditions of either Theorem 3.1 or Theorem 3.3, the solution of Eqs. (3.1)–(3.3) is strongly continuous in* $L_1(\Lambda_T; d\mu_T)$.

Under the conditions of either Theorem 3.1 or Theorem 3.3, the unique solution of the initial-boundary value problem

$$\begin{aligned} &\frac{\partial u_\lambda}{\partial t}(x,v,t) + v \cdot \frac{\partial u_\lambda}{\partial x} + a(x,v,t)\frac{\partial u_\lambda}{\partial v} \\ &\quad + \{h(x,v,t) + \lambda\}u_\lambda(x,v,t) = (Ju_\lambda)(x,v,t), \quad (x,v,t) \in \Omega \times \mathcal{V} \times (t_0, T); \end{aligned}$$

$$\begin{aligned} u_\lambda(x,v,t_0) &= g_{t_0,\lambda}(x,v), && (x,v) \in \Omega \times \mathcal{V}; \\ u_{-,\lambda}(x,v,t) &= (Ku_{+,\lambda})(x,v,t), && (x,v,t) \in \sum\nolimits_- \times (t_0, T), \end{aligned}$$

can be written as $u_\lambda(x,v,t) = [\mathcal{S}_\lambda(t,t_0)g_{t_0,\lambda}](x,v)$, where $\mathcal{S}_\lambda(t,t_0)$, $t \geq t_0 \geq 0$, is an evolution family on $L_1(\Lambda_T; d\mu_T)$. In particular, if a, h, J and K do not depend on time, the solution of Eqs. (3.1)–(3.3) is generated by a strongly continuous semigroup on $L_1(\Lambda_T; d\mu_T)$. It is easily verified that the same result is found in the L_p-case under the conditions of Theorem A.1.

If a, h, J and K do not depend on time, the proof that the solution of Eqs. (1.1)–(1.3) is generated by a strongly continuous semigroup can also be given using the Hille–Yosida theorem [23]. For that purpose one studies the boundary value problem

$$v\frac{\partial u_\lambda}{\partial x} + a(x,v)\frac{\partial u_\lambda}{\partial v} + \{h(x,v)+\lambda\}u_\lambda(x,v)$$
$$= (Ju_\lambda)(x,v) + f_\lambda(x,v), \quad (x,v) \in \Omega \times \mathcal{V};$$

$$u_{-,\lambda}(x,v) = (Ku_{+,\lambda})(x,v) + g_{-,\lambda}(x,v), \quad (x,v) \in \Sigma_-,$$

where $\lambda \geq 0$ and h, a, J and K are independent of t. Then defining X as in (1.4) and supposing that every maximal integral curve of X whose extension to the left or right is finite has a corresponding left or right endpoint belonging to the set

$$\{(x,v) \in \partial(\Omega \times \mathcal{V}) \colon a(x,v) \neq 0 \text{ if } v = 0\},$$

we have the Green's identity (1.5), where ϕ belongs to the test function space Φ of Borel functions on $\Omega \times \mathcal{V}$ such that (i) ϕ is continuously differentiable on each integral curve of X, (ii) ϕ and $X\phi$ are bounded, and (iii) the support of ϕ is bounded and the lengths of the integral curves meeting the support of ϕ is bounded away from zero. Defining $L_{p,\mathrm{loc}}(\Omega \times \mathcal{V}; d\mu)$ and the trace as in Section 2 we obtain, for $1 \leq p < +\infty$, the existence of a unique trace $u_\pm \in L_{p,\mathrm{loc}}(\Omega \times \mathcal{V}; d\mu)$ if $\{u, (X+h)\} \subset L_p(\Omega \times \mathcal{V}; d\mu)$; moreover, if $u_- \in L_p(\Sigma_-; d\nu_-)$, we have $u_+ \in L_p(\Sigma_+; d\nu_+)$, $\{h|u|^p, |u|^{p-1}Xu\} \subset L_1(\Omega \times \mathcal{V}; d\mu)$, and

$$\int_{\Sigma_+} |u_+|^p d\nu_+ + p\int\int_{\Omega\times\nu} h|u|^p \, d\mu$$
$$= \int_{\Sigma_-} |u_-|^p d\nu_- + p\int_{\Omega\times\nu} \mathrm{sgn}(u)|u|^{p-1}(X+h)u \, d\mu.$$

From that point the proof of the existence of a unique solution of Eqs. (1.1)–(1.3) appears to be completely analogous to the proof of Theorems 3.1 and 3.2 (if $p = 1$) or Theorem A.1 (if $1 < p < +\infty$), as has been given without detailed explanation in [3, 16]. However, the description of the integral curves of X may be quite different from the description of the characteristics of Y where each maximal integral curve is parametrized by $s \in (0, \ell(z))$ with $\ell(z) \in (0, T]$ for every $z \in \Sigma_-$. For instance, there may be semi-infinite integral curves with only a left endpoint or only a right endpoint, infinite integral curves without either endpoint, and closed integral curves. For instance, for the time dependent kinetic equation

$$\frac{\partial u_\lambda}{\partial t} + y\frac{\partial u_\lambda}{\partial x} - x\frac{\partial u_\lambda}{\partial y} + \{h(x,y,t)+\lambda\}u_\lambda(x,y,t) = f(x,y,t),$$

where $(x,y) \in \mathbb{R}^2$, $t > 0$ and $\lambda \geq 0$, the integral curves of the vector field $X = y(\partial/\partial x) - x(\partial/\partial y)$ are the circles

$$x = r\sin s, \qquad y = r\cos s,$$

where $r = \sqrt{x^2+y^2}$ is constant. Also other situations may occur. For this reason we have given above a different proof of the strong continuity of the solution of Eqs. (1.1)–(1.3) as a function of time. This proof has the advantage of avoiding the vector field X and of extending to evolution families.

We have

Theorem 4.2. *Suppose a, h, J and K do not depend on time, and suppose $J = J_1 + J_2$ with*

$$\|J_1u\|_1 \leq \delta\|hu\|_1, \quad \|J_2u\|_1 \leq j\|u\|_1, \quad \|Ku^+\|_1 \leq \kappa\|u^+\|_1,$$

for certain $\delta, \kappa \in [0,1)$ and $j \geq 0$. Then the solution u of the initial-boundary value problem (1.1)–(1.3) with $f = 0$ and $g_- = 0$ has the form $u = \mathcal{S}_K(t)g_0$, where $\{S_K(t)\}_{t\geq 0}$ is a strongly continuous semigroup on $L_1(\Omega\times\mathcal{V}; d\mu)$ with infinitesimal generator B_K defined by

$$D(B_K) = \left\{ u \in L_1(\Omega\times\mathcal{V};\ d\mu):\ \begin{array}{c} (X+h)u \in L_1(\Omega\times\mathcal{V}; d\mu) \\ u_\pm \in L_1\left(\sum_\pm; d\nu_\pm\right) \\ u_- = Ku_+ \end{array} \right\}; \quad (4.2)$$

$$(B_Ku)(x,v) = -(Xu)(x,v) - h(x,v)u(x,v) + (Ju)(x,v), \quad (4.3)$$

where $(x,v) \in \Omega\times\mathcal{V}$.

Proof. The only thing to verify is the definition of B_K. However, $D(B_K)$ coincides with the set of stationary solutions of Eq. (1.1)–(1.3) with $\lambda > j$ if $f = 0$ and $g^- = (g_0, 0)$. Such solutions u satisfy $(Y+h)u \in L_1(\Lambda_T; d\mu_T)$, $u^\pm \in L_1(\Sigma_T^\pm; d\nu^\pm)$ and $u^- = \mathcal{K}u^+$ with its temporal trace belonging to $L_1(\Omega\times\mathcal{V}; d\mu)$. Hence $u \in L_1(\Omega\times\mathcal{V}; d\mu)$, $u_\pm \in L_1(\Sigma_\pm; d\nu_\pm)$, $(X+h)u = (Y+h)u$ is in $L_1(\Omega\times\mathcal{V}; d\mu)$ and $u_- = Ku_+$.

If the conditions of Theorem 4.2 are satisfied, J_1 and J_2 and K are positive operators, $\delta \in [0,1]$ and $\kappa \in [0,1]$, a monotonicity argument yields that for $f = 0$ and $g_- = 0$ and a, h, J and K independent of t the solution of Eqs. (1.1)–(1.3) has the form $u = S_K(t)g_0$, where $\{S_K(t)\}_{t\geq 0}$ is a strongly continuous positive semigroup on $L_1(\Omega\times\mathcal{V}; d\mu)$. Its generator is the closure of the operator B_K defined by (4.2) and (4.3). It equals B_K only if for each solution u of Eqs. (1.1)–(1.3) the trace $u^{\pm} \in L_1(\Sigma_T^{\pm}; d\nu^{\pm})$ and $hu \in L_1\ (\Lambda_T;\ d\mu_T)$. In the L_p-case the same result (but for $\delta > 0$ instead of $\delta \in [0, 1]$) can be derived.

If a, h, J and K depend on time, then Proposition 4.1 implies that the solution u of the initial-boundary value problem (1.1)–(1.3) with $f = 0$ and $g_- = 0$ has the form $u = S_K(t, 0)g_0$, where $\{S_K(t, s)\}_{t \geq s}$ is an evolution family on $L_1(\Omega \times \mathcal{V}; d\mu)$, provided $\delta, \kappa \in [0, 1)$ and $j \geq 0$. Moreover, if J_1, J_2 and K are positive operators, $\delta \in [0, 1]$ and $\kappa \in [0, 1]$, a monotonicity argument based on Proposition B.1 yields that for $f = 0$ and $g_- = 0$ the solution of Eqs. (1.1)–(1.3) has the form $u = S_K(t, 0)g_0$, where $\{S_K(t, s)\}_{t \geq s}$ is an evolution family of positive operators on $L_1(\Omega \times \mathcal{V}; d\mu)$. In the L_p-case the same result (but for $\delta > 0$ instead of $\delta \in [0, 1]$) can be derived for $\kappa \in [0, 1)$, also using Proposition B.1.

5 WEIGHTED SPACES AND MULTIPLYING BOUNDARIES

Let us consider Eqs. (3.1)–(3.3) on the function space $L_p(\Lambda_T; w d\mu_T)$ with $1 \leq p < \infty$ and $\lambda \geq 0$, where w is a positive, time independent weight function such that $w^{1/p} \in \Phi_T$. Then if $\{w^{1/p}u, (Y + h + \lambda)(w^{1/p}u)\} \subset L_p(\Lambda_T; d\mu_T)$, $w^{1/p}u$ has a unique trace $w^{1/p}u^{\pm}$. Moreover, if $w^{1/p}u^- \in L_p(\Sigma_T^-; d\nu^-)$, then $w^{1/p}u^+ \in L_p(\Sigma_T^+; d\nu^+)$, $wh|u|^p$ and $w^{1/q}|u|^{p-1}Y(w^{1/p}u)$ with $q = p/(p-1)$ are μ_T-integrable and

$$
\begin{aligned}
&\|w|u^+|^p\|_1 + p\int_{\Lambda_T} w(h+\lambda)|u|^p d\mu_T \\
&\qquad = \|w|u^-|^p\|_1 + p\int_{\Lambda_T} \operatorname{sgn}(u) w^{1/q}|u|^{p-1}(Y + h + \lambda)(w^{1/p}u)\, d\mu_T,
\end{aligned}
$$

which may be rewritten as

$$
\begin{aligned}
&\|w|u^+|^p\|_1 + p\int_{\Lambda_T} w\left(h - \frac{Yw}{pw} + \lambda\right)|u|^p d\mu_T \\
&\qquad = \|w|u^-|^p\|_1 + p\int_{\Lambda_T} \operatorname{sgn}(u) w|u|^{p-1}(Y + h + \lambda)u\, d\mu_T, \qquad (5.1)
\end{aligned}
$$

If we also assume that Yw/w be bounded above, then there exists $\lambda_p \geq 0$ such that

$$
h_p = h - \frac{Yw}{pw} + \lambda_p \geq 0. \qquad (5.2)
$$

This identity then takes the form

$$\|u^+\|_{p,w}^p + p\|h_p|u|^p\|_{1,w} + (\lambda - \lambda_p)\|u\|_{p,w}^p = \|u^-\|_{p,w}^p + p\int_{\Lambda_T} \operatorname{sgn}(u)|u|^{p-1}(Y + h + \lambda)uw\, d\mu_T,$$

where $\|\cdot\|_{p,w}$ denotes the L_p-norm with respect to the measures μ_T and $\nu^\pm$ weighted by w. From this moment on the theory of Sections 2 and 3 and Appendix A can be repeated in full, with the sole exception of Theorem 3.3, provided λ is replaced by $\lambda - \lambda_p$. The fact that the vector field Y may not be divergence free with respect to the measure $wd\mu_T$ [i.e., there may exist a C^1-function v of compact support such that $\int_{\Lambda_T} Yv\, w\, d\mu_T \neq 0$], does not matter, because of the "renormalization" of h and λ. One should require, however, norm estimates on J_1, J_2 and K that involve the weighted function spaces rather than the original spaces. This means that δ, j and κ should be replaced by constants δ_w, j_w and κ_w satisfying

$$\|J_1 u\|_{p,w} \leq \delta_w \|h_p|u|^p\|_{1,w}^{1/p}, \quad \|J_2 u\|_{p,w} \leq j_w \|u\|_{p,w}, \quad \|\mathcal{K}u^+\|_{p,w} \leq \kappa_w \|u^+\|_{p,w}, \tag{5.3}$$

where $\delta_w \geq 0$ (but $\delta_w \in [0,1]$ if $p = 1$), $j_w \geq 0$ and $\kappa_w \in [0,1]$.

Choosing a particular weight proposed by Boulanouar [4], we now derive the following result under multiplying boundary conditions. For the sake of simplicity, we do not write $J = J_1 + J_2$.

Theorem 5.1. *Suppose the lengths $\ell(x)$ of the integral curves of Y have a positive lower bound, $\|K\| \geq 1$, and $h \in L_\infty(\Lambda_T; d\mu_T)$. Then for $1 \leq p < \infty$ the initial-boundary value problem (1.1)–(1.3) has a unique solution $u \in L_p(\Lambda_T; d\mu_T)$ for every $f \in L_p(\Lambda_T; d\mu_T)$ and $(g_0, g_-) \in L_p(\Sigma_T^-; d\nu_-)$, provided J is bounded on $L_p(\Lambda_T; d\mu_T)$. For this solution we have $h|u|^p \in L_1(\Lambda_T; d\mu_T)$ and $u^\pm \in L_p(\Sigma_T^\pm; d\nu_\pm)$.*

Proof. Let us introduce the weight function

$$w(z,s) = \|K\|^{ps/\ell(z)}, \quad 0 \leq s \leq \ell(z), \ z \in \Sigma_T^-.$$

Then $1 \leq w(z,s) \leq \|K\|^p$, $w = 1$ on Σ_T^-, $w = \|K\|^p$ on Σ_T^+, and

$$\frac{Yw}{pw} = \frac{\ln(\|K\|)}{\ell(z)} \leq \frac{\ln(\|K\|)}{\inf\{\ell(z) \colon z \in \Sigma_T^-\}} < +\infty.$$

Since the norms on $L_p(\Lambda_T; d\mu_T)$ and $L_p(\Lambda_T; wd\mu_T)$ are equivalent, $w = 1$ on Σ_T^- and $w = \|K\|^p$ on Σ_T^+, we obtain (5.3) (with $J_1 = 0$ and $J_2 = J$) for finite $j_w \geq 0$ and $\kappa_w \in [0,1]$. Since $h \in L_\infty(\Lambda_T; d\mu_T)$ and the lengths of the integral curves of Y have a positive lower bound, one may apply

Eq. (IX 3.11) of [16], which is based on an argument of [30], to prove that the trace $u^{\pm} \in L_p(\Sigma_T^{\pm}; d\nu_{\pm})$.

The condition on the lengths of the integral curves is needed to guarantee the existence of a constant λ_p such that h_p in (5.2) is nonnegative. The second assumption, $h \in L_\infty(\Lambda_T; d\mu_T)$, is only required to get the trace $u^{\pm} \in L_p(\Sigma_T^{\pm}; d\nu_{\pm})$, in accordance with Eq. (IX 3.11) of [16].

6 APPLICATIONS

In this section two illustrative kinetic theory applications are discussed. Both of them have been or still are the object of intensive research.

6.1 Electron Transport in Weakly Ionized Gases

The initial-boundary value problem for the electron distribution in a weakly ionized gas is given by

$$\frac{\partial u}{\partial t} + v \cdot \frac{\partial u}{\partial x} + a(x, t) \cdot \frac{\partial u}{\partial v} + \nu(x, v, t)u = (Ju)(x, v, t) + f(x, v, t), \tag{6.1}$$

$$u(x, v, 0) = u_0(x, v), \tag{6.2}$$

where $x \in \mathbb{R}^n$ is the spatial variable, $v \in \mathbb{R}^n$ is velocity, $n \in \mathbb{N}$ (only $n = 1$ and $n = 3$ are physically relevant), $a(x, v, t)$ is the electrostatic acceleration, $\nu(x, v, t)$ is the collision frequency, $(Ju)(x, v, t)$ describes scattering, and $f(x, v, t)$ accounts for internal sources (cf. [1, 15, 25, 26]). We assume that ν is integrable on every subset of $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+$ of finite Lebesgue measure and almost everywhere positive, $a(x, t)$ is Lipschitz continuous, and

$$(Ju)(x, v, t) = \int_{\mathbb{R}^n} k(x, v, \hat{v}, t)\, \nu(x, \hat{v}, t)\, u(x, \hat{v}, t)\, d\hat{v}$$

for some nonnegative measurable function $k(x, v, \hat{v}, t)$ satisfying

$$\int_{\mathbb{R}^n} k(x, v, \hat{v}, t)\, dv = 1.$$

The acceleration $a(x, t)$ should have the property that the integral curves of the vector field $(\partial/\partial t) + v(\partial/\partial x) + a(x, t)(\partial/\partial v)$ do not reach infinity in finite time. This is the case if $|a(x, t)| \leq C(1 + |x|)$ for all $(x, t) \in \mathbb{R}^n \times \mathbb{R}_+$ with C independent of (x, t). As a result, when considering Eqs. (6.1)–(6.2) for $t \in [0, T]$, the integral curves may be parametrized by $s \in [0, T]$. Let us define

$$C_T = \sup_{(x(0),\, x'(0)) \in \mathbb{R}^n \times \mathbb{R}^n} \int_0^T \nu(x(t), x'(t), t)\, dt, \tag{6.3}$$

where $x(t)$ is the solution of the characteristic equations $(dx/dt) = v$ and $(dv/dt) = a(x, t)$ given $(x(0), v(0)) \in \mathbb{R}^n \times \mathbb{R}^n$.

In the present context, $\Omega = \mathcal{V} = \mathbb{R}^n$ and the boundary $\Sigma = \emptyset$. For each $T > 0$, J is a positive operator on $Y_{0,T} = L_1(\mathbb{R}^n \times \mathbb{R}^n \times [0, T])$ satisfying $\|Ju\|_1 = \|\nu u\|_1$ for every $u \geq 0$ in $Y_{0,T}$. Thus for every $f \in Y_{0,T}$ and $u_0 \in X_0 = L_1(\mathbb{R}^n \times \mathbb{R}^n)$ there exists a unique solution u in $Y_{0,T}$. This solution is nonnegative if f and u_0 are nonnegative. Moreover, in X_0 we have

$$\|u\|_1 \leq \int_0^T \|f(\cdot, \cdot, \tau)\|_1\, d\tau + \|u_0\|_1,$$

where the equality sign holds if f and u_0 are nonnegative and $\nu u \in Y_{0,T}$ [which occurs if C_T given by (6.3) is finite]. Moreover, $u(\cdot, \cdot, t)$ is a strongly continuous function from $[0, T]$ into X_0. The solution can be written in the form

$$u(\cdot, \cdot, t) = S(t, 0)u_0 + \int_0^t S(t, \tau) f(\cdot, \cdot, \tau)\, d\tau, \quad 0 \leq t \leq T,$$

where $\{S(t, s)\}_{t \geq s}$ is an evolution family of contractions on X_0.

When studying Eqs. (6.1)–(6.2) on the weighted space $Y_{\sigma,T} = L_1(\mathbb{R}^n \times \mathbb{R}^n \times [0, T], (1 + v^2)^{\sigma/2} dx\, dv\, dt)$ with $w(v) = (1 + v^2)^{\sigma/2}$ for some $\sigma \geq 0$, we first observe that $(Yw/w) = \sigma(a(x, t) \cdot v)/(1 + v^2)$. Now let us assume (Yw/w) is bounded below. Then, if J is bounded on $Y_{\sigma,T}$ or if $\|Ju\|_{1,w} \leq \delta_\sigma \|\nu u\|_{1,w}$ for every $u \in Y_{\sigma,T}$ with $\delta_\sigma \in [0, 1]$, then (6.1)–(6.2) have a unique solution in $Y_{\sigma,T}$ for every initial condition $u_0 \in X_\sigma = L_1(\mathbb{R}^n \times \mathbb{R}^n, (1 + v^2)^{\sigma/2} dx\, dv)$. Moreover, $u(\cdot, \cdot, t)$ is a strongly continuous function from $[0, T]$ into X_σ. The solution can be written in the form

$$u(\cdot, \cdot, t) = S(t, 0)u_0 + \int_0^t S(t, \tau) f(\cdot, \cdot, \tau)\, d\tau, \quad 0 \leq t \leq T,$$

where $\{S(t, s)\}_{t \geq s}$ is an evolution family on X_σ.

6.2 The Linearized Boltzmann Equation

The initial-boundary value problem for the evolution of the distribution function $f(x, v, t)$ of the charged particles in a two-component gas mixture of mostly neutral and comparatively few particles, where the distribution function $F(x, v, t)$ of the neutral particles is known, is given by

$$\frac{\partial f}{\partial t} + v \cdot \frac{\partial f}{\partial x} + [\Gamma(x,t) + v \times \Xi(x,t)] \cdot \frac{\partial f}{\partial v} + \nu(x,v,t) f = (Jf)(x,v,t), \tag{6.4}$$

$$f(x, v, 0) = f_0(x, v). \tag{6.5}$$

Here the nonlinear Boltzmann equation has been linearized, $x \in \Omega \subset \mathbb{R}^3$ with Ω a bounded region with a piecewise C^1 boundary, $v \in \mathbb{R}^3$,

$$\nu(x,v,t) = 2\pi \int_{\mathbb{R}^3} \int_0^{\pi/2} F(x, v_*, t)\, B(\theta, v - v_*)\, dv_*\, d\theta$$

is the collision frequency, and

$$(Jf)(x,v,t) = \int_{\mathbb{R}^3} \int_0^{\pi/2} \int_0^{2\pi} f(x, v', t) F(x, v'_*, t)\, B(\theta, v - v_*)\, dv_*\, d\theta\, d\varepsilon,$$

with $v' = v - 2(1+\kappa)^{-1}((v - v_*) \cdot e)e$, $v'_* = v_* + 2\kappa(1+\kappa)^{-1}((v - v_*) \cdot e)e$, κ is the ratio of the masses of the charged and the neutral particles, and $e = (\sin\theta \cos\varepsilon, \sin\theta \sin\varepsilon, \cos\theta)$, is the collision term. Assuming an inverse power law interaction potential, we have

$$B(\theta, w) = w^{\gamma}\, b(\theta), \quad \theta \in \left[0, \frac{\pi}{2}\right), \quad w > 0,$$

where $\gamma = (k-5)/(k-1) \in (-3, 1)$ and $b(\theta)$ is a nonnegative function in $L_1(0, \pi/2)$ (cf. [6] for the details). We assume that $\Gamma(x,t)$ and $\Xi(x,t)$ are Lipschitz continuous on $\overline{\Omega} \times [0, T]$ and that $|\Gamma(x,t)| + |\Xi(x,t)| \leq C(1 + |x|)$ for some constant C and any $T > 0$. In this way we prevent the integral curves of the vector field $Y = (\partial/\partial t) + v(\partial/\partial x) + [\Gamma(x,t) + v \times \Xi(x,t)](\partial/\partial v)$ from running off to infinity in finite time.

Many of the results of [24, 12, 11] can now be reproduced as applications of the theory of Sections 2–5, under the simplifying conditions that $J = J_1 \geq 0$ and $\|Ju\|_1 = \|hu\|_1$ for $u \geq 0$. In these papers, Eqs. (6.4)–(6.5) were considered under hypotheses that are very similar to those made in the present article, as well as for the weight functions $w(x,v) = (1 + v^2)^{\sigma/2}$ for some $\sigma \geq 0$. One should, of course, assume that $(Yw/w) = \sigma(\Gamma(x,t) \cdot v)/(1 + v^2)$ is bounded below.

A THE L_p CASE

Theorem A.1. *For $1 < p < +\infty$ the initial-boundary value problem (3.1)–(3.3) has a unique solution $u_\lambda \in L_p(\Lambda_T; d\mu_T)$ for every $f \in L_p(\Lambda_T; d\mu_T)$*

and $(g_0, g_-) \in L_p(\Sigma_T^-; d\nu^-)$, *provided* $J = J_1 + J_2$ *with*

$$\|J_1 u_\lambda\|_p \leq \delta \|h|u_\lambda|^p\|_1^{1/p}, \quad \|J_2 u_\lambda\|_p \leq j\|u_\lambda\|_p, \quad \|\mathcal{K}u_\lambda^+\|_p \leq \kappa\|u_\lambda^+\|_p,$$

for certain $\kappa \in [0, 1)$, $j, \delta \geq 0$ *and* $\lambda > 0$ *satisfying* $j < \lambda(1-\kappa) - \delta(\lambda/p)^{1/p}$. *Then the solution* u_λ *satisfies* $h|u_\lambda|^p \in L_1(\Lambda_T; d\mu_T)$ *and* $u_\lambda^\pm \in L_p(\Sigma_T^\pm; d\nu^\pm)$, *and is nonnegative if* J_1, J_2, $\mathcal{K}$, f *and* (g_0, g_-) *are nonnegative.*

Notice that any $\lambda > 0$ satisfying $j < \lambda(1-\kappa) - \delta(\lambda/p)^{1/p}$ has the property that $\lambda > j$. Since $\lambda(1-\kappa) - \delta(\lambda/p)^{1/p} \to +\infty$ as $\lambda \to +\infty$ for any $p > 1$, the condition on λ is satisfied for λ large enough.

Proof. Integrating Eqs. (3.1)–(3.3) with $J = 0$ and $\mathcal{K} = 0$ along integral curves, we find a unique solution $u = S_\lambda(f_\lambda, g_\lambda^-)$, where $g_\lambda^- = (g_{0,\lambda}, g_{-,\lambda})$ and, for $q = p/(p-1)$,

$$\|S_\lambda(f_\lambda, g_\lambda^-)\|_p \leq \frac{1}{\lambda}\|f_\lambda\|_p + \frac{1}{\lambda^{1/p}}\|g_\lambda^-\|_p; \tag{A.1}$$

$$\|S_\lambda(f_\lambda, g_\lambda^-)^+\|_p \leq \frac{1}{\lambda^{1/q}}\|f_\lambda\|_p + \|g_\lambda^-\|_p; \tag{A.2}$$

$$\|h|S_\lambda(f_\lambda, g_\lambda^-)|^p\|_1^{1/p} \leq \frac{1}{p^{1/p}}\left(\frac{1}{\lambda^{1/q}}\|f_\lambda\|_p + \|g_\lambda^-\|_p\right). \tag{A.3}$$

Indeed, 2.1 implies

$$\begin{aligned}\|u_\lambda^+\|_p^p + p\|h|u_\lambda|^p\|_1 + p\lambda\|u_\lambda\|_p^p &= \|u_\lambda^-\|_p^p + p\int_{\Lambda_T} \operatorname{sgn}(u_\lambda)|u_\lambda|^{p-1}(Ju_\lambda + f_\lambda)\, d\mu_T \\ &\leq \|\mathcal{K}u_\lambda^+ + g_\lambda^-\|_p^p + (p-1)\lambda\|u_\lambda\|_p^p + \lambda^{1-p}\|Ju_\lambda + f_\lambda\|_p^p,\end{aligned}$$

so that

$$\|u_\lambda^+\|_p^p + p\|h|u_\lambda|^p\|_1 + \lambda\|u_\lambda\|_p^p \leq \|\mathcal{K}u_\lambda^+ + g_\lambda^-\|_p^p + \lambda^{1-p}\|Ju_\lambda + f_\lambda\|_p^p. \tag{A.4}$$

Equation (A.4) with $\mathcal{K} = 0$ and $J = 0$ immediately gives the estimates (A.1)–(A.3).

Now suppose $J = 0$ and $\kappa \in [0, 1)$. Then any solution of Eqs. (3.1)–(3.3) satisfies $u_\lambda = S_\lambda(f_\lambda, \mathcal{K}u_\lambda^+ + g_\lambda^-)$, where

$$u_\lambda^+ = S_\lambda(0, \mathcal{K}u_\lambda^+)^+ + S_\lambda(f_\lambda, g_\lambda^-)^+.$$

Since $\|S_\lambda(0, \mathcal{K}u_\lambda^+)^+\|_p \leq \|\mathcal{K}u_\lambda^+\|_p \leq \kappa\|u_\lambda^+\|_p$, a contraction mapping argument yields $u_\lambda^+ \in L_p(\Sigma_T^+; d\nu^+)$ uniquely. We denote the so-obtained solution by

$u_\lambda = Z_\lambda(f_\lambda, g_\lambda^-)$. We then have

$$\begin{aligned}\|Z_\lambda(f_\lambda, g_\lambda^-)\|_p &\le \lambda^{-1}\|f_\lambda\|_p + \lambda^{-1/p}\{\|g_\lambda^-\|_p + \kappa\|u_\lambda^+\|_p\}\\ &\le \frac{1}{1-\kappa}\left(\frac{1}{\lambda}\|f_\lambda\|_p + \frac{1}{\lambda^{1/p}}\|g_\lambda^-\|_p\right);\\ \|Z_\lambda(f_\lambda, g_\lambda^-)^+\|_p &\le \frac{1}{1-\kappa}\left(\frac{1}{\lambda^{1/q}}\|f_\lambda\|_p + \|g_\lambda^-\|_p\right);\\ \|h|Z_\lambda(f_\lambda, g_\lambda^-)|^p\|_1^{1/p} &\le \frac{1}{p^{1/p}}\left(\frac{1}{\lambda^{1/q}}\|f_\lambda\|_p + \|g_\lambda^-\|_p + \kappa\|u_\lambda^+\|_p\right)\\ &\le \frac{1}{p^{1/p}(1-\kappa)}\left(\frac{1}{\lambda^{1/q}}\|f_\lambda\|_p + \|g_\lambda^-\|_p\right).\end{aligned}$$

Let us now consider Eqs. (3.1)–(3.3) for $J = J_1$. Then any solution u_λ satisfies

$$u_\lambda = Z_\lambda(J_1 u_\lambda + f_\lambda, g_\lambda^-) = Z_\lambda(J_1 u_\lambda, 0) + Z_\lambda(f_\lambda, g_\lambda^-).$$

Moreover, since

$$\|h|Z_\lambda(J_1 u_\lambda, 0)|^p\|_1^{1/p} \le \frac{1}{p^{1/p}\,\lambda^{1/q}(1-\kappa)}\|J_1 u_\lambda\|_p \le \frac{\delta}{p^{1/p}\,\lambda^{1/q}(1-\kappa)}\|h|u_\lambda|^p\|_1^{1/p},$$

a contraction mapping argument yields the existence of u_λ if $\lambda > (\delta/p^{1/p}(1-\kappa))^q$. We denote the so-obtained solution by $u_\lambda = W_\lambda(f_\lambda, g_\lambda^-)$. Then

$$\begin{aligned}\|W_\lambda(f_\lambda, g_\lambda^-)\|_p &\le \frac{\delta\|h|u_\lambda|^p\|_1^{1/p} + \|f_\lambda\|_p + \lambda^{1/q}\|g_\lambda^-\|_p}{\lambda(1-\kappa)} \le \frac{\|f_\lambda\|_p + \lambda^{1/q}\|g_\lambda^-\|_p}{\lambda(1-\kappa) - \delta(\lambda/p)^{1/p}};\\ \|W_\lambda(f_\lambda, g_\lambda^-)^+\|_p &\le \frac{\delta\|h|u_\lambda|^p\|_1^{1/p} + \|f_\lambda\|_p + \lambda^{1/q}\|g_\lambda^-\|_p}{\lambda^{1/q}(1-\kappa)} \le \frac{\|f_\lambda\|_p + \lambda^{1/q}\|g_\lambda^-\|_p}{\lambda^{1/q}(1-\kappa) - (\delta/p^{1/p})};\\ \|h|W_\lambda(f_\lambda, g_\lambda^-)|^p\|_1^{1/p} &\le \frac{\delta\|h|u_\lambda|^p\|_1^{1/p} + \|f_\lambda\|_p + \lambda^{1/q}\|g_\lambda^-\|_p}{p^{1/p}\,\lambda^{1/q}(1-\kappa)} \le \frac{\|f_\lambda\|_p + \lambda^{1/q}\|g_\lambda^-\|_p}{p^{1/p}\lambda^{1/q}(1-\kappa) - \delta}.\end{aligned}$$

Considering the full problem (3.1)–(3.3), we must solve u_λ from the equation

$$u_\lambda = W_\lambda(J_2 u_\lambda, 0) + W_\lambda(f_\lambda, g_\lambda^-).$$

Since Eq. (3.11) implies that

$$\|W_\lambda(J_2 u_\lambda, 0)\|_p \le \frac{1}{\lambda(1-\kappa) - \delta(\lambda/p)^{1/p}} \|J_2 u_\lambda\|_p \le \frac{j}{\lambda(1-\kappa) - \delta(\lambda/p)^{1/p}} \|u_\lambda\|_p,$$

we apply a contraction mapping argument to obtain the solution, requiring that $j < \lambda(1-\kappa) - \delta(\lambda/p)^{1/p}$. This completes the proof.

The monotonicity argument of [3] and Section XI.5 of [16] used to prove the existence of a solution of Eqs. (3.1)–(3.3) for positive $\mathcal{K}$, J_1 and J_2 and nonnegative f and g^- for $\kappa = 1$ breaks down whenever $\delta > 0$. The reason is that $\lambda > 0$ has satisfy the inequality $j < \lambda(1-\kappa) - \delta(\lambda/p)^{1/p}$, which is impossible for some $\lambda > 0$ non depending on κ if we pass to the limit $\kappa \uparrow 1$. The monotonicity argument only goes through for $p > 1$ if one can choose $J_1 = 0$, i.e., if J is a bounded. However, the resulting existence theorem for the solution of Eqs. (3.1)–(3.3) for $\kappa = 1$ and nonnegative f_λ and g_λ^- has been obtained before in [3].

B STRONG LIMITS OF EVOLUTION FAMILIES

Throughout Appendix B, μ is a positive measure on E which is assumed to be σ-finite if $p = 1$.

Proposition B.1. *For $1 \le p < \infty$, let $\{U_n(t,s)\}_{t \ge s}$ be an increasing sequence of evolution families of positive operators on $L_p(E, d\mu)$ such that $\|U_n(t,s)\| \le Me^{\omega(t-s)}$ for $t \ge s$ and $n \in \mathbb{N}$. Then there exists a unique evolution family $\{U(t,s)\}_{t \ge s}$ on $L_p(E, d\mu)$ such that*

$$\lim_{n\to\infty} \|U(t,s)g - U_n(t,s)g\|_p = 0, \quad g \in L_p(E, d\mu).$$

Proof. It suffices to establish the strong continuity of $U(t,s)$ as a function of $(t,s) \in \Delta$. To do so, we define the evolution semigroups [22, 27, 10].

$$[T_n(t)f](\tau) = U_n(\tau, \tau - t) f(\tau - t), \quad [T(t)f](\tau) = U(\tau, \tau - t) f(\tau - t).$$

According to either [22], Proposition 3.1 (cf. [10], Proposition 3.11), $\{T_n(t)\}_{t\ge0}$ is a strongly continuous semigroup on the Banach space $Y = C_0(\mathbb{R}; L_p(E, d\mu))$ of strongly continuous functions $f : \mathbb{R} \to L_p(E, d\mu)$ vanishing strongly at infinity, endowed with the norm $\|f\|_Y = \sup_{\tau\in\mathbb{R}} \|f(\tau)\|_p$. Moreover, $\|T_n(t)\| \le Me^{\omega t}$. According to the same result in [27], it suffices to prove that $\{T(t)\}_{t\ge0}$ is a strongly continuous semigroup on Y.

To reduce the problem further, let us show that it suffices to prove that

$$\lim_{n\to\infty} < T_n(t)f, \varphi > = < T(t)f, \varphi >, \quad f \in Y, \ \varphi \in Y^*, \tag{27}$$

where Y^* is the dual of Y. Indeed, (B.1) would cause the function $t \mapsto T(t)f$ acting from $[0, \infty)$ into Y to be weakly measurable for every $f \in Y$. By Pettis' measurability theorem ([13], Theorem II.1.2) and the separability of Y, it would be strongly measurable. From (B.1) we would also find the semigroup properties $T(0) = I$ and $T(t_1)T(t_2) = T(t_1 + t_2)$ for $t_1, t_2 \geq 0$. Finally, Lemma VIII 1.3 of [14] would imply the strong continuity of $t \mapsto T(t)$.

The proof of (B.1) hinges upon a characterization of Y^* that allows one to write the pairings occurring in (B.1) as integrals. After that, monotone convergence will do the job.

First of all, $Y = C_0(\mathbb{R}) \otimes_\varepsilon L_p(E, d\mu)$, where $\otimes_\varepsilon$ denotes the injective tensor product [13]. The dual spaces of $C_0(\mathbb{R})$ and $L_p(E, d\mu)$ are the space $NBV(\mathbb{R})$ of regular Borel measures of bounded variation on $\mathbb{R}$ with the total variation norm and $L_q(E, d\mu)$ with $q = p/(p-1)$, respectively [14]. For $\nu \in NBV(\mathbb{R})$ and $h \in L_q(E, d\mu)$ the corresponding functionals have the form $< b, \nu > = \int_{-\infty}^{\infty} b \, d\nu$ and $< g, h > = \int_E gh \, d\mu$. Then the Cartesian product K of the closed unit spheres in $NBV(\mathbb{R})$ and $L_q(E, d\mu)$ endowed with their weak-* topologies is a compact Hausdorff space, as a result of Alaoglu's theorem. Then the dual space of $C(K)$ consists of the regular Borel measures on K. According to a result by Grothendieck ([13], Theorem VIII.2.5), Y^* consists of the continuous bilinear functionals ψ on $C_0(\mathbb{R}) \times L_p(E, d\mu)$ such that

$$\psi(b, g) = \int_K < b, \nu >< g, h > d\rho(\nu, h), \quad b \in C_0(\mathbb{R}), \ g \in L_p(E, d\mu),$$

for some regular Borel measure ρ on K.

Next, let $f \in Y$ and $\varphi \in Y^*$. Then

$$\begin{aligned} < T_n(t)f, \varphi > &= \int_K \int_E \int_{-\infty}^{\infty} [(T_n(t)f)(\tau)](z) \, h(z) \, d\nu(\tau) \, d\mu(z) \, d\rho(\nu, h) \\ &= \int_K \int_E \int_{-\infty}^{\infty} [U_n(\tau, \tau - t) f(\tau - t)](z) \, h(z) \, d\nu(\tau) \, d\mu(z) \, d\rho(\nu, h) \end{aligned}$$

for some regular Borel measure ρ on K. Monotone convergence will instantly give (B.1), which completes the proof.

For semigroups Proposition (B.1) follows easily by applying the Hille–Yosida theorem to the iterates of the resolvent of the limiting semigroup, or by applying Proposition A.1 of [32].

ACKNOWLEDGMENTS

The author is greatly indebted to Vladimir Protopopescu for valuable advice regarding the publishability of this article and to an unknown referee for valuable comments, one of which led to an improvement of Theorem 3.3.

Research supported in part by INDAM and MURST.

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Received September 1, 2000
Revised January 5, 2001
Accepted January 8, 2001