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# Stability of Stationary Transport Equations with Accretive Collision Operators

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In this paper we consider transport equations with accretive collision operators. We characterize when the equation has a unique solution and show that in this case the solution is stable under small perturbations of the collision operator and the initial value. In one case in which there is more than one solution we show how to make a special selection of a solution, which is then stable again under small perturbations of both the collision operator and the initial value. The results obtained here parallel those obtained earlier for the case where the collision operator is positive semidefinite. © 2000 Academic Press

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## 0. INTRODUCTION

Let $H$ be a complex Hilbert space with scalar product $\langle \cdot, \cdot \rangle$. Let $T$ be an injective (i.e., with zero kernel) selfadjoint operator on $H$ and let $Q_+$ and $Q_-$ be the orthogonal projections onto the maximal $T$-invariant subspaces of vectors $h$ for which $\langle Th, h \rangle \geqslant 0$ and $\langle Th, h \rangle \leqslant 0$, respectively. We do not assume that $T$ is bounded. Let $B$ be a compact operator on $H$ such that $A = I - B$ has a positive semidefinite real part whose kernel coincides with the kernel of $A$, i.e., $\operatorname{Ker}(\operatorname{Re} A) = \operatorname{Ker} A$. Here $I$ denotes the identity operator. Consider the vector-valued differential equation

$$(T\psi)'(x) = -A\psi(x), \qquad 0 < x < \infty, \tag{0.1}$$

with boundary conditions

$$\lim_{x \to 0} Q_+ \psi(x) = \varphi_+, \qquad \limsup_{x \to +\infty} \|\psi(x)\|_H < +\infty. \tag{0.2}$$

Here $\varphi_+ \in \operatorname{Im} Q_+$ is a given vector. The problem is to find the $H$-valued function $\psi(x)$. The derivative in (0.1) is to be interpreted as an $x$-derivative in the strong operator topology of $H$.

In this paper we study the stability of solutions of the boundary value problem (0.1), (0.2). Here, stability is understood in the sense of robustness; namely, a solution is said to be stable if every boundary value problem with coefficients sufficiently close to the coefficients of (0.1), (0.2) has a solution which is as close as we wish to the original solution, on any finite interval prescribed in advance. In particular, we identify the stable solutions, if such exist, of (0.1), (0.2).

The motivation for studying (0.1) with boundary conditions (0.2) stems from the stationary 1D transport equation which appears in radiative transfer, neutron physics, and rarefied gas dynamics (e.g., [6–8]). It has been the subject of intensive study by physicists, mathematicians, and engineers, resulting in thousands of articles and a variety of textbooks. The boundary value problem (0.1)–(0.2) has been described in detail in [15, 16]. In the most widely studied applications, $A$ is a positive semidefinite selfadjoint operator on a Hilbert space $H$. However, stationary transfer of polarized radiation in a plane-parallel atmosphere of infinite optical thickness leads to boundary value problems of the form (0.1)–(0.2), where $A$ is often no longer selfadjoint but instead has a positive semidefinite real part $\operatorname{Re} A = \frac{1}{2}(A + A^*)$, $A^*$ denoting the adjoint of $A$. Such an operator $A$ will be called *accretive*. For these polarized radiation models $A$ and $\operatorname{Re} A$ turn out to have the same at most one-dimensional null space. For this reason, we now study the model problem (0.1)–(0.2) with $A$ as indicated above.

The existence and uniqueness theory for the boundary value problem (0.1)–(0.2) is well developed and has been described in great detail in [15]. Essentially, there are two quite different families of transport problems. For the first family $B=I-A$ is a compact operator and the solution $\psi(x)$ is sought in the given Hilbert space $H$. For the second family $A$ is a bounded operator or a Sturm–Liouville differential operator and the solution $\psi(x)$ is sought in a suitable extension of the domain of $T$ in $H$. One strategy for tackling the first family of problems [19, 15] is

(1) to introduce the three complementary projections $P_+$, $P_-$, and $P_0$ commuting with the evolution operator $T^{-1}A$, where $P_+$ corresponds to the spectrum in the open right half plane, $P_-$ to the spectrum in the open left half plane, and $P_0$ to the zero spectrum,

(2) to write any solution of (0.1) in the form

$$\psi(x)=e^{-xT^{-1}A}P_+h+P_0h, \qquad P_0h\in \operatorname{Ker} A,$$

and

(3) to reduce (0.2) to the vector equation

$$Q_+V\psi(0)=\varphi_+,$$

where $\psi(0)=P_+h+P_0h$ and

$$V=Q_+[P_++P_0]+Q_-P_-.$$

The operator in this vector equation is then proved to be a compact perturbation of the identity. Another strategy for the first family of problems is to write (0.1)–(0.2) as a vector-valued convolution equation and to exploit Wiener–Hopf factorization and Fredholm theory. The strategy for solving the second family of problems [2, 3] is similar to the first strategy for the first family, except for a few notable differences. The operator $A$ must be positive semidefinite, but $B=I-A$ need not have any compactness properties. Moreover, the solution is sought in an extension of the domain $D(T)$ of the operator $T$ in the original Hilbert space $H$.

The existence and uniqueness of the solution of the stationary equation of transfer of polarized radiation were proved using invariance of the positive cone of functions having their values in the positive cone of Stokes vectors under the operators $\pm TQ_\pm$ and $B$, without using that $\operatorname{Re} A$ is positive semidefinite [21, 23]. In [24] its unique solvability was established using the accretiveness of $A$, with the help of the Fredholm alternative applied to the convolution integral equation version of (0.1)–(0.2). In [27] another approach, based on the indicator function, has been used to study the existence and uniqueness of the solution of a boundary value problem of the type (0.1)–(0.2); only the finite-dimensional case with positive semidefinite $A$ was considered there.

The stability of solutions of (0.1)–(0.2) has been proved in various situations. The case where $T$ is bounded and $A$ is positive semidefinite, with some restrictions on the structure of the null space of $A$, was established for the first family in [20] and for the second family in [22]. For the first family of problems similar results were obtained in [26] without making restrictive assumptions on the structure of the null space of $A$ and using general results on the stability of invariant subspaces of matrices. In [26] only $A$ and in the finite-dimensional case both $T$ and $A$ were perturbed, whereas the perturbations in [20, 22] only involved $A$.

The stability problem for the solution of the boundary value problem (0.1)–(0.2) under suitable perturbations on $T$ and for strictly positive selfadjoint $A$ or for $A$ having a strictly positive selfadjoint real part is standard [20, 26]. However, when $A$ has a nontrivial null space, the stability problem reduces to a stability problem for suitable subspaces of the range of $P_0$, which is a finite-dimensional space. Consequently, stability results for matrices can be applied. The present problem, where $A$ has a positive semidefinite real part but need not be selfadjoint itself, requires one to generalize the stability arguments used in [26].

The paper consists of three sections (besides the introduction) and an appendix. In Section 1 we study the finite dimensional case, which is basic in the sense that the problem of stability of the boundary value problem (0.1)–(0.2) will be eventually reduced to a finite dimensional problem. Section 2 is of a technical nature. There we introduce various spectral decompositions of the operators involved and develop their properties. Our main results, Theorems 3.3 and 3.5, are stated and proved in Section 3. Finally, in the Appendix, a variation of a well-known result concerning convergence of generators of strongly continuous contractive semigroups is given.

Standard notation is used: Ker $A$ and Im $A$ stand for the kernel and range of a linear operator $A$, respectively; Re $A = \frac{1}{2}(A + A^*)$ is the *real part* of $A$; the domain of a densely defined linear operator $A$ is denoted $D(A)$; $\mathbb{R}$ and $\mathbb{C}$ denote the real line and the complex plane, respectively.

## 1. THE FINITE DIMENSIONAL CASE

### 1.1. *Preliminaries*

In this section we consider the problem (0.1)–(0.2) in a finite dimensional setting. So $T$ is an invertible selfadjoint $n \times n$ matrix and $A$ is an accretive $n \times n$ matrix with Ker $A =$ Ker(Re $A$). A pair $(T, A)$ with these properties will be called an *accretive admissible pair* (on $\mathbb{C}^n$). We are looking at the problem

$$(T\psi)'(x) = -A\psi(x), \qquad 0 < x < +\infty, \tag{1.1}$$

with boundary conditions

$$\lim_{x \to 0} Q_+ \psi(x) = \varphi_+, \qquad \limsup_{x \to +\infty} \|\psi(x)\| < +\infty. \tag{1.2}$$

Observe that $iT^{-1}A$ is dissipative with respect to the indefinite scalar product generated by $T$. This observation allows us to use the results from [28, 29, 31].

We start with the spectral properties of $T^{-1}A$.

PROPOSITION 1.1. *Assume $T$ is an invertible selfadjoint $n \times n$ matrix and $A$ is an $n \times n$ matrix with positive semidefinite real part such that* $\operatorname{Ker}(\operatorname{Re} A) = \operatorname{Ker} A$. *Then the following statements hold*:

1. $\operatorname{Im} A = \operatorname{Im} A^*$ *and* $\operatorname{Ker} A = \operatorname{Ker} A^*$.
2. $T^{-1}A$ *does not have nonzero purely imaginary eigenvalues.*
3. $(\operatorname{Ker} T^{-1}A)^n = \operatorname{Ker}(T^{-1}A)^2$ *for* $n \geqslant 2$.
4. $\operatorname{Ker} A \cap \operatorname{Im}(T^{-1}A) = \operatorname{Ker} A^* \cap \operatorname{Im}(T^{-1}A^*)$. *Let us denote this subspace by* $N_0$.
5. *Let* $h \in \operatorname{Ker} A$. *Then* $h \in N_0$ *if and only if* $\langle Th, g \rangle = 0$ *for every* $g \in \operatorname{Ker} A$.

*Proof.* Since the kernels of $A$ and $\operatorname{Re} A$ coincide, the kernels of $A$ and $A^*$ coincide as well. Thus the ranges of $A$ and $A^*$ must necessarily coincide.

Let $\lambda$ be purely imaginary and let $h$ be a vector such that $Ah = \lambda Th$. Then

$$0 \leqslant \langle Ah, h \rangle + \langle h, Ah \rangle = (\lambda + \bar{\lambda}) \langle Th, h \rangle = 0,$$

so that $h \in \operatorname{Ker}(\operatorname{Re} A)$. But then $\lambda Th = Ah = 0$ and hence either $\lambda = 0$ or $h = 0$.

Next, suppose $g, k, \ell$ are vectors such that $Ag = Tk$, $Ak = T\ell$, and $A\ell = 0$. Then $(\operatorname{Re} A)\ell = 0$ and hence $A^*\ell = 2(\operatorname{Re} A)\ell - A\ell = 0$. Therefore,

$$\begin{aligned} 0 &\leqslant \langle Ak, k \rangle + \langle k, Ak \rangle = \langle T\ell, k \rangle + \langle k, T\ell \rangle = \langle \ell, Tk \rangle + \langle Tk, \ell \rangle \\ &= \langle \ell, Ag \rangle + \langle Ag, \ell \rangle = \langle A^*\ell, g \rangle + \langle g, A^*\ell \rangle = 0, \end{aligned}$$

whence $(\operatorname{Re} A)k = 0$ and thus $T\ell = Ak = 0$ and $\ell = 0$.

Next, if $h$ and $k$ are vectors such that $Ah = Tk$ and $Ak = 0$, then the first part of the proposition implies the existence of $h_* \in H$ such that $A^*h_* = Tk$ and $A^*k = 0$.

Finally, if $h \in N_0$, then $Ah = A^*h = 0$ and there exists $f$ such that $Af = Th$. Thus for any vector $g$ with $Ag = A^*g = 0$ we have $\langle Th, g \rangle = \langle Af, g \rangle = \langle f, A^*g \rangle = 0$. Conversely, if $\langle Th, g \rangle = 0$ for every $g \in \operatorname{Ker} A$, then $Th$ is orthogonal to $\operatorname{Ker} A$ and hence belongs to $\operatorname{Im} A^* = \operatorname{Im} A$. Thus there exists $f$ such that $Af = Th$, and hence $h \in N_0$. ∎

The subspace $N_0$ described in the fourth part of the above proposition consists of those vectors in $\operatorname{Ker} A$ that are part of a Jordan chain of $T^{-1}A$ of length two. This subspace also consists precisely of those vectors in $\operatorname{Ker} A^*$ that are part of a Jordan chain of $T^{-1}A^*$ of length two.

In view of Part 2 of the above proposition, we can define the spectral projections $P_+$, $P_-$, and $P_0$ of $T^{-1}A$ corresponding to its eigenvalues in the open right half plane, in the open left half plane, and at zero, respectively. Then $\operatorname{Im} P_+ \dot{+} \operatorname{Im} P_- \dot{+} \operatorname{Im} P_0 = \mathbb{C}^n$.

Next we prove an important property of the pair $(T, A)$ that is reminiscent of a property that holds for the case where $A$ is positive semidefinite. A $(T^{-1}A)$-invariant maximal $T$-nonnegative subspace $\mathscr{M}$ is called *stable* if for every $\varepsilon > 0$ there is a $\delta > 0$ such that for every accretive $\hat{A}$ and every selfadjoint $\hat{T}$ with $\|A - \hat{A}\| + \|T - \hat{T}\| < \delta$ there exists an $(\hat{T}^{-1}\hat{A})$-invariant maximal $\hat{T}$-nonnegative subspace $\mathscr{N}$ with $\operatorname{gap}(\mathscr{M}, \mathscr{N}) < \varepsilon$. Note that since $T$ is assumed to be invertible, the invertibility of $\hat{T}$ is guaranteed for $\delta$ sufficiently small. Here the well-known notion of a gap between two subspaces $\mathscr{M}$, $\mathscr{N}$ of $\mathbb{C}^n$ is used:

$$\operatorname{gap}(\mathscr{M}, \mathscr{N}) = \|P_{\mathscr{M}} - P_{\mathscr{N}}\|,$$

where $P_{\mathscr{M}}$, $P_{\mathscr{N}}$ is the orthogonal projection onto $\mathscr{M}$, $\mathscr{N}$, respectively. (See, e.g., [12, 14, 17] for more details.) Analogously, stability of invariant maximal nonpositive subspaces is defined.

THEOREM 1.1. *Let $(T, A)$ be an accretive admissible pair.*

(i) *Assume that* $\operatorname{Ker} A$ *is $T$-nonnegative, i.e., $\langle Tg, g \rangle \geq 0$ for all $g \in \operatorname{Ker} A$. Then there exist a unique $(T^{-1}A)$-invariant maximal $T$-nonnegative subspace $\mathscr{M}_+$ such that $\sigma(T^{-1}A|_{\mathscr{M}_+})$ is contained in the closed upper half plane and a unique $(T^{-1}A)$-invariant maximal $T$-nonpositive subspace $\mathscr{M}_-$ such that $\sigma(T^{-1}A|_{\mathscr{M}_-})$ is contained in the closed lower half plane. In fact,*

$$\mathscr{M}_+ = \operatorname{Ker} T^{-1}A \dot{+} \operatorname{Im} P_+, \qquad \mathscr{M}_- = N_0 \dot{+} \operatorname{Im} P_-.$$

*Both $\mathscr{M}_+$ and $\mathscr{M}_-$ are stable.*

(ii) *Assume that* Ker $A$ *is T-nonpositive, i.e.,* $\langle Tg, g\rangle \leqslant 0$ *for all* $g \in \operatorname{Ker} A$. *Then there exist a unique* $(T^{-1}A)$*-invariant maximal T-nonnegative subspace* $\mathcal{M}_+$ *such that* $\sigma(T^{-1}A|_{\mathcal{M}_+})$ *is contained in the closed upper half plane and a unique* $(T^{-1}A)$*-invariant maximal T-nonpositive subspace* $\mathcal{M}_-$ *such that* $\sigma(T^{-1}A|_{\mathcal{M}_-})$ *is contained in the closed lower half plane. In fact,*

$$\mathcal{M}_+ = N_0 \dot{+} \operatorname{Im} P_+, \qquad \mathcal{M}_- = \operatorname{Ker} T^{-1}A \dot{+} \operatorname{Im} P_-.$$

*Both* $\mathcal{M}_+$ *and* $\mathcal{M}_-$ *are stable.*

(iii) *If* Ker $A$ *is T-indefinite, then there exist infinitely many* $(T^{-1}A)$*-invariant T-nonnegative subspaces* $\mathcal{M}_+$ *such that* $\sigma(T^{-1}A|_{\mathcal{M}_+})$ *is contained in the closed upper half plane, as well as infinitely many* $(T^{-1}A)$*-invariant T-nonpositive subspaces* $\mathcal{M}_-$ *such that* $\sigma(T^{-1}A|_{\mathcal{M}_-})$ *is contained in the closed lower half plane. None of these is stable; more precisely, for every subspace* $\mathcal{M}_\pm$ *as above there exist* $\varepsilon_0 > 0$ *and a sequence* $\{B_m\}_{m=1}^{\infty}$ *of accretive matrices that converge to* $A$ *and such that* $\operatorname{gap}(\mathcal{M}_\pm, \mathcal{N}_\pm) \geqslant \varepsilon_0$ *for every* $T^{-1}B_m$*-invariant maximal T-nonnegative (or T-nonpositive, as the case may be) subspace* $\mathcal{N}_\pm$.

*Proof.* Let $\{x_{1,1}, ..., x_{r,1}, x_{r+1,1}, ..., x_{s,1}\}$ be a basis for the subspace Ker $T^{-1}A$, and let $\{x_{1,2}, ..., x_{r,2}\}$ be vectors such that $iT^{-1}Ax_{i,2} = x_{k,1}$, $(k = 1, ..., r)$, and

$$\operatorname{span}\{x_{k,j}\}_{k,j} = \operatorname{Ker}(T^{-1}A)^2.$$

Introduce the square matrices

$$CM_1 = (\langle Tx_{k,1}, x_{j,1}\rangle)_{k,j=r+1}^{s}$$

of order $s - r$ and

$$CM_2 = (\langle Tx_{k,1}, x_{j,2}\rangle)_{k,j=1}^{r}$$

of order $r$. Now consider the sets in the complex plane:

$$N_1 = \{\langle CM_1 x, x\rangle : 0 \neq x \in \mathbb{C}^{s-r}\};$$
$$N_2 = \{\langle CM_2 x, x\rangle : 0 \neq x \in \mathbb{C}^{r}\}. \tag{2.4}$$

We note (see [29, 31]) that both $CM_1$ and $CM_2$ are nonsingular and that the sets $N_1$ and $N_2$ do not depend on the particular choice of the Jordan basis in $\operatorname{Ker}(T^{-1}A)^2$. Moreover, $CM_1$ is hermitian, so $N_1 \subset \mathbb{R}$.

Corollary 3.2.2 and Theorem 3.2.6 in [31] now tell us that there exists a unique $(T^{-1}A)$-invariant maximal $T$-nonnegative subspace $\mathcal{M}_+$ such that $\sigma(T^{-1}A|_{\mathcal{M}_+})$ is contained in the closed upper half plane if and only if there is a unique $(T^{-1}A)$-invariant maximal $T$-nonpositive subspace $\mathscr{M}_-$ such that $\sigma(T^{-1}A|_{\mathscr{M}_-})$ is contained in the closed lower half plane if and only if both $0 \notin N_1$ and $0 \notin N_2$.

We shall show that the latter condition $0 \notin N_2$ is automatically fulfilled. Indeed, observe that for $k=1, ..., r$ we have $Tx_{k,1} = iAx_{k,2}$, so

$$CM_2 = (\langle iAx_{k,2}, x_{j,2}\rangle)_{k,j=1}^r.$$

Introduce $X = (x_{1,2}^* \cdots x_{r,2}^*)^*$, define the $nr \times nr$ matrix $\mathscr{A} = \bigoplus_{k=1}^r iA$, and put $\operatorname{Im} \mathscr{A} = (\mathscr{A} - \mathscr{A}^*)/2i$ for the imaginary part of $\mathscr{A}$. Then $CM_2 = X^* \mathscr{A} X$. Assume that $0 \in N_2$; then for some $x \neq 0$ we have $\langle CM_2 x, x\rangle = 0$. Consequently, also

$$0 = \operatorname{Im} \langle CM_2 x, x\rangle = \operatorname{Im}\langle \mathscr{A} Xx, Xx\rangle = \langle \operatorname{Im} \mathscr{A} Xx, Xx\rangle.$$

As $\operatorname{Re} A \geqslant 0$ also $\operatorname{Im} \mathscr{A} \geqslant 0$. Thus $(\operatorname{Im} \mathscr{A})\, Xx = 0$. Since $\operatorname{Ker}\, (\operatorname{Re} A) = \operatorname{Ker} A$ it follows that $\operatorname{Ker}(\operatorname{Im} \mathscr{A}) = \operatorname{Ker} \mathscr{A}$, and therefore $\mathscr{A} Xx = 0$. This implies that $CM_2 x = X^* \mathscr{A} Xx = 0$, and as $CM_2$ is invertible we arrive at $x = 0$, which is a contradiction.

The former of the two conditions above, i.e., $0 \notin N_1$, is equivalent to $\operatorname{Ker} A$ being $T$-definite. Indeed, $\operatorname{span}\{x_{1,1}, ..., x_{r,1}\}$ is the isotropic part of $\operatorname{Ker} A$ (i.e., $\operatorname{Ker} A \cap (T^{-1} \operatorname{Ker} A)^\perp$) and $\operatorname{span}\{x_{r+1,1}, ..., x_{s,1}\}$ is $T$-nondegenerate. If the latter subspace is indefinite then $0 \in N_1$; if it is definite then $0 \notin N_1$.

From [31, Corollary 3.2.2] (see also [29]) the uniqueness statements now follow. Moreover, in that case the unique invariant maximal semidefinite subspaces are stable under perturbations of $T$ and $A$ within the class of pairs $\{(H, C) : H = H^* \text{ invertible}, \operatorname{Re} C \geqslant 0\}$, so they are definitely stable under perturbations of $T$ and $A$ in the class of accretive admissible pairs. Part (iii) can also be derived easily from [31] (see also [28]). ∎

We shall say that the pair $(T, A)$ satisfies the *numerical range condition* if $0 \notin N_1$. When $A$ is selfadjoint, the numerical range condition reduces to the sign condition used in [26].

## 1.2. *Stability of Solutions*

In this section we first derive necessary and sufficient conditions on an accretive admissible pair $(T, A)$ for (1.1)–(1.2) to have a unique solution. We then go on to establish conditions under which such solutions are stable under perturbations of $T$ and $A$.

Let us first study the existence and uniqueness of solutions. Recalling the definitions of $P_0$, $P_+$, and $P_-$, the general solution of $(T\psi)' = -A\psi$ is given by

$$\psi(x) = e^{-xT^{-1}A} h,$$

where $h = \psi(0)$. The solution $\psi(x)$ is bounded for $0 < x < \infty$ if and only if

$$h \in \operatorname{Im} P_+ \dotplus \operatorname{Ker} A.$$

The condition $Q_+ \psi(0) = \varphi_+$ amounts to $Q_+ h = \varphi_+$.

PROPOSITION 1.2. *The subspace* $\operatorname{Im} P_+ \dotplus \operatorname{Ker} A$ *contains a* $(T^{-1}A)$-*invariant maximal T-non-negative subspace.*

*Proof.* This follows from the theory of dissipative operators in indefinite scalar product spaces (see [29, 31]). ∎

Denote by $Z_+$ the set of all $(T^{-1}A)$-invariant maximal $T$-nonnegative subspaces contained in $\operatorname{Im} P_+ \dotplus \operatorname{Ker} A$. We consider the following two cases.

*Case* 1. $\operatorname{Im} P_+ \dotplus \operatorname{Ker} A \in Z_+$, i.e., $\langle Tg, g \rangle \geqslant 0$ for all $g \in \operatorname{Ker} A$. Then the map

$$Q_+ : \operatorname{Im} P_+ \dotplus \operatorname{Ker} A \to \operatorname{Im} Q_+ \tag{1.3}$$

is one-to-one and onto. Therefore, the boundary value problem (1.1)–(1.2) has a unique solution given by

$$\psi(x) = e^{-xT^{-1}A}\, W_+ \varphi_+, \tag{1.4}$$

where $W_+$ is the inverse of the map (1.3).

*Case* 2. $\operatorname{Im} P_+ \dotplus \operatorname{Ker} A \notin Z_+$. Then the map (1.3) is onto, but has a nontrivial kernel. The boundary value problem (1.1)–(1.2) has infinitely many solutions. To describe some of them, for every fixed $\mathscr{M}_+ \in Z_+$, a unique solution is given by (1.4), where $W_+$ is the inverse of the map $Q_+ : \mathscr{M}_+ \to \operatorname{Im} Q_+$.

We summarize our findings in the following result (cf. [10]).

THEOREM 1.2. *Let* $(T, A)$ *be an accretive admissible pair. Then the boundary value problem* (1.1)–(1.2) *has at least one solution and the number of linearly independent solutions of its homogeneous counterpart equals the dimension of a maximal T-negative subspace of* $\operatorname{Ker} A$. *Moreover, to each maximal T-nonnegative subspace* $\mathscr{M}_+$ *of* $\operatorname{Ker} A + \operatorname{Im} P_+$ *there corresponds a parametrized family of solutions* $\psi$ *of the form* (1.4), *where* $W_+$ *is the inverse of the map* $Q_+ : \mathscr{M}_+ \to \operatorname{Im} Q_+$. *Thus* (1.1)–(1.2) *have a unique solution if and only if* $\operatorname{Ker} A$ *is T-nonnegative.*

Next we study the stability problem. In the following stability result, we will compare solutions of (1.1)–(1.2) for the accretive admissible pair $(T, A)$ to solutions of the boundary value problem

$$(\hat{T}\hat{\psi})'(x) = -\hat{A}\hat{\psi}(x), \qquad 0 < x < +\infty, \tag{1.5}$$

$$\lim_{x \to 0} \hat{Q}_+ \hat{\psi}(x) = \hat{\varphi}_+, \qquad \limsup_{x \to +\infty} \|\hat{\psi}(x)\| < +\infty, \tag{1.6}$$

for the accretive admissible pair $(\hat{T}, \hat{A})$. We will write hats for the corresponding quantities for the pair $(\hat{T}, \hat{A})$, often without further explanation.

THEOREM 1.3. *Let $(T, A)$ be an accretive admissible pair.*

(i) *Assume that* Ker $A$ *is $T$-non-negative, i.e., $\langle Tg, g \rangle \geqslant 0$ for all $g \in$* Ker $A$. *Then for every $\varepsilon > 0$ and $x_0 > 0$ there exists $\delta > 0$ such that*

$$\sup_{0 \leqslant x \leqslant x_0} \|e^{-xT^{-1}A} W_+ \varphi_+ - e^{-x\hat{T}^{-1}\hat{A}} \hat{W}_+ \hat{\varphi}_+\| < \varepsilon \tag{1.7}$$

*whenever $(\hat{T}, \hat{A})$ is an accretive admissible pair and $\hat{\varphi}_+$ is an initial vector satisfying*

$$\|\varphi_+ - \hat{\varphi}_+\| + \|T - \hat{T}\| + \|A - \hat{A}\| < \delta.$$

*Here $W_+ \varphi_+$ and $\hat{W}_+ \hat{\varphi}_+$ are the values in $x = 0$ of the unique solutions of the boundary value problems* (1.1)–(1.2) *and* (1.5)–(1.6).

(ii) *Assume that* Ker $A$ *is $T$-non-positive, i.e., $\langle Tg, g \rangle \leqslant 0$ for all $g \in$* Ker $A$. *Then for every $\varepsilon > 0$, $x_0 > 0$, and maximal $T$-non-negative $(T^{-1}A)$-invariant subspace $\mathscr{M}_+$ of* Im $P_+ \dot{+}$ Ker $A$ *there exist $\delta > 0$ and a maximal $\hat{T}$-non-negative $(\hat{T}^{-1}\hat{A})$-invariant subspace $\hat{\mathscr{M}}_+$ of* Im $\hat{P}_+ \dot{+}$ Ker $\hat{A}$ *such that*

$$\operatorname{gap}(\mathscr{M}_+, \hat{\mathscr{M}}_+) + \sup_{0 \leqslant x \leqslant x_0} \|e^{-xT^{-1}A} W_+ \varphi_+ - e^{-x\hat{T}^{-1}\hat{A}} \hat{W}_+ \hat{\varphi}_+\| < \varepsilon \tag{1.8}$$

*whenever $(\hat{T}, \hat{A})$ is an accretive admissible pair and $\hat{\varphi}_+$ is an initial vector satisfying*

$$\|\varphi_+ - \hat{\varphi}_+\| + \|T - \hat{T}\| + \|A - \hat{A}\| < \delta.$$

*Here $W_+ \varphi_+ \in$* Im $P_+ \dot{+}$ Ker $A$ *and $\hat{W}_+ \hat{\varphi}_+ \in$* Im $\hat{P}_+ \dot{+}$ Ker $\hat{A}$ *are the values in $x = 0$ of the unique solutions of the boundary value problems* (1.1)–(1.2) *and* (1.5)–(1.6) *with $W_+ \varphi_+ \in \mathscr{M}_+$ and $\hat{W}_+ \hat{\varphi}_+ \in \hat{\mathscr{M}}_+$.*

*Proof.* Theorem 1.1 contains the necessary results on the stable perturbation of $\mathscr{M}_+$ to $\hat{\mathscr{M}}_+$ in the gap topology. The stable perturbation of the group $e^{-xT^{-1}A}$, uniformly in $x$ on compact subsets of $[0, +\infty)$, is immediate. So it suffices to prove the continuity of the maps $W_+$ when perturbing $T$ and $A$. But this is clear, because, as a result of the gap topology continuity of $\mathscr{M}_+$, the orthogonal projection of $H$ onto $\mathscr{M}_+$ is stable under perturbation and $Q_+ : \mathscr{M}_+ \to \operatorname{Im} Q_+$ is invertible with inverse $W_+$. ∎

*Remark* 1.1. In (1.7) and (1.8) we can replace $\sup_{0 \leqslant x \leqslant x_0}$ by $\sup_{0 \leqslant x < \infty}$. The proof of this result is slightly more complicated, in particular in the case in which $A$ is not invertible. Compare Lemma 3.3 below, where this is proved in a more general setting for the case where $A$ is invertible.

## 2. SPECTRAL PROJECTIONS AND SUBSPACES

To facilitate formulating stability results and to stay in touch with the terminology used in [19, 20, 26], we define an *accretive admissible pair* on a Hilbert space $H$ as a pair $(T, A)$ of linear operators on $H$ such that

(i) $T$ is a (possibly unbounded) injective selfadjoint operator,

(ii) $A$ is a compact perturbation of the identity having positive semidefinite real part, and the kernel of the real part of $A$ coincides with Ker $A$,

(iii) there exists $\alpha > 0$ such that $\operatorname{Im}(I - A) \subset \operatorname{Im} |T|^\alpha \cap D(|T|^{1+\alpha})$, and

(iv) when $A$ is not invertible, there exist projections $P_0$ and $P_0^\dagger$ of the same finite rank such that $\operatorname{Im} P_0 \subset D(|T|^{2+\alpha})$ for some $\alpha > 0$, $TP_0 = P_0^\dagger T$, $AP_0 = P_0^\dagger A$, and $A[\operatorname{Ker} P_0] = \overline{T[\operatorname{Ker} P_0]} = \operatorname{Ker} P_0^\dagger$.

In [19, 20, 26] "admissible" pairs $(T, B)$, where $B = I - A$ (rather than $(T, A)$), were considered.

In this section various spectral projections and subspaces of $T^{-1}A$ are introduced and their properties are derived. Here the technical assumptions (iii) and the part $\operatorname{Im} P_0 \subset D(|T|^{2+\alpha})$ for some $\alpha > 0$ of (iv) do not play a role. Thus, throughout this section we assume that the pair $(T, A)$ satisfies the conditions (i), (ii), and (iv), with $\operatorname{Im} P_0 \subset D(T)$ instead of $\operatorname{Im} P_0 \subset D(|T|^{2+\alpha})$ for some $\alpha > 0$ above. Under these conditions a three-way decomposition of $H$ into closed $T^{-1}A$-invariant subspaces is obtained.

### 2.1. *Spectral Decomposition if A Is Invertible*

When $A$ has a strictly positive selfadjoint real part, then $T^{-1}A$ does not have any spectrum on the imaginary line. To verify this, compare with Proposition 1.1, Part (2), and recall that $A$ is a compact perturbation of the identity, which shows that the same proof as in Proposition 1.1 can be used. For $h \in H$, we then consider the vector-valued convolution equation [15, Chap. 7]

$$\psi_h(x) - \int_{-\infty}^{\infty} T^{-1} E(x-y; -T^{-1})(I-A)\, \psi_h(y)\, dy = E(x; -T^{-1})\, h, \tag{2.1}$$

where $0 \neq x \in \mathbb{R}$ and

$$E(x, -T^{-1}) = \begin{cases} -e^{-xT^{-1}} Q_+ = \int_0^\infty e^{-x/t}\, \sigma(dt), & x > 0, \\ -e^{-xT^{-1}} Q_- = -\int_{-\infty}^0 e^{-x/t}\, \sigma(dt), & x < 0, \end{cases}$$

is the *bisemigroup* generated by $-T^{-1}$ and $\sigma(\cdot)$ is the resolution of the identity of $T$. The notion of a bisemigroup arises naturally in linear transport theory (see [4, 9] and references in [9]). Equation (2.1) has a unique solution in $BC_{\bullet}(H) := BC((-\infty, 0]; H) \oplus BC([0, +\infty); H)$ that vanishes strongly as $x \to \pm\infty$. We denote here by $BC([0, +\infty); H)$ the set of $H$-valued bounded continuous functions on $[0, +\infty)$. We define $\mathrm{BC}((-\infty, 0]; H)$ analogously. We then define the bisemigroup generated by $-T^{-1}A$ by

$$E(x; -T^{-1}A)\, h = \psi_h(x), \qquad 0 \neq x \in \mathbb{R},$$

and the spectral projections $P_+$ and $P_-$ of $T^{-1}A$ by

$$P_+ h = \psi_h(0^+), \qquad P_- h = -\psi_h(0^-).$$

It is known [15, Lemma VII.2.1] that the constituent semigroups are strongly vanishing as $x \to \pm\infty$, and that $E(x; -T^{-1}A) - E(x; -T^{-1})$ is a compact operator for every $x \in \mathbb{R}$ (and also for $x = 0^{\pm}$). Further, if $T$ is bounded, $E(x; -T^{-1}A)$ is an exponentially decaying bisemigroup in the sense of [4].

Likewise, $AT^{-1}$ also does not have any spectrum on the imaginary line, and we can define the bisemigroup generated by $-AT^{-1}$ in a similar way. We introduce the spectral projections $P_+^{\dagger}$ and $P_-^{\dagger}$ by

$$P_{\pm}^{\dagger} h = \pm E(0^{\pm}; -AT^{-1})\, h,$$

and derive the relations

$$P_{\pm}^{\dagger} T = T P_{\pm},$$

valid in the sense that $P_{\pm}$ leave $D(T)$ invariant and on $D(T)$ the above relations hold. In a similar way we have

$$E(x; -AT^{-1})\, T = TE(x; -T^{-1}A), \qquad 0 \neq x \in \mathbb{R}.$$

## 2.2. *Spectral Decomposition if A Is Non-Invertible*

When $A$ has a positive semidefinite real part whose kernel is nontrivial and coincides with Ker $A$, the situation is more involved. For future use we note the following easily verified fact:

PROPOSITION 2.1. *For every compression of $A$, i.e., operator of the form $\pi_0 A \pi_0: H_0 \to H_0$, where $\pi_0$ is the orthogonal projection onto a (closed) subspace $H_0$ of $H$, the kernel of $\pi_0 A \pi_0$ is contained in the kernel of $A$.*

First let $T$ be bounded. Consider the operator polynomial $L(\lambda) = A - \lambda T$.

PROPOSITION 2.2. *$L(\lambda)$ is invertible for $0 < |\lambda| \leqslant \varepsilon$ for some $\varepsilon > 0$ (independent of $\lambda$).*

*Proof.* We represent $A$ and $T$ as block operator matrices with respect to the orthogonal decomposition

$$H = ((\text{Ker } A) \cap (\text{Ker } T_0)) \oplus ((\text{Ker } A) \cap (\text{Im } T_0)) \oplus (\text{Ker } A)^{\perp},$$

where $T_0$ is the compression of the operator $T$ on the finite dimensional subspace Ker $A$:

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & A_3 \end{bmatrix}, \qquad T = \begin{bmatrix} 0 & 0 & T_{13} \\ 0 & T_{22} & T_{23} \\ T_{13}^* & T_{23}^* & T_{33} \end{bmatrix}.$$

In these formulas, the operators $A_3: (\text{Ker } A)^{\perp} \to (\text{Ker } A)^{\perp}$ and

$$T_{22}: ((\text{Ker } A) \cap (\text{Im } T_0)) \to ((\text{Ker } A) \cap (\text{Im } T_0))$$

are invertible and $T_{13}$ is onto (i.e., surjective); the latter condition follows because $T$ is assumed to have a zero kernel. Using a Schur complement, it is now easy to see that, for $\lambda \neq 0$, $L(\lambda)$ is invertible provided both $A_3 - \lambda T_{33}$ and $T_{13}(A_3 - \lambda T_{33})^{-1} T_{13}^*$ are invertible. Since $A_3$ is invertible, the invertibility of $A_3 - \lambda T_{33}$ is guaranteed for $0 < |\lambda| \leqslant \varepsilon$ for $\varepsilon > 0$ sufficiently small. Consider the compression $A_{3c}$ of $A_3$ on the subspace $(\text{Ker } T_{13})^{\perp}$. It is easily seen that

$$\dim(\text{Ker } T_{13})^{\perp} \leqslant \dim(\text{Ker } A \cap \text{Ker } T_0).$$

It follows in view of Proposition 2.1 that $A_{3c}$ has zero kernel and, being a finite dimensional operator, is therefore invertible. Thus, the invertibility of $T_{13}(A_3-\lambda T_{33})^{-1}\, T_{13}^*$ for $0<|\lambda|\leqslant\varepsilon$ is also guaranteed. ▮

Let $\Gamma$ denote the positively oriented circle with center 0 and radius $\varepsilon$, where $\varepsilon$ is taken from Proposition 2.2. Then the operators

$$P_0=-\frac{1}{2\pi i}\int_\Gamma L(\lambda)^{-1}\,T\,d\lambda, \qquad P_0^\dagger=-\frac{1}{2\pi i}\int_\Gamma TL(\lambda)^{-1}\,d\lambda$$

are projections of finite and equal rank such that $TP_0=P_0^\dagger\, T$, $AP_0=P_0^\dagger A$, and $A[\operatorname{Ker} P_0]=\overline{T[\operatorname{Ker} P_0]}=\operatorname{Ker} P_0^\dagger$ (see, e.g., [11, 30]).

If $T$ is unbounded, we must *assume* the existence of projections $P_0$ and $P_0^\dagger$ of the same finite rank such that $\operatorname{Im} P_0\subset D(T)$, $TP_0=P_0^\dagger\, T$, $AP_0=P_0^\dagger\, A$, and $A[\operatorname{Ker} P_0]=\overline{T[\operatorname{Ker} P_0]}=\operatorname{Ker} P_0^\dagger$.

In either case all statements of Proposition 1.1 hold. This implies $\operatorname{Im} P_0=\operatorname{Ker}(T^{-1}A)^2$ and $\operatorname{Im} P_0^\dagger=\operatorname{Ker}(AT^{-1})^2$. Hence $\operatorname{Ker} P_0$ is invariant under $T^{-1}A$ and $\operatorname{Ker} P_0^\dagger$ is invariant under $AT^{-1}$.

To define a three-way spectral decomposition of $T^{-1}A$, we consider an operator $\beta$ on $\operatorname{Im} P_0$ without any zero or purely imaginary eigenvalues. We then define

$$A_\beta=T\beta^{-1}P_0+A(I-P_0). \tag{2.2}$$

Then with respect to the decomposition $H=\operatorname{Im} P_0\oplus\operatorname{Ker} P_0$ we have

$$A_\beta{}^{-1}T=\beta\oplus(T^{-1}A\,|_{\operatorname{Ker} P_0})^{-1}. \tag{2.3}$$

In the following, we shall choose $\beta^{-1}$ to be $T$-accretive, i.e., $\beta^{-1}P_0=T^{-1}X$ for some accretive $X$, in order to get that $A_\beta$ is $T$-accretive as well.

Observe that both terms in the decomposition (2.3) are bounded. It is easily seen (using the fact that $A-I$ is compact) that any point in the spectrum of $T^{-1}A_\beta$ on the imaginary axis must be an eigenvalue, and from (2.3) it is then clear that $T^{-1}A_\beta$ does not have a spectrum on the imaginary axis. Next we consider the vector-valued convolution integral equation

$$\psi_h(x)-\int_{-\infty}^{\infty} T^{-1}E(x-y;\, -T^{-1})(I-A_\beta)\,\psi_h(y)\,dy=E(x;\, -T^{-1})\,h,$$

where $0\neq x\in\mathbb{R}$, $h\in H$, and we put $E(x;\, -T^{-1}A_\beta)\,h=\psi_h(x)$. We then define

$$E(x;\, -T^{-1}A)=(I-xT^{-1}A)\,P_0+E(x,\, -T^{-1}A_\beta)(I-P_0),$$

which does not depend on the particular choice of $\beta$. Further, we define $P_\pm$ by

$$P_+ = E(0^+; -T^{-1}A)(I - P_0), \qquad P_- = -E(0^-; -T^{-1}A)(I - P_0).$$

2.3. *Signs of Scalar Products on the Spectral Subspaces*

Let $H_T$ be the complex Hilbert space obtained by closing $D(T)$ with respect to the scalar product $\langle h, g\rangle_{|T|} = \langle |T|\, h, g\rangle$. Then the projections $Q_\pm$, which leave invariant $D(T)$, have unique extensions from their restrictions on $D(T)$ to complementary orthogonal projections on $H_T$. Then $H_T$ is a Krein space with respect to the indefinite scalar product

$$\langle h, g\rangle_T = \langle (Q_+ - Q_-)\, h, g\rangle_{|T|}. \tag{2.4}$$

This scalar product satisfies $\langle h, g\rangle_T = \langle Th, g\rangle$ for $h, g \in D(T)$.

Now note that $i\,T^{-1}A$ is dissipative in the Krein space $H_T$ in the following sense: $\operatorname{Im}\langle i\,T^{-1}Ah, h\rangle_T = \operatorname{Re}\langle Ah, h\rangle \geqslant 0$ for every $h \in D(T^{-1}A)$. In analogy with a well-known result on dissipative operators [1], we have

Proposition 2.3. *For every* $h \in \operatorname{Im} P_\pm \cap D(T)$ *we have* $\pm\langle Th, h\rangle \geqslant 0$.

*Proof.* First let $A$ be invertible and $h \in \operatorname{Im} P_+ \cap D(T)$. Then $\psi(x) = E(x; -T^{-1}A)\, h$ is the unique solution of the boundary value problem (0.1)–(0.2) with $\varphi_+ = Q_+ h$. Hence, using again that $A - I$ is compact, and hence that $\operatorname{Re} A$ has closed range, we have

$$\langle h, h\rangle_T = \langle T\psi(0), \psi(0)\rangle = -\lim_{c \to +\infty} \int_0^c \frac{d}{dx} \langle T\psi(x), \psi(x)\rangle\, dx$$

$$= 2 \lim_{c \to +\infty} \int_0^c \langle (\operatorname{Re} A)\, \psi(x), \psi(x)\rangle\, dx \geqslant \varepsilon \int_0^\infty \|\psi(x)\|_H^2\, dx \geqslant 0$$

for some $\varepsilon > 0$, where the expression vanishes if and only if $h = 0$. So, we even show here that if $A$ is invertible, then $\operatorname{Im} P_+ \cap D(T)$ is uniformly $T$-positive.

The proposition follows for noninvertible $A$ by considering $A_\beta$ and $h \in \operatorname{Im} P_+$ and observing that $E(x; -T^{-1}A_\beta)\, h$ satisfies the boundary value problem (0.1)–(0.2) with boundary vector $\varphi_+ = Q_+ h$.

Replacing $T$ and $Q_+$ by $-T$ and $Q_-$ in (0.1) and (0.2), one gets the analogous result for $\operatorname{Im} P_-$. ∎

Proposition 2.4. *The subspace* $\operatorname{Im} P_0$ *is nondegenerate with respect to the indefinite scalar product in* $H_T$.

*Proof.* Let $h \in \operatorname{Im} P_0$ satisfy $\langle Th, g\rangle = 0$ for every $g \in \operatorname{Im} P_0$. Then $Th$ is orthogonal to $\operatorname{Ker} A^*$ and therefore $Th = Af$ for some $f$. Then

$$2\langle (\operatorname{Re} A) f, f\rangle = \langle Th, f\rangle + \langle f, Th\rangle = 0,$$

so that $(\operatorname{Re} A) f = 0$ and hence $Th = Af = 0$, implying $h = 0$. ▮

To complicate life, we introduce the three complementary projections $P_{0,*}$, $P_{+,*}$, and $P_{-,*}$ reducing $T^{-1}A^*$ and their counterparts $P^{\dagger}_{0,*}$, $P^{\dagger}_{+,*}$, and $P^{\dagger}_{-,*}$ reducing $A^*T^{-1}$. Then we have the following adjoint relations:

$$(P_0)^* = P^{\dagger}_{0,*}, \qquad (P_+)^* = P^{\dagger}_{+,*}, \qquad (P_-)^* = P^{\dagger}_{-,*};$$
$$(P^{\dagger}_0)^* = P_{0,*}, \qquad (P^{\dagger}_+)^* = P_{+,*}, \qquad (P^{\dagger}_-)^* = P_{-,*}.$$

PROPOSITION 2.5. *Put* $\mathscr{M}_+ = [\operatorname{Im} P_{-,*} \oplus \operatorname{Im} Q_+] \cap \operatorname{Im} P_0$ *and* $\mathscr{M}_- = [\operatorname{Im} P_+ \oplus \operatorname{Im} Q_-] \cap \operatorname{Im} P_0$. *Then* $\mathscr{M}_+$ *is a strictly positive subspace of* $\operatorname{Im} P_{0,*}$ *and* $\mathscr{M}_-$ *is a strictly negative subspace of* $\operatorname{Im} P_0$ *with respect to the indefinite scalar product of* $H_T$. *Moreover,*

$$[\mathscr{M}_+ \cap \operatorname{Ker} A] \oplus [\mathscr{M}_- \cap \operatorname{Ker} A] \oplus N_0 = \operatorname{Ker} A, \tag{2.5}$$

*and hence* $[\mathscr{M}_+ \cap \operatorname{Ker} A] \oplus N_0$ *is a maximal positive and* $[\mathscr{M}_- \cap \operatorname{Ker} A] \oplus N_0$ *is a maximal negative subspace of* $\operatorname{Im} P_0$ *which is contained in* $\operatorname{Ker} A$.

*Proof.* One immediately sees that $\langle Tf, g\rangle = 0$ for $f \in \operatorname{Im} P_0$ and $g \in \operatorname{Im} P_{-,*}$, because in that case $Tf \in \operatorname{Im} P^{\dagger}_0 = [\operatorname{Ker} P_{0,*}]^{\perp} \subset [\operatorname{Im} P_{-,*}]^{\perp}$.

Now let $f \in \mathscr{M}_+$. Then there exist unique $g \in \operatorname{Im} P_{-,*}$ and $h \in \operatorname{Im} Q_+$ such that $f = g + h$. Then, as $f \in \operatorname{Im} P_0$

$$0 \leqslant \langle Th, h\rangle = \langle Tf, f\rangle + \langle Tg, g\rangle,$$

where $\langle Tg, g\rangle \leqslant 0$ (see Proposition 2.3). Thus $\langle Tf, f\rangle \geqslant 0$. Further, $\langle Tf, f\rangle = 0$ implies $\langle Th, h\rangle = \langle Tg, g\rangle = 0$ and hence, applying the Cauchy-Schwartz inequality on $\operatorname{Im} Q_+$ and $\operatorname{Im} P_{-,*}$, which we can do by Proposition 2.3, we have $g = h = 0$, implying $f = 0$. Therefore $\mathscr{M}_+$ is a strictly positive subspace of $\operatorname{Im} P_0$. Similarly, $\mathscr{M}_-$ is a strictly negative subspace of $\operatorname{Im} P_0$. As a result, $\mathscr{M}_+ \cap \mathscr{M}_- = \{0\}$.

We now compute

$$\begin{aligned}(T[\mathscr{M}_+ \cap \operatorname{Ker} A])^{\perp} &= \overline{[(\operatorname{Im} P^{\dagger}_{-,*})^{\perp} \cap (\operatorname{Im} Q_+)^{\perp}] + (\operatorname{Ker} A^*T^{-1})^{\perp}}\\ &= \overline{[(\operatorname{Ker} P_-) \cap \operatorname{Im} Q_-] + \overline{\operatorname{Im} T^{-1}A}}\\ &= \overline{[(\operatorname{Ker} P_-) \cap \operatorname{Im} Q_-] + N_0 \oplus \operatorname{Ker} P_0}\\ &= \mathscr{M}_- \oplus N_0 \oplus \operatorname{Ker} P_0,\end{aligned}$$

where $\mathscr{M}_- \oplus N_0$ is a negative subspace of $\operatorname{Im} P_0$. As a result, we obtain (2.5). ∎

When $A$ is positive semidefinite, the situation simplifies because one can prove that $(T[\mathscr{M}_+])^\perp = \mathscr{M}_- \oplus \operatorname{Ker} P_0$, which makes $\mathscr{M}_+$ into a maximal strictly positive (and $\mathscr{M}_-$ into a maximal strictly negative) subspace of $\operatorname{Im} P_0$ (see [15, 19]). However, the crux of the matter is proving that $\operatorname{Ker} A$ contains both a maximal positive and a maximal negative subspace of $\operatorname{Im} P_0$, and this can be accomplished also if $A$ only has a positive semidefinite real part whose kernel coincides with $\operatorname{Ker} A$.

Let us now find the adjoint of the operator $A_\beta$ defined by (2.2). First of all, $A(I-P_0)$ has adjoint $A^*(I-P_{0,*})$. Writing $\beta^{[*]}$ for the unique operator on $\operatorname{Im} P_{0,*}$ such that $\langle T\beta h_0, g_{0,*}\rangle = \langle Th_0, \beta^{[*]} g_{0,*}\rangle$ for all $h_0 \in \operatorname{Im} P_0$ and $g_{0,*} \in \operatorname{Im} P_{0,*}$, we get

$$(A_\beta)^* = (A^*)_{\beta^{[*]}} = A^*(I-P_{0,*}) + T(\beta^{[*]})^{-1} P_{0,*}.$$

We now easily see that $A_\beta$ has a positive semidefinite real part if and only if

$$\operatorname{Re}\langle T\beta^{-1} P_0 h, h\rangle \geqslant 0,$$

i.e., if and only if $T\beta^{-1}P_0$ has a positive semidefinite real part (in $H$). This is equivalent to requiring that $T(\beta^{[*]})^{-1}P_{0,*}$ has a positive semidefinite real part. This occurs, for instance, if $\beta$ is a positive operator on $\operatorname{Im} P_0$ with respect to the indefinite scalar product $\langle f, g\rangle_T = \langle Tf, g\rangle$.

### 2.4. *Spectral Decompositions in an Extended Hilbert Space*

First let $A$ be invertible, and put

$$V = Q_+ P_+ + Q_- P_-, \qquad V^\dagger = Q_+ P_+^\dagger + Q_- P_-^\dagger. \tag{2.6}$$

Then $V[D(T)] \subset D(T)$ and $TV = V^\dagger T$. Also, using the scalar product (2.4) on the extension space $H_T$ of $D(T)$ we obtain

$$\langle Vh, k\rangle_{|T|} = \langle h, \tilde{V}k\rangle, \qquad h, k \in D(T),$$

where $\tilde{V} = (Q_+ - Q_-)\, V^\dagger (Q_+ - Q_-)$. With the help of the identity

$$2V - I = (Q_+ - Q_-)(P_+ - P_-),$$

we easily obtain the important identity

$$2\langle Vh, g\rangle_{|T|} = \langle h, g\rangle_{|T|} + \langle h, g\rangle_{|S|}, \qquad h, g \in D(T), \tag{2.7}$$

where

$$\langle h, g \rangle_{|S|} = \langle T(P_+ - P_-) h, g \rangle. \tag{2.8}$$

The following result is due to Beals [2] when $A$ is positive semidefinite. For this case the present proof has been given before in [15].

PROPOSITION 2.6. *Let $A$ be invertible. Then* (2.8) *can be extended to a positive semi-definite scalar product on $H_T$ that is equivalent with $(\cdot, \cdot)_{|T|}$. Moreover, $V$ extends to a boundedly invertible, positive selfadjoint operator on $H_T$.*

*Proof.* Let us equip $D(T)$ with the graph scalar product

$$\langle h, g \rangle_{GT} = \langle h, g \rangle + \langle Th, Tg \rangle, \qquad h, g \in D(T). \tag{2.9}$$

Then the intertwining relation $V[D(T)] \subset D(T)$ and $TV = V^\dagger T$ and the boundedness of $V$ and $V^\dagger$ on $H$ imply that $V$ is bounded with respect to the norm induced by (2.9). Putting $\tilde{V} = (Q_+ - Q_-)(V^\dagger)^* (Q_+ - Q_-)$, we see that

$$\langle Vh, g \rangle_{|T|} = \langle h, \tilde{V} g \rangle_{|T|}, \qquad h, g \in D(T),$$

while $\tilde{V} g = Vg$ for every $g \in D(T)$. According to a well-known result by M. G. Krein [18], $V$ extends to a bounded linear operator on $H_T$. Moreover, as a result of (2.7), $V$ is positive selfadjoint on $H_T$. Its invertibility on $H_T$ follows from the inequality $\|Vh\|_{|T|} \geqslant \frac{1}{2} \|h\|_{|T|}$ for every $h \in H_T$ (cf. (2.7)). Since $2P_+ - I = I - 2P_- = P_+ - P_- = (Q_+ - Q_-)(2V - I)$ on $D(T)$, the projections $P_+$ and $P_-$ extend to bounded projections on $H_T$ and $P_+ - P_-$ is invertible. As a result, $2V - I$ is strictly positive selfadjoint on $H_T$; hence, by (2.7), the scalar product (2.8) extends to a positive semi-definite scalar product on $H_T$ that is equivalent to $(\cdot, \cdot)_{|T|}$. ▮

When $A$ is not invertible, by using the operator $A_\beta$ defined by (2.2) with $\beta$ an invertible operator on Im $P_0$ that is $T$-positive, as well as the spectral projections of $T^{-1} A_\beta$, one easily proves that $P_+$, $P_-$, and $P_0$ extend to bounded projections on $H_T$. The solutions of the boundary value problems (0.1)–(0.2) can then be extended to $H_T$, where $\varphi_+ \in Q_+[H_T]$ and $\varphi_- \in Q_-[H_T]$. As in [2] for positive semidefinite $A$, these results can be shown to hold under the sole assumptions that $T$ is injective selfadjoint, $A$ is accretive and bounded and has closed range, and, when $A$ is not invertible, there exist projections $P_0$ and $P_0^\dagger$ of the same finite rank such that Im $P_0 \subset D(T)$, $TP_0 = P_0^\dagger T$, $AP_0 = P_0^\dagger A$, and $A[\text{Ker } P_0] = \overline{T[\text{Ker } P_0]} = \text{Ker } P_0^\dagger$.

## 3. STABILITY PROBLEM: THE INFINITE-DIMENSIONAL CASE

This section is devoted to the stability problem for accretive admissible pairs $(T, A)$ on an infinite dimensional Hilbert space $H$. First we consider the case where $A$ is invertible. Next, exclusively for pairs $(T, A)$ with $T$ bounded, we reduce the infinite dimensional stability problem to a finite dimensional stability problem. The latter is dealt with using the results of Section 1.

### 3.1. *Stability Problem for* $(T, A)$ *with* $A$ *Invertible*

Given an accretive admissible pair $(T, A)$ on $H$, with $A$ invertible, we consider the convolution equation

$$\psi(x) - \int_{-\infty}^{\infty} T^{-1}E(x-y; -T^{-1})(I-A)\, \psi(y)\, dy = \omega(x), \qquad x \in \mathbb{R}. \quad (3.1)$$

Then the integral operator $\mathscr{L}_{(T,A)}$ appearing in (3.1) is bounded in any of the Banach spaces $L_p(\mathbb{R}; H)$ $(1 \leqslant p \leqslant +\infty)$, the Banach spaces of strongly measurable $H$-valued functions, and $BC_{\bullet}(H)$ with norm bounded above by

$$\int_{-\infty}^{\infty} \|T^{-1}E(x; -T^{-1})(I-A)\|\, dx, \quad (3.2)$$

where the norm under the integral sign is the operator norm on $H$ (see [15]).

Let $(T, A)$ be an accretive admissible pair on $H$ where $A = I - B$ is invertible. Then (3.1) is uniquely solvable in any of the Banach spaces mentioned above, in particular in $BC_{\bullet}(H)$. In fact, one can find an explicit expression for the inverse by either applying Fourier transformation plus simple algebra or reducing the integral equation to a vector-valued differential equation in the case of a strongly continuous $\omega$ with a jump discontinuity in $x = 0$ and with values in $D(T)$ such that $T\omega$ has a strong derivative in $BC_{\bullet}(H)$, followed by a continuous extension of the solution formula obtained. The unique solution is given by

$$\psi(x) = \omega(x) + \int_{-\infty}^{\infty} T^{-1}AE(x-y; -T^{-1}A)\, A^{-1}B\, \omega(y)\, dy$$

$$= \omega(x) + \int_{-\infty}^{\infty} T^{-1}E(x-y; -AT^{-1})\, B\, \omega(y)\, dy.$$

From (3.2) it follows [5, 13] that

$$\int_{-\infty}^{\infty} \|T^{-1}E(x; -AT^{-1})\, B\|\, dx < +\infty.$$

As a result, in any of the above Banach function spaces we have

$$\|(I - \mathscr{L}_{(T,A)})^{-1} - I\| \leqslant \int_{-\infty}^{\infty} \|T^{-1}E(x; -AT^{-1})\, B\|\, dx < +\infty.$$

PROPOSITION 3.1. *Let $(T, A)$ be an accretive admissible pair on $H$ where $A = I - B$ is invertible. Then for any accretive admissible pair $(\hat{T}, \hat{A})$ for which*

$$\varDelta := \frac{\int_{-\infty}^{\infty} \|T^{-1}E(x; -T^{-1})\, B - \hat{T}^{-1}E(x; -\hat{T}^{-1})\, \hat{B}\|\, dx}{1 + \int_{-\infty}^{\infty} \|T^{-1}E(x; -AT^{-1})\, B\|\, dx} < 1, \tag{3.3}$$

*the operator $\hat{A} = I - \hat{B}$ is invertible, while*

$$\|(I - \mathscr{L}_{(T,A)})^{-1} - (I - \mathscr{L}_{(\hat{T},\hat{A})})^{-1}\| \leqslant \frac{\varDelta}{1-\varDelta}\left[1 + \int_{-\infty}^{\infty} \|T^{-1}E(x; -AT^{-1})\, B\|\, dx\right]^{-1}. \tag{3.4}$$

*Proof.* Let us write $\mathscr{L} = \mathscr{L}_{(T,A)}$ and $\hat{\mathscr{L}} = \mathscr{L}_{(\hat{T},\hat{A})}$, and let us write (3.1) and the analogous equations for the pair $(\hat{T}, \hat{A})$ in the form

$$\begin{cases} \psi - \mathscr{L}\psi = \omega, \\ \hat{\psi} - \hat{\mathscr{L}}\hat{\psi} = \hat{\omega}. \end{cases} \tag{3.5}$$

Then for $\|\mathscr{L} - \hat{\mathscr{L}}\| < \|(I - \mathscr{L})^{-1}\|^{-1}$ the operator $I - \hat{\mathscr{L}}$ is invertible and we have the well-known estimate

$$\|(I - \mathscr{L})^{-1} - (I - \hat{\mathscr{L}})^{-1}\| \leqslant \frac{\|\mathscr{L} - \hat{\mathscr{L}}\|\, \|(I - \mathscr{L})^{-1}\|}{1 - \|\mathscr{L} - \hat{\mathscr{L}}\|\, \|(I - \mathscr{L})^{-1}\|}\, \|(I - \mathscr{L})^{-1}\|.$$

Using the various upper bounds for the norms appearing in the right-hand side we obtain (3.3) and (3.4). ∎

We now derive an easy perturbation result if $A = I - B$ is invertible and only $A$ is perturbed. Clearly, the ($T$-dependent) norm $B \mapsto \int_{-\infty}^{\infty} \|T^{-1}E(x; -T^{-1})\, B\|\, dx$ is weaker than the usual operator norm. Indeed, for every $h \in H$ we have

$$\int_{-\infty}^{\infty} T^{-1}E(x-y; -T^{-1})\, Bh\, dy = Bh, \qquad x \in \mathbb{R}.$$

As a result,

$$\|Bh\| \leqslant \left( \int_{-\infty}^{\infty} \|T^{-1}E(x; -T^{-1})\, B\|\, dy \right) \|h\|, \qquad h \in H,$$

which proves the assertion.

THEOREM 3.1. *Let $(T, A)$ be an accretive admissible pair on $H$ where $A = I - B$ is invertible. Then for any accretive admissible pair $(T, \hat{A})$ for which*

$$\int_{-\infty}^{\infty} \|T^{-1}E(x; -T^{-1})[B - \hat{B}]\|\, dx < 1 + \int_{-\infty}^{\infty} \|T^{-1}E(x; -AT^{-1})\, B\|\, dx,$$

*we have the estimates*

$$\|P_{\pm} - \hat{P}_{\pm}\| \leqslant \frac{\Delta'}{(1 + \mathscr{E} - \Delta')(1 + \mathscr{E})},$$

$$\|E(x; -T^{-1}A) - E(x; -T^{-1}\hat{A})\| \leqslant \frac{\Delta'}{(1 + \mathscr{E} - \Delta')(1 + \mathscr{E})}, \qquad 0 \neq x \in \mathbb{R},$$

*where*

$$\Delta' = \int_{-\infty}^{\infty} \|T^{-1}E(x; -T^{-1})[B - \hat{B}]\|\, dx,$$

$$\mathscr{E} = \int_{-\infty}^{\infty} \|T^{-1}E(x; -AT^{-1})\, B\|\, dx.$$

*Proof.* The theorem follows immediately from Proposition 3.1, when taking $\omega(x) = E(x; -T^{-1})\, h$ as in (2.1) and observing that $\omega$ has norm $\|h\|_H$ in $\mathrm{BC}_{\bullet}(H)$. ∎

Following the terminology of [26], a family of accretive admissible pairs $\{(T, \hat{A})\colon \hat{A} \in \mathscr{A}\}$, where $\mathscr{A}$ is a set of operators $A$ that satisfy conditions (ii), (iii), and (iv) of Section 2 (for the same $T$), is called *uniform* if there exists a common $\alpha > 0$ such that

$$\operatorname{Im} \hat{B} \subset \operatorname{Im} |T|^{\alpha} \cap D(|T|^{2+\alpha}), \qquad \hat{A} = I - \hat{B} \in \mathscr{A}, \tag{3.6}$$

and there exists a finite constant $\mu$ (not depending on $\hat{A}$) such that

$$\max(\|\, |T|^{-\alpha} \hat{B}\|, \|\, |T|^{2+\alpha} \hat{B}\|) \leqslant \mu, \qquad \hat{A} = I - \hat{B} \in \mathscr{A}. \tag{3.7}$$

In that case, for any $\gamma \in (0, \alpha)$, $\hat{A} \in \mathscr{A}$, and $h \in H$ we have the estimates

$$\| |T|^{-\gamma} \hat{B}h\| \leqslant \| |T|^{-\alpha} \hat{B}h\|^{\gamma/\alpha} \|\hat{B}h\|^{(\alpha-\gamma)/\alpha};$$
$$\| |T|^{2+\gamma} \hat{B}h\| \leqslant \| |T|^{2+\alpha} \hat{B}h\|^{(2+\gamma)/(2+\alpha)} \|\hat{B}h\|^{(\alpha-\gamma)/(2+\alpha)}. \tag{3.8}$$

To obtain (4.9), write

$$\begin{aligned}\| |T|^{-\gamma} \hat{B}h\|^2 &= \int |t|^{-2\gamma} \langle \sigma(dt)\, \hat{B}h, \hat{B}h\rangle \\ &\leqslant \left[\int (|t|^{-2\gamma})^{\alpha/\gamma} \langle \sigma(dt)\, \hat{B}h, \hat{B}h\rangle\right]^{\gamma/\alpha} \\ &\quad\times \left[\int 1^{\alpha/(\alpha-\gamma)} \langle \sigma(dt)\, \hat{B}h, \hat{B}h\rangle\right]^{1-\gamma/\alpha}\end{aligned}$$

where Hölder's inequality was applied to $\int |t|^{-2\gamma} \langle \sigma(dt)\, \hat{B}h, \hat{B}h\rangle$, and analogously

$$\begin{aligned}\| |T|^{2+\gamma} \hat{B}h\|^2 &= \int |t|^{4+2\gamma} \langle \sigma(dt)\, \hat{B}h, \hat{B}h\rangle \\ &\leqslant \left[\int (|t|^{4+2\gamma})^{(2+\alpha)/(2+\gamma)} \langle \sigma(dt)\, \hat{B}h, \hat{B}h\rangle\right]^{(2+\gamma)/(2+\alpha)} \\ &\quad\times \left[\int 1^{(2+\alpha)/(\alpha-\gamma)} \langle \sigma(dt)\, \hat{B}h, \hat{B}h\rangle\right]^{(\alpha-\gamma)/(2+\alpha)}.\end{aligned}$$

Estimates (3.8) can be employed to derive from Theorem 3.1 the following corollary:

COROLLARY 3.1. *Let $(T, A)$ be an accretive admissible pair on $H$ where $A = I - B$ is invertible. Let $\{(T, \hat{A}) : \hat{A} \in \mathscr{A}\}$ be a uniform family of accretive admissible pairs and let $\alpha \in (0, 1)$ be the constant in (3.7). Then for every $c \in (0, \alpha/(2+\alpha))$ there exist constants $\delta > 0$ and $M$ only depending on $(T, A)$ and $c$ such that*

$$\|P_\pm - \hat{P}_\pm\| \leqslant M \|B - \hat{B}\|^c,$$
$$\|E(x; -T^{-1}A) - E(x; -T^{-1}\hat{A})\| \leqslant M \|B - \hat{B}\|^c, \qquad 0 \neq x \in \mathbb{R},$$

*whenever $\hat{A} \in \mathscr{A}$ and $\|A - \hat{A}\| < \delta$.*

*Proof.* For $\gamma = (1-c)\,\alpha$, (3.7) and the first part of (3.8) can be applied yielding

$$\|T^{-1}E(x; -T^{-1})[B-\hat{B}]\| \leqslant (2\mu)^{1-c} \,\| |T|^{-1+c\alpha} E(x; -T^{-1})\| \, \|B-\hat{B}\|^{c}. \tag{3.9}$$

In a similar way, (3.7) and the second part of (3.8) with $\gamma = (1-c)\,\alpha - 2c$ imply

$$\begin{aligned} &\|T^{-1}E(x; -T^{-1})[B-\hat{B}]\| \\ &\qquad \leqslant (2\mu)^{1-c} \,\| |T|^{-3+2c-(1-c)\alpha} E(x; -T^{-1})\| \, \|B-\hat{B}\|^{c}. \end{aligned} \tag{3.10}$$

Using (3.9) for $0 < |x| \leqslant 1$ and (3.10) for $|x| > 1$ to estimate the quantity $\Delta'$ appearing in the statement of Theorem 3.1, Corollary 3.1 is immediate. ▮

So far we have only allowed perturbations of $A$ while leaving $T$ invariant. When both $T$ and $A$ are varied, we must derive suitable continuity properties of $\omega(x) = E(x; -T^{-1})\,h$ as $T$ varies [cf. (3.5)]. To formulate the crucial Lemma 3.1 below, we need the concept of *generalized convergence* of densely defined closed operators, which is based on the norm

$$\delta(T, S) = \operatorname{gap}(G(T), G(S)),$$

where $G(T)$ and $G(S)$ are the graphs of $T$ and $S$ and gap denotes the gap between closed subspaces of $H \oplus H$ (cf. [12, 14, 17]). When $T$ (and hence $T^*$) is a closed and densely defined operator on a Hilbert space $H$, we find for the orthogonal projection of $H \oplus H$ onto $G(T)$

$$P_{G(T)} = \begin{bmatrix} (I+T^*T)^{-1} & (I+T^*T)^{-1}\,T^* \\ T(I+T^*T)^{-1} & T(I+T^*T)^{-1}\,T^* \end{bmatrix}.$$

Thus restricted to the algebra of bounded operators, generalized convergence is equivalent to convergence in the norm [17, Theorem IV.2.13]. Moreover, for two selfadjoint operators $T$ and $\hat{T}$,

$$\begin{aligned} &\operatorname{gap}(T, \hat{T}) \\ &= \left\| \begin{bmatrix} (I+T^2)^{-1} - (I+\hat{T}^2)^{-1} & T(I+T^2)^{-1} - \hat{T}(I+\hat{T}^2)^{-1} \\ T(I+T^2)^{-1} - \hat{T}(I+\hat{T}^2)^{-1} & -(I+T^2)^{-1} + (I+\hat{T}^2)^{-1} \end{bmatrix} \right\|. \end{aligned}$$

LEMMA 3.1. *Let $T_n$ and $T$ be injective positive selfadjoint operators on $H$, and let $T_n$ converge to $T$ in the generalized sense. Then*

$$\lim_{n\to\infty} \|[E(x; -T_n^{-1}) - E(x; -T^{-1})]\, h\| = 0, \qquad h \in H,$$

*uniformly in $0 \neq x \in \mathbb{R}$. When $T$ is boundedly invertible, we have*

$$\lim_{n\to\infty} \|E(x; -T_n^{-1}) - E(x; -T^{-1})\| = 0, \tag{3.11}$$

*uniformly in $0 \neq x \in \mathbb{R}$.*

Details of the proof of this lemma are given in the appendix. The lemma follows essentially from variants of either [17, Theorem IX.2.16, or from [25, Theorem III.4.2]. The precise variation that we need of the latter theorem is stated and proved in the Appendix.

We now immediately have the following result.

THEOREM 3.2. *Let $(T, A)$ be an accretive admissible pair on $H$ where $A = I - B$ is invertible. Then for any accretive admissible pair $(\hat{T}, \hat{B})$ for which*

$$\Delta((T, A), (\hat{T}, \hat{A})) := \int_{-\infty}^{\infty} \|T^{-1}E(x; -T^{-1})\, B - \hat{T}^{-1}E(x; -\hat{T}^{-1})\, \hat{B}\|\, dx$$

*and* $\operatorname{gap}(T, \hat{T})$ *are small enough, and for every $h \in H$ there exists a constant $C = C(h)$ such that*

$$\|[P_\pm - \hat{P}_\pm]\, h\| \leqslant C[\operatorname{gap}(T, \hat{T}) + \Delta((T, A), (\hat{T}, \hat{A}))],$$

$$\|[E(x; -T^{-1}A) - E(x; -\hat{T}^{-1}\hat{A})]\, h\| \leqslant C[\operatorname{gap}(T, \hat{T}) + \Delta((T, A), (\hat{T}, \hat{A}))],$$

*where $0 \neq x \in \mathbb{R}$.*

*Proof.* Under the hypotheses of Theorem 3.2, there exists for every $h \in H$ a constant $C'$ (depending on $h$) such that

$$\|\omega_h - \hat{\omega}_h\|_{BC_\bullet(H)} \leqslant C' \operatorname{gap}(T, \hat{T}),$$

where $\omega_h(x) = E(x; -T^{-1})\, h$ and $\hat{\omega}_h(x) = E(x; -\hat{T}^{-1})\, h$. Theorem 3.2 now follows directly from Theorem 3.1. ∎

COROLLARY 3.2. *Let $(T, A)$ be an accretive admissible pair on $H$ where $T$ and $A = I - B$ are boundedly invertible. Then for any accretive admissible* *pair* $(\hat{T}, \hat{A})$ *for which* $\|T-\hat{T}\| + \|B-\hat{B}\|$ *is small enough, there exists a constant* $C$ *such that*

$$\|P_{\pm} - \hat{P}_{\pm}\| \leqslant C[\|T-\hat{T}\| + \|B-\hat{B}\|],$$

$$\|E(x; -T^{-1}A) - E(x; -\hat{T}^{-1}\hat{A})\| \leqslant C[\|T-\hat{T}\| + \|B-\hat{B}\|],$$

*where* $0 \neq x \in \mathbb{R}$.

*Proof.* When $T$ is invertible and $\|T-\hat{T}\| < (1/\|T^{-1}\|)$, (3.11) is true and hence the right-hand sides of (3.5) satisfy

$$\|\omega - \hat{\omega}\|_{BC_{\bullet}(H)} \leqslant C' \|T-\hat{T}\|$$

for some constant $C'$. Further, if $T$ is invertible and hence also $\hat{T}$ for $\|T-\hat{T}\| < (1/\|T^{-1}\|)$, we easily derive that

$$\int_{-\infty}^{\infty} \|T^{-1}E(x; -T^{-1}) - \hat{T}^{-1}E(x; -\hat{T}^{-1})\| \, dx \leqslant C'' \|T-\hat{T}\|$$

for some constant $C''$. The corollary now follows easily by using (3.3) and (3.4). ▮

### 3.2. *Reduction to a Stability Problem of Finite Dimension*

In this section we will generalize Theorem 3.1 and Corollary 3.1 to accretive admissible pairs $(T, A)$ where $A = I - B$ is not necessarily invertible. In that case the convolution integral equation is not uniquely solvable. Thus the simple arguments used to derive the results of Subsection 3.1 can no longer be applied. Moreover, we confine ourselves to accretive admissible pairs $(T, A)$ with *bounded* $T$ to avoid perturbing the spectral projection $P_0$ of $T^{-1}A$ corresponding to a nonisolated zero eigenvalue.

Let $(T, A)$ be an accretive admissible pair in $H$ for which $A = I - B$ is not invertible and $T$ is bounded. Let $\Gamma$ be a simple positively oriented Jordan contour enclosing zero such that the rest of the spectrum of $T^{-1}A$ is contained in the exterior domain of $\Gamma$. Then there exists $\varepsilon > 0$ such that $\Gamma$ separates the spectrum of $T^{-1}\hat{A}$ for all accretive admissible pairs $(T, \hat{A})$, with $\hat{A} = I - \hat{B}$, for which $\|B-\hat{B}\| < \varepsilon$. Put

$$\hat{P}_{\Gamma} = -\frac{1}{2\pi i} \int_{\Gamma} (\hat{A} - \lambda T)^{-1} T \, d\lambda, \qquad \hat{P}_{\Gamma}^{\dagger} = -\frac{1}{2\pi i} \int_{\Gamma} T(\hat{A} - \lambda T)^{-1} \, d\lambda,$$

thus generalizing $P_0$ and $P_0^\dagger$. Then $\hat{P}_\Gamma$ and $\hat{P}_\Gamma^\dagger$ are projections of finite and equal rank such that

$$T\hat{P}_\Gamma = \hat{P}_\Gamma^\dagger T, \qquad \hat{A}\hat{P}_\Gamma = \hat{P}_\Gamma^\dagger \hat{A}, \qquad \text{and}$$
$$\hat{A}[\operatorname{Ker} \hat{P}_\Gamma] = \overline{T[\operatorname{Ker} \hat{P}_\Gamma]} = \operatorname{Ker} \hat{P}_\Gamma^\dagger;$$

see, e.g., [11, 30].

PROPOSITION 3.2. *Let $(T, A)$ be an accretive admissible pair where $T$ is bounded, and let $M_\Gamma = \max_{\lambda \in \Gamma} \|(A - \lambda T)^{-1}\|$. Then for every accretive admissible pair $(T, \hat{A})$ satisfying $\|B - \hat{B}\| < (1/M_\Gamma)$ we have*

$$\max(\|P_0 - \hat{P}_\Gamma\|, \|P_0^\dagger - \hat{P}_\Gamma^\dagger\|) \leqslant \frac{\ell(\Gamma)\, M_\Gamma^2\, \|T\|\, \|B - \hat{B}\|}{2\pi(1 - M_\Gamma \|B - \hat{B}\|)},$$

*where $\ell(\Gamma)$ is the length of $\Gamma$.*

*Proof.* The proof is immediate from the estimate

$$\|(\hat{A} - \lambda T)^{-1} - (A - \lambda T)^{-1}\| \leqslant \frac{\|(A - \lambda T)^{-1}\|^2\, \|B - \hat{B}\|}{1 - \|(A - \lambda T)^{-1}\|\, \|B - \hat{B}\|}, \qquad \lambda \in \Gamma,$$

which is valid if $\|B - \hat{B}\| < (1/M_\Gamma)$. ∎

We now reduce the stability problem to a stability problem on the finite dimensional space $\operatorname{Im} P_0$. First we choose an arbitrary operator $\hat{\beta}$ on $\operatorname{Im} \hat{P}_\Gamma$ without zero or purely imaginary eigenvalues and modify the accretive admissible pair $(T, \hat{A})$ by putting

$$\hat{A}_{\hat{\beta}} = T\hat{\beta}^{-1}\hat{P}_\Gamma + \hat{A}(I - \hat{P}_\Gamma), \qquad \hat{B}_{\hat{\beta}} = I - \hat{A}_{\hat{\beta}}.$$

Then we have the decomposition

$$\hat{A}_{\hat{\beta}}^{-1}\, T = \hat{\beta} \oplus (T^{-1}\hat{A}\,|_{\operatorname{Ker} \hat{P}_\Gamma})^{-1},$$

and hence $T^{-1}\hat{A}_{\hat{\beta}}$ does not have zero or purely imaginary eigenvalues. We then obtain

$$E(x; -T^{-1}\hat{A}) = e^{-xT^{-1}\hat{A}}\hat{P}_\Gamma + E(x, -T^{-1}\hat{A}_{\hat{\beta}})(I - \hat{P}_\Gamma),$$

which does not depend on the particular choice of $\hat{\beta}$ and where the group $e^{-xT^{-1}\hat{A}}$ is defined on a finite dimensional subspace of $H$. We then have

$$\hat{P}_+(I - \hat{P}_\Gamma) = E(0^+; -T^{-1}\hat{A})(I - \hat{P}_\Gamma),$$
$$\hat{P}_-(I - \hat{P}_\Gamma) = -E(0^-; -T^{-1}\hat{A})(I - \hat{P}_\Gamma).$$

Since $\hat{\beta}$ can be chosen in such a way that $(T, \hat{A}_{\hat{\beta}})$ is an accretive admissible pair on $H$ with $\hat{A}_{\hat{\beta}}$ invertible, Theorem 3.1 and Corollary 3.1 can be applied. As a result, we have

PROPOSITION 3.3. *Let $(T, A)$ be an accretive admissible pair on $H$ where $A = I - B$ is not invertible and $T$ is bounded. Then for any accretive admissible pair $(T, \hat{A})$ for which*

$$\Delta' = \int_{-\infty}^{\infty} \|T^{-1}E(x; -T^{-1})[B - \hat{B}]\| \, dx$$

*is small enough, there exists a constant $C$ such that*

$$\begin{cases} \|P_{\pm} - \hat{P}_{\pm}(I - \hat{P}_{\Gamma})\| \leqslant C\Delta', \\ \|E(x; -T^{-1}A)(I - P_0) - E(x; -T^{-1}\hat{A})(I - \hat{P}_{\Gamma})\| \leqslant C\Delta', \end{cases}$$

*where $0 \neq x \in \mathbb{R}$. Moreover, if $\{(T, \hat{A}) : \hat{A} \in \mathscr{A}\}$ is a uniform family of accretive admissible pairs and $\alpha$ is the constant in (3.6), then for every $c \in (0, \alpha/(2 + \alpha))$ there exist constants $\delta > 0$ and $M$ depending only on $(T, A)$ and $c$ such that*

$$\|P_{\pm} - \hat{P}_{\pm}(I - \hat{P}_{\Gamma})\| \leqslant M \, \|B - \hat{B}\|^c,$$

$$\|E(x; -T^{-1}A)(I - P_0) - E(x; -T^{-1}\hat{A})(I - \hat{P}_{\Gamma})\| \leqslant M \, \|B - \hat{B}\|^c,$$

*whenever $0 \neq x \in \mathbb{R}$, $\hat{A} \in \mathscr{A}$, and $\|B - \hat{B}\| < \delta$.*

We have now taken care of the stability issues involving the complement of the finite dimensional spaces $\operatorname{Im} \hat{P}_{\Gamma}$, where $\hat{P}_{\Gamma}$ is Lipschitz stable under perturbations of $B$ and tends to $P_0$ as $\hat{B}$ tends to $B$. We now exploit the stability of the family of subspaces $\operatorname{Im} \hat{P}_{\Gamma}$ to reduce the remaining stability issue to a stability problem on the finite-dimensional subspace $\operatorname{Im} P_0$.

If

$$\|B - \hat{B}\| < \left[ M_{\Gamma} \, \|2P_0 - I\| \left( 1 + \frac{\ell(\Gamma)}{2\pi} M_{\Gamma} \, \|T\| \right) \right]^{-1},$$

then $\|P_0 - \hat{P}_{\Gamma}\| < 1/\|2P_0 - I\|$ and hence $\|I - V_{\Gamma}\| < 1$, where

$$V_{\Gamma} = P_0 \hat{P}_{\Gamma} + (I - P_0)(I - \hat{P}_{\Gamma}).$$

Indeed, $I - V_{\Gamma} = (2P_0 - I)(P_0 - \hat{P}_{\Gamma})$. Then the invertibility of $V_{\Gamma}$ implies that

$$\operatorname{Im} P_0 \oplus \operatorname{Ker} \hat{P}_{\Gamma} = \operatorname{Ker} P_0 \oplus \operatorname{Im} \hat{P}_{\Gamma} = H.$$

We now define the families $S_\Gamma$ and $H_\Gamma$ of operators on $\operatorname{Im} P_0$:

$$\begin{aligned} S_\Gamma &= V_\Gamma (T^{-1}\hat{A}|_{\operatorname{Im} \hat{P}_\Gamma}) V_\Gamma^{-1}, \\ \langle H_\Gamma h, g\rangle &= \langle T V_\Gamma^{-1} h, V_\Gamma^{-1} g\rangle, \qquad h, g \in \operatorname{Im} P_0. \end{aligned} \tag{3.12}$$

Then $H_\Gamma$ is an invertible selfadjoint operator on $\operatorname{Im} P_0$, while

$$\langle H_\Gamma h, g\rangle = \langle T V_\Gamma^{-1} h, V_\Gamma^{-1} g\rangle, \qquad h, g \in \operatorname{Im} P_0, \tag{3.13}$$

is a nondegenerate indefinite scalar product on $\operatorname{Im} P_0$. Moreover,

$$\operatorname{Re}\langle H_\Gamma S_\Gamma h, h\rangle = \operatorname{Re}\langle \hat{A} V_\Gamma^{-1} h, V_\Gamma^{-1} h\rangle \geqslant 0, \qquad h \in \operatorname{Im} P_0.$$

Hence $iS_\Gamma$ is dissipative with respect to the scalar product (3.13). Putting $A_\Gamma = H_\Gamma S_\Gamma$, we immediately have

LEMMA 3.2. *The pair* $(H_\Gamma, A_\Gamma)$ *is an accretive admissible pair on* $\operatorname{Im} P_0$. *Moreover,* $\langle Tg, g\rangle$ *is positive (zero, negative) on* $\operatorname{Ker} \hat{A}$ *if and only if, for* $h = V_\Gamma g$, $\langle H_\Gamma h, h\rangle$ *is positive (zero, negative) on* $\operatorname{Ker} A_\Gamma$.

The following result is immediate from Theorem 2.4, Proposition 3.3, and Lemma 3.2. We leave its detailed proof to the reader.

THEOREM 3.3. *Let* $(T, A)$ *be an accretive admissible pair on* $H$ *where* $T$ *is bounded and* $A$ *is not invertible. Then the following statements are true*:

(i) *Let* $\operatorname{Ker} A$ *be* $T$*-nonnegative, i.e.,* $\langle Tg, g\rangle \geqslant 0$ *for all* $g \in \operatorname{Ker} A$. *Then there exist a unique* $(T^{-1}A)$*-invariant* $T$*-nonnegative subspace* $\mathcal{M}_+$ *given by* $\operatorname{Ker} T^{-1}A \dotplus \operatorname{Im} P_+$ *and a unique* $(T^{-1}A)$*-invariant* $T$*-nonpositive subspace* $\mathcal{M}_-$ *given by* $N_0 \dotplus \operatorname{Im} P_-$. *Moreover, for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for all accretive admissible pairs* $(T, \hat{A})$ *on* $H$ *satisfying*

$$\int_{-\infty}^{\infty} \|T^{-1} E(x; -T^{-1})[B - \hat{B}]\| \, dx < \delta \tag{3.14}$$

*there exist a unique* $(T^{-1}\hat{A})$*-invariant* $T$*-nonpositive subspace* $\hat{\mathcal{M}}_-$ *given by* $\hat{N}_0 \dotplus \operatorname{Im} \hat{P}_-$ *and a unique* $(T^{-1}\hat{A})$*-invariant* $T$*-nonnegative subspace* $\hat{\mathcal{M}}_+$ *given by* $\operatorname{Ker} T^{-1}\hat{A} \dotplus \operatorname{Im} \hat{P}_+$ *satisfying*

$$\operatorname{gap}(\mathcal{M}_+, \hat{\mathcal{M}}_+) + \operatorname{gap}(\mathcal{M}_-, \hat{\mathcal{M}}_-) < \varepsilon, \tag{3.15}$$

*where* $\hat{N}_0$ *denotes the* $T$*-isotropic part of* $\operatorname{Ker} \hat{A}$.

(ii) *Let* $\operatorname{Ker} A$ *be* $T$*-nonpositive, i.e.,* $\langle Tg, g\rangle \leqslant 0$ *for all* $g \in \operatorname{Ker} A$. *Then there exist a unique* $(T^{-1}A)$*-invariant* $T$*-nonnegative subspace* $\mathcal{M}_+$ *given by* $N_0 \dotplus \operatorname{Im} P_+$ *and a unique* $(T^{-1}A)$*-invariant* $T$*-nonpositive subspace*

$\mathcal{M}_-$ *given by* $\operatorname{Ker} T^{-1}A \dot{+} \operatorname{Im} P_-$. *Moreover, for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for all accretive admissible pairs* $(T, \hat{A})$ *on* $H$ *satisfying* (3.14) *there exist a unique* $(T^{-1}\hat{A})$*-invariant T-nonnegative subspace* $\hat{\mathcal{M}}_+$ *given by* $\hat{N}_0 \dot{+} \operatorname{Im} \hat{P}_+$ *and a unique* $(T^{-1}\hat{A})$*-invariant T-nonpositive subspace* $\hat{\mathcal{M}}_-$ *given by* $\operatorname{Ker} T^{-1}\hat{A} \dot{+} \operatorname{Im} \hat{P}_-$ *satisfying* (3.15), *where* $\hat{N}_0$ *denotes the T-isotropic part of* $\operatorname{Ker} \hat{A}$.

## 3.3. *Stability of Solutions*

In this section we combine the results of the previous sections to arrive at stability results for solutions of (1.1)–(1.2). We assume throughout the section that $(T, A)$ is an accretive admissible pair on $H$ satisfying either of the following sets of assumptions:

(i) $A$ is invertible, and $T$ may be either bounded or unbounded,

(ii) $A$ is not invertible, but $T$ is bounded.

In the second case, we shall say that $(T, A)$ satisfies the *positive* (*negative*) *numerical range condition* if $\langle Tg, g \rangle$ is nonnegative (nonpositive) for every $g \in \operatorname{Ker} A$, respectively. We say that $(T, A)$ satisfies the *numerical range condition* if $\langle Tg, g \rangle$ does not change sign for $g \in \operatorname{Ker} A$. From Lemma 3.2 it is clear that the perturbed accretive admissible pair $(T, \hat{A})$ satisfies the positive (negative) numerical range condition if and only if the pair $(H_\Gamma, A_\Gamma)$ satisfies the positive (negative) numerical range condition, and that $(T, \hat{A})$ satisfies the numerical range condition if and only if $(H_\Gamma, A_\Gamma)$ satisfies the numerical range condition.

Recall that the solution of (1.1)–(1.2) is given by

$$\psi(x) = e^{-xT^{-1}A} P_+ h + P_0 h = e^{-xT^{-1}A}(P_+ + P_0)\, h,$$

where $h \in \operatorname{Ker} A$ and $Q_+(P_+ + P_0)\, h = \varphi_+$.

Our first result focuses on existence and uniqueness of the solution. To formulate it, in analogy with the finite dimensional case, we denote by $Z_+$ the set of all closed $(T^{-1}A)$-invariant maximal $T$-nonnegative subspaces contained in $\operatorname{Im} P_+ \dot{+} \operatorname{Ker} A$ and consider the following two cases.

*Case* 1. $\operatorname{Im} P_+ \dot{+} \operatorname{Ker} A \in Z_+$, i.e., $\langle Tg, g \rangle \geqslant 0$ for all $g \in \operatorname{Ker} A$. Then the map (1.3) is one-to-one and onto. Therefore, the boundary value problem (1.1)–(1.2) has a unique solution given by (1.4), where $W_+$ is the inverse of the map (1.3).

*Case* 2. $\operatorname{Im} P_+ \dot{+} \operatorname{Ker} A \notin Z_+$. Then the map (1.3) is onto, but has a nontrivial kernel. The boundary value problem (1.1)–(1.2) has infinitely many solutions. For every fixed $\mathcal{M}_+ \in Z_+$, a solution is given by (1.4), where $W_+$ is the inverse of the map $Q_+ : \mathcal{M}_+ \to \operatorname{Im} Q_+$.

THEOREM 3.4. *Let* $(T, A)$ *be an accretive admissible pair on* $H$. *Then the boundary value problem* (1.1)–(1.2) *has at least one solution and the number of linearly independent solutions of its homogeneous counterpart equals the dimension of a maximal* $T$*-negative subspace of* Ker $A$. *Moreover, to each maximal* $T$*-nonnegative closed* $(T^{-1}A)$*-invariant subspace* $\mathscr{M}_+$ *of* $\operatorname{Ker} A \dotplus \operatorname{Im} P_+$ *there corresponds a parametrized family of solutions* $\psi$ *of the form* (1.4), *where* $W_+$ *is the inverse of the map* $Q_+ : \mathscr{M}_+ \to \operatorname{Im} Q_+$. *Thus* (1.1)–(1.2) *has a unique solution if and only if* $(T, A)$ *satisfies the positive numerical range condition.*

Next we study the stability problem. In the following stability result, we will compare solutions of (1.1)–(1.2) for the accretive admissible pair $(T, A)$ to solutions of the boundary value problem (1.5)–(1.6) (with $T$ and $Q_+$ instead of $\hat{T}$ and $\hat{Q}_+$) for the accretive admissible pair $(T, \hat{A})$. We will write hats for the corresponding quantities for the pair $(T, \hat{A})$, often without further explanation. First we deal with the comparatively easy case when $A$ is invertible.

LEMMA 3.3 *Let* $(T, A)$ *be an accretive admissible pair on* $H$ *where* $A$ *is invertible. Then for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$\sup_{0 \leqslant x < +\infty} \| e^{-xT^{-1}A} W_+ \varphi_+ - e^{-xT^{-1}\hat{A}} \hat{W}_+ \hat{\varphi}_+ \| < \varepsilon$$

*whenever* $(T, \hat{A})$ *is an accretive admissible pair on* $H$ *and* $\hat{\varphi}_+$ *is an initial vector satisfying*

$$\| \varphi_+ - \hat{\varphi}_+ \| + \int_{-\infty}^{\infty} \| T^{-1} E(x; -T^{-1}) [B - \hat{B}] \| \, dx < \delta.$$

*Proof.* Theorem 3.1 gives perturbation results on the bisemigroup $E(x; -T^{-1}A)$ and the spectral projection $P_+$. Further, we have

$$W_+ = [Q_+ P_+ + Q_- P_-]^{-1} Q_+, \qquad \hat{W}_+ = [Q_+ \hat{P}_+ + Q_- \hat{P}_-]^{-1} Q_+,$$

from which the lemma follows. ∎

The following result is easily obtained using Theorem 1.3, Proposition 3.3, and Lemmas 3.2 and 3.3. We leave its proof to the reader.

THEOREM 3.5. *Let* $(T, A)$ *be an accretive admissible pair on* $H$.

(i) *Assume that* Ker $A$ *is* $T$*-nonnegative. Then for every* $\varepsilon > 0$ *and* $x_0 > 0$ *there exists* $\delta > 0$ *such that*

$$\sup_{0 \leqslant x \leqslant x_0} \| e^{-xT^{-1}A} W_+ \varphi_+ - e^{-xT^{-1}\hat{A}} \hat{W}_+ \hat{\varphi}_+ \| < \varepsilon$$

*whenever* $(T, \hat{A})$ *is an accretive admissible pair and* $\hat{\varphi}_+$ *is an initial vector satisfying*

$$\| \varphi_+ - \hat{\varphi}_+ \| + \| A - \hat{A} \| < \delta.$$

*Here* $W_+ \varphi_+$ *and* $\hat{W}_+ \hat{\varphi}_+$ *are the values in* $x = 0$ *of the unique solutions of the boundary value problems* (1.1)–(1.2) *and* (1.5)–(1.6).

(ii) *Assume that* Ker $A$ *is* $T$*-nonpositive. Then for every* $\varepsilon > 0$, $x_0 > 0$, *and maximal* $T$*-nonnegative subspace* $\mathscr{M}_+$ *of* Ker $A$ *there exist* $\delta > 0$ *and a maximal* $T$*-nonnegative subspace* $\hat{\mathscr{M}}_+$ *of* Ker $\hat{A}$ *such that*

$$\operatorname{gap}(\operatorname{Im} P_+ \dot{+} \mathscr{M}_+, \operatorname{Im} \hat{P}_+ \dot{+} \hat{\mathscr{M}}_+) + \sup_{0 \leqslant x \leqslant x_0} \| e^{-xT^{-1}A} W_+ \varphi_+ - e^{-xT^{-1}\hat{A}} \hat{W}_+ \hat{\varphi}_+ \| < \varepsilon$$

*whenever* $(T, \hat{A})$ *is an accretive admissible pair and* $\hat{\varphi}_+$ *is an initial vector satisfying*

$$\| \varphi_+ - \hat{\varphi}_+ \| + \| A - \hat{A} \| < \delta.$$

*Here* $W_+ \varphi_+ \in \operatorname{Im} P_+ \dot{+} \operatorname{Ker} A$ *and* $\hat{W}_+ \hat{\varphi}_+ \in \operatorname{Im} \hat{P}_+ \dot{+} \operatorname{Ker} \hat{A}$ *are the values in* $x = 0$ *of the unique solutions of the boundary value problems* (1.1)–(1.2) *and* (1.5)–(1.6) *with* $W_+ \varphi_+ \in \mathscr{M}_+$, $\hat{W}_+ \hat{\varphi}_+ \in \hat{\mathscr{M}}_+$.

We remark that in the case $T$ is bounded we can replace $\sup_{0 \leqslant x \leqslant x_0}$ in both parts of the theorem by $\sup_{0 \leqslant x < \infty}$. This is based on Lemma 3.3 and the remark following Theorem 1.3. The case where $A$ is non-invertible only involves technicalities which are more difficult than for the case where $A$ is invertible, and requires $T$ to be bounded. We leave the details to the interested reader.

## APPENDIX A

In what follows, we denote by $R(\lambda : A) = (\lambda I - A)^{-1}$ the resolvent of an operator $A$.

THEOREM A.1. *Let* $A, A_n$ *be the generators of the strongly continuous contraction semigroups* $T(t)$ *and* $T_n(t)$ *of linear operators on a Banach space*

*$X$, respectively. Assume that $\int_0^\infty \|T(t)\, z\|\, dt$ is finite for every $z \in D(A)$. Then the following are equivalent:*

(a) *For every $x \in X$ and for a fixed $\lambda \in \mathbb{C}$ with* $\operatorname{Re} \lambda > 0$, *$R(\lambda : A_n)\, x \to R(\lambda : A)\, x$ as $n \to \infty$.*

(b) *For every $x \in X$ and $t \geqslant 0$, $T_n(t)\, x \to T(t)\, x$ as $n \to \infty$.*

*Moreover, the convergence in Part* (b) *is uniform in $t \in [0, +\infty)$.*

*Proof.* We start by showing that (a) $\Rightarrow$ (b). Fix $x \in X$ and consider

$$\begin{aligned} \|(T_n(t) - T(t))\, R(\lambda : A)\, x\| &\leqslant \|T_n(t)(R(\lambda : A) - R(\lambda : A_n))\, x\| \\ &\quad + \|R(\lambda : A_n)(T_n(t) - T(t))\, x\| \\ &\quad + \|(R(\lambda : A_n) - R(\lambda : A))\, T(t)\, x\| \\ &= D_1 + D_2 + D_3. \end{aligned} \tag{A.1}$$

Since $\|T_n(t)\| \leqslant 1$ for $t \geqslant 0$ it follows from (a) that $D_1 \to 0$ as $n \to \infty$ uniformly on $[0, +\infty)$.

We consider next $D_3$. First observe that the contractivity and strong integrability of $T(t)$ imply

$$\lim_{t \to \infty} \|T(t)\, x\| = 0$$

for every $x \in X$. Since the function $s \to T(s/(1-s))\, x$ extends to a continuous function from $[0, 1]$ into $X$, the set $\{T(t)\, x : 0 \leqslant t < \infty\}$ is relatively compact in $X$. Since the strong convergence as described in (a) holds uniformly on (relatively) compact subsets of $X$, it follows that $D_3 \to 0$ as $n \to \infty$ uniformly on $[0, \infty)$.

Finally, using Lemma III.4.1 of [25] with $B = A_n$ we have

$$\begin{aligned} &\|R(\lambda : A_n)[T_n(t) - T(t)]\, R(\lambda : A)\, x\| \\ &\quad \leqslant \int_0^t \|T_n(t-s)\|\, \|[R(\lambda : A) - R(\lambda : A_n)]\, T(s)\, x\|\, ds \\ &\quad \leqslant \int_0^\infty \|[R(\lambda : A) - R(\lambda : A_n)]\, T(s)\, x\|\, ds. \end{aligned} \tag{A.2}$$

The integrand in the right-hand side of (A.2) is bounded by $2\, \|x\|/(\operatorname{Re} \lambda)$ and it tends to zero as $n \to \infty$. By Lebesgue's dominated convergence theorem, and assuming in addition that $x \in D(A)$, the right-hand side tends to zero as $n \to \infty$ and therefore

$$\lim_{n \to \infty} \|R(\lambda : A_n)[T_n(t) - T(t)]\, R(\lambda : A)\, x\| = 0, \tag{A.3}$$

and the limit in (A.3) is uniform on $[0, +\infty)$. Since every $x \in D(A^2)$ can be written as $x = R(\lambda : A)\, y$ for some $y \in D(A)$ it follows from (A.3) that for $x \in D(A^2)$, $D_2 \to 0$ as $n \to \infty$ uniformly on $[0, +\infty)$. From (A.1) we then deduce that for $x \in D(A^3)$

$$\lim_{n \to \infty} \|(T_n(t) - T(t))\, x\| = 0 \tag{A.4}$$

and the limit in (A.4) is uniform on $[0, +\infty)$. Since $\|T_n(t) - T(t)\|$ are uniformly bounded on $[0, +\infty)$ and since $D(A^3)$ is dense in $X$, it follows that (A.4) holds for every $x \in X$ uniformly on $[0, +\infty)$ and that (a) $\Rightarrow$ (b).

Assume now that (b) holds and $\operatorname{Re} \lambda > 0$. Then

$$\|R(\lambda : A_n)\, x - R(\lambda : A)\, x\| \leqslant \int_0^\infty e^{-(\operatorname{Re} \lambda)\, t}\, \|(T_n(t) - T(t))\, x\|\, dt. \tag{A.5}$$

The right-hand side of (A.5) vanishes as $n \to \infty$ by (b) and Lebesgue's dominated convergence theorem and therefore (b) $\Rightarrow$ (a).

Note that the implication (a) $\Rightarrow$ (b) holds for every $x \in X$ for which $T(\cdot)\, x$ is integrable on $[0, \infty)$. Notice also that the equivalence of parts (a) and (b) holds for subsets $F$ of vectors $x \in D(A)$ that are orbit closed in the sense that $F = \{T(t)\, x \colon x \in F, t \geqslant 0\} \subset D(A)$.

*Proof of Lemma* 3.1. We first show that the conditions of the theorem above are satisfied in our case. As $T = T^*$ and $T_n = T_n^*$ the semigroups $E(x, -T_n^{-1})\, Q_{n,+}$ (for $x > 0$) and $E(-x, T_n^{-1})\, Q_{n,-}$ (for $x < 0$) as well as $E(x, -T^{-1})\, Q_+$ (for $x > 0$) and $E(-x, T^{-1})\, Q_-$ (for $x < 0$) are contraction semigroups.

To get the integrability condition, when $T$ is bounded $\|E(x; -T^{-1})\|$ is exponentially decreasing as $x \to \pm\infty$ and in that case Lemma 3.1 follows directly from Theorem A.1. When $T$ is unbounded, the vector $x$ in the proof of the transition (a) $\Rightarrow$ (b) of the theorem above should be taken in the range of $\sigma(E)$, where $\sigma(\cdot)$ is the resolution of the identity of $T$ and $E$ is bounded. Note that the set of such $x$ is orbit closed. Under these conditions $T(s)\, x$ (still in the notation of [25]) is exponentially decreasing and the theorem of dominated convergence can be applied to prove (A.3) above, which implies (A.4) uniformly on $[0, +\infty)$. Moreover, for such $x$ we can prove as above that $D_3 \to 0$ uniformly on $[0, \infty)$ as $n \to \infty$. When

$T$ is boundedly invertible, the bisemigroups involved allow a Cauchy integral representation over a contour enclosing the spectrum of $T$ and $T_n$ (for $n$ large enough) in either the right or the left halfplane. In that case, (3.11) is immediate.

Finally, condition (a) of the theorem above is satisfied under the assumptions of Lemma 3.1. Indeed, the graph of $T$ (respectively, of $T_n$) is the same as the graph of the resolvent of $T$ (respectively, of $T_n$). Thus convergence in the generalized sense implies convergence of the resolvents. So we can apply Theorem A.1 to prove Lemma 3.1.

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